Refined broccoli invariants

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Bibliography

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Contents

Tropical refined Severi degree

Let Δ be a lattice polytope, $X = X(\Delta)$ be the associated projective toric surface and $L = L(\Delta)$ be the associated tautological toric line bundle and $\delta \geq 0$.

Recall that Block-Göttsche defined a tropical refined Severi degree

$$N_{trop}^{(X,L),\delta}(y) := N_{trop}^{(X,L),\delta}(y,\mathcal{P}) = \sum_{C} mult(C;y),$$

where the sum is taken over certain simple δ-nodal marked curves of degree Δ through points $\mathcal{P}$ in tropically general position, such that

- the refined curve multiplicity $mult(C;y)$ is a product of vertex multiplicities and a Laurent polynomial in $\mathbb{N}[y^{\pm\frac{1}{2}}]$
- $N_{trop}^{(X,L),\delta}(y)$ is a symmetric Laurent polynomial in $\mathbb{N}[y^{\pm 1}]$

Interpolation

Furthermore, one has

- $mult(C;1) = mult_{\mathbb{C}}(C)$
- $mult(C;-1) = mult_{\mathbb{R}}(C)$,

where $mult_{\mathbb{C}}(C)$ and $mult_{\mathbb{R}}(C)$ are Mikhalkin's complex and real curve multiplicities.

If $X = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown up at up to 3 points and $\delta = |\Delta^\circ \cap \mathbb{Z}^2| - 1$, then

- $N_{trop}^{(X,L),\delta}(1) = N^{(X,L),\delta}$ toric Severi degree
- $N_{trop}^{(X,L),\delta}(-1) = W^{X,L}$ *totally real* Welschinger number of real rational curves in X through $|\partial\Delta \cap \mathbb{Z}^2| - 1$ points

Natural question. What about the *non-totally real* case?

Non-totally real case

The tropical curves considered for non-totally real tropical Welschinger invariants are the *Shustin curves*.

However, Shustin could only prove via correspondence theorems that the weighted sum of Shustin curves through a point configuration in tropically general position is independent of the chosen point configuration (*invariance*).

A purely tropical proof could only be achieved by the detour via *broccoli curves*: The *bridge algorithm* shows that the weighted sum of broccoli curves coincides with the weighted sum of Shustin curves.

Gathmann's and Markwig's methods apply in order to give a purely tropical proof of invariance for broccoli invariants.

Refined broccoli invariants

Idea. Define *refined broccoli curves* with *refined multiplicities* $m_C(y)$ giving rise to *refined broccoli invariants* $N^{rB}_{(r,s)}(y, \Delta, F)$ such that

- the refined curve multiplicity $m_C(y)$ is (more or less) a product of vertex multiplicities and a symmetric Laurent polynomial in $\mathbb{Q}[y^{\pm 1}]$
- $N^{rB}_{(r,s)}(y, \Delta, F)$ is a symmetric Laurent polynomial in $\mathbb{Q}[y^{\pm 1}]$
- $N^{rB}_{(r,s)}(1, \Delta, F)$ recovers (more or less) tropical descendant Gromov-Witten invariants
- $N^{rB}_{(r,s)}(-1, \Delta, F)$ is the broccoli invariant $N^{B}_{(r,s)}(\Delta, F)$.

(r, s)-marked curves

Definition

Let $r, s \in \mathbb{N}$. An (r, s)-marked (plane tropical rational parametrized) curve is a tuple $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ for some $n \in \mathbb{N}$ such that:

a) Γ is a metric graph whose connected components are rational. The unbounded edges of Γ will be called *ends*.

b) $h : \Gamma \to \mathbb{R}^2$ is a continuous map which is integer affine linear on each edge of Γ, i.e. on each edge E it is of the form $h(t) = a + tv$ for some $a \in \mathbb{R}^2$ and $v \in \mathbb{Z}^2$. If we parametrize E starting at the vertex V, then the vector v above will be denoted $v(E, V)$ or just $v(E)$ for ends. An edge is called *contracted* if it has direction 0.

(r,s)-marked curves

Definition (continued)

c) At each vertex V of Γ, the *balancing condition*

$$\sum_{V \in E} v(E,V) = 0$$

holds.

d) $x_1, \ldots, x_{r+s}$ is a labeling of the contracted ends of Γ; they are called *markings* of C. Moreover, $y_1, \ldots, y_n$ is a labeling of the non-contracted ends of Γ; they are called *unmarked ends*. The collection of their directions $(v(y_1), \ldots, v(y_n))$ is called the degree of C, its number of vectors will be denoted by $|\Delta| = n$.

(r, s)-marked curves

Definition

The weight $\omega(E)$ of an edge E in an (r, s)-marked curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ is the greatest common divisor of the entries of its direction vector. The edge E is called *even* if its weight is even; otherwise E is called *odd*.

Definition

An (r, s)-marked curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ is connected if Γ is connected. We denote by $M_{(r,s)}(\Delta)$ the set of isomorphism classes of connected (r, s)-marked curves.

Combinatorial types

Definition

The combinatorial type of an (r,s)-marked curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ is the data of the non-metric graph Γ together with the labelings $x_1, \ldots, x_{r+s}$ and $y_1, \ldots, y_n$ as well as the directions of all edges. For any combinatorial type α, we denote by $M^{\alpha}_{(r,s)}(\Delta)$ the subspace of $M_{(r,s)}(\Delta)$ of all marked curves of type α.

Conditions in general position

Definition

Let C be an (r,s)-marked curve of degree $\Delta = (v(y_1),\ldots,v(y_n))$ and let $F \subset \{1,\ldots,n\}$. The ends y_i with $i \in F$ are called *fixed ends*. The evaluation map with respect to F is given by

$$ev_F : M_{(r,s)}(\Delta) \to (\mathbb{R}^2)^{(r+s)} \times \prod_{i\in F} \mathbb{R}^2/\langle v(y_i)\rangle,$$

$$C \mapsto (h(x_1),\ldots,h(x_{r+s}),(h(y_i))_{i\in F})$$

A collection of conditions for ev_F is a tuple $\mathcal{P} = (P_1,\ldots,P_{r+s},(Q_i)_{i\in F})$ of points $P_i \in \mathbb{R}^2$ and lines $Q_i \in \mathbb{R}^2/\langle v(y_i)\rangle$. The *locus of conditions in general position* is the complement in $\mathbb{R}^{2(r+s)+|F|}$ of the union $\bigcup_\alpha ev_F(M^\alpha_{(r,s)}(\Delta))$ over all cells $M^\alpha_{(r,s)}(\Delta)$ in $M_{(r,s)}(\Delta)$ of dimension at most $2(r+s)+|F|-1$.

Oriented marked curves

Definition

An *oriented* (r,s)-marked curve consists of the data of an (r,s)-marked curve C and of a unique orientation of each unmarked edge of C. The set of *fixed ends* $F \subset \{1,\dots,n\}$ is the set of indices i of all the unmarked ends y_i oriented inwards. We denote by $M^{or}_{(r,s)}(\Delta,F)$ the set of oriented (r,s)-marked curves of degree Δ with set of fixed ends F. There is a forgetful map

$$ft : M^{or}_{(r,s)}(\Delta,F) \to M_{(r,s)}(\Delta)$$

which forgets the orientation of unmarked edges.

Remark

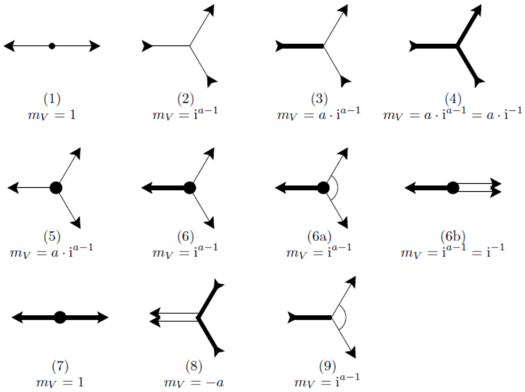
In the oriented case, there is an analogous evaluation map ev_F defined on $M^{or}_{(r,s)}(\Delta,F)$ and defining conditions in general position.

Vertex types

Conventions

- An oriented/non-oriented (r, s)-marked curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ will always be drawn as $h(\Gamma) \subset \mathbb{R}^2$.
- Even edges are displayed in bold and odd edges are drawn as thin lines.
- The images $h(x_i)$ of the markings are drawn as small dots if $i = 1, \ldots, r$ and as big dots if $i = r + 1, \ldots, r + s$.
- If two ends adjacent to the same vertex have the same direction, these ends will be drawn as two edges parallel to each other with a small distance in between; if they have not the same direction, this is indicated by an arc.

Vertex types



Here, *a* mostly denotes Mikhalkin's vertex multiplicity; in case (8), *a* denotes the absolute value of the determinant of the even direction vectors.

Broccoli and Welschinger curves

Definition

Let $C \in M^{or}_{(r,s)}(\Delta, F)$ be an oriented (r,s)-marked curve only composed of vertices as above. Then its multiplicity m_C is given by

$$m_C = \prod_{k=1}^{n} i^{\omega(y_k)-1} \prod_{V} m_V.$$

Definition

Let $C \in M^{or}_{(r,s)}(\Delta, F)$ be an oriented (r,s)-marked curve. Then C is called

a) an *oriented broccoli curve* if it is only composed of vertices of type (1) to (6).

b) an *oriented Welschinger curve* if it is only composed of vertices of type (1) to (5), (6b), (7) and (8).

Broccoli invariant

Definition

Let $M^B_{(r,s)}(\Delta, F)$ be the closure of broccoli curves in $M^{or}_{(r,s)}(\Delta, F)$ (which is a polyhedral subcomplex) and let $r + 2s + |F| = |\Delta| - 1$. If $\mathcal{P}$ is a collection of conditions in general position for ev_F, then the *broccoli invariant* (with respect to $\mathcal{P}$) is defined as

$$N^B_{(r,s)}(\Delta, F, \mathcal{P}) = \frac{1}{|G(\Delta, F)|} \sum_C m_C,$$

where the sum is taken over all broccoli curves C in $M^B_{(r,s)}(\Delta, F)$ such that $ev_F(C) = \mathcal{P}$ and $G(\Delta, F)$ is the subgroup of S_n of all permutations σ such that $\sigma(i) = i$ for $i \in F$ and $v(y_i) = v(y_{\sigma(i)})$ for all i.

Welschinger number

Definition

Let $M^W_{(r,s)}(\Delta, F)$ be the closure of Welschinger curves in $M^{or}_{(r,s)}(\Delta, F)$ (which is a polyhedral subcomplex) and let $r + 2s + |F| = |\Delta| - 1$. If $\mathcal{P}$ is a collection of conditions in general position for ev_F, then the tropical *Welschinger number* (with respect to $\mathcal{P}$) is defined as

$$N^W_{(r,s)}(\Delta, F, \mathcal{P}) = \frac{1}{|G(\Delta, F)|} \sum_C m_C,$$

where the sum is taken over all Welschinger curves C in $M^W_{(r,s)}(\Delta, F)$ such that $ev_F(C) = \mathcal{P}$ and $G(\Delta, F)$ is the subgroup of S_n of all permutations σ such that $\sigma(i) = i$ for $i \in F$ and $v(y_i) = v(y_{\sigma(i)})$ for all i.

Invariance

Theorem (Gathmann-Markwig-Schroeter)

The broccoli invariants are independent of the collection of conditions $\mathcal{P}$ and therefore we simply write them as $N^B_{(r,s)}(\Delta, F)$ or $N^B_{(r,s)}(\Delta)$ if $F = \varnothing$.

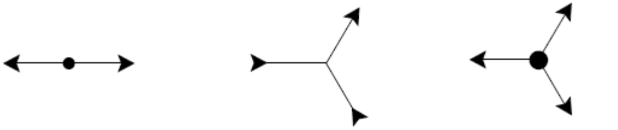
Theorem (Gathmann-Markwig-Schroeter)

If Δ is a toric del Pezzo degree, then $N^B_{(r,s)}(\Delta, F) = N^W_{(r,s)}(\Delta, F)$. In this case the Welschinger numbers are therefore independent of the collection of conditions $\mathcal{P}$ and we simply write them as $N^W_{(r,s)}(\Delta, F)$ or $N^W_{(r,s)}(\Delta)$ if $F = \varnothing$.

Oriented refined broccoli curves

Definition

An oriented (r, s)-marked curve is called *oriented refined broccoli curve* if it is composed of the vertex types (I), (II) and (III) below, where any parity of the adjacent edges is allowed. The *refined vertex multiplicity* is written below each vertex.



(I) $m_V(y) = 1$

(II) $m_V(y) = \frac{y^{a/2} - y^{-a/2}}{y^{1/2} - y^{-1/2}}$

(III) $m_V(y) = \frac{y^{a/2} + y^{-a/2}}{y^{1/2} + y^{-1/2}}$

Here, a denotes Mikhalkin's vertex multiplicity of the respective vertex type.

Oriented refined broccoli curves

Definition

Let $y_1, \dots, y_n$ be the unmarked ends of an oriented refined broccoli curve C. Then we let

- $m_{y_i}(y) = \dfrac{y^{\omega(y_i)/2} + (-1)^{\omega(y_i)} y^{-\omega(y_i)/2}}{y^{1/2} + (-1)^{\omega(y_i)} y^{-1/2}}$ if y_i is fixed
- $m_{y_i}(y) = \dfrac{y^{\omega(y_i)/2} - (-1)^{\omega(y_i)} y^{-\omega(y_i)/2}}{\omega(y_i)(y^{1/2} - (-1)^{\omega(y_i)} y^{-1/2})}$ if y_i is non-fixed.

We define the *refined multiplicity* of C as

$$m_C(y) = \prod_{i=1}^{n} m_{y_i}(y) \prod_{V \in C} m_V(y)$$

Terminology: Oriented broccoli curves in the sense of Gathmann-Markwig-Schroeter are called *old broccoli curves*.

Relationship with old broccoli curves

Theorem (Göttsche-Schroeter)

a) If C is an old broccoli curve, then it is also an oriented refined broccoli curve. In this case its multiplicity m_C equals the specialization $m_C(-1)$ of its refined multiplicity at $y = -1$.

b) Conversely, every oriented refined broccoli curve which is not an old broccoli curve satisfies $m_C(-1) = 0$.

Definition

Let C be a refined broccoli curve. The *broccoli index* of C is defined as $i_B(C) = -\#V_{cm} - \#E_f + \#V_{wcm} + \#E_n$, where

- E_f (resp. E_n) is the set of fixed (non-fixed) even ends
- V_{cm} (resp. V_{wcm}) is the set of vertices of even Mikhalkin multiplicity such that a complex marking (no complex marking) is adjacent to them.

Properties of the refined curve multiplicities

Theorem (Göttsche-Schroeter)

Let C be an oriented refined broccoli curve and let $\bar{N}$ be the set of non-fixed ends of C.

a) The refined curve multiplicity $m_C(y)$ is a symmetric Laurent polynomial in y, more precisely $m_C(y)\prod_{i\in\bar{N}}\omega(y_i)\in\mathbb{Z}[y^{\pm 1}]$ and $m_C(y)=m_C(y^{-1})$. In particular, if $\bar{N}=\varnothing$, then $m_C(y)\in\mathbb{Z}[y^{\pm 1}]$.

b) If C is an old broccoli curve, then $m_C(-1)\neq 0$.

c) If C is not an old broccoli curve, then $i_B(C)$ is even and strictly positive and $m_C(y)=(y^{\frac{1}{2}}+y^{\frac{-1}{2}})^{i_B(C)}f(y)$ for $f\in\mathbb{Q}[y^{\pm 1}]$ with $f(-1)\neq 0$.

Oriented refined broccoli invariant

Definition

Let $M^{rB}_{(r,s)}(\Delta, F)$ be the closure of refined broccoli curves in $M^{or}_{(r,s)}(\Delta, F)$ (which is a polyhedral subcomplex) and let $r + 2s + |F| = |\Delta| - 1$. If $\mathcal{P}$ is a collection of conditions in general position for ev_F, then the *refined broccoli invariant* (with respect to $\mathcal{P}$) is defined as

$$N^{rB}_{(r,s)}(y, \Delta, F, \mathcal{P}) = \frac{1}{|G(\Delta, F)|} \sum_C m_C(y),$$

where the sum is taken over all oriented refined broccoli curves C in $M^{rB}_{(r,s)}(\Delta, F)$ such that $ev_F(C) = \mathcal{P}$ and $G(\Delta, F)$ is the subgroup of S_n of all permutations σ such that $\sigma(i) = i$ for $i \in F$ and $v(y_i) = v(y_{\sigma(i)})$ for all i.

Invariance

Corollary (Göttsche-Schroeter)

We obtain the broccoli invariant $N^{B}_{(r,s)}(\Delta, F, \mathcal{P})$ if we set $y = -1$ in the refined broccoli invariant $N^{rB}_{(r,s)}(y, \Delta, F, \mathcal{P})$.

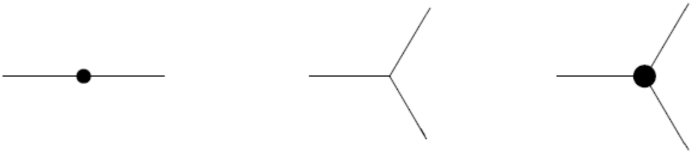
Theorem (Göttsche-Schroeter)

The refined broccoli invariants are independent of the collection of conditions $\mathcal{P}$ and therefore we simply write them as $N^{rB}_{(r,s)}(y, \Delta, F)$ or $N^{rB}_{(r,s)}(y, \Delta)$ if $F = \varnothing$.

Unoriented refined broccoli curves

Definition

An (r, s)-marked curve is called *unoriented refined broccoli curve* if it is composed of the vertex types (I'), (II') and (III') below, where any parity of the adjacent edges is allowed. The *refined vertex multiplicity* is written below each vertex.



(I') $m_V(y) = 1$

(II') $m_V(y) = \frac{y^{a/2} - y^{-a/2}}{y^{1/2} - y^{-1/2}}$

(III') $m_V(y) = \frac{y^{a/2} + y^{-a/2}}{y^{1/2} + y^{-1/2}}$

Here, a denotes Mikhalkin's vertex multiplicity of the respective vertex type.

Unoriented refined broccoli curves

Definition

Let $y_1, \ldots, y_n$ be the unmarked ends of a refined broccoli curve C and let $F \subset \{1, \ldots, n\}$ be such that $r + 2s + |F| = |\Delta| - 1$. Then we let

- $m_{y_i,F}(y) = \dfrac{y^{\omega(y_i)/2} + (-1)^{\omega(y_i)} y^{-\omega(y_i)/2}}{y^{1/2} + (-1)^{\omega(y_i)} y^{-1/2}}$ if $y_i \in F$
- $m_{y_i,F}(y) = \dfrac{y^{\omega(y_i)/2} - (-1)^{\omega(y_i)} y^{-\omega(y_i)/2}}{\omega(y_i)(y^{1/2} - (-1)^{\omega(y_i)} y^{-1/2})}$ if $y_i \notin F$.

We define the *refined multiplicity* of C as

$$m_{C,F}(y) = \prod_{i=1}^{n} m_{y_i,F}(y) \prod_{V \in C} m_V(y)$$

Unoriented refined broccoli invariant

Definition

Let $F \subset \{1, \ldots, n\}$ be such that $r + 2s + |F| = |\Delta| - 1$ and let $M^{urB}_{(r,s)}(\Delta)$ be the closure of unoriented refined broccoli curves in $M_{(r,s)}(\Delta)$ (which is a polyhedral subcomplex). If $\mathcal{P}$ is a collection of conditions in general position for ev_F, then the *unoriented refined broccoli invariant* (with respect to $\mathcal{P}$) is defined as

$$N^{urB}_{(r,s)}(y, \Delta, F, \mathcal{P}) = \frac{1}{|G(\Delta, F)|} \sum_C m_C(y),$$

where the sum is taken over all unoriented refined broccoli curves C in $M^{urB}_{(r,s)}(\Delta, F)$ such that $ev_F(C) = \mathcal{P}$ and $G(\Delta, F)$ is the subgroup of S_n of all permutations σ such that $\sigma(i) = i$ for $i \in F$ and $v(y_i) = v(y_{\sigma(i)})$ for all i.

Relationship to oriented refined broccoli curves

Lemma (Göttsche-Schroeter)

Let $r, s \geq 0$, let $\Delta = (v_1, \ldots, v_n)$ be a collection of vectors in $\mathbb{Z}^2 \setminus 0$ and let $F \subset \{1, \ldots, n\}$ be such that $r + 2s + |F| = |\Delta| - 1$. Moreover, let $\mathcal{P}$ be a collection of conditions in general position. Then there is a bijection between unoriented and oriented refined broccoli curves through $\mathcal{P}$ with degree Δ and set of fixed ends F.

Corollary (Göttsche-Schroeter)

We have $N^{urB}_{(r,s)}(y, \Delta, F, \mathcal{P}) = N^{rB}_{(r,s)}(y, \Delta, F, \mathcal{P})$.

Rational tropical descendant Gromov-Witten invariants

Definition

Let C be an m-marked rational tropical curve of degree $\Delta = (v_1, \dots, v_1, v_2, \dots, v_2, \dots, v_u, \dots, v_u)$, where v_i are distinct for $i = 1, 2, \dots, u$ and all the vertices not adjacent to a marking are 3-valent. Let $|\Delta| = n$ and let $F \subset \{1, \dots, n\}$.

- $\alpha = (\alpha_1, \alpha_2, \dots)$ is the sequence defined by $\alpha_i = \#\{v(y_j) \in \Delta | \; j \in F \textit{ and } v(y_j) \textit{ has weight } i\}$
- $I^\alpha = \prod_i i^{\alpha_i}$
- $m_C^{desc} = \prod_V m_V$, where the product is taken over all 3-valent vertices of C not adjacent to a marking.

Rational tropical descendant Gromov-Witten invariants

Definition (continued)

Let $\mathbf{k} = (k_1, k_2, \dots)$ be a sequence of non-negative integers with $\sum_i k_i = m$ and $0k_0 + 1(\alpha_1 + k_1) + \dots = |\Delta| - m - 1$ and choose a vector $\mathbf{a} = (a_1, \dots, a_m)$ containing the number i exactly k_i-times. Then the *rational tropical descendant Gromov-Witten invariant* is defined as

$$\tilde{N}^{trop}_{\Delta,\mathbf{k}}(\alpha) = \frac{1}{I^\alpha} \frac{1}{|G(\Delta, F)|} \sum_C m_C^{desc}$$

where the sum is taken over all m-marked rational tropical curves of degree Δ with markings $x_1, \dots, x_m$ such that $ev_F(C) = \mathcal{P}$ and x_i is adjacent to a vertex of valence $a_i + 3$ for all $i = 1, \dots, m$.

Remark

$\tilde{N}^{trop}_{\Delta,\mathbf{k}}(\alpha)$ does not depend on the choice of $\mathbf{a}$.

Tropical refined descendant curves

Definition

Let $\Delta = (v_1, \dots, v_n)$ and $F \subset \{1, \dots, n\}$. An (r,s)-marked curve $C \in M_{(r,s)}(\Delta)$ with set F of fixed ends is called *tropical refined descendant curve* if its markings satisfy the following conditions

- each of the r real markings has to be adjacent to a 3-valent vertex of the underlying graph Γ
- each of the s complex markings has to be adjacent to a 4-valent vertex of Γ.

Lemma (Göttsche-Schroeter)

Let $r, s \geq 0$, $\Delta = (v_1, \dots, v_n)$ and $F \subset \{1, \dots, n\}$ with $r + 2s + |F| = |\Delta| - 1$. For any collection of conditions $\mathcal{P}$ in general position, there is a bijection between unoriented refined broccoli curves and refined descendant curves through $\mathcal{P}$.

Tropical refined descendant invariants

Definition

Let $F \subset \{1, \ldots, n\}$ be such that $r + 2s + |F| = |\Delta| - 1$ and let $M^{desc}_{(r,s)}(\Delta, F)$ be the closure of refined descendant curves in $M_{(r,s)}(\Delta)$. If $\mathcal{P}$ is a collection of conditions in general position for ev_F, then the *tropical refined descendant invariant* is defined as

$$N^{desc}_{(r,s)}(y, \Delta, F, \mathcal{P}) = \frac{1}{|G(\Delta, F)|} \sum_C m_C(y),$$

where the sum is taken over all refined descendant curves C in $M^{desc}_{(r,s)}(\Delta, F)$ such that $ev_F(C) = \mathcal{P}$; the multiplicities are obtained from the lemma.

Corollary (Göttsche-Schroeter)

$N^{desc}_{(r,s)}(y, \Delta, F, \mathcal{P}) = N^{urB}_{(r,s)}(y, \Delta, F, \mathcal{P})$.

Relationship to tropical descendant Gromov-Witten invariants

Theorem (Göttsche-Schroeter)

One has

$$N^{desc}_{(r,s)}(1,\Delta,F,\mathcal{P}) = \frac{\prod_{i\in O_f}\omega(y_i)}{\prod_{i\in O_n}\omega(y_i)}\tilde{N}^{trop}_{\Delta,(r,s,0,\dots)}(\alpha)$$

where

- $\alpha = (\alpha_1, \alpha_2, \dots)$ is the sequence defined by $\alpha_i = \#\{v(y_j) \in \Delta | \; j \in F \textit{ and } v(y_j) \textit{ has weight } i\}$
- $O_f = \{i \in F | \; \omega(y_i) \textit{ is odd}\}$
- $O_n = \{i \notin F | \; \omega(y_i) \textit{ is odd}\}$

Other refinements

Question. Is there a way to define *refined Welschinger curves*?

The easiest definition for a refined Welschinger curve could be that it is a non-refined Welschinger curve, where a vertex type

- (1) counts with multiplicity 1
- (2), (3) and (4) are a special case of vertex type (II) and hence count with multiplicity $\dfrac{y^{a/2} - y^{-a/2}}{y^{1/2} - y^{-1/2}}$
- (5) or (6b) are a special case of vertex type (III) and hence count with multiplicity $\dfrac{y^{a/2} + y^{-a/2}}{y^{1/2} + y^{-1/2}}$
- (7) counts with multiplicity 1
- (8) counts with multiplicitiy $\dfrac{2(y^{a/2} - y^{-a/2})}{y - y^{-1}}$

Other refinements

The refined curve multiplicties would then specialize to the non-refined multiplicities.

However, it is not clear whether the *refined Welschinger numbers* - defined in the usual way - would be invariant, i.e. would not depend on a collection of conditions in general position.

This would probably require a *refined bridge algorithm* proving the equality of refined broccoli invariants and refined Welschinger numbers for toric del Pezzo degrees.

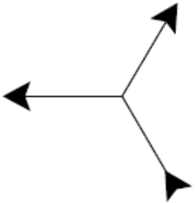
Without such an algorithm, refined Welschinger numbers could actually yield completely new refined invariants with a different enumerative meaning.

Other refined invariants

Question. Is there a way to define *refined bridge curves*?

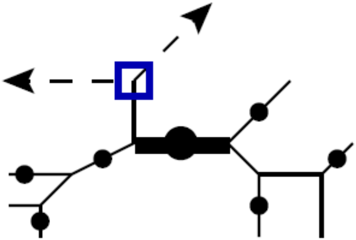
One should define a refined version of vertex type (9).

A natural choice would be to remove the restrictions on the parity of the weights and define a refined vertex type (9) of the form



However, if one uses this definition, the bridge algorithm does not terminate in general:

Other refined invariants



The string (dashed) can be moved up without hitting any other vertex of the curve. Hence the bridge algorithm would not work.

Even if one sticks to the old definition of vertex type (9), problems with counts of bridge curves come up.

Altogether, there is no obvious way how to define refined bridge curves or a refined bridge algorithm.

Example

Question: How is it possible to compute $N^{rB}_{(r,s)}(y,\Delta)$?

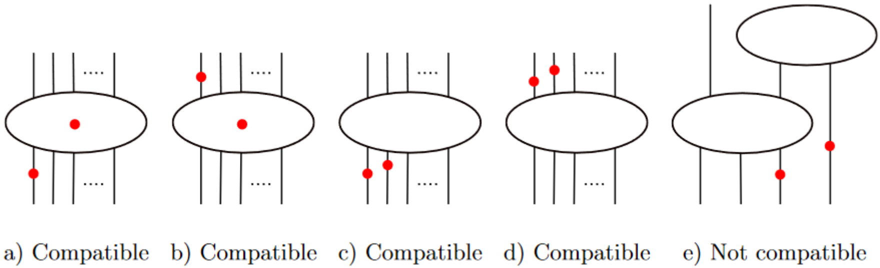
Idea: Systematically study floor diagrams and define refined multiplicities!

One considers the floor diagrams of tropical curves of degree Δ with $r+2s$ points: A pairing of order s on the $\{1,\ldots,r+2s\}$ is a set S of s disjoint pairs $\{i,i+1\}\subset\{1,\ldots,r+2s\}$.

A marking m of a floor diagram is called *compatible* with S if for any $\{i,i+1\}\in S$, the set $\{m(i),m(i+1)\}$ has the following form:

- an elevator and an adjacent floor
- two elevators that have a common adjacent floor such that both are either ending on this floor or emanating from this floor

Example


....
a) Compatible
b) Compatible
c) Compatible
d) Compatible
e) Not compatible

Example

Given a marked floor diagram $(\mathcal{D}, m)$ and a pairing S of order s on $\{1, \dots, r+2s\}$, one can define a refined S-multiplicity

$$\mu_S(\mathcal{D}, m)(y) \in \mathbb{Z}_{>0}[y^{\pm 1}],$$

which is by definition 0 if $(\mathcal{D}, m)$ is not compatible with S.

Theorem (Brugallé-Jaramillo Puentes)

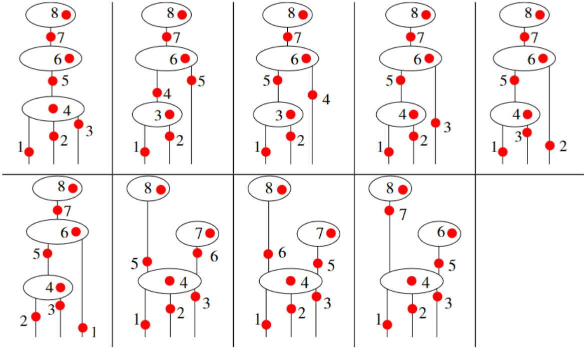
If $s \geq 0$, $r + 2s = |\partial\Delta \cap \mathbb{Z}^2| - 1$ and Δ is an h-transverse polygon and S is any pairing of order s on $\{1, \dots, r+2s\}$, then

$$N^{rB}_{(r,s)}(y, \Delta) = \sum_{(\mathcal{D},m)} \mu_S(\mathcal{D}, m)(y)$$

where the sum is taken over all isomorphism classes of marked floor diagrams with Newton polygon Δ and genus 0.

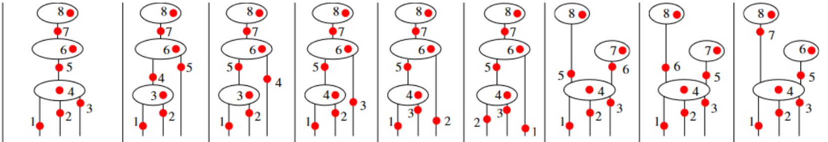
Example

For example, let $\Delta = \Delta_3$ and consider all the isomorphism classes of marked floor diagrams of rational cubics through 8 points.



Example

- $S = \varnothing$
- $S_1 = \{(7,8)\}$
- $S_2 = \{(5,6),(7,8)\}$
- $S_3 = \{(3,4),(5,6),(7,8)\}$
- $S_4 = \{(1,2),(3,4),(5,6),(7,8)\}$



μ	$q+2+q^{-1}$	1	1	1	1	1	1	1	1
μ_{S_1}	$q+2+q^{-1}$	1	1	1	1	1	0	0	1
μ_{S_2}	$q+q^{-1}$	1	1	1	1	1	0	0	1
μ_{S_3}	$q+q^{-1}$	1	0	0	1	1	0	0	1
μ_{S_4}	$q+q^{-1}$	1	0	0	0	0	0	0	1

Example

Theorem (Brugallé-Jaramillo Puentes)

- $N^{rB}_{(8,0)}(y,\Delta_3) = y + 10 + y^{-1}$
- $N^{rB}_{(6,1)}(y,\Delta_3) = y + 8 + y^{-1}$
- $N^{rB}_{(4,2)}(y,\Delta_3) = y + 6 + y^{-1}$
- $N^{rB}_{(2,3)}(y,\Delta_3) = y + 4 + y^{-1}$
- $N^{rB}_{(0,4)}(y,\Delta_3) = y + 2 + y^{-1}$