SYLLABUS FOR WORKSHOP ON UNSTABLE $\mathbb{A}^1$-HOMOTOPY.

There will be three courses consisting of three one hour lectures apiece. Each will be supported by some exercises. There will be an hour scheduled each day to work on these exercises and ask questions.

COURSE ONE: INTRODUCTION TO UNSTABLE MOTIVIC HOMOTOPY THEORY

This first course sets up the machinery necessary to do unstable motivic homotopy theory, and gives some basic theorems, such as the Purity theorem. For course 3, we introduce classifying spaces and BGL_n.

Lecture 1. Define simplicial presheaves, and simplicial sets if need be. Define simplicial model categories and infinity categories. Introduce Bousfield localization in both contexts. Introduce the Nisnevich topology, and Zariski and étale topologies if need be. Give the corresponding local rings, in particular, mentioning henselization. Give examples of important differences between these topologies (e.g. functors with Nisnevich but not étale descent). Then define (a few) simplicial model categories and the underlying infinity categories where we can do unstable motivic homotopy theory. Introduce local objects. With this work, we can import techniques from algebraic topology. References include [Jar87], [Bla01], [Isa05], and [DHI04], [Dug01], [Rob13], [GS09].

Lecture 2. Introduce bigraded spheres, with examples including $\mathbb{P}^1$ and $\mathbb{A}^n - \{0\}$ as push-outs. (If there are algebraic geometry folks, I'll even define the smash product.) Construct realization functors, and give examples such as the complex and real realizations of the Hopf map. Define Thom spaces. Give the construction of the deformation to the normal cone and prove the Purity theorem.

Lecture 3. Introduce K_*^{MW} and $GW(k)$. Construct the degree homomorphism, with examples. Define homotopy sheaves, and associated long exact sequences. State the computation of the unstable homotopy groups of spheres. Introduce classifying spaces and the difference between classifying étale and Nisnevich principal G-bundles. Do the example of BGL_n for Course 3, and state Morel/Asok–Hoyois–Wendt $\mathbb{A}^1$-representability theorems for vector bundles.

COURSE TWO: THE UNSTABLE CONNECTIVITY THEOREM

This part aims to cover the technical parts of [Mor12] necessary to state [Mor12, Theorem 6.38]. It will not be possible to cover this important theorem in full in the time we have, so we will content ourselves with outlining certain parts of the proof and how it fits together.

Lecture 1. In this lecture we will introduce the concept of strongly and strictly $\mathbb{A}^1$-invariant sheaves, the concept of contraction $M \mapsto M_{-1}$, and start to outline the Rost–Schmidt complex of a strongly $\mathbb{A}^1$-invariant sheaf of groups M on $\mathbf{Sm}_k$ and a scheme X.

Lecture 2. We will explain how the Rost–Schmidt complex implies that strong $\mathbb{A}^1$ invariance for a sheaf of abelian groups implies strict $\mathbb{A}^1$-invariance. We will present conditions sufficient to show a sheaf is strongly $\mathbb{A}^1$-invariant, and we will show that the $\mathbb{A}^1$-homotopy sheaves of a space are strongly or strictly $\mathbb{A}^1$-invariant where these terms apply.

Lecture 3. We will deduce necessary and sufficient conditions on homotopy sheaves for testing $\mathbb{A}^1$-locality, and we will deduce that $\mathbb{A}^1$-localization does not decrease connectivity of a space (the $\mathbb{A}^1$-connectivity theorem). We will begin the study of $\mathbb{A}^1$ fiber sequences, based on [AWW15, Section 2.3].

COURSE THREE: COMPUTATIONS AND APPLICATIONS

The goal of the third course will be to discuss applications of unstable homotopy theory to the classification of vector bundles and other torsors over smooth affine schemes. The main tools, which we will discuss below in greater detail, are Postnikov towers in the setting of unstable motivic homotopy theory, representability results after Morel and Asok-Hoyois-Wendt, stable range computations for homotopy groups of classifying spaces, unramified sheaves, and the various symplectic and orthogonal forms of K-theory that appear at the boundary of the stable range. Our aim is the following theorem due to Asok and Fasel [AF14].

Theorem 0.1. *Let X be a smooth affine 3-fold over an algebraically closed field of characteristic not 2. Then, The first and second Chern class maps induce a bijection*

$$\mathrm{Vect}_2(X) \simeq \mathrm{CH}^1(X) \times \mathrm{CH}^2(X),$$

where $\mathrm{Vect}_2(X)$ is the set of isomorphism classes of rank 2 vector bundles on X.

We describe the contents of the course in more detail below.

Lecture 1. In this lecture, we will state the theorem above of Asok and Fasel, placing the result in the context of a long line of work of Bass, Quillen, Suslin, Lindel, Popescu and many others on the classification of projective modules over regular commutative rings. Then, we will discuss, following Asok–Hoyois–Wendt [AHW15], a result of Morel that says that rank-n vector bundles on a smooth affine scheme over a field k are given by $\mathbb{A}^1$-homotopy classes of maps $X \to BGL_n$.

Lecture 2. This lecture will be dedicated to as complete as possible a computation of the low-degree homotopy sheaves of BGL_2, which we will need for the proof. Necessary ingredients include the theory of unramified sheaves as well as various symplectic forms of K-theory.

Lecture 3. We put the ingredients above together to give a detailed guided tour of the $\mathbb{A}^1$-Postnikov tower of BGL_n and a proof of the theorem. Time permitting, we will discuss some further problems related to applying the techniques of Asok and Fasel to the classification of torsors for non-special algebraic groups on smooth affine schemes. In particular, we will discuss the open problem of $\mathbb{A}^1$-homotopy invariance for PGL_n-torsors on smooth affine schemes over $\mathbb{C}$.

References

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