ON CHOW MOTIVES OF SURFACES

STEFAN GILLE

Abstract. We show that the Rost nilpotence principle holds for all surfaces over fields of characteristic zero in the category of Chow motives with integral coefficients.

0. Introduction

Let $k$ be a field. We denote by $\mathcal{C}h\text{om}(k, R)$ the category of Chow motives over $k$ with coefficients in a commutative ring $R$. In [12] it has been shown that if $\text{char } k = 0$ then Rost nilpotence holds for geometrically rational surfaces in $\mathcal{C}h\text{om}(k, \mathbb{Z})$. Recall what this means. If $\alpha$ is a correspondence of degree 0 on such a $k$-surface $S$, i.e. an element of the Chow group $\text{CH}_2(S \times_k S)$, such that $\alpha_E = 0$ in $\text{CH}_2(S_E \times_E S_E)$ for some field extension $E/k$ then there is an integer $N \geq 1$, such that $\alpha \circ N = 0$, where $\circ$ denotes the product of correspondences.

The aim of this work is to extend this result in two directions. First we want to show that the restriction to fields of characteristic zero is not necessary, and that the result is also true if the coefficient ring is a finite field. Second we want to prove the following generalization:

Theorem. Let $k$ be a field of characteristic 0 and $S$ a (smooth projective and geometrically integral) $k$-surface. Then Rost nilpotence is true for $S$ in the category of Chow motives with integral coefficients.

More precisely we show, that if $\alpha$ is a correspondence of degree 0 on a $k$-surface $S$, such that $\alpha_E = E \times_k \alpha = 0$ for some field extension $E/k$, then $\alpha \circ 9 = 0$.

The proof of this result uses a generalization of Bloch's [1] exact sequence (which has been used in [12] to prove the same result for geometrically rational surfaces) due to Colliot-Thélène and Raskind [7], as well as the “trivial” short exact sequence

$$0 \to \mathbb{K}_2^M(k) \to \mathbb{K}_2^M(k(W)) \to \mathbb{K}_2^M(k(W))/\mathbb{K}_2^M(k) \to 0,$$

where $W$ is an integral $k$-scheme and $\mathbb{K}_2^M(W)$ is the unramified $\mathbb{K}_2^M$-cohomology group of $W$.

The main labor is to show that these two sequences behave well with respect to push-forwards along correspondences of degree 0. This and the proof the main theorem above is done in Sections 1 and 2. Since by Coombes [8] any geometrically rational surface has a Galois splitting field we get as byproduct a verification of Rost nilpotence for geometrically rational surfaces over fields of positive characteristic.

Date: July 27, 2011.

2000 Mathematics Subject Classification. Primary: 14J26; Secondary: 14C15.

Key words and phrases. Chow motive, surface.

This work has been partially supported by the Deutsche Forschungsgemeinschaft, GI 706/1-2.
In Section 3 we show that Rost nilpotence is true for a geometrically rational surface $S$ in the category of Chow motives with a finite field $\mathbb{F}_q$, $q = p^r$ for $p$ a prime number, as coefficient ring. This follows from the integral coefficient case by elementary algebra using the fact that the Chow groups of $S_{\bar{k}} = k \times_k S$, where $\bar{k}$ is an algebraic closure of $k$, are free abelian groups of finite rank. Therefore this argument does not apply to arbitrary surfaces.

As a consequence we get that the Krull-Schmidt principle holds for geometrically rational surfaces in $\mathcal{ Chow}(k, \mathbb{F}_q)$ which allows us to define the upper $q$-motive of a geometrically rational surface. It turns out that the upper $q$-motive is a birational invariant of geometrically rational surfaces, see Theorem 6.

Acknowledgement. I would like to thank Burt Totaro for pointing out to me that the restriction to characteristic zero in [12] is not necessary. I would further like to thank Sasha Vishik and Kirill Zainoulline for useful discussions.

1. Preliminaries on cycle complexes

1.1. Notations and conventions. Let $k$ be a field. We denote by $\bar{k}$ an algebraic closure of $k$.

All schemes in this work are assumed to be of finite type over a field. If $L/k$ is a field extension and $X$ is a $k$-scheme we set $X_L := L \times_k X$.

By a $k$-surface we understand a smooth projective and geometrically integral $k$-scheme of dimension 2.

For a $k$-scheme $X$ we denote by $\mathbb{Z}(X)$ the free abelian group of dimension $i$ cycles on $X$, and by $\text{CH}_i(X)$ Chow group of dimension $i$ cycles modulo rational equivalence as defined in the book of Fulton [11]. If $L/k$ is a field extension the base change $X \mapsto X_L$ induces a restriction morphism $\text{res}_{L/k} : \mathbb{Z}(X) \to \mathbb{Z}(X_L)$.

The image of a cycle $\alpha$ under $\text{res}_{L/k}$ is denoted by $\alpha_L$. By abuse of notation we denote also by $\alpha$ the class of $\alpha \in \mathbb{Z}(X)$ in $\text{CH}_i(X)$.

If $Z \subseteq X$ is a closed subscheme we denote by $[Z]$ its class in the group of cycles on $X$, where $[Z]$ is defined as in [11, Sect. 1.5].

If the $k$-scheme $X$ is equidimensional we set $\text{CH}^i(X) := \text{CH}_{\dim X-i}(X)$.

The $n$-th Milnor $K$-group of a field $E$ is denoted by $K_n^M(E)$. Recall that $K_n^M(E) = \mathbb{Z}$, and $K_1^M(E) = E^\times$ is the multiplicative group of units of the field $E$.

If $E/F$ is a field extension we denote (following [17]) by $r_{E/F}$ the restriction homomorphism $K_n^M(F) \to K_n^M(E)$ and if the extension $E/F$ is finite by $e_{E/F}$ the corestriction or transfer map $K_n^M(E) \to K_n^M(F)$.

1.2. The complex $C^*(X, n)$. Let $X$ be a equidimensional $k$-scheme. We denote by $k(x)$ the residue field of $x \in X$ and set $X^{(i)} := \{ x \in X \mid \text{codim} |x| = i \}$. We consider the following (cohomological) cycle complexes $C^*(X, n)$, $n \in \mathbb{N}$:

$$\bigoplus_{x \in X^{(n)}} K_n^M(k(x)) \xrightarrow{d^M_k} \bigoplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \xrightarrow{d^M_k} \bigoplus_{x \in X^{(2)}} K_{n-2}^M(k(x)) \xrightarrow{d^M_k} \ldots ,$$
where the direct sum $\bigoplus_{x \in X^{(i)}} K^M_{n-i}(k(x))$ is in degree $i$ and the differentials are induced by second residue maps, see [14] and [17], or also the book [9], for the definition of this complex. We denote $C^*(X, n)[r]$ the shifted complex: $C^*(X, n)[r]^i = C^{i+r}(X, n)$ with no sign change in the differential, i.e. $d_X^r [r] = d_X^{r+r}$.

If $h : C^*(X, n) \rightarrow C^*(Y, m)[r]$ is a morphism of complexes we denote by $h^b$ the induced homomorphism $C^*(X, n) \rightarrow C^*(Y, m)$ in degree $i$. The cohomology group in degree $i$ of $C^*(X, n)$ is denoted by $H^i(X, K^M)$.

Recall that there is a natural isomorphism $H^i(X, K^M) \simeq CH^i(X)$.

1.3. Functorial properties of $C^*(X, n)$. Let $X, Y$ and $Z$ be three equidimensional $k$-schemes with $Y$ and $Z$ smooth and integral. Set $d = \dim Z$ and $c = \dim X$. Let further $f : X \rightarrow Z$ be a proper and $g : X \rightarrow Y$ be any morphism. Then we have, see [17, Sect. 4 and 11] or [9, Chap. IX],

(i) a push-forward morphism of complexes

$$f_* : C^*(X, n) \rightarrow C^*(Z, n + (d - c))[d - c],$$

and

(ii) a pull-back morphism of complexes $g^* : C^*(Y, n) \rightarrow C^*(X, n)$ which depends on the choice of a coordination $\tau$ of the tangent bundle $TY$ of the smooth $k$-scheme $Y$. (Recall that a coordination $\tau = \{X_i, \tau_i\}_{i=0}^{r}$ of an affine bundle $g : V \rightarrow X$ of rank $n$ consists of a filtration of closed subschemes $X = X_0 \supset X_1 \supset \ldots \supset X_r \supset X_{r+1} = \emptyset$ and trivialization maps $\tau_i : g^{-1}(X_i \setminus X_{i+1}) \rightarrow k^r \times_k (X_i \setminus X_{i+1})$ for $0 \leq i \leq r$, see [17, p. 371].)

If $g$ is flat of constant relative dimension the pull-back $g^*$ is independent of $\tau$ by [17, Prop. 12.2] and will be denoted by $g^*$ in this case.

The morphism of cohomology groups $H^i(Y, K^M) \rightarrow H^i(X, K^M)$ induced by $g^*$ also does not depend on the coordination $\tau$ and will be denoted by $g^*$ as well.

1.4. The (intersection) product on cycle complexes. If $Y$ is smooth $k$-scheme there is a product

$$H^i(Y, K^M) \times H^j(Y, K^M) \rightarrow H^{i+j}(Y, K^M), \quad (\alpha, \beta) \mapsto \alpha \cdot \beta,$$

which induces the usual (intersection) product on $CH^*(X) = H^*(X, K^M)$, see [9, Chap. 56].

We will have need for the following special case of the projection formula. Let $g : X \rightarrow Y$ be a proper morphism with $X$ and $Y$ equidimensional $k$-schemes, and $Y$ moreover smooth (as above). Then we have

$$g_*(g^*([X]) \cdot \alpha)$$

for all $\alpha \in H^i(Y, K^M)$, see [9, Prop. 56.11]. Note that it is not assumed $X$ to be smooth (an assumption which seems to be necessary for the general case of the projection formula).

1.5. We prove now a technical lemma which we need in 1.7 below.

To formulate it we denote for a $k$-scheme $W$ by $q_W$ the structure morphism $W \rightarrow \text{Spec } k$. This is a flat morphism and induces a pull-back morphism of
complexes $q^*_W : C^*(\text{Spec } k, n) \to C^*(W, n)$ for any integer $n \geq 0$. Note that $H^0(\text{Spec } k, K^M_n) = K^M_n(k)$, and that the induced homomorphism in degree zero

$$(q^*_W)^{[0]} : K^M_n(k) \to C^0(W, n) = \bigoplus_{w \in W^{(0)}} K^M_n(k(w))$$

is equal to $\sum_{w \in W^{(0)}} n_w r_{k(w)/k}$ for some integers $n_w \geq 0$, respectively to $r_{k(W)/k}$ if $W$ is integral, see [17, Sect. 3.5].

**Lemma.** Let $f : X \to Z$ be a proper and $g : X \to Y$ be an arbitrary morphism of $k$-schemes as in 1.3 (i.e. in particular that $Y$ and $Z$ are $k$-smooth and integral). Assume that $\dim X = \dim Z$. Then we have

(i) $\text{Im} \left( f^*[0] \circ \sum_{x \in X^{(0)}} r_{k(x)/k} \right) \subseteq \text{Im} r_{k(Z)/k}$, and

(ii) $\text{Im} \left( (g^*_Y)^{[0]} \circ r_{k(Y)/k} \right) \subseteq \text{Im} \left( \sum_{x \in X^{(0)}} r_{k(x)/k} \right)$ for any coordination $\tau$ of $TY$.

If $X$ is also integral then we have the equation $(g^*_Y)^{[0]} \circ r_{k(Y)/k} = r_{k(X)/k}$.

**Proof.** By the very definition, see [17, Sect. 3.4], the homomorphism $f^*[0]$ is either zero on $K^M_n(k(x))$ (if $\dim f([x]) < \dim Z$) or equal the corestriction $c_{k(x)/k(Z)}$. Hence we have (i).

To prove (ii) we consider the following diagram

$$
\begin{array}{ccc}
H^0(\text{Spec } k, K^M_n) &=& K^M_n(k) \\
&\xrightarrow{q^*_W}& H^0(Y, K^M_n) \\
\text{Im} (f^*[0]) &\subseteq& \text{Im} r_{k(Z)/k}, \quad \text{and} \\
&\xrightarrow{\tau^*}& H^0(X, K^M_n) \\
&\xrightarrow{(g^*_Y)^{[0]}}& \bigoplus_{x \in X^{(0)}} K^M_n(k(x)).
\end{array}
$$

This diagram commutes. The right hand square by the very definition and the left hand square by [17, Thm. 12.1]. Since the composition of the morphisms in the rows are equal to $(q^*_X)^{[0]} = r_{k(Y)/k}$ and to $(q^*_X)^{[0]} = \sum_{x \in X^{(0)}} n_x \cdot r_{k(x)/k}$ for some integers $n_x \geq 0$, respectively, we get the claimed inclusion.

The last assertion follows also from this diagram since in this case we have that $(q^*_X)^{[0]} = r_{k(X)/k}$. \hfill \Box

1.6. **Galois descent for $C^*(X, 2)$**. Let $X, Y$ and $Z$ be equidimensional $k$-schemes with $\dim X = \dim Z$, $f : X \to Z$ be a proper and $g : X \to Y$ an arbitrary morphism of $k$-schemes. We assume that $Y$ and $Z$ are smooth projective and geometrically integral. Let further $L$ be a Galois extension of $k$ with Galois group $G = \text{Gal}(L/k)$.

Let $\tau$ be a coordination of the tangent bundle $TY$ of the smooth $k$-scheme $Y$. The pull-back $\tau_L$ of this coordination along the projection $\pi_Y : Y_L \to Y$ is then a coordination of the tangent bundle of the smooth $L$-scheme $Y_L$.

Then it follows from the results of Rost [17], see [12, Sect. 3] for details, that $(g^*_Y)^{[1]} : C^i(Y_L, 2) \to C^i(X_L, 2)$ and $(f^*_x)^{[i]} : C^i(X_L, 2) \to C^i(Z_L, 2)$ are homomorphisms of $G$-modules. (Note that in [12, Sect. 3.6 and Thm. 3.7] the assumption
that $X$ is smooth is not necessary.) Moreover the following diagram commutes:

\[
\begin{array}{cccccc}
C^*(Y_L, 2) & & & & C^*(X_L, 2) & \xrightarrow{f_L} & C^*(Z_L, 2) \\
\pi_X & & & & \pi_X & & \pi_X \\
\downarrow & & & & \downarrow & & \downarrow \\
C^*(Y, 2) & & & & C^*(X, 2) & \xrightarrow{f} & C^*(Z, 2) ,
\end{array}
\]

where $\pi_X : X_L \rightarrow X$ and $\pi_Z : Z_L \rightarrow Z$ are the respective projections.

By the assumptions on $Y$ and $Z$ the pull-back morphisms

\[(\pi_Y)^*[i] : C^i(Y, 2) \rightarrow C^i(Y_L, 2) \text{ and } (\pi_Z)^*[i] : C^i(Z, 2) \rightarrow C^i(Z_L, 2)
\]

induce isomorphisms $C^i(Y, 2) \xrightarrow{\sim} C^i(Y_L, 2)^G$ and $C^i(Z, 2) \xrightarrow{\sim} C^i(Z_L, 2)^G$ for $i = 1, 2$, and so by the commutative diagram (2) above we have for $i = 1$ and $i = 2$:

\[
(f_L \circ \tau)^*[i] \circ (g_L^*[i])^G = (f_\tau)^*[i] \circ (g_L^*[i]).
\]

**1.7. An exact sequence.** We continue with the notation above, and denote for a $k$-scheme $W$ by $3_W$ and $3_W$ the kernel respectively the image of the differential $d_W^1 : C^1(W, 2) \rightarrow C^2(W, 2)$, i.e. we have an exact sequence

\[
0 \rightarrow 3_W \rightarrow C^1(W, 2) \xrightarrow{d_W^1} 3_W \rightarrow 0.
\]

With this notation we get from the upper row of the commutative diagram (2) a commutative diagram of $G$-homomorphisms with exact rows

\[
\begin{array}{cccccc}
0 & & & & K_2^M(L(Y))/H^0(Y_L, K_2^M) & \xrightarrow{f_L \circ \tau} & 3_Y_L \xrightarrow{(f_L \circ \tau)([0])} H^1(Y_L, K_2^M) \rightarrow 0 \\
& & & & (f_L \circ \tau)([1]) & \xrightarrow{f_L \circ \tau} & (f_L \circ \tau)([1]) \\
0 & & & & K_2^M(L(Z))/H^0(Z_L, K_2^M) & \xrightarrow{3_Z_L} & H^1(Z_L, K_2^M) \rightarrow 0
\end{array}
\]

for any coordination $\tau$ of $TY$. We remark that the column arrow on the right hand side $f_L \circ \tau : H^1(Y_L, K_2^M) \rightarrow H^1(Z_L, K_2^M)$ does not depend on the coordination $\tau$.

Taking Galois cohomology we get another commutative diagram with exact rows (which are part of the exact sequence of Colliot-Thélène and Raskind [7, Prop. 3.6])

\[
\begin{array}{cccccc}
H^1(G, K_2^M(L(Y)))/H^0(Y_L, K_2^M) & \xrightarrow{H^1(G, f_L \circ \tau)([0])} & H^1(G, 3_Y_L) & \xrightarrow{H^1(G, f_L \circ \tau)([1])} & H^1(G, H^1(Y_L, K_2^M)) \\
H^1(G, f_L \circ \tau)([0]) & \xrightarrow{H^1(G, f_L \circ \tau)([1])} & H^1(G, 3_Y_L) & \xrightarrow{H^1(G, f_L \circ \tau)([1])} & H^1(G, H^1(Y_L, K_2^M)) \\
H^1(G, K_2^M(L(Z)))/H^0(Z_L, K_2^M) & \xrightarrow{H^1(G, 3_Z_L)} & H^1(G, 3_Z_L) & \xrightarrow{H^1(G, 3_Z_L)} & H^1(G, H^1(Z_L, K_2^M))
\end{array}
\]

(4)

(here $G = \text{Gal}(L/k)$ and $H^i(G, -)$ denotes the Galois cohomology functor).

**1.8. Another exact sequence.** For $W$ a geometrically integral $k$-scheme we set

\[
B(W) := H^0(W_L, K_2^M)/K_2^M(L) \text{ and } D(W) := K_2^M(L(W))/H^0(W_L, K_2^M).
\]
By the lemma in 1.5 the homomorphism \((f_L \circ g^{\tau}_L)^{[0]} : K^M_2(L(Y)) \to K^M_2(L(Z))\) induces a morphism of exact sequences

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & B(Y) & \longrightarrow & K^M_2(L(Y))/K^M_2(L) & \longrightarrow & D(Y) & \longrightarrow & 0 \\
& & (f_L \circ g^{\tau}_L) & \downarrow & (f_L \circ g^{\tau}_L)^{[0]} & \downarrow & (f_L \circ g^{\tau}_L)^{[0]} & & \\
0 & \longrightarrow & B(Z) & \longrightarrow & K^M_2(L(Z))/K^M_2(L) & \longrightarrow & D(Z) & \longrightarrow & 0 \\
\end{array}
\] (5)

for any coordination \(\tau\) of \(TY\). Note here that the morphism

\[f_L \circ g^{\tau}_L : B(Y) = H^0(Y_L, K^M_2(L)) \to H^0(Z_L, K^M_2(L)) = B(Z)\]
does not depend on the coordination \(\tau\).

By a result of Colliot-Thélène [6, Thm. 1 (ii) and Rem. 5.2] we know that \(H^1(G, K^M_2(L(W))/K^M_2(L)) = 0\) if \(W/k \neq \emptyset\) for any smooth and geometrically integral \(k\)-scheme \(W\). Therefore we get from the commutative diagram (5) above by Galois cohomology the following fact.

**Lemma.** Let \(Y, Z\) and \(f, g\) be as in 1.6. If \(Y(k) \neq \emptyset \neq Z(k)\) then the following diagram commutes and has exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(G, K^M_2(L(Y))/H^0(Y_L, K^M_2(L)) & \to & H^2(G, H^0(Y_L, K^M_2(L))/K^M_2(L)) & \longrightarrow & 0 \\
& & H^1(G, (f_L \circ g^{\tau}_L)^{[0]}) & \downarrow & H^2(G, (f_L \circ g^{\tau}_L)^{[0]}) & \longrightarrow & 0 \\
0 & \longrightarrow & H^1(G, K^M_2(L(Z))/H^0(Z_L, K^M_2(L)) & \to & H^2(G, H^0(Z_L, K^M_2(L))/K^M_2(L)) & \longrightarrow & 0 \\
\end{array}
\] (6)

1.9. As observed in [7, Prop. 3.6] there is an isomorphism

\[\rho_W : A^2(W) := \text{Ker}(\text{res}_{L/k} : \text{CH}^2(W) \to \text{CH}^2(W_L)) \cong H^1(G, \mathfrak{I}_W)\]

for any smooth and geometrically integral \(k\)-scheme \(W\), where \(L/k\) is a Galois extension with group \(G\). We recall it’s construction.

We have, cf. 1.2, the equality \(H^2(W, K^M_2) = \text{CH}^2(W)\), and therefore a commutative diagram with exact rows (recall that we have set \(\mathfrak{I}_W := \text{Im}(C^1(W, 2) \xrightarrow{\delta^W_1} C^2(W, 2))\))

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{I}_W & \longrightarrow & C^2(W, 2) & \longrightarrow & \text{CH}^2(W) & \longrightarrow & 0 \\
& & (\pi^W_1)^{[2]} & \downarrow & (\pi^W_1)^{[2]} & \cong & \pi^W_1 & \downarrow & \\
0 & \longrightarrow & \mathfrak{I}^G_{W_L} & \longrightarrow & (C^2(W_L, 2))^G & \longrightarrow & \text{CH}^2(W_L)^G & ,
\end{array}
\] (7)

where \(\pi_W : W_L \to W\) denotes the projection, and the middle column is an isomorphism, see 1.6. We denote this diagram by \(\text{Diag}_1(W)\). By the snake lemma we get from it a natural isomorphism (the connecting homomorphism)

\[\delta^W_1 : A^2(W) \cong \text{Coker}(\pi^W_1) : \mathfrak{I}_W \to \mathfrak{I}^G_{W_L}\].

We apply this now to the situation of 1.6. As recalled there the morphisms of complexes \(f_{L*}\) and \(g^{\tau}_L\) are both \(G\)-equivariant and commute with the pullbacks along the base change morphisms, see (2). Therefore we have a morphism of
diagrams

\[ f_* \circ g_{L, \tau_L} : \text{Diag}_1(Y) \longrightarrow \text{Diag}_1(Z), \]

and so by the functorial properties of the connecting homomorphism we arrive at the commutative diagram

\[
\begin{array}{c}
\Lambda^2(Y) \xrightarrow{\delta^Y_2} \text{Coker} \left( (\pi^*_Y)^{[2]} : \mathcal{I}_Y \longrightarrow \mathcal{I}^G_{Y_L} \right) \\
f_* \circ g^* \downarrow \quad \downarrow (f_L \circ g^*_{L, \tau_L})^{[2]} \\
\Lambda^2(Z) \xrightarrow{\delta^Z_2} \text{Coker} \left( (\pi^*_Z)^{[2]} : \mathcal{I}_Z \longrightarrow \mathcal{I}^G_{Z_L} \right),
\end{array}
\]

where \((f_L \circ g^*_{L, \tau_L})^{[2]}\) denotes the homomorphism induced by \((f_L \circ g^*_{L, \tau_L})^{[2]} : \mathcal{I}^G_{Y_L} \longrightarrow \mathcal{I}^G_{Z_L} \).

On the other hand, applying the Galois cohomology functor to the exact \(G\)-module sequence

\[
0 \longrightarrow \mathfrak{I}_{W_L} \longrightarrow C^1(W_L, 2) \xrightarrow{d_{W_L}} \mathfrak{I}_{W_L} \longrightarrow 0
\]

we get an exact sequence

\[
0 \longrightarrow \mathfrak{I}^G_{W_L} \longrightarrow C^1(W_L, 2)^G \xrightarrow{(d_{W_L})^G} \mathfrak{I}^G_{W_L} \xrightarrow{\partial} H^1(G, \mathfrak{I}_{W_L}),
\]

and the homomorphism \(\partial\) is surjective since by Shapiro’s lemma and Hilbert 90 we have \(H^1(G, C^1(W_L, 2)) = 0\), cf. [7, Proof of Prop. 3.6]. Hence we have the following commutative diagram with exact rows

\[
\begin{array}{c}
0 \longrightarrow \mathfrak{I}_{W_L} \longrightarrow C^1(W_L, 2) \xrightarrow{d_{W_L}} \mathfrak{I}_{W_L} \longrightarrow 0 \\
(\pi^*_W)^{[1]} \downarrow \quad \downarrow (\pi^*_W)^{[1]} \quad (\pi^*_W)^{[2]} \downarrow \\
0 \longrightarrow \mathfrak{I}^G_{W_L} \longrightarrow C^1(W_L, 2)^G \xrightarrow{(d_{W_L})^G} \mathfrak{I}^G_{W_L} \xrightarrow{\partial} H^1(G, \mathfrak{I}_{W_L}),
\end{array}
\]

where the middle column arrow \((\pi^*_W)^{[1]} : C^1(W, 2) \longrightarrow C^1(W_L, 2)^G\) is an isomorphism (again by Shapiro’s lemma). We denote this diagram by Diag\(_2(W)\).

The connecting homomorphism \(\partial\) in Diag\(_2(W)\) induces an isomorphism

\[ \delta^W_2 : \text{Coker} \left( (\pi^*_W)^{[2]} : \mathcal{I}_W \longrightarrow \mathcal{I}^G_{W_L} \right) \cong H^1(G, \mathfrak{I}_{W_L}). \]

As above we get then in the situation of 1.6 a commutative diagram

\[
\begin{array}{c}
\text{Coker} \left( (\pi^*_Y)^{[2]} : \mathcal{I}_Y \longrightarrow \mathcal{I}^G_{Y_L} \right) \xrightarrow{\delta^Y_2 \cong} H^1(G, \mathfrak{I}_{Y_L}) \\
(f_L \circ g^*_{L, \tau_L})^{[1]} \downarrow \quad \downarrow (f_L \circ g^*_{L, \tau_L})^{[1]} \\
\text{Coker} \left( (\pi^*_Z)^{[2]} : \mathcal{I}_Z \longrightarrow \mathcal{I}^G_{Z_L} \right) \xrightarrow{\delta^Z_2 \cong} H^1(G, \mathfrak{I}_{Z_L}),
\end{array}
\]

The isomorphism \(\rho_W\) of [7, Prop. 3.6] is then defined as

\[ \rho_W := \delta^W_2 \cdot \delta^Y_2 : \Lambda^2(W) \cong H^1(G, \mathfrak{I}_{W_L}). \]

Putting diagrams (8) and (10) together we have proven
Lemma. Let $X, Y$ and $Z$ be equidimensional $k$-schemes with $Y$ and $Z$ smooth and geometrically integral, and $L/k$ a Galois extension with group $G$. Let further $f : X \to Z$ be a proper and $g : X \to Y$ an arbitrary $k$-morphism. Then the following diagram commutes

$$
\begin{array}{ccc}
A^2(Y) & \xrightarrow{\rho_Y} & H^1(G, \mathcal{F}_Y) \\
\downarrow f^* \cdot g^* & & \downarrow H^1(G, (f_L \cdot g_L)_L)^{(1)} \\
A^2(Z) & \xrightarrow{\rho_Z} & H^1(G, \mathcal{F}_Z)
\end{array}
$$

for any coordination $\tau$ of $TY$. In particular, the column map on the right hand side also does not depend on the coordination $\tau$ of $TY$.

1.10. We are in position to state and prove the main result of this section. Let $X_i$, $i = 1, 2, 3$, be three smooth projective and geometrically integral $k$-schemes. We assume that $d = \text{dim } X_2 = \text{dim } X_3$, and set $X_{ij} := X_i \times_k X_j$ for $1 \leq i < j \leq 3$. Let further $L/k$ be a Galois extension with group $G = \text{Gal}(L/k)$, and $\alpha \in \mathbb{Z}_d(X_2 \times_k X_3)$ a cycle. We can write

$$\alpha = \sum_{j=1}^r [W_j]$$

with $\iota_j : W_j \hookrightarrow X_2 \times_k X_3$ an integral subscheme of dimension $d$ for $j = 1, \ldots, r$.

We define morphisms $f(j)$ and $g(j)$ by the diagram

$$
\begin{array}{ccc}
X_1 \times_k W_j & \xrightarrow{f(j)} & X_1 \times_k X_2 \times_k X_3 \\
\downarrow g(j) & & \downarrow p_{13} \\
X_1 \times_k X_2 & \xrightarrow{\iota_j} & X_1 \times_k X_3,
\end{array}
$$

where the $p_{ij}$'s are the respective projections, for $j = 1, \ldots, r$. Note that $f(j)$ is proper for all $1 \leq j \leq r$.

Since $\alpha$ is a correspondence of degree 0 it induces a push-forward

$$\alpha_* : H^i(X_{12}, K_{12}^M) \to H^i(X_{13}, K_{13}^M)$$

for all $i \geq 0$ which does not depend on the class of $\alpha$ in $\text{CH}_d(X_2 \times_k X_3)$. By the projection formula we have

$$\alpha_* = \sum_{j=1}^r [W_j]_* = \sum_{j=1}^r f(j)_* \cdot g(j)_*^{(1)},$$

and since $[W_j L] = [W_j]_L$, see [11, Prop. 1.7], we have also

$$\alpha_{L*} = \sum_{j=1}^r f(j)_{L*} \cdot g(j)_L^{(1)}.$$
Theorem 1. With above notations we have for any coordination \( \tau \) of the tangent bundle of the smooth scheme \( X_{12} = X_1 \times_k X_2 \):

(i) The following diagram commutes and has exact rows

\[
\begin{array}{cccc}
\text{H}^1 \left( G, K^M_2 \left( L(X_{12}) \right) / \text{H}^0 \left( X_{12L}, K^M_2 \right) \right) & \longrightarrow & \text{A}^2 \left( X_{12} \right) & \longrightarrow & \text{H}^1 \left( G, \text{H}^1 \left( X_{12L}, K^M_2 \right) \right) \\
\downarrow \text{H}^1 \left( G, \sum_{j=1}^r (f(j) \cdot g(j) \Gamma L \tau L)^{[0]} \right) & & & & \downarrow \text{H}^1 \left( G, \alpha \left( \tau L \right) \right) \\
\text{H}^1 \left( G, K^M_2 \left( L(X_{13}) \right) / \text{H}^0 \left( X_{13L}, K^M_2 \right) \right) & \longrightarrow & \text{A}^2 \left( X_{13} \right) & \longrightarrow & \text{H}^1 \left( G, \text{H}^1 \left( X_{13L}, K^M_2 \right) \right) \\
\end{array}
\]

and

(ii) the diagram below commutes

\[
\begin{array}{cccc}
0 & \longrightarrow & \text{H}^1 \left( G, K^M_2 \left( L(X_{12}) \right) / \text{H}^0 \left( X_{12L}, K^M_2 \right) \right) & \longrightarrow & \text{H}^2 \left( G, \text{H}^0 \left( X_{12L}, K^M_2 / K^M_2 \right) \right) \\
\downarrow \text{H}^1 \left( G, \sum_{j=1}^r (f(j) \cdot g(j) \Gamma L \tau L)^{[0]} \right) & & & & \downarrow \text{H}^2 \left( G, \alpha \left( \tau L \right) \right) \\
0 & \longrightarrow & \text{H}^1 \left( G, K^M_2 \left( L(X_{13}) \right) / \text{H}^0 \left( X_{13L}, K^M_2 \right) \right) & \longrightarrow & \text{H}^2 \left( G, \text{H}^0 \left( X_{13L}, K^M_2 / K^M_2 \right) \right) \\
\end{array}
\]

and has moreover exact rows if \( (X_1 \times_k X_2 \times_k X_3)(k) \neq \emptyset \).

In particular, if \( X_2 = X_3 \) and \( (X_1 \times_k X_2)(k) \neq \emptyset \) then we have

\[
\alpha_*(\alpha_*(\beta)) = 0
\]

for all \( \alpha \in Z_m(X_2 \times_k X_3) \) with \( \alpha \Gamma L = 0 \) in \( \text{CH}_d((X_2 \times_k X_3)_L) \), and for all \( \beta \in \text{CH}^2(X_1 \times_k X_2) \) with \( \beta \Gamma L = 0 \) in \( \text{CH}^2((X_1 \times_k X_2)_L) \).

Proof. The claimed commutative diagrams are a direct consequence of 1.7, 1.8, and 1.9, and the remarks above.

The proof of the last assertion follows from (i) and (ii) by a diagram chase. \( \square \)

2. Chow-motives and the Rost nilpotence theorem for surfaces

2.1. We denote \( \mathcal{C}^{\text{Chow}}(k, R) \) the category of effective Chow motives with coefficients in some commutative ring \( R \). This is the idempotent completion of the category \( \mathcal{C}^0(k, R) \) of correspondences of degree 0. Recall that the objects of \( \mathcal{C}^0(k, R) \) are smooth and projective \( k \)-schemes and the morphisms are given by

\[
\text{Mor}_{\mathcal{C}^0(k, R)}(X, Y) := \bigoplus_{i=1}^s R \otimes_{\mathbb{Z}} \text{CH}_{\dim X_i}(X_i \times_k Y),
\]

where \( X_1, \ldots, X_s \) are the irreducible (and so connected) components of \( X \).

If \( M \) and \( N \) are objects in \( \mathcal{C}^{\text{Chow}}(k, R) \) then \( \text{Hom}_k(M, N)_R \) denotes the group of homomorphisms, i.e. correspondences of degree 0, between \( M \) and \( N \). If \( R = \mathbb{Z} \) we suppress the (subscript) \( R \) and write \( \mathcal{C}^{\text{Chow}}(k) \) and \( \text{Hom}_k(M, N) \) instead of \( \mathcal{C}^{\text{Chow}}(k, \mathbb{Z}) \) and \( \text{Hom}_k(M, N)_{\mathbb{Z}} \), respectively.
The twisted Tate motives in $\text{Chow}(k, R)$ are denoted $R(i)$ for $i \geq 0$ an integer.

2.2. Example. If $k = \bar{k}$ is an algebraically closed field, and $S$ a rational $k$-surface. Then

$$S \simeq R \oplus R(1) \oplus R(2)$$

in $\text{Chow}(k, R)$, where $r$ is the rank of the Picard group of $S$. This follows from the blow-up formula for Chow motives, see [15], or from the fact that the intersection pairings $\text{CH}_i(S) \times \text{CH}^i(S) \to \text{CH}_0(S) = \mathbb{Z}$ are perfect pairings for $i = 0, 1, 2$, see [12] for details and references.

2.3. Two restriction functors.

(i) Let $E/k$ be a field extension. The base change functor $X \mapsto X_E := E \times_k X$ induces the restriction functor

$$\text{res}_{E/k} : \text{Chow}(k, R) \to \text{Chow}(E, R), \ M \mapsto M_E \quad \text{and} \quad f \mapsto f_E,$$

where $f \in \text{Hom}_k(M, N)_R$.

(ii) Let $f : R \to S$ be a homomorphism of commutative rings. The canonical homomorphism $R \otimes_{\mathbb{Z}} \text{CH}_i(X) \to S \otimes_{\mathbb{Z}} \text{CH}_i(X)$ induced by $f$ gives rise to another restriction functor

$$\text{res}_{S/R} : \text{Chow}(k, R) \to \text{Chow}(k, S).$$

The main example here is $R = \mathbb{Z}$, $p$ a prime number, and $f : \mathbb{Z} \to \mathbb{Z}/\mathbb{Z}$ the quotient map.

2.4. Definition. We say that Rost nilpotence holds for $M$ in $\text{Chow}(k, R)$ if the kernel of the restriction morphism $\text{res}_{E/k} : \text{End}_k(M)_R \to \text{End}_E(M_E)_R$ consists of nilpotent correspondences for all field extensions $E/k$.

2.5. The Rost lemma. Rost nilpotence has been shown for projective quadrics by Rost [18] and holds more general for projective homogeneous varieties (arbitrary coefficient ring $R$), see [4] and also [3]. It is true for any $k$-scheme if $R$ is a field of characteristic $0$ since then the restriction homomorphism $\text{res}_{E/k} : R \otimes_{\mathbb{Z}} \text{CH}_i(X) \to R \otimes_{\mathbb{Z}} \text{CH}_i(X_E)$ is injective for all schemes $X$ and all field extensions $E/k$.

The proofs of Rost nilpotence for quadrics and projective homogeneous varieties in [3, 4, 18] use the following lemma of Rost [18] for which Brosnan [2] has given another (spectral sequence free) proof. Note however that Vishik [19, Lem. 3.10] has given another proof of Rost nilpotence for quadrics which does not use this lemma of Rost but Voevodsky’s triangulated category of motives.

Lemma. Let $W$ be a smooth projective $k$-scheme and $\alpha \in \text{End}_k(W)$. If

$$\alpha_k(w) \ast \left( \text{CH}_{\text{codim } \overline{w}}(W_{\overline{k}(w)}) \right) = 0$$

for all $w \in W$, then $\alpha \circ \text{dim } W + 1 = 0$.

To apply this lemma to geometrically rational surfaces we will make use of the following observation.

2.6. Lemma. Let $X$ be a smooth and projective $k$-scheme, and $\alpha \in \text{End}_k(X)$ be given, such that $\alpha_E = 0$ for some field extension $E/k$. If $\text{CH}_i(X)$ is torsion free then $\alpha_s(\text{CH}_i(X)) = 0$. 
Proof. Follows from the restriction-corestriction formula and the fact that Chow groups do not change under purely transcendental field extensions by [10, Prop. 2.1.8], see [12, Lem. 1.6] for details.

2.7. The Rost nilpotence theorem for surfaces. If char $k = 0$ then part (ii) of the following theorem is the main result of [12]. However the argument there works also in positive characteristics.

More precisely, in the proof of [12, Thm. 4.8] it is not necessary to use resolution of singularities in dimension two, since the pull-back of Chow groups and cycle complexes along a morphism $g : X \rightarrow Y$ is already defined if $Y$ is a smooth $k$-scheme, and the special case of the projection formula $g_*(g^*(\alpha)) = g_*(\alpha) = 0$ is true in this case as well as long as the $k$-scheme $X$ is equidimensional, cf. 1.4. Hence [12, Cor. 4.9] is true also for a non perfect field, and so in turn Rost nilpotence for geometrically rational surfaces in $\text{Chow}(k)$ for any field $k$. Nevertheless we give here another proof.

Theorem 2. Let $k$ be a field and $S$ a $k$-surface. Then Rost nilpotence is true for $S$ in $\text{Chow}(k)$, if

(i) $\text{char } k = 0$, or

(ii) the surface $S$ is geometrically rational.

2.8. Proof of Theorem 2. We first prove Theorem 2 in a special case and then deduce from it the general case.

Theorem 3. Let $k$ be a field of any characteristic, $S$ a $k$-surface and $\alpha \in \text{End}_k(S)$. If $S(k) \neq \emptyset$ and if there exists a Galois extension $L/k$ with $\alpha_L = 0$ then $\alpha^\circ 3 = 0$.

Proof. This is a straightforward application of Theorem 1 with $X_i = S$ for all $1 \leq i \leq 3$ and $G = \text{Gal}(L/k)$ since $\alpha^\circ 3 = \alpha_*(\alpha_*(\alpha))$.

From this the general case follows. Let $S$ be a $k$-surface, and $\alpha \in \text{End}_k(S)$, such that $\alpha_E = 0$ for some field extension $E/k$.

We assume first that $\text{char } k = 0$. We have a tower of fields $E \supseteq F \supseteq k$ with $F/k$ purely transcendental and $E/F$ separable (since $\text{char } k = 0$) algebraic. Since the restriction morphism

$$\text{res}_{F/k} : \text{End}_k(S) \rightarrow \text{End}_F(S_F)$$

is an isomorphism, see for instance [10, Prop. 2.1.8], we can assume that there exists an algebraic extension $F/k$, such that $\alpha_F = 0$.

Let $x \in S$. Then $\alpha_{F,k(x)} = 0$, where $F \cdot k(x)$ is the composition of $F$ and $k(x)$. Since $\text{char } k(x) = \text{char } k = 0$ there exists a Galois extension $L/k(x)$, such that $(\alpha_{k(x)})_L = 0$.

If $\text{char } k > 0$ and $S$ is a geometrically rational $k$-surface then by the main result of Coombes [8] there exists a Galois extension $L/k(x)$, such that $S_L$ is rational. By 2.11 below we know that then $\alpha_L = 0$.

Hence in any case there exists a Galois extension $L/k(x)$, such that $\alpha_L = 0$ for all $x \in S$. By Theorem 3 we have then $\alpha^{\circ 3}_{k(x)} = 0$ for all $x \in S$, and so by the Rost lemma, see 2.4, we get $\alpha^{\circ 9} = 0$. We are done.
In the course of proof we have shown the following more precise nilpotent exponent.

2.9. Corollary. Let \( S \) be a \( k \)-surface, which we assume to be geometrically rational if \( \text{char } k > 0 \), and \( \alpha \in \text{End}_k(S) \) a correspondence of degree 0 with \( \alpha_E = 0 \) for some field extension \( E/k \). Then we have

\[
\alpha^9 = 0.
\]

2.10. Remark. We don not know whether for a general surface this nilpotence exponent is sharp or not. However if \( S \) is a geometrically rational surface the proof of Rost nilpotence in [12] shows that if \( \alpha \in \text{End}_k(S) \) is given with \( \alpha_E = 0 \) for some field extension \( E/k \) then \( \alpha^6 = 0 \).

We have used the following (well-known?) fact.

2.11. Lemma. Let \( S \) be a rational \( k \)-surface, and \( E/k \) a field extension. Then for any (commutative) coefficient ring \( R \) the following restriction homomorphisms are injective:

- (i) \( \text{res}_{E/k} : R \otimes \mathbb{Z} \text{CH}_i(S) \rightarrow R \otimes \mathbb{Z} \text{CH}_i(S_E), i = 0, 1, 2, \text{ and} \)
- (ii) \( \text{res}_{E/k} : \text{End}_k(S)_R \rightarrow \text{End}_E(S_E)_R. \)

Proof. If \( S = \mathbb{P}^2_k \) this is a consequence of the projective bundle theorem. Otherwise we have a birational map \( \varphi : S \dashrightarrow \mathbb{P}^2_k \). By the results in [16, Thm. 21.1] there exists a resolution of the singularities for \( \varphi \), i.e. there is a commutative (where it makes sense) diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & S \\
\downarrow & & \downarrow \varphi \\
\mathbb{P}^2_k & \xrightarrow{\psi} & \mathbb{P}^2_k
\end{array}
\]

such that \( \phi = \phi_1 \circ \phi_2 \circ \ldots \circ \phi_r \) and \( \psi = \psi_1 \circ \psi_2 \circ \ldots \circ \psi_s \) for monoidal transformations \( \phi_i, 1 \leq i \leq r \), and \( \psi_j, 1 \leq j \leq s \), with centers in closed points. Hence it is enough to show:

*Let \( S \) be a \( k \)-rational surface and \( S' \) a blow-up of \( S \) in closed points of \( S \). Then the assertions of the lemma are true for \( S \) if and only if they are true for \( S' \).*

This is a consequence of the blow-up formula for Chow groups [11, Prop. 6.7] and the fact that if \( F/k \) is a finite field extension and \( E/k \) an arbitrary field extension then the restriction homomorphism

\[
\text{res}_{E/k} : R \otimes \mathbb{Z} \text{CH}_0(F) \rightarrow R \otimes \mathbb{Z} \text{CH}_0(E \otimes_k F)
\]

is injective for any commutative ring \( R \) (with 1).

\[\square\]

3. Rost nilpotence and the Krull-Schmidt principle for geometrically rational surfaces

3.1. We extend now Theorem 2 (ii) by means of elementary algebra to finite coefficients rings \( \mathbb{Z}/\mathbb{Z}m \) and finite fields as coefficients.

**Theorem 4.** Let \( k \) be a field, \( m \) an integer \( \geq 2 \), and \( S \) a geometrically rational \( k \)-surface. Then Rost nilpotence is true for \( S \) in \( \text{Chow}(k, \mathbb{Z}/\mathbb{Z}m) \).
Proof. By the Chinese reminder theorem it is enough to show that Rost nilpotence is true for \( S \) in \( \mathfrak{Chow}(k, \mathbb{Z}/\mathbb{Z}'p^l) \) for all prime numbers \( p \) and all integer \( l \geq 1 \).

If \( E/k \) is a field extension we set \( \text{end}_E(S_E) := \text{End}_E(S_E)\mathbb{Z}/\mathbb{Z}'p^l \) and denote by \( \bar{\alpha} \) the image of \( \alpha \in \text{End}_E(S_E) \) under the restriction map \( \text{res}_{S/\mathbb{Z}'}L : \text{End}_E(S_E) \rightarrow \text{End}_E(S_E)\mathbb{Z}/\mathbb{Z}'p^l \).

Let \( \bar{\alpha} \in \text{end}_L(S) \) be an endomorphism, such that \( \bar{\alpha}_E = 0 \) for some field extension \( E/k \). If \( L \) is a finite Galois splitting field of \( S \) (which exists by Coombes [8]), i.e. \( S_L \) is a rational surface, then we have \( \bar{\alpha}_L = 0 \) in \( \mathfrak{Chow}(L, \mathbb{Z}/\mathbb{Z}'p^l) \) by 2.11. Let \( G = \text{Gal}(L/k) \) be the Galois group.

Denote by \( F/k \) the fixed field of the \( p \)-Sylow subgroup of \( G \). Assume \( \bar{\alpha}_F = 0 \) in \( \mathfrak{Chow}(F, \mathbb{Z}/\mathbb{Z}'p^l) \). Then we have by the projection formula \( [F : k] \cdot \alpha = 0 \) in \( \text{end}_k(S) \), and so \( \bar{\alpha} = 0 \) since \( p \) does not divide \( [F : k] \). Therefore we can assume \( [L : k] = p^s \) for some integer \( s \geq 1 \).

We have then \( \alpha_L = p^l \cdot \beta \) for some \( \beta \in \text{End}_L(S_L) \). Since \( \text{End}_L(S_L) \) is a free abelian group of finite rank, by 2.2 and 2.11, the endomorphism ring \( \text{End}_L(S_L) = \text{End}_L(S_L)\mathbb{Z}/\mathbb{Z}'p^l \) is finite and therefore some power of \( \beta \) is an idempotent in this ring. If \( \beta^m \) idempotent also \( \beta^{nm} \) is so for all natural numbers \( n \) and therefore we can choose \( m \in \mathbb{N} \), such that (i) \( \beta^m \) is an idempotent, and (ii) we have \( 2lm - s \geq l \).

Then we have \( \beta^{2m} = \beta^m + p^l \cdot \gamma \) for some \( \gamma \in \text{End}_L(S_L) \), and so

\[
(\alpha^{2m})_L = \alpha^{2m} = p^{2lm} \cdot \beta^m + p^{l+2lm} \cdot \gamma.
\]

By applying corestriction and then restriction along \( L/k \) we get

\[
p^s \cdot [(\alpha^{2m})_L - p^{2lm - s} \cdot \text{cor}_{L/k}(\beta^m)_L - p^{l+2lm-s} \cdot \text{cor}_{L/k}(\gamma)_L] = 0
\]

in \( \text{End}_L(S_L) \). Since the \( \mathbb{Z} \)-module \( \text{End}_L(S_L) \) is torsion free this gives

\[
\text{res}_{L/k} \left( \alpha^{2m} - p^{2lm - s} \cdot \text{cor}_{L/k}(\beta^m) - p^{l+2lm-s} \cdot \text{cor}_{L/k}(\gamma) \right) = 0,
\]

and so by Theorem 2 (ii) we have

\[
(\alpha^{2m} - p^l \cdot \delta)^6 = 0,
\]

in \( \text{End}_k(S) \), where \( \delta = p^{2lm - s - l} \cdot \text{cor}_{L/k}(\beta^m) + p^{2lm - s} \cdot \text{cor}_{L/k}(\gamma) \). Therefore \( \alpha^{12m} = 0 \) in \( \text{end}_k(S) = \mathbb{Z}/\mathbb{Z}'p^l \otimes_{\mathbb{Z}} CH_2(S \times_k S) \). We are done. \( \square \)

Let \( \mathbb{F}_q \) be the field with \( q = p^f \), \( p \) a prime number, elements. Then the morphism \( \mathbb{F}_p \simeq \mathbb{Z}/\mathbb{Z}'p \twoheadrightarrow \mathbb{F}_q \) is flat and so Theorem 4 has the following corollary.

**Corollary.** Let \( k \) be a field (of any characteristic) and \( \mathbb{F}_q \) the field with \( q = p^f \), \( p \) a prime number, elements. Then Rost nilpotence is true for any geometrically rational surface in \( \mathfrak{Chow}(k, \mathbb{F}_q) \).

### 3.2. The Krull-Schmidt principle

Let \( M \in \mathfrak{Chow}(k, R) \). We say that the **Krull-Schmidt principle** holds for \( M \) if (i) \( M \) has a direct sum decomposition into indecomposable motives, and (ii) any such decomposition is unique up to permutation of summands and isomorphisms.

Karpenko [13, Sect. 2] observed that the arguments of Chernousov and Merkurjev [5] give the following result.
Theorem 5. Assume the coefficient ring $R$ is finite. Then the Krull-Schmidt principle holds in the pseudo-abelian subcategory of $\text{Chow}(k, R)$ which is generated by all motives for which Rost-nilpotence is true.

In particular, by Theorem 4 above the Krull-Schmidt principle holds for geometrically rational surfaces in $\text{Chow}(k, \mathbb{Z}/2m)$ for all integers $m \geq 2$, and in $\text{Chow}(k, \mathbb{F}_q)$ for all finite fields $\mathbb{F}_q$.

3.3. Upper motives. We start by recalling a definition. The multiplicity $\text{mult} \alpha$ of a correspondence $\alpha \in \text{Hom}_k(X, Y)_R$, where $X$ is an irreducible variety, is the ring element

$$\text{pr}_1^*(\alpha) \in R \otimes_{\mathbb{Z}} \text{CH}_{\dim X}(X) = R,$$

where $\text{pr}_1 : X \times_k Y \rightarrow X$ is the projection to the first factor. Note that $\text{mult} \alpha_L = \text{mult} \alpha$ for all field extensions $L/k$.

Definition. Let $X$ be a smooth projective and irreducible $k$-scheme. A summand $M = (X, \pi)$ of the motive of $X$ in $\text{Chow}(k, R)$ is called upper or an upper motive if $\text{mult} \pi = 1$.

3.4. The upper motive of a geometrically rational surface. Let now $S$ be a geometrically rational $k$-surface, and $R = \mathbb{F}_q$ for some prime power $q = p^r$. Then by Theorem 5 above the Krull-Schmidt principle holds for the motive of $S$ in $\text{Chow}(k, \mathbb{F}_q)$, and by 2.2 we have $\text{Pic} S_k \cong \mathbb{F}_q \oplus \mathbb{F}_q(1)^r \oplus \mathbb{F}_q(2)$ in $\text{Chow}(k, \mathbb{F}_q)$, where $r$ is the rank of $\text{Pic} S_k$. Therefore there is a unique indecomposable summand $M = (S, \pi)$ of $S$ in $\text{Chow}(k, \mathbb{F}_q)$, such that $M_k \cong \mathbb{F}_q \oplus N$ for some $N \in \text{Chow}(k, \mathbb{F}_q)$. This motive $M$ is an upper motive, and is called the upper $q$-motive of $S$ in $\text{Chow}(k, \mathbb{F}_q)$, or shorter (if $q$ is clear from the context) the upper motive of $S$. We denote it by $U_q(S)$.

Let now $S'$ be another $k$-surface which is birational to $S$. Then if the field $k$ is perfect, see e.g. [16, Sect. 21], there is a smooth surface $T$ and morphisms $\phi : T \rightarrow S$ and $\psi : T \rightarrow S'$, which are both iterations of monoidal transformations in closed points. If now $M$ is the upper $q$-motive of $S$ then by the blow-up formula, see [15], the motive $M$ is isomorphic (in $\text{Chow}(k, \mathbb{F}_q)$) to the upper $q$-motive of $T$ and so also to the one of $S'$. We have proven:

Theorem 6. Let $k$ be a perfect field and $q = p^l$ a prime power. Then the upper $q$-motive is a birational invariant of geometrically rational $k$-surfaces.

References

ON CHOW MOTIVES OF SURFACES


Stefan Gille, Mathematisches Institut, Universität München, Theresienstrasse 39, 80333 München, Germany

E-mail address: gille@mathematik.uni-muenchen.de