

# On the Categorification problem for Motivic Donaldson-Thomas invariants

Marco Robalo (Jussieu)

Work in progress joint with B. Hennion (Orsay) and J. Holstein (  
Hamburg)

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## CY varieties

- $Y$  a Calabi-Yau variety of dimension 3 over  $\mathbb{C}$ , ie,  $\omega_Y \simeq \mathcal{O}_Y$ .
- Example: The Fermat quintic

$$X_5 = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subseteq \mathbb{P}_{\mathbb{C}}^4$$

or more generally, any smooth quintic in  $\mathbb{P}_{\mathbb{C}}^4$ ,  $\omega_Y \simeq \mathcal{O}(5 - 4 - 1) \simeq \mathcal{O}_Y$

- Motivation from string theory  
[Candelas-Horowitz-Strominger-Witten]: Spacetime is assumed to be of the form  $\mathbb{R}^4 \times Y$  where  $Y$  is a CY-manifold of real dimension 6, with size the order of the Planck length. Infinitely many choices of  $Y$ .
- Paths/interactions of string-particles through spacetime, define 2-dimensional real surfaces (1-dimensional algebraic curves) of genus  $g$  in  $Y$ .



# Counting algebraic curves in a Calabi-Yau

- Counting **parametrized** curves  $f : C \rightarrow Y$  (**GW-invariants**)

$\overline{\mathcal{M}}_{g,n}(Y, \beta)$       quasi-smooth,  $Vol = \int_{virt. class} \in \mathbb{Q}$  ✓  
moduli space (stack) of stable maps

- Counting **embedded** curves  $C \subseteq Y$ :

$\text{Hilb}_{\text{codim } 2}(Y)$       not quasi-smooth,  $Vol$  ✗  
Hilbert scheme of codim 2 subschemes

- Counting **ideal sheaves**  $I_C \in \text{Coh}(Y)$  (**DT-invariants**)

$\overline{\mathcal{M}}\text{Coh}(Y)$       quasi-smooth,  $Vol = \int_{virt. class} \in \mathbb{Z}$  ✓  
Moduli of coherent sheaves      CY + Serre duality

## Behrend approach to DT-invariants

**Observation:** Serre duality + CY condition imposes a **symmetry** on the deformation theory of  $\mathcal{M}Coh(Y)$ : symmetric obstruction theory

$$\{1^{st\ order} \text{ def. of } E \in Coh(Y)\} \simeq \{\text{Obstructions to def. of } E \in Coh(Y)\}^\vee$$

### Theorem (K. Behrend)

Let  $X$  be a quasi-smooth algebraic stack with a **symmetric obstruction theory** (Ex:  $X = \mathcal{M}Coh(Y)$ ). Then there is a function  $\nu_{Behrend} : X \rightarrow \mathbb{Z}$  such that

$$Vol(X) := \int_{[X]^{vir}} = \chi(X, \nu_{Behrend}) = \sum_n n \cdot \chi(\nu_{Behrend} = n)$$

**Behind the scenes:** This extra symmetry is a shadow of a ***(-1)-shifted symplectic form*** on  $\mathcal{M}Coh(Y)$  [Pantev-Toën-Vaquié-Vezzosi].

**In this talk:** DT-theory  $\leftrightarrow$  (-1)-shifted symplectic geometry

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# Shifted Symplectic Geometry

- Deformation theory of a stack/derived stack  $X$  is controlled by its *cotangent complex*  $\mathbb{L}_X \in \mathrm{DQcoh}_\infty(X)$ . When  $X$  is locally of finite presentation,  $\mathbb{L}_X$  is a *perfect complex*, with dual  $\mathbb{T}_X = \mathbb{L}_X^\vee$  the *tangent complex*.
- $X$  Smooth,  $\mathbb{L}_X = \Omega_X^1$ .
- $n$ -shifted 2-forms =  $\{\mathcal{O}_X \rightarrow \mathbb{L}_X \wedge \mathbb{L}_X[n]\} = \{\mathbb{T}_X \wedge \mathbb{T}_X \rightarrow \mathcal{O}_X[n]\}$
- de Rham diff.  $DR(X) := [\mathcal{O}_X \xrightarrow{d_R} \mathbb{L}_X \xrightarrow{d_R} \mathbb{L}_X \wedge \mathbb{L}_X \longrightarrow \dots]$   
[Connes, Toën-Vezzosi]:  $d_R$  should not be understood as an internal differential but rather as the **action of an extra operator  $\epsilon$  of degree 1**
- $n$ -shifted **closed** 2-forms: Need homotopy  $d_R(\omega) \sim 0$

$$\{\text{maps } \underbrace{k(2)[-2n-1]}_{\epsilon=0 \text{ action}} \rightarrow \underbrace{DR(X)}_{\epsilon=d_R}\}$$



# Shifted Symplectic Geometry

## Definition (Pantev-Toën-Vaquié-Vezzosi)

An  $n$ -shifted symplectic form on  $X$  is a  $n$ -shifted closed 2-form such that its underlying 2-form  $\mathbb{T}_X \wedge \mathbb{T}_X \rightarrow \mathcal{O}_X[n]$  is non-degenerate, ie, induces an equivalence

$$\mathbb{T}_X \simeq \mathbb{L}_X[n]$$

- $X = T^*\mathbb{A}^1 = \mathbb{A}^2$  has 0-shifted symplectic form given by  $\omega = dx \wedge dy$ .
- $X = Perf$  the derived stack classifying perfect complexes has a 2-shifted symplectic form.

$$\mathbb{T}_{E, Perf} = R\text{End}(E)[1] \simeq E \otimes E^\vee[1]$$

$$\mathbb{T}_{E, Perf} \wedge \mathbb{T}_{E, Perf} \simeq E \otimes E^\vee[1] \otimes E \otimes E^\vee[1] \rightarrow \mathcal{O}[2] \quad \text{evaluation map}$$

- (PTVV)  $Y$  a CY of dimension 3 over  $k$ . Then  $X := \text{Map}(\underbrace{Y}_3, \underbrace{Perf}_2)$

is  $(2-3=-1)$ -symplectic. In particular,

$\mathcal{M}Coh(Y) \subseteq \text{Map}(Y, Perf)$  is  $-1$ -symplectic ( $\Rightarrow$  Behrend Symmetry)

# Shifted Symplectic Geometry

- $U$  a smooth  $k$ -scheme with a function  $f : U \rightarrow \mathbb{A}_k^1$ . The derived critical locus  $X = d\text{Crit}(f)$  is the derived fiber product

$$\begin{array}{ccc}
 X := d\text{Crit}(f) & \xrightarrow{i} & U \\
 \downarrow i & & \downarrow df \\
 U & \xrightarrow{0} & T^*U
 \end{array}$$

*coh.deg*

- 1

0

1

$$i^*T_U \xrightarrow{\text{Hess}(f)} i^*\mathbb{L}_U = T_X$$

$$i^*T_U \xrightarrow{H(f)^\vee} i^*\mathbb{L}_U = \mathbb{L}_X$$

symmetry of the Hessian  $\Rightarrow T_X \simeq \mathbb{L}_X[-1]$  is the underlying 2-form of a (-1)-shifted symplectic structure on  $X$ .

**Example:**  $(U, f) = (\mathbb{A}^1, x^3)$   $d\text{Crit} = \text{Spec } k[x]/(f' = 3x^2)$

## Joyce's approach to DT-invariants

All examples are locally of this form:

### Theorem (Brav-Bussi-Joyce (Darboux Lemma))

*Let  $X$  be a  $(-1)$ -symplectic derived scheme. Then Zariski locally  $X$  is symplectomorphic to a derived critical locus  $d\text{Crit}(U, f)$  with  $U$  smooth.*

**Consequence:** Locally on  $X$  it makes sense to analyse the singularities of the function  $f$  on  $U$  via the perverse sheaf of **vanishing cycles**

$$P_{U,f} \in \text{Perv}_{d\text{Crit}(f)}(U) = \text{Perv}(d\text{Crit}(f)) = \text{Perv}(\text{Crit}(f))$$

**Problem:** **Ambiguity** in the choice of local presentations:

$$d\text{Crit}(\mathbb{A}^1, x^3) = \text{Spec } k[x]/(3x^2) \simeq \text{Spec } k[x, y]/(3x^2, 2y) = d\text{Crit}(\mathbb{A}^2, x^3 + y^2)$$

$P_{(\mathbb{A}^1, x^3)}$  and  $P_{(\mathbb{A}^2, x^3 + y^2)}$  **non-canonically** isomorphic.

## Joyce's approach to DT-invariants

### Theorem (Brav-Bussi-Dupont-Joyce-Szendroi (BBDJS))

Let  $X$  be a  $(-1)$ -symplectic derived scheme. Assume that there exists a line bundle  $L$  together with an equivalence  $L \otimes L \simeq \det(\mathbb{T}_X)$  (aka *orientation data*). Then:

- The locally defined perverse sheaves of vanishing cycles  $P_{U,f}$  glue to a globally defined perverse sheaf  $P \in \text{Perv}(X)$ .
- $\chi(P) = \nu_{\text{Behrend}}$  computing locally the Euler characteristic of vanishing cycles. Gives back DT-counting.

**Proof:** Glue by hand using local presentations of the underlying classical scheme as classical critical loci.

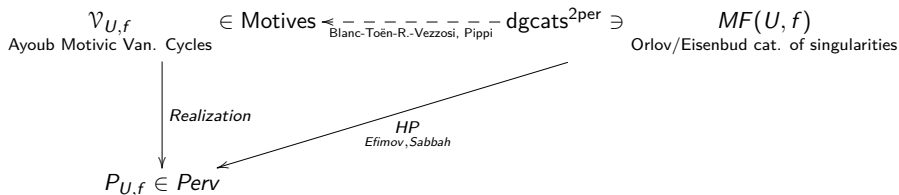
- method does not see the full derived structure.
- strategy works for perverse sheaves because:
  - ▶ they form a **1-category** (no higher homotopies needed to glue).
  - ▶ they have the  $\mathbb{A}^1$ -**homotopy invariance** property.

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# Motivic DT and categorification

Different invariants capture vanishing cycles of  $f$  on  $U$ :



**MF:**  $U_0 := f^{-1}(0)$ ,  $M \in \text{Coh}(U_0)$ , infinite resolution by projective modules becomes eventually **2-periodic** [Serre-Auslander-Buchsbaum-Eisenbud]

$$\underbrace{\dots \rightarrow F \rightarrow Q \rightarrow F \rightarrow Q}_{\in MF(U, f)} \rightarrow \underbrace{P_n \rightarrow \dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0}_{\in \text{Perf}(U_0)} \rightarrow M$$

## Motivic DT and categorification

**Gluing Problem:** Given a  $(-1)$ -symplectic derived scheme  $X$ , can we glue the Darboux locally defined dg-categories  $MF(U, f)$  as a sheaf of dg-categories on  $X$ ? Is Joyce's orientation data enough?

**Rmk:** Version of the gluing problem for the Fukaya category (Seidel, Kontsevich, Nadler, Shende, Ganatra, Pardon,...).

**Complications:** The gluing no longer takes place in a 1-category but in an  $\infty$ -category. Complicated coherences are required. Need a gluing mechanism.

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# The Darboux Stack

**Classical Picture:**  $X$  a classical symplectic manifold, then locally  $X$  is symplectomorphic to some  $T^*M$  (Darboux's lemma). We can analyse the moduli of such Darboux parametrizations:

$$\text{Darb}_X : S \subseteq X \text{ open} \mapsto \{M \text{ smooth manifold, } S \simeq T^*M \text{ symplectic}\}$$

The data of a symplectomorphism  $S \simeq T^*M$  in particular implies:

- The fibers of the projection  $S \simeq T^*M \rightarrow M$  define a **smooth Lagrangian foliation**  $\mathcal{F}$  on  $S$  (ie,  $\omega|_{\text{fibers}} = 0$ ).
- The symplectic form on  $S$  is **exact** ie, there exists a 1-form  $\alpha$  (Liouville form on  $T^*M$ ) with  $d_R(\alpha) = \omega$ .

We call such  $(\mathcal{F}, \alpha)$  a **Darboux datum** on  $S$ .

# The Darboux Stack

**(-1)-shifted geometry:** These notions make sense thanks to the work of Toën-Vezzosi on [derived foliations](#).

## Theorem (Pantev-Toën)

*$S$  a  $(-1)$ -symplectic derived scheme. Then the following data are equivalent:*

- *Darboux data on  $S$ , ie a globally defined smooth derived Lagrangian foliation  $\mathcal{F}$  on  $S$  + an exact structure  $\alpha$ .*
- *the data of an exact structure on  $S$  + a smooth [formal](#) scheme  $\mathcal{U}$ , and a [Lagrangian fibration](#)  $S \rightarrow \mathcal{U}$  given by a closed immersion with  $S_{\text{red}} = \mathcal{U}_{\text{red}}$ .*
- *the data of a smooth [formal](#) scheme  $\mathcal{U}$ , a function  $f$  on  $\mathcal{U}$  and a symplectomorphism  $S \simeq d\text{Crit}(\mathcal{U}, f)$*

**Classical Picture:** Darboux data on  $S \Leftrightarrow [S \subseteq T^*M \rightarrow M]$ .

**(-1)- picture :** Darboux data on  $S \Leftrightarrow [S \simeq d\text{Crit}(\mathcal{U}, f) \hookrightarrow \mathcal{U}]$ .

**Idea:**  $\mathcal{U} := S/\mathcal{F}$  the formal leaf space.  $f =$  exact struct. - isotropic struct.

# The Darboux Stack

**Example:**  $(\widehat{\mathbb{A}^1}, x^3)$  gives Darboux data

$$d\text{Crit}(x^3) = \text{Spec}(k[x]/(3x^2)) \hookrightarrow \widehat{\mathbb{A}^1}$$

## Construction

*The assignment:*

$$S \rightarrow X \text{ étale} \mapsto \{(\alpha, \mathcal{F}) : \text{Exact structure } \alpha + \text{smooth Lag. fol. } \mathcal{F} \text{ on } S\}$$

*defines a stack on the small étale site of a  $n$ -shifted symplectic derived scheme  $X$ . We call it the Darboux stack  $\text{Darb}_X$ .*

**Remark:**  $\text{Darb}_X := \text{Exact}_X \times \text{LagFol}_X^{\text{sm}}$

**Comment:** In the case where  $X$  is  $(-2)$ -symplectic, this recovers the local data used by Borisov-Joyce and Oh-Thomas to glue DT-invariants for Calabi-Yau 4-folds.

# The Darboux Stack

## Construction

Both MF and Joyce's construction have  $\text{Darb}_X$  as a natural domain, and define natural transformations of sheaves on the small étale site of  $X$ :

$$P : \text{Darb}_X(S) \ni (\mathcal{U}, f) \rightarrow P_{\mathcal{U}, f} \in \text{Perv}_X(S) := \text{Perv}(S)^\simeq$$

$$MF : \text{Darb}_X(S) \ni (\mathcal{U}, f) \rightarrow MF(\mathcal{U}, f) \in \text{dgc}at_{X_{dR}}^{2per}(S) := \underbrace{(\text{dgc}at_{S_{dR}}^{2per})^\simeq}_{\text{categorical crystals}}$$

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## Action of Quadratic Bundles

**Ambiguity Problem:** in the choice of local presentations:

$$d\text{Crit}(\mathbb{A}^1, x^3) = \text{Spec } k[x]/(3x^2) \simeq \text{Spec } k[x, y]/(3x^2, 2y) = d\text{Crit}(\mathbb{A}^2, x^3 + y^2)$$

### Definition

$$\text{Quad}_{dR}(S) := \{(Q, q) : (\text{loc. trivial}) \text{ quadratic vector bundles on } S_{dR}\}$$

### Construction

$X$  a  $(-1)$ -symplectic derived scheme. Then:

- The assignment  $S/X \text{ étale} \mapsto \text{Quad}_{dR}(S)$  defines a sheaf of monoids  $\text{Quad}_{X_{dR}}$  on  $X_{\text{ét}}$  for the sum of quadratic bundles,
- $\text{Quad}_{X_{dR}}(S)$  acts on  $\text{Darb}_X(S)$ ,

$$d\text{Crit}(\mathcal{U}, f) \simeq S \simeq d\text{Crit}(\mathcal{U} \times_{S_{dR}} Q, f + q)$$

**Rmk:** Every morphism of Darboux foliations on  $X$ ,  $(\mathcal{U}, f) \rightarrow (\mathcal{V}, g)$  is étale locally of the form  $(\mathcal{U}, f) \rightarrow (\mathcal{U} \times_{S_{dR}} Q, f + q)$

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## Recovering the perverse gluing of BBDJS:

**Fact:**  $(M, q) \in \text{Quad}_{X_{dR}}(S)$  then  $\det(M)$  is a **2-torsion line bundle** over  $S$ , ie,  $\det(M)^2 \simeq \mathcal{O}_S$ . This follows from the non-degeneracy of the Hessian.

### Construction

$X$  a  $(-1)$ -symplectic derived scheme. Then:

- $\det : \text{Quad}_{X_{dR}} \rightarrow B\mu_{2,X} = \text{fib}(B\mathbb{G}_{m,X} \xrightarrow{2} B\mathbb{G}_{m,X})$  is a map of monoids.
- $P : \text{Darb}_X \rightarrow \text{Perv}_X$  comes with **homotopy coherent data** rendering the actions compatible on both sides ( on the right the action of  $B\mu_2$  is defined by BBDJS).
- In particular,  $P$  descends to the quotients

$$\bar{P} : \text{Darb}_X / \text{Quad}_{X_{dR}} \rightarrow \text{Perv}_X / B\mu_{2,X}$$



## Recovering the perverse gluing of BBDJS:

**Warning:**  $Quad_{X_dR}$  is not a group but  $B\mu_{2,X}$  is. The construction factors through the [group completion](#)

$$Quad_{X_dR}^+, \quad Darb_X / Quad_{X_dR} := Darb_X^+ / Quad_{X_dR}^+$$

Theorem (Hennion-Holstein-R. as a reformulation of BBDJS )

$X$  a  $(-1)$ -symplectic derived scheme. Then the quotient map

$$\bar{P} : Darb_X / Quad_{X_dR} \rightarrow Perv_X / B\mu_{2,X}$$

is **null-homotopic**, ie, it admits a factorization through  $X$  (seen as a the final object of the étale topos  $X$ ):

$$\begin{array}{ccc} Darb_X / Quad_{X_dR} & \xrightarrow{\bar{P}} & Perv_X / B\mu_{2,X} \\ \text{final} \downarrow & \nearrow & \\ X & & \end{array}$$

In other words, the gluing of the perverse sheaves  $P_{U,f}$  is always well-defined in the quotient  $Perv_X / B\mu_{2,X}$ .

## Recovering the perverse gluing of BBDJS:

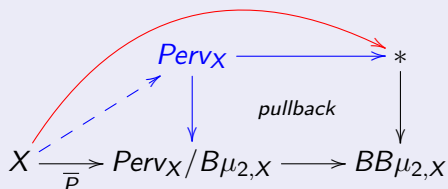
### Remark

*The composition*

$$X \rightarrow \text{Perv}_X / B\mu_{2,X} \rightarrow * / B\mu_{2,X} = BB\mu_{2,X}$$

is the class in  $H^2(X, \mathbb{Z}/2\mathbb{Z})$  of the bundle classifying square roots of  $\det(\mathbb{T}_X)$ .

An orientation data of BBDJS corresponds precisely to the choice of a **null-homotopy** of this composition



Such a null-homotopy provides a lifting through the fiber product and defines a well-defined glued perverse sheaf.

## What about gluing MF:

**Fact:**  $(M, q) \in \text{Quad}_{X_{dR}}(S)$  then  $MF(M, q)$  has a structure of **2-torsion 2-periodic derived Azumaya algebra** over  $S_{dR}$ . This is a consequence of **Preygel-Thom-Sebastiani** followed by **Knörrer periodicity**

$$MF(M, q) \otimes MF(M, q) \simeq MF(M \times M, q \boxplus -q) \simeq MF(S_{dR}, 0)$$

### Construction

$X$  a  $(-1)$ -symplectic derived scheme. Then:

- $MF : \text{Quad}_{X_{dR}} \rightarrow \text{Az}_{X_{dR}}^{2\text{per}, 2\text{-tor}}$  is a map of monoids.
- $MF : \text{Darb}_X \rightarrow \text{dgc}at_{X_{dR}}^{2\text{per}}$  comes with **homotopy coherent data** rendering the actions compatible on both sides (on the right the action of  $\text{Az}_{X_{dR}}^{2\text{per}, 2\text{-tor}}$  is given by tensor products of dg-categories).
- In particular,  $MF$  descends to the quotients

$$\overline{MF} : \text{Darb}_X / \text{Quad}_{X_{dR}} \rightarrow \text{dgc}at_{X_{dR}}^{2\text{per}} / \text{Az}_{X_{dR}}^{2\text{per}, 2\text{-tor}}$$

## What about gluing MF:

### Dishonest Corollary

Let  $X$  be a  $(-1)$ -shifted symplectic derived scheme. Assume  $X$  is equipped with a section

$$X \rightarrow \text{Darb}_X / \text{Quad}_{X_{dR}}$$

Then the locally defined categories  $\text{MF}(\mathcal{U}, f)$  glue as a sheaf of 2-periodic dg-categories on  $X$  under the prescription of a *category orientation data*, ie, a trivialization of the composition

$$X \rightarrow \text{Darb}_X / \text{Quad}_{X_{dR}} \rightarrow \text{dgcats}_{X_{dR}}^{2\text{per}} / \text{Az}_{X_{dR}}^{2\text{per}, 2\text{-tor}} \rightarrow \text{BAz}_{X_{dR}}^{2\text{per}, 2\text{-tor}}$$

## New Orientation data

The orientation data of BBDJS is (a priori) not enough to glue  $MF$ . A categorical orientation provides new obstruction classes coming from the fibration sequence

$$Az_{X_{dR}}^{2per, 2-tor} \rightarrow Az_{X_{dR}}^{2per} \xrightarrow{2} Az_{X_{dR}}^{2per}$$

- $\pi_0(Az_{X_{dR}}^{2per, 2-tor}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \rightarrow \{MF(*, 0), MF(\mathbb{A}^1, x^2)\}$ .
- $\pi_1(Az_{X_{dR}}^{2per, 2-tor}) \simeq \underbrace{\mathbb{Z}/2\mathbb{Z}}_{BBDJS} \simeq \{Id, [1]\}$
- $\pi_2(Az_{X_{dR}}^{2per, 2-tor}) = \mathbb{Z}/2\mathbb{Z} \simeq \text{Ker}(z^2 : \mathbb{C}^* \rightarrow \mathbb{C}^*)$
- $\pi_n(Az_{X_{dR}}^{2per, 2-tor}) = 0 \quad n \geq 3,$

## What about gluing MF:

**Honest** gluing of MF: requires a factorization

$$\begin{array}{ccc} \text{Darb}_X / \text{Quad}_{X_{dR}} & \xrightarrow{\overline{MF}} & \text{dgc}at_{X_{dR}}^{2\text{per}} / \text{Az}_{X_{dR}}^{2\text{per}, 2\text{-tor}} \\ \text{final} \downarrow & \nearrow & \\ X & & \end{array}$$

### Intermediate Results (Obstructions)

- $\pi_0^{\text{sheaf}}(\text{Darb}_X / \text{Quad}_{X_{dR}}) = \text{Exact}_X$  (transitivity of the action).
- $\pi_1(\text{Darb}_X / \text{Quad}_{X_{dR}}) \neq 0$  in general
- eventually truncated because  $X$  is quasi-smooth
- The map  $\overline{MF}$  is zero on all higher  $\pi_n$  for  $n \geq 2$ .
- The map  $\overline{MF}$  is zero on  $\pi_1$ .

# Gluing

Work in progress

*Explaining the null-homotopy.*

**Question:** What happens through Mirror symmetry? Relation to the works of Nadler, Shende, Ganatra, Pardon gluing Fukaya categories of 0-shifted Weinstein manifolds via Kashiwara-Schapira microlocal sheaves?

**Question:** Exponential Motives over any base  $\rightsquigarrow$  motives of foliations.

Thank you for your time.