Semi-topological theories and algebraic equivalence

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Harnessing Motivic Invariants (Essen)

Semi-topological theories

$Z_r(X)$, the group completion of the Chow monoid $\mathcal{C}_r(X) = \coprod_e C_{r,e}(X)$ of effective r-**cycles** on X.
Lawson homology: $L_r H_n(X) \ = \ \pi_{n-2r}(Z_r(X)^{an})$

$Mor(X, (\mathcal{C}_0(\mathbb{P}^s))$, the monoid of effective s-**cocycles** on X.
Morphic cohomology: $L^s H^m(X) \ = \ \pi_{2s-m}(Mor(X,(\mathcal{C}_0(\mathbb{P}^s))^{an})^+/(Mor(X,(\mathcal{C}_0(\mathbb{P}^s))^{an})^+))$.

Semi-topological K-theory:
$K_i^{semi}(X) \ = \ \pi_i(Mor(X, Grass^+)^{an})$.

Motivation

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- $K^{semi}(X)$ is a geometric invariant, ignoring arithmetic aspects of X.
- Semi-topolgoical theories interpolate well between motivic and analytic theories.
- Cycle spaces are "small" models, colimits of projective varieties with their analytic topology.
- Enable integral formulations of classical conjectures about algebraic cycles.
- Technology of Lawson suspension, $(-)^{sst}$, operations might be useful in other contexts.

Algebraic equivalence

Definition

Two r-cycles γ_1, γ_2 on X are algebraically equivalent if there exists a smooth projective curve C with points c_1, c_2 and an $r+1$ cycle Γ on $X \times C$ such that γ_i is the specialization of Γ at c_i.

$\pi_0(Z_r(X)^{an}) \;=\; \pi_0(Z_r(X))$ equals $(r - cycles)/\sim_{alg}$.

Hence, if $d = dim(X)$, then $L_{d-1}H_{2d-2}(X) = NS(X)$.

Cycle map: $Z_r(X) \;\to\; CH_r(X) \to\; \pi_0(Z_r(X)) \;\to\; H_{2r}(X^{an})$.

First values

Lawson: $L_r H_n(\mathbb{P}^N) \; = \; H_n((\mathbb{P}^N)^{an}), \;\; 0 \leq r \leq N/2.$

Dold-Thom: $L_0 H_*(X^{an}) \; \equiv \; \pi_*(Z_0(X)^{an}) \; = \; H_*(X^{an})$ if X projective; otherwise, use Borel-Moore homology of X^{an}.

$$L_{d-1} H_{2d-1}(X) \; \equiv \; \pi_1(Z_{d-1}(X)^{an}) \; = \; H_{2d-1}(X^{an}, \mathbb{Z});$$
$$L_{d-1} H_{2d-2+i}(X^{an}) \; \equiv \; \pi_i(Z_{d-1}(X)^{an}) \; = \; 0, \quad i > 2.$$

$K_0^{semi}(X)$ is given by (virtual) algebraic vector bundles modulo algebraic equivalence.

Consequences of Lawson Moving Construction

Key thorem:

Lawson suspension: $\Sigma : Z_r(X)^{an} \to Z_{r+1}(\Sigma X)^{an}$ is a homotopy equivalence; proof uses **Lawson's moving construction** for algebraic cycles on $\Sigma X \equiv *\#X$.

algebraic join: $X \subset \mathbb{P}^M, \; Y \subset \mathbb{P}^n \;\Rightarrow\; X\#Y \subset \mathbb{P}^{n+m+1}$.

F-Gabber intersection product: For U quasi-projective,smooth, $\exists$ continuous pairing $Z_r(U)^{an} \times Z_s(U)^{an} \to Z_{r+s-d}(U)^{an}$.

$Z_r(U)^{an} \equiv (Z_r(X) \backslash Z_r(X_\infty))^{an}$, the "semi-topological motive" of U.

F- Lawson Duality $L^sH^m(X) \;\to\; L_{d-s}H_{2d-m}(X)$ is an isomorphism for X smooth, compatible with Poincaré duality.

F-Walker technology

Using the left Kan extension for $\epsilon : (CW) \to (Var/\mathbb{C})$, a presheaf F on $Var/\mathbb{C}$ determines presheaf
$$F^{sst} : X \mapsto F(X \times \Delta^{\bullet}_{top})$$

$\mathcal{K}^{sst}(X) \ \equiv \ \mathcal{K}(X \times \Delta^{\bullet}_{top}) \stackrel{\approx}{\to} K^{semi}(X)$ for X projective.

$H^{2q-*}_{M}(X \times \Delta^{\bullet}_{top}, \mathbb{Z}(q)) \ \stackrel{\sim}{\to} \ L^q H^{2q-*}(X)$ for X smooth, quasi-projective.

$\pi_{*-2r}(z_{equi}(X,r)(\Delta^{\bullet}_{top})) \ \stackrel{\sim}{\to} \ L_r H_*(X)$ for X quasi-projective.

F -Walker: If $F \to G$ induces iso of sheafifications $F_h \to G_h$, then $F(\Delta^{\bullet}_{top}) \to G(\Delta^{\bullet}_{top})$ is a weak equivalence.

F-Walker interpolation

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F-Walker establish natural (w.r.t X) maps of (topological) spectra

$$\mathcal{K}(X) \ \to \ \mathcal{K}^{sst}(X) \ \to \ kU(X^{an})$$

mapping via total Segre class to triple of spectra for motivic cohomology, morphic cohomology, and singular cohomology.

For X smoooth, quasi-projective $\mathcal{K}(X, \mathbb{Z}/n) \ \stackrel{\sim}{\to} \ \mathcal{K}^{sst}(X, \mathbb{Z}/n)$, using Suslin rigidity.

$\mathcal{K}^{sst}(X)[\beta^{-1}] \ \stackrel{\sim}{\to} \ KU(X^{an}) = kU(X^{an})[\beta^{-1}]$, where β is the Bott element $\beta \in Mor(\mathbb{P}^1, Grass^1(\mathbb{P}^\infty))$.

Comparison Theorems

For X a smooth, quasi-projective variety, F-Haesemeyer-Walker construct **maps** between **Atiyah-Hirzeburch spectral sequences**

$$E_2^{p,q}(alg) = H_{\mathcal{M}}^{p-q}(X; A(-q)) \Rightarrow K^{alg}_{-p-q}(X; A)$$

$$E_2^{p,q}(sst) = L^{-q}H^{p-q}(X; A) \Rightarrow K^{sst}_{-p-q}(X; A)$$

$$E_2^{p,q}(top) = H^{p-q}(X; A) \Rightarrow ku^{p+q}(X^{an}; A).$$

For $A = \mathbb{Z}/n$, the map between the top and middle spectral sequences is an isomorphism.

After localizing w.r.t. the Bott element, the map between the middle and bottom spectral sequences is an isomorphism.

For $A = \mathbb{Q}$, each spectral sequence degenerates with degeneration given by the Chern character.

Computations

F-Hasemeyer-Walker verify that $\mathcal{K}^{sst}(X) \to \mathcal{K}^{top}(X)$ is a weak equivalence for various X:

a.) A smooth projective curve.
b.) A smooth projective surface S with $H^2(S^{an})$ is algebraic.
c.) A smooth, projective, rational 3-fold.
d.) A smooth, projective, linear variety.
e.) A projective smooth toric fibration over a.), b.), c), or d.)
f.) A smooth, projective, rational 4-fold (on connected components).

For quasi-projective versions of a.), b.), c.), d.), e.) F-H-W refine the Deligne, Gillet-Soulé weight filtration to prove
$\pi_i(\mathcal{K}^{sst}(X)) \;\simeq\; \pi_i(\mathcal{K}^{top}(X^{an})), \quad i \geq dim(X) - 1.$

F-Mazur operations: h, s

Join pairing:
$Z_r(X)^{an} \wedge Z_0(\mathbb{P}^1)^{an} \;\to\; Z_{r+1}(\Sigma^2 X)^{an} \simeq Z_{r-1}(X)^{an}$
Consider maps on homotopy groups.

$h : L_r H_n(X) \to L_{r-1} H_{n-2}(X)$, induced by intersection with a hyperplane section of $\Sigma^2 X$.

$s : L_r H_n(X) \;\to\; L_{r-1} H_n(X)$, induced by intersection with a $\mathbb{P}^1$-family hyperplane sections.

Factoring the cycle map: $s^{\circ r} \circ \pi_0 : Z_r(X) \to \pi_0(Z_r(X))$
$\pi_2(Z_{r-1}(X)^{an}) \to \cdots \to \pi_{2r}(Z_0(X)^{an}) = H_{2r}(X^{an})$.

Semi-top filtration: $T_i H_n(X^{an})$

This is an decreasing filtration introduced on the homology of a projective variety.

$$T_i H_n(X^{an}) \equiv im\{s^{\circ i} : L_i H_n(X) \to L_0 H_n(X) = H_n(X^{an})\}.$$

F-Mazur show that $T_i H_n(X^{an}) \subset H_n(X^{an})$ is generated by images of classes in $H_{n-2i}(Y^{an})$ via **algebraic correspondences** $\Gamma \subset Y \times X$ of relative dimension i.

Conjecturally equal to Grothendieck's niveau filtration $\{G_i H_n(X^{an})\}$, where $G_i H_n(X^) \supset T_i H_n(X^{an})$ is generated by images $H_n(W^{an}) \to H_n(X^{an})$ with $dim(W) \leq n - i$.

Heuristic description of classes generating $T_i H_n(X^{an})$: cycles with i-algebraic parameters and $n - 2i$ topological parameters.

Conjectures

Integral formulations of **classical conjectures** on cycles:

Grothendieck's Standard Conjectures imply that $\{T_iH_n(X^{an})\}$ equals $\{G_iH_n(X)\}$.

Generalized Hodge Conjecture. Lawson homology groups have a natural mixed Hodge structure preserved by the s-map. Does $\{T_iH_n(X^{an}, \mathbb{Q})\}$ equal Grothendieck's "corrected" Hodge filtration on homology? YES, for many of the examples for which the G.H.C. has been proved.

Beilinson-Soulè Vanishing Conjecture is equivalent (thanks to Bloch-Kato) to $L^{d-r}H^m(X) = 0, \quad m < 0.$

Weaker Conjecture: $L_rH_n(X) = 0$ for $n >> 0$.

Suslin's Conjecture

Suslin's Conjecture: Let X be a smooth, quasi-projective variety. Then

$$L^q H^m(X) \ \to \ H^m(X^{an}; \mathbb{Z})$$

iso for $m \leq q$ and mono for $m = q + 1$.

This implies that $L_r H_{2r+i}(X) \ \equiv \ \pi_i(Z_r(X)^{an})$ should be an interesting new invariant only for $i < d - r$.

Suslin Conjecture plus Quillen-Lichtenbaum Conjecture implies Semi-topological Quillen-Lichtenbaum Conjecture: $K_n^{sst}(X) \to K^n_{top}(X^{an})$ iso for $n \geq d - 1$, mono for $n = d - 2$.

For X projective, smooth, Beilinson proved surjectivity of the map of Suslin's Conjecture is equivalent to Grothendieck's Standard Conjectures.

Sheaf-theoretic Suslin's Conjecture

F-Haesemayer-Walker: $tr_{\leq q}\mathbb{Z}(q)^{sst} \cong \mathbb{Z}(q)^{sst}$.

This enables Suslin's Conjecture to be formulated as an integral form of the Beilinson-Lichtenbaum conjecture:

Suslin's Conjecture is the equivalent to the assertion that

$$\mathbb{Z}(q)^{sst} \;\to\; tr_{\leq q}\mathbb{R}\epsilon_*\mathbb{Z}$$

is a quasi-isomorphism of complexes of Zariski sheaves on X.

Such a sheaf formulation was useful in proving Bloch-Kato, since often easier to prove map is iso.

S-filtration: $S^j Z_r(X)$; Griffiths group

This is an increasing filtration introduced on algebraic r-cycles on a quasi-projective variety.

$$S_j Z_r(X) \equiv ker\{s^{\circ j} \circ \pi_0 : Z_r(X) \to L_j H_{2r}(X)\}.$$

Thus, $S_r Z_r(X)/S_0 Z_r(X)$ is the mysterious **Griffiths group**.

Nori's constructions using variations of Mixed Hodge structures enables construction of r-cycles γ on suitable varieties X such that γ lies in $S_j(X) \backslash S_{j-1}(X)$.

If r-cycle on X lies in $S_j(X) \backslash S_{j-1}(X)$, then

$$s^{j-1}(\zeta) \in ker\{L_{j-1} H^{2r}(X) \to H_{2r}(X^{an}\}.$$

This gives data for the range of **interesting geometric invariants** allowed by Suslin's Conjecture.

?? infinitely divisible elements ??

QUESTION: Are there infinitely divisible (by all primes p) elements of Lawson homology groups $L_r H_n(X)$?

These groups are **countable**. So far, no counter-examples and some intuition.

Desired properties over other fields F

1. For $F = \mathbb{C}$ or $\mathbb{R}$, then agrees with earlier constructions.
2. Functorial on (Sm/F), admits base change for $F \to F'$.
3. $\pi_*(\mathcal{K}^{sst}(X), \mathbb{Z}/n) \ = \ K_i(X, \mathbb{Z}/n)$.
4. $F \to F'$ extension of algebraically closed fields, then $\mathcal{K}^{sst}(X) \to \mathcal{K}^{sst}(X_{F'})$ is an equivalence.
5. $\pi_0(\mathcal{K}^{sst}(X))$ is the Grothendieck group of vector bundles on X modulo algebraic equivalence.

Uncertain properties:

- Semi-topological homology has no infinitely divisible elements.
- Behavior with respect to colimits of base fields?

Non-archimedean base field

Fanciful idea. Use rigid analytic spaces to define semi-topological theories for varieties over a non-archimedan base field (e.g., $\mathbb{Q}_p$).

For a Noetherian K-scheme X over the non-archimedan field F, use the left Kan extension associated to (rigid analytic spaces/F) $\to$ (Sch/F) to define

$$\mathcal{K}^{sst}(X) \equiv Hom(X \times \mathbb{B}^\bullet, \underline{Grass})$$

where $n \mapsto \mathbb{B}^n$ is the cosimplical rigid analytic space consisting of unit balls and where $\underline{Grass}$ represents some construction involving Grassmannians of projective spaces.

We might obtain maps of spectra

$$\mathcal{K}(X) \to \mathcal{K}^{sst}(X) \to \mathcal{K}^{rigid}(X^{rigid})$$

with reasonable properties.