HOMOTOPY INVARIANCE AND GERSTEN COMPLEXES OVER BASE SCHEMES

1. Introduction

Work in progress, joint with A. Druzhinin and P.A Østvær.

From Morel, Suslin and Voevodsky's work in motivic homotopy theory we know that if $\mathcal{F}$ is a homotopy invariant presheaf with transfers, then the associated Gersten complex is exact:

$$0 \to \mathcal{F}(X) \to \mathcal{F}(k(X)) \to \bigoplus_{x \in X^{(1)}} \mathcal{F}_{-1}(k(x)) \to \bigoplus_{x \in X^{(2)}} \mathcal{F}_{-2}(k(x)) \to \cdots .$$

for any essentially smooth local scheme X over a perfect field k. This follows from Voevodsky's analysis of the cohomological properties of homotopy invariant presheaves with transfers, in particular his *strict homotopy invariance theorem.*

This is also true if $\mathcal{F}$ has framed transfers. Inserting K-theory for $\mathcal{F}$ we then get Gersten's conjecture in the geometric case, originally proved by Quillen, Panin.

More general bases has later been studied by e.g.:

- Schmidt–Strunk: conditional exactness over Dedekind schemes with infinite residue fields.
- Deshmukh–Kulkarni–Yadav: exactness over J-2 Noetherian irreducible bases of finite type (that S is J-2 means that for any X over S, the regular locus of X is open).

In this talk we aim to explore a little bit about what happens with properties like strict homotopy invariance and exactness of the Gersten complex over more general base schemes. We will see that:

(1) SHI is false over positive dimensional base schemes, and look at Ayoub's counterexample. Then we'll have a look on a modified version of SHI which goes through.
(2) The Gersten complex, or Cousin complex, associated with a cohomology theory represented in $\mathbf{SH}(B)$, is not exact. However, we will see that the complex with support on fibers is indeed exact.

2. Strict homotopy invariance

Homotopy invariant presheaves $\mathcal{F}$ on smooth varieties over a field k equipped with some sort of *transfer structure* were shown by Voevodsky to satisfy several remarkable properties.

By a *transfer structure* we mean that $\mathcal{F}$ should have pullback maps along correspondences α of the form

$$X \xleftarrow{\ f\ } Z \xrightarrow{\ g\ } Y$$

with some conditions on f. For instance f can be required to be:

- Finite and surjective over a component of X, which yields Suslin and Voevodsky's category of finite correspondences Cor_k.

- Framed correspondences: Have f finite, together with the data of a framing of Z, i.e., embedding of Z in $\mathbf{A}^n_X$ and at étale neighborhood U of Z in $\mathbf{A}^n_X$:

$$\begin{array}{ccc} U & \longrightarrow & \mathbf{A}^n_X \\ & \searrow\!\!\!\!\nearrow_{\overline{g}} & \\ Z & & Y \end{array}$$

such that $Z = \phi^{-1}(0)$ for a $\phi\colon U \to \mathbf{A}^n$.

Example 2.1. There is a level 1 framed correspondence σ from X to X given by $Z = \{0\}$, $U = \mathbf{A}^1_X$ and taking $\overline{g}$ as the projection $\mathbf{A}^1_X \to X$.

Theorem 2.2 (Voevodsky's strict homotopy invariance theorem)**.** *Let k be a perfect field. If $\mathcal{F}\colon \mathrm{Cor}_k^{\mathrm{op}} \to \mathrm{Ab}$ is a homotopy invariant PST, then so are the associated Nisnevich cohomology presheaves $H^i_{\mathrm{nis}}(-, \mathcal{F}_{\mathrm{nis}})$ for all $i \geq 0$.*

Some consequences of this result, and the ingredients of its proof, include:

(a) We get formula for motivic localization of presheaves with transfers: for

$$\begin{array}{ccc} \mathrm{PSh}_\Sigma(\mathrm{Cor}_k) & \xrightarrow{L_{\mathrm{nis}}} & L_{\mathrm{nis}}\mathrm{PSh}_\Sigma(\mathrm{Cor}_k) \\ {\scriptstyle L_{\mathbf{A}^1}}\downarrow & \searrow^{L_{\mathrm{mot}}} & \downarrow \\ L_{\mathbf{A}^1}\mathrm{PSh}_\Sigma(\mathrm{Cor}_k) & \longrightarrow & \mathbf{DM}^{\mathrm{eff}}(k) \end{array}$$

we have $L_{\mathrm{mot}} = L_{\mathrm{nis}} L_{\mathbf{A}^1}\colon \mathrm{PSh}_\Sigma(\mathrm{Cor}_k) \circlearrowleft$ (if we instead consider these localization functors as endofunctors).

(b) $L_{\mathbf{A}^1}\mathcal{F}$ has the explicit description $C^{\mathrm{Sus}}_*(\mathcal{F}) = (\cdots \to \mathcal{F}(- \times \Delta^n) \to \mathcal{F}(- \times \Delta^{n-1}) \to \cdots)$.

(c) If U is a henselian local scheme, then

$$H^i_{\mathrm{nis}}(\mathbf{A}^1_U, \mathcal{F}_{\mathrm{nis}}) \cong \begin{cases} \mathcal{F}_{\mathrm{nis}}(U) & i = 0 \\ 0 & i > 0. \end{cases}$$

2.1. **Other base schemes.** A natural question is to extend this theory to other base schemes. However, Ayoub showed that the results of SHI and cancellation are no longer true if the base scheme B has positive dimension.

Example 2.3 (Ayoub in case of surfaces)**.** Suppose that $\dim B \geq 1$ and let's see what happens with SHI. We will show that property (c) above fails, i.e., that there is a homotopy invariant PST over B with $H^1_{\mathrm{nis}}(\mathbf{A}^1_U, \mathcal{F}_{\mathrm{nis}}) \neq 0$.

So suppose that B is a henselian local positive dimensional scheme. Let $f \in \mathcal{O}(B)$ such that $\varnothing \neq Z(f) \subsetneq B$, and let $r = ft - 1 \in \mathcal{O}(\mathbf{A}^1_B) = \mathcal{O}(B)[t]$. Consider the following Zariski covering of $\mathbf{A}^1_B$:

$$\begin{array}{ccc} W = \mathbf{A}^1 \setminus Z(fr) & \longrightarrow & \mathbf{A}^1_B \setminus Z(r) \\ {\scriptstyle j}\downarrow & & \downarrow \\ \mathbf{A}^1_B \setminus Z(f) & \longrightarrow & \mathbf{A}^1_B. \end{array}$$

Let $\mathcal{F} = \overline{\mathrm{Cor}}_B(-, W) = \mathrm{coker}\left(\mathrm{Cor}_B(- \times \mathbf{A}^1, W) \xrightarrow{i_0^* - i_1^*} \mathrm{Cor}_B(-, W)\right)$.

Then we have the Mayer–Vietoris sequence

$$\cdots \to H^0(\mathbf{A}^1_B \setminus Z(f), \mathcal{F}) \oplus H^0(\mathbf{A}^1_B \setminus Z(r), \mathcal{F}) \to H^0(W, \mathcal{F}) \xrightarrow{\delta} H^1(\mathbf{A}^1_B, \mathcal{F}) \to \cdots$$

We have $\mathrm{id} \in H^0(W, \mathcal{F}) = \overline{\mathrm{Cor}}_B(W, W)$ and claim $\delta(\mathrm{id}) \neq 0$ in $H^1(\mathbf{A}^1_B, \mathcal{F})$.

Suppose $\delta(\mathrm{id}) = 0$. Then there is

$$\xi \in H^0(\mathbf{A}^1_B \setminus Z(f), \mathcal{F}) \oplus H^0(\mathbf{A}^1_B \setminus Z(r), \mathcal{F})$$

mapping to id.

Exercise 2.4. $\mathrm{Cor}_B(\mathbf{A}^1_B \setminus Z(r), W) = 0$.

Hence

$$\xi \in H^0(\mathbf{A}^1_B \setminus Z(f), \mathcal{F}).$$

In other words, there is a correspondence

$$\xi \in \mathrm{Cor}_B(\mathbf{A}^1_B \setminus Z(f), W)$$

such that $\xi \circ j \sim_{\mathbf{A}^1} \mathrm{id}_W$.

$$\begin{array}{ccc} W & \overset{j}{\hookrightarrow} & \mathbf{A}^1_B \setminus Z(f) \\ & \searrow & \downarrow \xi \\ & & W \end{array}$$

But then, for any homotopy invariant PST $\mathcal{E}$ over B we have $j^* \circ \xi^* = \mathrm{id}\colon \mathcal{E}(W) \to \mathcal{E}(W)$, i.e., j^* is surjective. But this fails for instance for $\mathcal{E} = \mathcal{O}^*$.

3. The tf-topology

Quasi-theorem (DKØ): "The SHI theorem holds if we discard the examples of the type above".

Definition 3.1. Let B be a base scheme. The tf-topology (trivial fiber topology) on Sm_B is the subtopology of the Nisnevich topology on Sm_B generated by Nisnevich squares of the form

$$\begin{array}{ccc} X' \times_B (B \setminus Z) & \longrightarrow & X' \\ \downarrow & & \downarrow p \\ X \times_B (B \setminus Z) & \longrightarrow & X \end{array}$$

where

- p is étale and affine
- Z is closed in B.

Example 3.2.

- The tf-topology is the trivial topology if $B = \mathrm{Spec}(k)$, where by trivial topology we mean $H^{\geq 1}_{tf} = 0$.
- $\mathbf{A}^1_B = (\mathbf{A}^1_B \setminus Z(f)) \cup (\mathbf{A}^1_B \setminus Z(ft-1))$ is a tf-covering, while $\mathbf{A}^1_B = (\mathbf{A}^1_B \setminus 0) \cup (\mathbf{A}^1_B \setminus 1)$ is not.

Definition 3.3. Let $\mathcal{F}$ be a PST over B and let τ be a topology on Sm_B. We say that $\mathcal{F}$ is τ-strictly homotopy invariant if the τ-cohomology presheaves $H^i_\tau(-, \mathcal{F}_\tau)$ are homotopy invariant for all $i \geq 0$.

Theorem 3.4 (DKØ)**.** *Let $\mathcal{F}$ be a PST over B. Then $\mathcal{F}$ is tf-SHI $\implies$ $\mathcal{F}$ is Nis-SHI.*

Remark 3.5. We can say that SHI over k is the statement that τ-SHI $\implies$ Nis-SHI, where τ is the trivial topology. The natural extension to positive dimensional bases is $\tau = \mathrm{tf}$.

Sketch of proof. We need to show that the localization

$$L^{\mathrm{tf}}_{\mathrm{nis}} \colon \mathbf{SH}^{\mathrm{fr}}_{\mathrm{tf}}(B) \to \mathbf{SH}^{\mathrm{fr}}_{\mathrm{nis}}(B)$$

preserves $\mathbf{A}^1$-local objects. We prove this via a localization theorem: For

$$\mathbf{SH}^{\mathrm{fr}}_{\mathrm{tf}}(Z) \rightleftarrows \mathbf{SH}^{\mathrm{fr}}_{\mathrm{tf}}(B) \rightleftarrows \mathbf{SH}^{\mathrm{fr}}_{\mathrm{tf}}(B \setminus Z)$$

we show that SHI for the outer ones implies SHI for the middle.

For $\mathbf{SH}^{\mathrm{fr}}_{\mathrm{tf}}(Z)$: Reduce to the case for $\mathbf{SH}^{\mathrm{fr}}_{\mathrm{tf}}(B, Z)$, where $\mathrm{Sm}_{B,Z} = \langle X^h_{X \times_B Z} : X \in \mathrm{Sm}_B \rangle$. Prove there is an adjunction of presheaf categories

$$\mathrm{PSh}^{\mathrm{fr}}(\mathrm{Sm}_Z, \mathrm{Spt}) \rightleftarrows \mathrm{PSh}^{\mathrm{fr}}(\mathrm{Sm}_{B,Z}, \mathrm{Spt})$$

which defines an equivalence on homotopy categories. We study this adjunction, and in particular show that the involved functors preserve $\mathbf{A}^1$-local objects. For this we need a moving lemma of a type that we will come back to later in the talk. □

4. Gersten- and Cousin complexes

Let $A^{*,*}$ be a bigraded cohomology theory on Sm_B, for instance $A \in \mathbf{SH}(B)$ and

$$A^{p,q}_Z(X) = [\Sigma^\infty X/(X \setminus Z), \Sigma^{p,q} A]_{\mathbf{SH}(B)}.$$

We can then form the associated Cousin complex

$$0 \to A^{p,0}(X) \to \bigoplus_{x \in X^{(0)}} A^{p,0}_x(X) \to \bigoplus_{x \in X^{(1)}} A^{p+1,0}_x(X) \to \cdots \to \bigoplus_{x \in X^{(\dim X)}} A^{p+\dim X,0}_x(X) \to 0,$$

If we are in a situation where we have Gysin isomorphisms (e.g., immersions of regular schemes), so that $A^{p+1,0}_x(X) \cong A^{p,-1}(x)$ etc., then we get the Gersten complex by replacing the groups with support with the cohomology of the points x. However, the Cousin complex is always defined, so we will stick with that.

Example 4.1. If $\dim(B) \geq 1$, and $U \in \mathrm{EssSm}_B$ local, the Cousin complex of U need not be exact. Indeed, if U is integral with generic point η, then exactness ensures in particular the injectivity of the map

$$\mathcal{F}(U) \to \mathcal{F}(\eta)$$

for, say, any homotopy invariant PST $\mathcal{F}$. This injectivity result implies the vanishing of $H^i_{\mathrm{nis}}(\mathbf{A}^1_U, \mathcal{F}_{\mathrm{nis}})$ for $i \geq 1$, which we saw is false in general without some tf-locality assumptions.

The main point is that we do have exactness for the Cousin complex with support on fibers:

Theorem 4.2. *Let U be an essentially smooth local scheme over B, and let $A^{*,*}$ be a cohomology theory represented by $A \in \mathbf{SH}(B)$.*

Then for all $z \in B$ and $p \in \mathbb{Z}$, the Cousin complex with support on the fibers

$$0 \to A^{p,0}_{U_z}(U) \to \bigoplus_{x \in U_z^{(0)}} A^{p,0}_x(U) \to \bigoplus_{x \in U_z^{(1)}} A^{p+1,0}_x(U) \to \cdots \to \bigoplus_{x \in U_z^{(\dim U_z)}} A^{p+\dim U_z,0}_x(U) \to 0 \tag{4.1}$$

is exact. Here $U_z = U \times_B z$.

Sketch of proof. The proof is similar to what Voevodsky did, the main novelty is a different moving lemma.

For simplicity we assume $z =$ generic point of B. We show the following slightly more general result. For $\mathcal{F}$ a homotopy invariant framed radditive sheaf of S^1-spectra, define the Cousin complex of $\mathcal{F}$ on U as

$$C^\bullet(U, \mathcal{F}) = \left(\bigoplus_{x \in U^{(0)}} \mathcal{F}(U_x) \to \bigoplus_{x \in U^{(1)}} \mathcal{F}_x(U)[1] \to \cdots \to \bigoplus_{x \in U^{(d)}} \mathcal{F}_x(U)[d] \right).$$

We want to show that there is a stable equivalence in the sense of Bousfield–Friedlander:

$$\mathcal{F}(U_z) \xrightarrow{\simeq} \mathrm{Tot}(C^\bullet(U_z, \mathcal{F})).$$

To show the equivalence, first define, for $Z \subseteq U$ a closed subscheme,

$$\mathcal{F}_Z(U) = \mathrm{fib}(\mathcal{F}(U_Z^h) \to \mathcal{F}(U_Z^h \setminus Z)).$$

Write $U = X_{(x)} \in \mathrm{EssSm}_B$.

Now the situation is as follows. Consider:

- $Y \subseteq X_z$ a closed subscheme.
- For $Y \subseteq Y'$ in X_z, get maps $\mathcal{F}_Y(X_z) \xrightarrow{i^*} \mathcal{F}_{Y' \times_{X_z} U}(U_z)$ induced by the inclusion $i \colon U_z \to X_z$.

The strategy now is similar to what Voevodsky did: We want to show that maps of the above type, which reduce codimension of support, induce 0 on homotopy groups. If we know this, then the long exact sequence on homotopy from the localization sequence

$$\varinjlim_{\mathrm{codim}_{U_z} Y' = c} \mathcal{F}_{Y'}(U_z) \to \bigoplus_{y \in U_z^{(c)}} \mathcal{F}_y(U_z) \to \varinjlim_{\mathrm{codim}_{U_z} Y = c+1} \mathcal{F}_Y(U_z)[1]$$

splits into short exact sequences, which splices to a long exact Cousin complex.

Key geometric input is a moving lemma using framed correspondences: the existence of a framed correspondence $r \colon U_z \times \mathbf{A}^1 \to X_z$ such that $r_0 = \sigma^n \circ i$ and $r_1 = j \circ r'$, where

$$U_z \xrightarrow{r'} X_z \setminus Y \xrightarrow{j} X_z.$$

This shows that the maps $\mathcal{F}_Y(X_z) \to \mathcal{F}_{Y' \times_{X_z} U}(U_z)$ factor as

$$\begin{array}{ccc} \mathcal{F}_Y(X_z) & \xrightarrow{\quad i^* \quad} & \mathcal{F}_{Y' \times_{X_z} U}(U_z) \\ & {\scriptstyle j^*} \searrow \quad \nearrow {\scriptstyle (r')^*} & \\ & \mathcal{F}_Y(X_z \setminus Y) & \end{array}$$

and are hence trivial on homotopy. □

5. tf-Cousin complexes

We define the tf-Cousin complex of $A^{*,*}$ as

$$0 \to A^{p,0}(X) \to \bigoplus_{z \in B^{(0)}} A^{p,0}_{X_z}(X) \to \bigoplus_{z \in B^{(1)}} A^{p+1,0}_{X_z}(X) \to \cdots \to \bigoplus_{z \in B^{(\dim B)}} A^{p+\dim B,0}_{X_z}(X) \to 0.$$

These are complexes of length $\dim(B) + 1$, as opposed to the case for the ordinary Cousin complex of X, which has $\dim(X) + 1$ terms. By the result above, we have:

Theorem 5.1. *The tf-Cousin complex is equivalent to the Cousin complex of $A^{*,*}$ on any essentially smooth local U over B.*

Example 5.2. For $\Sigma^\infty B/(B \setminus z) \in \mathbf{SH}(B)$, the associated tf-Cousin complex is not exact in degree $\dim(B)$.

We may form the "Cousinian SH" subcategory of $\mathbf{SH}(B)$, spanned by those $A \in \mathbf{SH}(B)$ whose Cousin complex is exact. By definition, it contains the objects for which the Gersten conjecture holds.

Thus, by the above example,

$$\Sigma^\infty B/(B \setminus z) \notin \mathbf{SH}^{\mathrm{Cous}}.$$

A conjecture is:

$$\mathbf{1} \in \mathbf{SH}^{\mathrm{Cous}}(B)$$

for B a finite dimensional regular Noetherian separated scheme. Another question is $\mathsf{KGL} \in \mathbf{SH}^{\mathrm{Cous}}(B)$, which would imply the Gersten conjecture over B.

By our results, $\mathbf{SH}^{\mathrm{Cous}}$ can be defined equivalently via tf-Cousin exactness. This definition has advantages with respect to functoriality.