MOTIVES SEMINAR-SS 2018-PROGRAM

The end goal for the motives seminar is to prove the following theorems of Clausen-Matthew-Morrow [3, Theorem A, B, C]. For a ring R, let $K^{inv}(R)$ denote the fiber of the trace map $K(R) \to TC(R)$.

Theorem 1 (Theorem A)**.** *Let (R, I) be a henselian pair. Then for all n, the map $K^{inv}(R)/n \to K^{inv}(R/I)/n$ is an equivalence.*

Theorem 2 (Theorem B)**.** *Let R be a ring henselian along (p) and such that R/p has finite Krull dimension. Let $d = sup_{x\in \mathrm{Spec}\,(R/p)} log_p[k(x) : k(x)^p]$ where $k(x)$ denotes the residue field at x. Then the map $K(R)/p^i \to TC(R)/p^i$ is a equivalence in degrees $\geq max(d, 1)$ for each $i \geq 1$.*

Theorem 3 (Theorem C: Étale K-theory is TC at points of characteristic p)**.** . *Let R be a strictly henselian local ring with residue field of characteristic $p > 0$. Then $K^{inv}(R)/p = 0$, i.e., the map $K(R) \to TC(R)$ is a p-adic equivalence.*

To prove these results, we will take a bit of a tour through a series of earlier results relating relative K-theory with relative (topological) cyclic homology, and computing mod p^ν-K-theory in characteristic p.

Preliminary program

Lecture 1. 17.04. Mod p Bloch-Kato and Beilinson-Lichtenbaum in characteristic p.
i. Present the theorem of Bloch-Kato [2, Theorem 2.1] (see also [8, §1]): For F a field of characteristic $p > 0$, the dlog map $dlog : K^M_n(F)/p \to \Omega^n_F$ is injective with image the étale sheaf of logarithmic forms, that is, the kernel of $1 - C^{-1}$, C the Cartier operator. If time permits, you can give a sketch of the proof.
ii. Present the theorem of Geisser-Levine [6, Theorem 1.1] and a sketch of the proof: For F a field of characteristic $p > 0$, the mod p motivic cohomology of F is given by

$$H^q(F, \mathbb{Z}/p(n)) = \begin{cases} 0 & \text{for } q \neq n \\ K^M_n(F)/p & \text{for } q = n \end{cases}$$

Give the consequences for mod p K-theory: For X a smooth finite type scheme over a field k of characteristic $p > 0$, the map of sheaves $\mathcal{K}^M_n/p^\nu \to \mathcal{K}_n(X; \mathbb{Z}/p^\nu)$ is an isomorphism for all $\nu \geq 1$. Furthermore, $K_n(X; \mathbb{Z}/p^\nu) = 0$ for $n > \dim_k X$.

Lectures 2, 3. The theorems of Dundas-Goodwillie-McCarthy on relative

K-theory and cyclic homology.

Lecture 2. 24.04 Present the theorem of Dundas-Goodwillie [7, Main Theorem]:

Theorem 4. *Let (R, I) be a commutative (simplicial) $\mathbb{Q}$-algebra with nilpotent ideal I. Then the trace map $K(R, I) \to HC^-(R, I)$ is an equivalence.*

This includes the construction of the trace map to cycle homology.

Lecture 3. ??.05 (01.05 is May Day) Discuss the McCarthy's refinement [9, Main Theorem] of Dundas-Goodwillie:

Theorem 5. *Let (R, I) be a commutative (simplicial) ring with nilpotent ideal I. Then the trace map $K(R, I) \to TC(R, I)$ is an equivalence.*

Lecture 4, 5, 6 Present some of the results of [1], restricting to characteristic p rings, that is, replacing "perfectoid" with "perfect" throughout. The goal is to recover the Geisser-Hesselholt computation of mod p^n K-theory in characteristic p.

Lecture 4. 08.05. [1, §2, §3] Section 2 sets up some foundations of HH, THH, TC^-, TP. Section 3 shows that these theories satisfy faithfully flat descent.

Lecture 5. ??.05. (15.05 is the conference "Homotopy theory and Refined Enumerative Geometry and 22.05 is the Pfingsten holiday) [1, §6.1, 6.3]. We ignore sections 4-5. Section 6.1 is written uniformly for perfectoids R in both characteristic p and mixed characteristic, but you should just stick to characteristic p, which means that R is a perfect $\mathbb{F}_p$-algebra. This simplifies some things, for example you only need the first half of Theorem 6.1. Also, it means that you should replace A_{inf} and its element ξ by the Witt vectors $W(R)$ and its element p. In fact, the calculations of section 6.1 for perfect R all collapse formally to the case $R = \mathbb{F}_p$, which is treated in [10, §IV.4]. You can ignore section 6.2.

In section 6.3 you calculate THH, TC^-, TP of smooth algebras over a perfectoid R; again, just assume R is a perfect ring of characteristic p. The calculations formally reduce to HH, rather than going through Hesselholts hard work

Lecture 6. 29.05. [1, §8, part of §7]. §8 is is written only for char p (though it must sometimes refer back to section 7, which is again written uniformly for all perfectoids, but needs only be treated for perfect R in characteristic p).

By the time you get to the end of Section 8.4, you will have seen that TC_n is given (étale locally on smooth varieties) by $W\Omega_{log}$, which reproves Geisser-Hesselholt [5].

Lectures 7-13 These lectures present the Clausen-Matthew-Morrow paper.

Lecture 7. 05.06 [3, Overview+§2.1, §2.2]. The finiteness property of TC Part I

Lecture 8. 12.06 [3, §2.3]. The finiteness property of TC Part II

Lecture 9. 19.06 [3, §3]. This is a series of definitions and basic facts about henselian pairs. You should spend much of the time on Gabber's rigidity theorem and say a word about the proof.

Lecture 10. 26.06 [3, §4.1, 4.2] Main rigidity results. Part I

Lecture 11. 03.07 [3, §4.3, 4.4] Main rigidity results. Part II

Lecture 12. 10.07 [3, §5.1, 5.2] Applications. Part I

Lecture 13. 17.07 [3, §5.3, 5.4] Applications. Part II.

References

[1] Bhargav Bhatt, Matthew Morrow, Peter Scholze, *Topological Hochschild homology and integral p-adic Hodge theory* `https://arxiv.org/abs/1802.03261v1`

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[3] D. Clausen, A. Matthew, M. Morrow. *K-Theory and topological cyclic homology of henselian pairs.* `https://arxiv.org/abs/1803.10897v1`

[4] O. Gabber. *K-theory of Henselian local rings and Henselian pairs.* In Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), volume 126 of Contemp. Math., pages 59–70. Amer. Math. Soc., Providence, RI, 1992.

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[7] Goodwillie, Thomas G. *Relative algebraic K-theory and cyclic homology*. Ann. of Math. (2) 124 (1986), no. 2, 347402.

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[10] T. Nikolaus, P. Scholze. *On topological cyclic homology* `https://arxiv.org/abs/1707.01799v1`