

ADDITIVE HIGHER CHOW GROUPS OF SCHEMES

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ABSTRACT. We show how to make the additive Chow groups of Bloch-Esnault, Rülling and Park into a graded module for Bloch's higher Chow groups, in the case of a smooth projective variety over a field. This yields a projective bundle formula as well as a blow-up formula for the additive Chow groups of a smooth projective variety.

In case the base-field admits resolution of singularities, these properties allow us to apply the technique of Guillén and Navarro Aznar to define the additive Chow groups “with log poles at infinity” for an arbitrary finite-type k -scheme X . This theory has all the usual properties of a Borel-Moore theory on finite type k -schemes: it is covariantly functorial for projective morphisms, contravariantly functorial for morphisms of smooth schemes, and has a projective bundle formula, homotopy property, Mayer-Vietoris and localization sequences.

Finally, we show that the regulator map defined by Park from the additive Chow groups of 1-cycles to the modules of absolute Kähler differentials of an algebraically closed field of characteristic zero is surjective, giving an evidence of conjectured isomorphism between these two groups.

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INTRODUCTION

Since Bloch's introduction of his higher Chow groups over last twenty years ago, Voevodsky's construction of a good triangulated category of motives, with the higher Chow groups as the corresponding *motivic cohomology* has given what was at first a provisional definition a solid foundation. However, when compared with algebraic K -theory, this theory of motivic cohomology still has at least one serious deficiency: it fails to see the difference between a non-reduced scheme X and its associated reduced closed subscheme X_{red} .

One important case of a non-reduced scheme is of course that of the infinitesimal thickening of a reduced scheme. S. Bloch and H. Esnault have introduced in [7] the first attempt at a cycle-theoretic description of this motivic cohomology 'with modulus', defining groups which are

supposed to describe the part given by zero-cycles for $k[t]/t^2$. They showed that these groups coincide with the absolute differential forms Ω_k^n . In [8] they defined a cubical version of these *additive* Chow groups, extending the definition to zero-cycles for $k[t]/t^{m+1}$, and showed that for modulus $m = 1$ these groups agreed with the ones defined earlier.

K. Rülling [29] generalized these computations to arbitrary modulus m , showing that the zero-cycles with modulus m leads to a cycle theoretic explanation the generalized deRham-Witt complex of Hasselholt-Madsen. Park [27] extended the definition of zero-cycles with modulus m to a full cycle complex, and made some computations for 1-cycles.

Our aim in this paper is to understand the structural properties of Park's additive cycle complexes for arbitrary modulus. For a general smooth quasi-projective variety, these complexes do not have the nice behavior of Bloch's cycle complexes. The homotopy property quite obviously fails, but more seriously, these complexes do not have any reasonable localization property with respect the closed subschemes (even smooth ones). However, for smooth projective varieties, they seem to have interesting properties. Our first main technical result is the construction of a module structure for the additive Chow groups (with arbitrary modulus) of X over Chow ring of X leading immediately to an action of correspondences (for smooth projective schemes over a smooth k -scheme S) on the additive Chow groups. This gives us by standard constructions a projective bundle formula for arbitrary smooth X and a blow-up formula for the additive Chow groups of a smooth projective X .

If we now assume that our base field k admits resolution of singularities, we may apply the method of Guillén and Navarro Aznar to define the additive Chow groups "with log poles at infinity" for an arbitrary finite-type k -scheme X . This construction requires a detour through an integral version of Hanamura's triangulated category of motives. The resulting theory is a Borel-Moore theory, in that it is covariantly functorial for projective morphisms in \mathbf{Sch}_k . We add to the general theory of Guillén and Navarro Aznar and its application by Hanamura in that we construct in addition pull-back (Gysin) maps for morphisms of smooth quasi-projective varieties. This theory of additive Chow groups (for arbitrary modulus) with log poles has all nice structural properties one could hope for: projective bundle formula, homotopy property, Mayer-Vietoris and localization sequences.

As an application of this action of 'higher correspondences', we show that the regulator maps [27] from the additive Chow groups of 1-cycles to the modules of absolute Kähler differentials of an algebraically closed

field of characteristic zero is surjective, giving an evidence of conjectured isomorphism between these two groups.

Our hope is that the Atiyah-Hirzebruch spectral sequence for the algebraic K -theory of a smooth k -scheme X :

$$E_2^{p,q} := \mathrm{CH}^{-q}(X, -p - q) \implies K_{-p-q}(X)$$

(see [9, 12, 23]) has an analog with modulus and log poles:

$$E_2^{p,q} := \mathrm{TCH}_{\log}^{-q}(X, -p - q; m) \implies K_{-p-q}^{\log}(X; m);$$

here $\mathrm{TCH}_{\log}^{-q}(X, -p - q; m)$ is the logarithmic additive Chow group with modulus m (see § 6.2 for details). For X smooth and projective over k , $K^{\log}(X; m)$ is defined as the (-1 connected) homotopy fiber of the restriction map

$$K(X \times \mathrm{Spec} k[t]) \rightarrow K(X \times \mathrm{Spec} k[t]/(t^{m+1}));$$

in general, the logarithmic K -groups are at present not even defined, so the “correct” definition is part of the conjecture. At the very least, the groups $K_n^{\log}(X; m)$ should be the homotopy groups of a spectrum $K^{\log}(X; m)$, the assignment $X \mapsto K^{\log}(X; m)$ should extend to a functor on at least smooth quasi-projective k -schemes, covariant for projective maps and contravariant for arbitrary maps (where we consider $K^{\log}(X; m)$ as an object in the stable homotopy category of spectra). In addition, the spectra $K^{\log}(X; m)$ should satisfy an \mathbb{A}^1 homotopy invariance property, Nisnevich excision, a projective bundle formula and a blow-up formula.

Our paper is organized as follows. We begin with a general discussion of cubical objects in an additive or abelian category, then recall the definition and basic properties of the cubical version of Bloch’s cycle complex. In section 1.4 we recall the “Chow’s moving lemma” for Bloch’s cycle complex, as the techniques used to prove this result will form the core of the technical results here. Next, in section 1.5 we show how to refine the moving lemma to take into account behavior on the closure of the algebraic n -cubes, as is needed for applications to the additive cycle complexes. We recall the classical category of Chow motives and Hanamura’s construction of a triangulated category of motives in section 2; we also adapt Hanamura’s construction to our purposes in this section. We also show at this point how the constructions of Guillén and Navarro Aznar allow one to define the (Borel-Moore) motive of an arbitrary finite-type k -scheme, assuming one has resolution of singularities, and we finish the section by construction of the pull-back maps for morphisms in the category \mathbf{Sm}/k of smooth quasi-projective varieties over k .

In section 3, we introduce our main object of study, the additive cycle complexes for arbitrary modulus, together with their elementary properties. We construct the action of higher Chow groups in section 4, and extend this in section 5 to define the additive higher Chow groups of a Chow S -motive. This leads to the projective bundle formula and the blow-up formula. In section 6, we use the method of Guillén and Navarro Aznar to define the logarithmic additive higher Chow groups of a scheme of finite type over k , assuming k admits resolution of singularities; the theory is covariant for projective maps and contravariant for arbitrary maps of smooth varieties, satisfies a projective bundle formula, homotopy property, localization and Mayer-Vietoris and a blow-up formula. Finally, in section 7 we study the regulator maps from the additive Chow group of 1-dimensional cycles to the modules of absolute Kähler differentials of a characteristic zero field, as constructed by Park [27]. We establish some properties of the regulator maps and then use the results of section 4 to show the surjectivity of these regulator maps.

1. CYCLE COMPLEXES

1.1. Cubical objects. We introduce the “cubical category” **Cube**. This is the subcategory of **Sets** with objects $\underline{n} := \{0, \infty\}^n$, $n = 0, 1, 2, \dots$, and morphisms generated by

1. Inclusions: $\eta_{n,i,\epsilon} : \underline{n-1} \rightarrow \underline{n}$, $\epsilon = 0, \infty$, $i = 1, \dots, n$

$$\eta_{n,i,\epsilon}(y_1, \dots, y_{n-1}) = (y_1, \dots, y_{i-1}, \epsilon, y_i, \dots, y_{n-1})$$

2. Projections: $p_i : \underline{n} \rightarrow \underline{n-1}$, $i = 1, \dots, n$.

3. Permutations of factors: $(\epsilon_1, \dots, \epsilon_n) \mapsto (\epsilon_{\sigma(1)}, \dots, \epsilon_{\sigma(n)})$ for $\sigma \in S_n$.

4. Involutions: $\tau_{n,i}$ exchanging 0 and ∞ in the i th factor of \underline{n} .

Clearly all the Hom-sets in **Cube** are finite. For a category \mathcal{A} , we call a functor $F : \mathbf{Cube}^{\text{op}} \rightarrow \mathcal{A}$ a *cubical object* of \mathcal{A} .

Example 1.1. Let k be a field and set $\mathbb{P}^1 := \text{Proj}k[Y_0, Y_1]$, and let $y := Y_1/Y_0$ be the standard coordinate function on \mathbb{P}^1 . We set $\square^n := (\mathbb{P}^1 \setminus \{1\})^n$.

Letting $p_i : (\mathbb{P}^1)^n \rightarrow (\mathbb{P}^1)^{n-1}$ be the projection, we use the (rational) coordinate system (y_1, \dots, y_{n-1}) on \square^n , with $y_i := y \circ p_i$.

A *face* of \square^n is a subscheme F defined by equations of the form

$$y_{i_1} = \epsilon_1, \dots, y_{i_s} = \epsilon_s; \quad \epsilon_j \in \{0, \infty\}.$$

Let $\eta_{n,i,\epsilon} : \square^{n-1} \rightarrow \square^n$ be the inclusion

$$\eta_{n,i,\epsilon}(y_1, \dots, y_{n-1}) = (y_1, \dots, y_{i-1}, \epsilon, y_i, \dots, y_{n-1})$$

Thus $n \mapsto \square^n$ is a functor $\square : \mathbf{Cube} \rightarrow \mathbf{Sm}/k$. □

Let \mathcal{A} be an abelian category, $\underline{A} : \mathbf{Cube} \rightarrow \mathcal{A}$ be a cubical object. Let $\pi_{n,i} : \underline{A}(n) \rightarrow \underline{A}(n)$ be the endomorphism $p_i^* \circ \eta_{n,i,\infty}^*$, and set

$$\pi_n := (\text{id} - \pi_{n,n}) \circ \dots \circ (\text{id} - \pi_{n,1}).$$

Let

$$\underline{A}(n)_0 := \bigcap_{i=1}^n \ker \eta_{n,i,\infty}^* \subset \underline{A}(n)$$

and

$$\underline{A}(n)_{\text{degn}} := \sum_{i=1}^n p_i^*(\underline{A}(n-1)) \subset \underline{A}(n).$$

Finally let (\underline{A}_*, d) be the complex with $\underline{A}_n := \underline{A}(n)$ and with

$$d_n := \sum_{i=1}^n (-1)^i (\eta_{n,i,\infty}^* - \eta_{n,i,0}^*).$$

The following result is the basis of all ‘‘cubical’’ constructions; the proof is elementary and is left to the reader.

Lemma 1.2. *Let $\underline{A} : \mathbf{Cube} \rightarrow \mathcal{A}$ be a cubical object in an abelian category \mathcal{A} . Then*

1. For each n , π_n maps $\underline{A}(n)$ to $\underline{A}(n)_0$ and defines a splitting

$$\underline{A}(n) = \underline{A}(n)_{\text{degn}} \oplus \underline{A}(n)_0.$$

2. $d_n(\underline{A}(n)_{\text{degn}}) = 0$, $d_n(\underline{A}(n)_0) \subset \underline{A}(n-1)_0$

Definition 1.3. Let $\underline{A} : \mathbf{Cube} \rightarrow \mathcal{A}$ be a cubical object in an abelian category \mathcal{A} . Define the complex (\underline{A}_*, d) to be the quotient complex $n \mapsto (\underline{A}(n)/\underline{A}(n)_{\text{degn}}, d_n)$ of \underline{A}_* , d .

The lemma shows that (\underline{A}_*, d) is well-defined and is isomorphic to the subcomplex $n \mapsto (\underline{A}(n)_0, d_n)$ of \underline{A}_* .

1.2. Bloch’s cycle complex. We recall the cubical version of Bloch’s cycle complexes.

Definition 1.4 (cf. [6]). Let X be k -scheme of finite type. For $r \in \mathbb{Z}$, $n \geq 0$, we let $\underline{z}_r(X, n)$ be the free abelian group on integral closed subschemes Z of $X \times \square^n$ of dimension $r + n$ over k such that:

(Good position) For each face F of \square^n , Z intersects $X \times F$ properly:

$$\dim_k Z \cap X \times F \leq r + \dim_k F$$

it follows directly from the definition of $z_r(X, n)$ that for $\epsilon = 0, \infty$, $\partial_{n,i}^\epsilon := \eta_{n,i,\epsilon}^*$ gives a well-defined map

$$\partial_{n,i}^\epsilon : z_r(X, n) \rightarrow z_r(X, n-1),$$

so we have a cubical abelian group $\underline{n} \mapsto z_r(X, n)$.

Definition 1.5. Let

$$z_r(X, n) := z_r(X, n) / z_r(X, n)_{\text{degn}}.$$

The complex $(z_r(X, *), d)$ associated to the cubical abelian group

$$n \mapsto z_r(X, n)$$

is the *cubical cycle complex* of dimension r cycles on X . Set

$$\text{CH}_r(X, n) := H_n(z_r(X, *)).$$

If X is equi-dimensional of dimension d over k , set

$$z^i(X, *) := z_{i-r}(X, *)$$

and

$$\text{CH}^i(X, n) := H_n(z^i(X, *)).$$

Recall from [6] the original simplicial version of this construction, Bloch's cycle complex $\mathcal{Z}_r(X, *)$ and the *higher Chow groups*

$$\text{CH}_r(X, n) := H_n(\mathcal{Z}_r(X, *)).$$

Theorem 1.6 ([24]). *There is a natural isomorphism in $D^-(\mathbf{Ab})$*

$$z_r(X, *) \cong \mathcal{Z}_r(X, *),$$

hence a natural isomorphism

$$\text{CH}_r(X, n) \cong H_n(\mathcal{Z}_r(X, *)).$$

Remark 1.7. Both cycle complexes $\mathcal{Z}_r(X, *)$ and $z_r(X, *)$ are covariantly functorial for proper morphisms, and contravariantly functorial for flat morphisms (with a shift in r). The naturality in the above comparison theorem is with respect to these two functorialities. \square

1.3. Products. The main advantage the cubical complexes have over the simplicial version is the existence of an associative external product on the level of complexes.

For $X, Y \in \mathbf{Sch}_k$, let

$$\tau : X \times \square^n \times Y \times \square^m \rightarrow X \times Y \times \square^n \times \square^m$$

be the exchange-of-factors isomorphism. For integral cycles Z on $X \times \square^n$, W on $Y \times \square^m$, define

$$Z \boxtimes W := \tau_*(Z \times W)$$

where $Z \times W$ the cycle associated to the subscheme $Z \times_k W$. Extending by linearity gives us

$$\boxtimes_{n,m} : \underline{z}_r(X, n) \otimes \underline{z}_s(Y, m) \rightarrow \underline{z}_{r+s}(X \times_k Y, n + m).$$

Lemma 1.8 (cf. [24]). *The maps $\boxtimes_{n,m}$ define a map of complexes*

$$\boxtimes_{X,Y} : z_r(X, *) \otimes z_s(Y, *) \rightarrow z_{r+s}(X \times_k Y, *)$$

called external product. The external product is associative. Up to a natural homotopy, the external product is commutative.

Proof. It is clear that

$$\begin{aligned} \boxtimes_{n,m} (\underline{z}_r(X, n)_{\text{degn}} \otimes \underline{z}_s(Y, m) + \underline{z}_r(X, n)_{\text{degn}} \otimes \underline{z}_s(Y, m)) \\ \subset \underline{z}_{r+s}(X \times_k Y, n + m)_{\text{degn}}. \end{aligned}$$

The fact that \boxtimes is a map of complexes follows directly from the definition of the boundary maps. the associativity is obvious from the definition.

The homotopy commutativity is proved in [24]. \square

1.4. Moving lemmas. We recall the application of the classical moving lemma of Chow to Bloch's cycle complex. For $X \in \mathbf{Sch}_k$, let X_{sm} be the largest open subscheme of X which is smooth over k .

Definition 1.9. For $X \in \mathbf{Sch}_k$ equi-dimensional over k , let \mathcal{C} be a finite set of locally closed irreducible subsets of X_{sm} and let $e : \mathcal{C} \rightarrow \mathbb{Z}_+$ be a function. We define $\underline{z}^r(X, n)_{\mathcal{C},e} \subset \underline{z}^r(X, n)$ to be the subgroup generated by integral closed subschemes $Z \subset X \times \square^n$ such that, for each $C \in \mathcal{C}$ and each face F of \square^n ,

$$\text{codim}_{C \times F} Z \cap (C \times F) \geq r - e(C).$$

The $\underline{z}^r(X, n)_{\mathcal{C},e}$ form a subcomplex $\underline{z}^r(X, *)_{\mathcal{C},e}$ of $\underline{z}^r(X, n)$. In addition, for $z_1, \dots, z_n \in \underline{z}^r(X, n-1)$,

$$\sum_i (\text{id} \times p_i)^*(z_i) \in \underline{z}^r(X, n)_{\mathcal{C},e} \Leftrightarrow z_i \in \underline{z}^r(X, n-1)_{\mathcal{C},e} \text{ for all } i = 1, \dots, n.$$

Thus, with the evident definition of $\underline{z}^r(X, *)_{\mathcal{C},e,\text{degn}} \subset \underline{z}^r(X, *)_{\mathcal{C},e}$, the map

$$\frac{\underline{z}^r(X, *)_{\mathcal{C},e}}{\underline{z}^r(X, *)_{\mathcal{C},e,\text{degn}}} \rightarrow \frac{\underline{z}^r(X, *)}{\underline{z}^r(X, *)_{\text{degn}}} = z^r(X, *)$$

is injective, and we have the subcomplex $z^r(X, *)_{\mathcal{C},e}$ of $z^r(X, *)$.

The second moving lemma is

Theorem 1.10. *Let X be a quasi-projective k -scheme, equi-dimensional over k , \mathcal{C} a finite set of locally closed subsets of X_{sm} and $e : \mathcal{C} \rightarrow \mathbb{Z}_+$ a function. Then the inclusion $z^r(X, *)_{\mathcal{C}, e} \rightarrow z^r(X, *)$ is a quasi-isomorphism.*

As we will be using these techniques later on, we recall the idea of the proof, which goes in three steps:

Step 1: $X = \mathbb{P}^n$. In this case, one use the action of $G := \mathrm{SL}_{n+1}$ on \mathbb{P}^n . Let $K = k(G)$, a pure transcendental extension of k . Since K is a field, we can express the generic matrix $\eta \in \mathrm{SL}_{n+1}$ over k as a product of elementary matrices with K -coefficients.

Thus, there is a map

$$\phi : \mathbb{A}_K^1 \rightarrow G$$

with $\phi(0) = \mathrm{id}$, $\phi(1) = \eta$. Via the action of G on \mathbb{P}^n , we have the map

$$\mu_\phi : \mathbb{P}^n \times \mathbb{A}_K^1 \rightarrow \mathbb{P}^n.$$

Identifying $(\mathbb{A}^1, 0, 1)$ with $(\square^1, 0, \infty)$, pull-back by the maps

$$\mu_\phi \times \mathrm{id} : \mathbb{P}^n \times \square_K^{n+1} \rightarrow \mathbb{P}^n \times \square^n$$

define a degree 1 map of complexes

$$H_\phi : z^r(\mathbb{P}^n, *) \rightarrow z^r(\mathbb{P}_K^n, *)$$

that gives a homotopy between the base-extension $\pi : z^r(\mathbb{P}^n, *) \rightarrow z^r(\mathbb{P}_K^n, *)$ and T_η^* , where $T_\eta : \mathbb{P}^n \rightarrow \mathbb{P}_K^n$ is translation by η .

Both H_ϕ and T_η^* map $z^r(\mathbb{P}^n, *)_{\mathcal{C}, e}$ to $z^r(\mathbb{P}_K^n, *)_{\mathcal{C}, e}$. Since η is generic over k , T_η^* maps $z^r(\mathbb{P}^n, *)$ to $z^r(\mathbb{P}_K^n, *)_{\mathcal{C}, e}$. As in the proof of the localization theorem, this implies that base-extension

$$\pi : \frac{z^r(\mathbb{P}^n, *)}{z^r(\mathbb{P}^n, *)_{\mathcal{C}, e}} \rightarrow \frac{z^r(\mathbb{P}_K^n, *)}{z^r(\mathbb{P}_K^n, *)_{\mathcal{C}, e}}$$

is both injective and 0 on homology, whence the result in this case.

Step 2. $X \subset \mathbb{P}^N$ **projective.** This uses the method of the projecting cone. Suppose X has dimension n and let $L \subset \mathbb{P}^N$ be a dimension $N - n - 1$ linear subspace with $L \cap X = \emptyset$. Projection from L defines a finite morphism

$$\pi_L : X \rightarrow \mathbb{P}^n.$$

For Z a cycle on X , define

$$\tilde{L}(Z) := \pi_L^*(\pi_{L*}(Z)) - Z.$$

Since π_L is finite and \mathbb{P}^n is smooth, \tilde{L} is well-defined. In addition, $\tilde{L}(Z) \geq 0$ if $Z \geq 0$.

For a closed subset $W \subset X_{sm}$, define $\text{codim}_{X_{sm}} W$ to be the minimum of $\text{codim}_{X_{sm}} Z$ as Z runs over the irreducible components of W .

Definition 1.11. For A, B closed subsets of X of pure codimension a, b , respectively, define the *excess*

$$e(A, B) := a + b - \text{codim}_{X_{sm}}(A \cap B \cap X_{sm})$$

if $A \cap B \cap X_{sm} = \emptyset$, define the excess to be 0. If Z and W are cycles on X , define

$$e(Z, W) := e(\text{supp}(Z), \text{supp}(W)).$$

Let $\text{Gr}(N - n - 1, N)$ be the Grassmannian of \mathbb{P}^{N-n-1} 's in \mathbb{P}^N , and let $\text{Gr}(N - n - 1, N)_X$ be the open subscheme of L with $L \cap X = \emptyset$. The classical moving lemma of Chow is

Lemma 1.12. *Let Z and W be cycles on X . Then there is a non-empty open subscheme $U_{Z,W} \subset \text{Gr}(N - n - 1, N)_X$ such that, for each field $K \supset k$ and each K -point L of $U_{Z,W}$,*

$$e(\tilde{L}(Z), W) \leq \max(e(Z, W) - 1, 0).$$

For each $L \in \text{Gr}(N - n - 1, N)_X$, we have the maps

$$\pi_L \times \text{id} : X \times \square^n \rightarrow \mathbb{P}^n \times \square^n.$$

Define

$$\tilde{L}_n : z^r(X, *) \rightarrow z^r(X, *)$$

by

$$\tilde{L}_n(Z) := \pi_L \times \text{id}^*(\pi_L \times \text{id}_*(Z)) - Z$$

The maps \tilde{L}_n define the map of complexes

$$\tilde{L}_* : z^r(X, *) \rightarrow z^r(X, *).$$

With a little bit of work (see e.g. [5] or [22] for details) Chow's moving lemma implies

Lemma 1.13. *Fix a finite set \mathcal{C} of locally closed subsets of X_{sm} and a function $e : \mathcal{C} \rightarrow \mathbb{Z}_+$. Define $e - 1 : \mathcal{C} \rightarrow \mathbb{Z}_+$ by*

$$(e - 1)(C) := \max(e(C) - 1, 0).$$

Let $K = k(\text{Gr}(N - n - 1, N))$ and let $L_{gen} \in \text{Gr}(N - n - 1, N)(K)$ be the generic point of $\text{Gr}(N - n - 1, N)$. Then

$$\tilde{L}_{gen*} : z^r(X, *) \rightarrow z^r(X_K, *).$$

*maps $z^r(X, *)_{\mathcal{C}, e}$ to $z^r(X_K, *)_{\mathcal{C}, e-1}$.*

Sketch of proof. Let $|Z| \subset X \times \square^n$ be the support of a cycle Z in $z^r(X, n)$. For each face F of \square^n , let

$$Z(F, d) = \{x \in X \mid \dim_k x \times F \cap |Z| \geq d\}$$

where the inequality is satisfied if there is an irreducible component of $x \times F \cap Z$ of dimension $\geq d$ over k . $Z(F, d)$ is a constructible set, and Z is in $z^r(X, n)_{\mathcal{C}, e}$ if and only if

$$(1.1) \quad \text{codim}_{\mathcal{C}} C \cap [Z(F, d) \setminus Z(F, d-1)] \geq r - e(C) - \dim_k F + d$$

for each C, F and d . We have a similar condition for membership in $z^r(X, n)_{\mathcal{C}, e-1}$.

One has the operation $A \mapsto \tilde{L}(A)$ on constructible subsets A of X by setting

$$\tilde{L}(A) := \overline{(\pi_L^{-1}(\pi_L(A)) \setminus A)}$$

where the closure is in $\pi_L^{-1}(\pi_L(A))$. One shows that

$$\tilde{L}(Z)(F, d) \subset \tilde{L}(Z(F, d)).$$

Using the moving lemma 1.12, one shows that the inequality (1.1) for the subsets $Z(F, d)$ implies the inequality (1.1) for the subsets $\tilde{L}_{\text{gen}}(Z(F, d))$ with e replaced by $e - 1$, which proves the result. \square

To use this lemma, we proceed as follows: For each $C \in \mathcal{C}$, we have the constructible subset $\pi_{L_{\text{gen}}}(C)$. Write

$$\pi_{L_{\text{gen}}}(C) = C'_1 \cup \dots \cup C'_r$$

with each C'_i locally closed in \mathbb{P}_K^n . Let $d_i = \text{codim}_{\mathbb{P}^n}(C'_i) - \text{codim}_X C$ and define

$$e'(C'_i) := e(C) + d_i.$$

Let $\mathcal{C}' = \{C'_j \mid C \in \mathcal{C}\}$.

Then $\pi_{L_{\text{gen}}^*} : z^r(X, *) \rightarrow z^r(\mathbb{P}_K^n, *)$ maps $z^r(X, *)_{\mathcal{C}, e}$ to $z^r(\mathbb{P}_K^n, *)_{\mathcal{C}', e'}$ and $\pi_{L_{\text{gen}}^*}^* : z^r(\mathbb{P}_K^n, *) \rightarrow z^r(X_K, *)$ maps $z^r(\mathbb{P}_K^n, *)_{\mathcal{C}', e'}$ to $z^r(X_K, *)_{\mathcal{C}, e}$.

Lemma 1.13 implies that

$$\tilde{L}_* = \pi_{L_{\text{gen}}^*}^* \circ \pi_{L_{\text{gen}}^*} - \pi : \frac{z^r(X, *)_{\mathcal{C}, e}}{z^r(X, *)_{\mathcal{C}, e-1}} \rightarrow \frac{z^r(X_K, *)_{\mathcal{C}, e}}{z^r(X_K, *)_{\mathcal{C}, e-1}}$$

is zero, where π is the base-extension map. Thus $\pi_{L_{\text{gen}}^*}^* \circ \pi_{L_{\text{gen}}^*}$ is the same as base-extension on this quotient complex. However, we can factor $\pi_{L_{\text{gen}}^*}^* \circ \pi_{L_{\text{gen}}^*}$ as

$$\frac{z^r(X, *)_{\mathcal{C}, e}}{z^r(X, *)_{\mathcal{C}, e-1}} \xrightarrow{\pi_{L_{\text{gen}}^*}} \frac{z^r(\mathbb{P}_K^n, *)_{\mathcal{C}, e}}{z^r(\mathbb{P}_K^n, *)_{\mathcal{C}, e-1}} \xrightarrow{\pi_{L_{\text{gen}}^*}^*} \frac{z^r(X_K, *)_{\mathcal{C}, e}}{z^r(X_K, *)_{\mathcal{C}, e-1}}$$

By Step 1, the complex in the middle is acyclic, so π induces zero on homology. Since K is a pure transcendental extension of k , the quotient complex $z^r(X, *)_{\mathcal{C}, e} / z^r(X, *)_{\mathcal{C}, e-1}$ is acyclic. Since $z^r(X, *)_{\mathcal{C}, e-r} = z^r(X, *)$, the result follows by induction.

Step 3: X quasi-projective. This uses Bloch's moving lemma [4] involving blowing up faces of \square^n . We omit the details.

1.5. A new moving lemma. To allow for an action of “higher correspondences” on the additive cycle complexes with modulus, we will need an extension of Chow's moving lemma that takes into account the closures of the cycles.

We have the open immersion $\square^n \subset (\mathbb{P}^1)^n$; write $\overline{\square}^n$ for $(\mathbb{P}^1)^n$. For an integral subscheme or closed subset Z of $X \times \square^n$, we let \overline{Z} denote the closure in $X \times \overline{\square}^n$. Extending linearly, we have the operation

$$\overline{(-)} : z_*(X \times \square^n) \rightarrow z_*(X \times \overline{\square}^n).$$

Definition 1.14. For a set \mathcal{C} of locally closed subsets of X and a function $e : \mathcal{C} \rightarrow \mathbb{Z}_+$, let $z^r(X, n)_{\overline{\mathcal{C}}, e} \subset z^r(X, n)_{\mathcal{C}, e}$ be the subgroup generated by integral $Z \in z^r(X, n)$ such that, for each face F of \square^n ,

$$\text{codim}_{C \times \overline{F}}[\overline{Z \cap X \times F}] \cap C \times \overline{F} \geq r - e(C).$$

The $z^r(X, n)_{\overline{\mathcal{C}}, e}$ form a cubical abelian group, hence we have the subcomplex $z^r(X, *)_{\overline{\mathcal{C}}, e}$ of $z^r(X, *)_{\mathcal{C}, e}$.

Lemma 1.15. *Let X be a quasi-projective k -scheme, \mathcal{C} a finite set of locally closed subsets of X_{sm} , $e : \mathcal{C} \rightarrow \mathbb{Z}_+$ a function.*

(1) *If X is projective, the inclusion $z_r(X, *)_{\overline{\mathcal{C}}, e} \rightarrow z_r(X, *)$ is a quasi-isomorphism.*

(2) *In general, the inclusion $z_r(X, *)_{\overline{\mathcal{C}}, e} \rightarrow z_r(X, *)$ induces an isomorphism on H_0 .*

Proof. The proof is essentially the same as theorem 1.10. We indicate the modifications.

Step 1. $X = \mathbb{P}^n$ In modifying our earlier Step 1, since we need to keep track of closures, we use a finite sequence of “generic translations” rather than writing the generic matrix as a product of elementary matrices.

In detail: Let $H \subset \mathbb{P}^n$ be the generic hyperplane, defined over a pure transcendental extension K_1 of k . Let X_0, \dots, X_n be homogeneous coordinates on $\mathbb{P}_{K_1}^n$ such that H is defined by $X_0 = 0$. Let $x := (x_1, \dots, x_n)$ be the generic point of the affine space $\mathbb{A}_{K_1}^n := \mathbb{P}^n \setminus H$,

let $K = K_1(x_1, \dots, x_n)$ and let $\phi(t)$ be the $K[t]$ -valued point of GL_{n+1} defined linear transformation

$$(X_0, X_1, \dots, X_n) \mapsto (X_0, X_1 + tx_1X_0, \dots, X_n + tx_nX_0).$$

This gives us the action $\phi : \mathbb{A}^1 \times \mathbb{P}_K^n \rightarrow \mathbb{P}^n$.

For Z a cycle on $\mathbb{P}^n \times \overline{\square}^n$ we have the cycle $\phi^*(Z)$ on $\mathbb{A}^1 \times \mathbb{P}_K^n \times \overline{\square}^n$. Let $\overline{\phi^*(Z)}$ be the closure of $\phi^*(Z)$ on $\mathbb{P}^1 \times \mathbb{P}_K^n \times \overline{\square}^n$. One can easily compute the intersection $\overline{\phi^*(Z)} \cap \infty \times \mathbb{P}_K^n \times \overline{\square}^n$. This cycle is the join of $(x_1 : \dots : x_n) \times \overline{\square}^n \in H \times \overline{\square}^n$ with the the intersection $Z \cap H \times \overline{\square}^n$.

Since H is generic over k , it is easy to check that sending $Z \in \underline{z}_r(X, n)$ to $\phi^*(Z) \in \underline{z}^r(X_K, n+1)$ defines a homotopy H_ϕ of $\phi(1)^*$ with the base-change map on the complex $z_r(X, *)$, and in addition H_ϕ restricts to give a degree one map

$$H_\phi : \underline{z}_r(X, n)_{\overline{\mathcal{C}}, e} \rightarrow \underline{z}_r(X, n+1)_{\overline{\mathcal{C}}, e}$$

giving a homotopy between the base-change on $\phi(1)^*$ on the subcomplex $z_r(X, *)_{\overline{\mathcal{C}}, e}$.

Since the generic projective linear transformation is a composition of $n+1$ such translations, the argument we used for Step 1 above goes through to show prove the theorem for $X = \mathbb{P}^n$.

Step 2. X projective. We use exactly the same argument, replacing lemma 1.13 with

Lemma 1.16. *Fix a finite set \mathcal{C} of locally closed subsets of X_{sm} and a function $e : \mathcal{C} \rightarrow \mathbb{Z}_+$. Define $e-1 : \mathcal{C} \rightarrow \mathbb{Z}_+$ by*

$$(e-1)(C) := \max(e(C) - 1, 0).$$

Let $K = k(\mathrm{Gr}(N-n-1, N))$ and let $L_{gen} \in \mathrm{Gr}(N-n-1, N)(K)$ be the generic point of $\mathrm{Gr}(N-n-1, N)$. Then

$$\widetilde{L}_{gen*} : z_r(X, *) \rightarrow z_r(X_K, *).$$

*maps $z_r(X, *)_{\overline{\mathcal{C}}, e}$ to $z_r(X_K, *)_{\overline{\mathcal{C}}, e-1}$.*

The proof is exactly the same, where we replace the sets $Z(F, d)$ with

$$\overline{Z}(F, d) = \{x \in X \mid \dim_k x \times \overline{F} \cap \overline{|Z|} \cap X \times \overline{F} \geq d\}$$

Step 3. X quasi-projective. Here the ‘‘moving by blow-ups’’ technique does not pass well to the closures $\overline{\square}^n$.

However, if $j : X \rightarrow \overline{X}$ is a projective closure of X , then let $\underline{z}^r(X_{\overline{X}}, *)_{\overline{\mathcal{C}}, e} \subset \underline{z}^r(X, *)_{\overline{\mathcal{C}}, e}$ be the image of $j^* : \underline{z}^r(\overline{X}, *)_{\overline{\mathcal{C}}, e} \rightarrow \underline{z}^r(X, *)_{\overline{\mathcal{C}}, e}$. If Z is in $\underline{z}^r(X, n)_{\overline{\mathcal{C}}, e}$, then the closure \overline{Z} is in $\underline{z}^r(\overline{X}, *)_{\overline{\mathcal{C}}, e}$ for $n = 0, 1$,

Thus, letting $i : W \rightarrow \bar{X}$ be the complement of X , we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathrm{CH}_r(W, 0) & \xrightarrow{i_*} & H_0(z_r(\bar{X}, *)_{\bar{C}, e}) & \xrightarrow{j^*} & H_0(z_r(X, *)_{\bar{C}, e}) & \longrightarrow & 0 \\ \parallel & & \alpha \downarrow & & \downarrow \beta & & \\ \mathrm{CH}_{\dim X - r}(W, 0) & \xrightarrow{i_*} & \mathrm{CH}_r(\bar{X}, 0) & \xrightarrow{j^*} & \mathrm{CH}_r(X, 0) & \longrightarrow & 0 \end{array}$$

The map α is an isomorphism by Step 2, hence β is an isomorphism. \square

2. CATEGORIES OF MOTIVES

2.1. Chow motives over S . Let S be in \mathbf{Sm}/k . We recall the construction of a category of (homological) Chow motives over S .

We first assume that S is irreducible. Let \mathbf{SmProj}/S denote the full subcategory of \mathbf{Sm}/S consisting of the (smooth) projective S schemes. For irreducible $X, Y \in \mathbf{SmProj}/S$ set

$$\mathrm{Cor}_S^n(X, Y) := \mathrm{CH}_{\dim_S X + n}(X \times_S Y)$$

Extend the definition to arbitrary X, Y by taking the direct sum over the pairs of irreducible components of X and Y ; proceed similarly if S is not irreducible.

Define the category Cor_S with objects (X, n) , $X \in \mathbf{SmProj}/k$, $n \in \mathbb{Z}$, morphisms

$$\mathrm{Hom}_{\mathrm{Cor}_S}((X, n), (Y, m)) := \mathrm{Cor}_S^{m-n}(X, Y)$$

and composition law

$$\beta \circ \alpha := p_{XZ}^{XYZ}(p_{YZ}^{XYZ*}(\beta) \cup p_{XY}^{XYZ*}(\alpha)).$$

for $\alpha \in \mathrm{Cor}_S^*(X, Y)$, $\beta \in \mathrm{Cor}_S^*(Y, Z)$ and where, for instance, $p_{XZ}^{XYZ} : X \times_S Y \times_S Z \rightarrow X \times_S Z$ is the projection.

The product over S makes Cor_S a tensor category, with

$$(X, n) \otimes (Y, m) := (X \times_S Y, n + m)$$

and, for $\alpha : (X, n) \rightarrow (Y, m)$, $\alpha' : (X', n') \rightarrow (Y', m')$,

$$\alpha \otimes \alpha' := p_{XY}^{XX'YY'*}(\alpha) \cdot p_{X'Y'}^{XX'YY'*}(\alpha').$$

The unit is $1 := (S, 0)$. Note that

$$\mathrm{Hom}_{\mathrm{Cor}_S}(1, (X, n)) = \mathrm{CH}_n(X),$$

so sending (X, n) to $\mathrm{CH}_n(X)$ defines a functor

$$\mathrm{CH} : \mathrm{Cor}_S \rightarrow \mathbf{Ab}.$$

For each $r \in \mathbb{Z}$, we have the functor

$$m_{\text{Cor}}(r) : \mathbf{SmProj}/S \rightarrow \text{Cor}_S$$

sending X to (X, r) and $f : X \rightarrow Y$ to the class of the graph $\Gamma_f \subset X \times_S Y$. We write m_{Cor} for $m_{\text{Cor}}(0)$.

The category of Chow motives over S , $\text{Mot}(S)$, is the pseudo-abelian hull of Cor_S , i.e., objects are (X, n, α) with $\alpha \in \text{End}_{\text{Cor}_S}(X)$ idempotent: $\alpha^2 = \alpha$, and with

$$\text{Hom}_{\text{Mot}(S)}((X, n, \alpha), (Y, m, \beta)) := \beta \text{Hom}_{\text{Cor}_S}((X, n), (Y, m)) \alpha$$

$\text{Mot}(S)$ is a tensor category with $(X, n, \alpha) \otimes (Y, m, \beta) := ((X \times_S Y, n + m, \alpha \otimes \beta)$. $\text{Mot}(S)$ is a rigid tensor category with $(X, n, \alpha)^\vee := (X, \dim_S X - n, {}^t\alpha)$; the unit $\delta : 1 \rightarrow (X, n, \alpha) \otimes (X, n, \alpha)^\vee$ and the co-unit $\epsilon : (X, n, \alpha) \otimes (X, n, \alpha)^\vee \rightarrow 1$ are both the image of the class of the diagonal $[\Delta] \in \text{CH}^{\dim_k X}(X \times_S X)$.

Sending (X, n) to (X, n, id) defines a full tensor embedding $\text{Cor}_S \rightarrow \text{Mot}(S)$; we let

$$m(r) : \mathbf{SmProj}/S \rightarrow \text{Mot}(S)$$

be the composition of $m_{\text{Cor}}(r)$ with this embedding. We write $m(X)(n)$ for the object (X, n, id) .

2.2. Highly distinguished complexes. Central to Hanamura's construction of motives is the use of "distinguished subcomplexes" of various cycle complexes. These are defined by properness conditions on various intersections with faces on \square^n . We refine these conditions to pass to the closed n -cube $\overline{\square}^n$, leading to the "highly distinguished" subcomplexes.

Definition 2.1. Fix $X \in \mathbf{SmProj}/k$ and let \mathcal{W} be a finite set of irreducible closed subsets $W_n \subset X \times T_n$, $n = 1, \dots, N$ for schemes $T_n \in \mathbf{Sm}/k$. For $F \subset \square^n$ a face, let

$$p_{F,n} : X \times \overline{F} \times T_n \rightarrow X \times T_n$$

be the projection. Let $\underline{z}^r(X, *)_{\overline{\mathcal{W}}} \subset \underline{z}^r(X, *)$ be the subcomplex generated by irreducible $Z \subset X \times \square^n$ such that

- (1) Z is in $\underline{z}^r(X, n)$
- (2) For each face F of \square^n , $p_{F,n}^{-1}(W_n)$ and $\overline{(Z \cap X \times F)} \times T_n$ intersect properly on $X \times \overline{F} \times T_n$.

A subcomplex $z^r(X, *)' \subset z^r(X, *)$ that is the image of a subcomplex of the form $\underline{z}^r(X, *)_{\overline{\mathcal{W}}}$ is called *highly distinguished*.

Proposition 2.2. *The inclusion of a highly distinguished subcomplex $z^r(X, *)' \subset z^r(X, *)$ is a quasi-isomorphism.*

Proof. For each $W_n \subset X \times T_n$, form the contractible subsets

$$C_{n,d} = \{x \in X \mid x \times T_n \cap W_n \text{ contains a component of dimension } \geq d\}$$

Write each $C_{n,d} \setminus C_{n,d-1}$ as a union of irreducible locally closed subsets $C_{n,d}^j$. Set

$$e(C_{n,d}^j) := -\dim W_n - d - \dim C_{n,d}^j$$

Note that $e(C_{n,d}^j) \geq 0$. Let $\mathcal{C} := \{C_{n,d}^j\}$. It is an easy exercise to show that

$$\underline{z}^r(X, *)_{\overline{\mathcal{C}, e}} = \underline{z}^r(X, *)_{\overline{\mathcal{W}}}$$

The proposition thus follows from lemma 1.15. \square

Example 2.3. Let X, Y be in \mathbf{SmProj}/k . Suppose we are given irreducible closed subsets $W_i \subset X \times Y \times \square^{\ell_i}$, $i = 1, \dots, N$, which intersect all faces properly, i.e., each W_i is the support of some cycle in $\underline{z}^*(X \times Y, \ell_i)$. Let \mathcal{W} be the set of all irreducible components of all intersections $W_i \cap X \times Y \times F \subset X \times Y \times F$, as the F run over all faces of \square^{ℓ_i} . This gives us the distinguished subcomplex $z^r(X, *)_{\overline{\mathcal{W}}}$. Letting $W = \{W_1, \dots, W_N\}$, Hanamura has defined the *distinguished subcomplex* $z_W^r(X, *) \subset z^r(X, *)$ (see [17], p. 6); it follows directly from the definitions that

$$z^r(X, *)_{\overline{\mathcal{W}}} \subset z_W^r(X, *).$$

From our proposition 2.2 and [17], Proposition 1.3, this inclusion is a quasi-isomorphism. \square

In short:

Lemma 2.4. *Let X be in \mathbf{SmProj}/k . Then each distinguished subcomplex of $z^r(X, *)$, in the sense of [17], contains a highly distinguished subcomplex.*

2.3. The construction of $\mathrm{DM}^H(k)$. Hanamura considers three partially defined operations:

- (1) Take $X \in \mathbf{SmProj}/k$. The partially defined *cup product*

$$\cup_X : z^r(X, *) \otimes z^s(X, *) \dashrightarrow z^{r+s}(X, *)$$

is $\cup_X := \delta_X^* \circ \boxtimes$, where $\delta_X : X \rightarrow X \times X$ is the diagonal.

- (2) Take $X, Y \in \mathbf{SmProj}/k$ and $f \in z^s(X \times Y, \ell)$. The partially defined *push-forward by a correspondence* f is

$$\begin{aligned} f_* : z^r(X, n) &\dashrightarrow z^{r+s}(Y, n + \ell) \\ Z &\mapsto p_{Y*}(f \cup_{X \times Y} p_X^*(Z)) \end{aligned}$$

- (3) Take $X, Y, Z \in \mathbf{SmProj}/k$. The partially defined *composition of correspondences* is

$$\begin{aligned} \circ : z^r(X \times Y, *) \otimes z^s(Y \times Z, *) &\dashrightarrow z^{r+s}(X \times Z, *) \\ u \otimes v &\mapsto v \circ u; \\ v \circ u &:= p_{XZ*}(p_{YZ}^*(v) \cup_{XYZ} p_{XY}^*(u)) \end{aligned}$$

where the maps p_{XZ} , etc., are the projections, $p_{XZ} : X \times Y \times Z \rightarrow X \times Z$, etc.

In [17], propositions 1.4, 1.5, Hanamura shows that, given a distinguished subcomplex $z(-)'$ for the target of one of these operations, there is a distinguished subcomplex of the domain for which the operation is well-defined and the image lands in the given distinguished subcomplex $z(-)'$. Exactly the same argument, with the help of the dictionary given in example 2.3, proves the analog for the highly distinguished subcomplexes:

Proposition 2.5 (analog of [17], propositions 1.4, 1.5). *Take X, Y, Z and W in \mathbf{SmProj}/k .*

(1a) *Take $f \in z^s(X \times Y, \ell)$. There is a highly distinguished subcomplex $z^r(X, *)'$ such that f_* is defined on $z^r(X, *)'$.*

(1b) *Given further an element $g \in z^t(Y \times Z, m)$ such that $g \circ f$ is defined, there are highly distinguished subcomplexes $z^r(X, *)'$ and $z^{r+s}(Y, *)'$ such that f_* and $(g \circ f)_*$ are defined on $z^r(X, *)'$, g_* is defined on $z^{r+s}(Y, *)'$, $f_*(z^r(X, *)') \subset z^{r+s}(Y, *)'$ and $g_* \circ f_* = (g \circ f)_*$ on $z^r(X, *)'$.*

(2) *Take $w \in z^s(X, \ell)$. There is a highly distinguished subcomplex $z^r(X, *)'$ such that $w \cup_X (-)$ is defined on $z^r(X, *)'$.*

(3a) *Take $v \in z^s(Y \times Z, \ell)$. There is a highly distinguished subcomplex $z^r(X \times Y, *)'$ such that $v \circ (-)$ is defined on $z^r(X \times Y, *)'$.*

(3b) *Given further an element $w \in z^t(Z \times W, m)$ such that $w \circ v$ is defined, there is a highly distinguished subcomplexes $z^r(X \times Y, *)'$ and $z^{r+s}(X \times Z, *)'$ such that $v \circ (-)$ and $(w \circ v) \circ (-)$ are defined on $z^r(X \times Y, *)'$, $w \circ (-)$ is defined on $z^{r+s}(X \times Z, *)'$, $v \circ z^r(X \times Y, *)' \subset z^{r+s}(X \times Z, *)'$ and $w \circ (v \circ (-)) = (w \circ v) \circ (-)$ on $z^r(X \times Y, *)'$.*

(4) *These results generalize in the evident manner to any finite number of compositions of the operations f_* , $w \cup_X (-)$ and $v \circ (-)$.*

We follow Hanamura's construction of $\mathcal{D}_{finite}(k)$ with three modifications:

- (1) We construct a homological category of motives rather than a cohomological one, in that $DM^H(k)$ comes with a functor

$$m : \mathbf{SmProj}/k \rightarrow DM^H(k)$$

rather than a functor from \mathbf{SmProj}/k^{op} .

- (2) We replace “distinguished subcomplex” with “highly distinguished subcomplex”
- (3) We replace Hanamura's use of complexes of \mathbb{Q} -vector spaces $z^*(-, *)_{\mathbb{Q}}$ with the integral complexes $z^*(-, *)$. Structurally, the only price we pay is that the external product is now only commutative up to natural homotopy, rather than being strictly commutative. This means we need to be a bit more careful in making sure that we use a consistent order in our products. This also prevents us from giving the integral construction a tensor structure.

With these changes, one has the triangulated category $DM^H(k)$. We briefly recall the construction from [17] and show how to adapt it to our situation.

For integral $X, Y \in \mathbf{SmProj}/k$ and integers r, s , we set

$$C^n((X, r), (Y, s)) := z^{r-s+\dim Y}(X \times Y, -n) := z_{s-r+\dim X}(X \times Y, -n).$$

Clearly the differential in $z^{r-s+\dim Y}(X \times Y, *)$ makes $C^*((X, r), (Y, s))$ a cohomological complex; we write the differential in $C^*((X, r), (Y, s))$ as ∂ . We extend to finite formal sums of symbols $(X, r), (Y, s)$ by making C^* additive in each variable.

The objects of $DM^H(k)$ are built from *finite diagrams*: A finite diagram $(K^m, f^{n,m})$ consists of

- (1) finite formal symbols $K^m := \sum_i (X_{im}, r_{im})$, where each X_{im} is in \mathbf{SmProj}/k , integral, and each r_{im} is an integer. The sum is finite, and $K^m = 0$ except for finitely many m .
- (2) correspondences $f^{n,m} = \sum_{ij} f_{ij}^{n,m}$ with $m < n$ and with

$$f_{ij}^{n,m} \in C^{1+m-n}((X_{im}, r_{im}), (X_{jn}, r_{jn}))$$

In addition, we assume that all compositions

$$f^{m_n, m_{n-1}} \circ \dots \circ f^{m_2, m_1}$$

are defined.

- (3) The identity

$$\partial f^{n,m} + \sum_{m < \ell < n} f^{n,\ell} \circ f^{\ell,m} = 0$$

is satisfied for all $m < n$.

Given two objects (K, f) , (L, g) , one defines a filtered system of quasi-isomorphic *function complexes* $\mathcal{H}om(K, L)'$ as follows: Choose highly distinguished subcomplexes

$$C^*(K^m, L^n)' \subset C^*(K^m, L^n)$$

such that, for each n, m, n', m' the maps

$$\begin{aligned} (-) \circ f_{m'}^{m,n} &: C^*(K^m, L^{m'})'[m - m'] \rightarrow C^*(K^n, L^{m'})'[1 + n - m'] \\ g_n^{m',n'} \circ (-) &: C^*(K^n, L^{n'})'[n - n'] \rightarrow C^*(K^n, L^{m'})'[1 + n - m'] \end{aligned}$$

are defined. Let $C(K, M)'_p := \bigoplus_{n-m=p} C^*(K^m, L^n)'$ and let

$$h^{q,p} : C(K, M)'_p[-p] \rightarrow C(K, M)'_q[-q]$$

be the degree +1 map $\sum_{m,n,m',n'} g_n^{m',m} \circ (-) - \epsilon \cdot (-) \circ f_m^{n,n'}$, where the factor ϵ is $(-1)^N$ on $(C(K, M)'_p[-p])^N$. We have the total complex $\text{Tot}(C(K, M)')$ with

$$\text{Tot}(C(K, M)') := \bigoplus_p C(K, M)'_p[-p]$$

and differential $\sum_{p,q} \partial_{C(K, M)'_p[-p]} + h^{q,p}$. We set

$$\mathcal{H}om(K, L)' := \text{Tot}(C(K, M)').$$

This construction follows the formula for the category $C^b(\mathcal{A})$ of complexes over a d.g. category \mathcal{A} , defined by Kapranov [20].

The complex $\mathcal{H}om(K, L)'$ depends on the choice of the highly distinguished subcomplexes $C^*(K^m, L^n)'$, but if we make another choice $C^*(K^m, L^n)'' \subset C^*(K^m, L^n)'$, the induced inclusion map

$$\mathcal{H}om(K, L)'' \rightarrow \mathcal{H}om(K, L)'$$

is a quasi-isomorphism. Since the intersection of two highly distinguished subcomplexes is again highly distinguished, this gives us a well-defined filtered system of quasi-isomorphic complexes $\mathcal{H}om(K, L)'$. In particular, the cohomology $H^0(\mathcal{H}om(K, L)')$ is canonically defined. We set

$$\text{Hom}(K, L) := H^0(\mathcal{H}om(K, L)').$$

The composition law $\text{Hom}(K, L) \otimes \text{Hom}(L, M) \rightarrow \text{Hom}(K, M)$ is defined as follows: First fix an element $h_1 \in \text{Hom}(K, L)$. Choose a collection of highly distinguished subcomplexes $C^*(K^m, L^n)'$ and a representative $(h_1^{n,m})$ in $Z^0(\mathcal{H}om(K, L)')$ for h_1 . By proposition 2.5, there is a collection of highly distinguished subcomplexes $C^*(L^{m'}, M^{n'})'$

and $C^*(K^m, M^{n'})'$ for which $\mathcal{H}om(L, M)'$ and $\mathcal{H}om(K, M)'$ are defined and for which the operation

$$(-) \circ (h_1^{n,m}) : \mathcal{H}om(L, M)' \rightarrow \mathcal{H}om(K, M)'$$

is defined. This gives us the operation

$$(-) \circ h_1 : \text{Hom}(L, M) \rightarrow \text{Hom}(K, M),$$

which is independent of the choice of highly distinguished subcomplexes $C^*(L^{m'}, M^{n'})'$ and $C^*(K^m, M^{n'})'$.

If we now change the representative $(h_1^{n,m})$ to $(h_1^{n,m})'$, there is an element $(H_1^{n,m}) \in C^{-1}(\mathcal{H}om(K, L)')$ with

$$d(H_1^{n,m}) = (h_1^{n,m}) - (h_1^{n,m})'$$

By replacing our choices $C^*(L^{m'}, M^{n'})'$ and $C^*(K^m, M^{n'})'$ with smaller highly distinguished subcomplexes if necessary, we may assume that

$$(-) \circ * : \mathcal{H}om(L, M)' \rightarrow \mathcal{H}om(K, M)'$$

is defined on the subcomplex of $\mathcal{H}om(K, L)'$ generated by all irreducible components of all the representatives $(h_1^{n,m})$, $(h_1^{n,m})'$ and $(H_1^{n,m})$. Thus the maps $(-)\circ(h_1^{n,m})$ and $(-)\circ(h_1^{n,m})'$ are homotopic, hence the composition $(-)\circ h_1$ is independent of the choice of representative for h_1 in $Z^0(\mathcal{H}om(K, L)')$.

The proof that $(-)\circ h_1$ is independent of the choice of highly distinguished subcomplexes $C^*(K^m, L^n)'$ is similar as is the argument that the composition law is associative. This gives us the additive category $\widetilde{\text{DM}}^H(k)$.

Using the construction of [20] as extended by [17] one defines the cone of a morphism $h : K \rightarrow L$ in $\widetilde{\text{DM}}^H(k)$, or more precisely, the cone of a choice of representative of h in some $Z^0(\mathcal{H}om(K, L)')$. The collection of cone sequences gives $\widetilde{\text{DM}}^H(k)$ the structure of a triangulated category; the argument of Hanamura copied word for word proves this result.

Finally, we let $\text{DM}^H(k)$ denote the pseudo-abelian hull of $\widetilde{\text{DM}}^H(k)$. The main theorem of [2] says that $\widetilde{\text{DM}}^H(k)$ inherits a canonical structure of a triangulated category from $\widetilde{\text{DM}}^H(k)$.

Fix an integer r . We have the functor

$$m(r) : \mathbf{SmProj}/k \rightarrow \text{DM}^H(k)$$

sending X to the complex $K^0 = (X, r)$, $K^m = 0$ for $m \neq 0$, and sending $f : Y \rightarrow X$ to the graph Γ_f in $z^{\dim Y}(X \times Y)$.

Lemma 2.6. *The functor $m(r)$ extends to a full embedding*

$$i : \text{Mot}(k) \rightarrow \text{DM}^H(k).$$

Proof. Since $\mathcal{H}om((X, r), (Y, s)) = z_{\dim X + s - r}(X \times Y, *)$ we have

$$\begin{aligned} \text{Hom}_{\text{DM}^H(k)}((X, r), (Y, s)) &= \text{CH}_{\dim X + s - r}(X \times Y) \\ &= \text{Hom}_{\text{Mot}(k)}((X, r), (Y, s)), \end{aligned}$$

giving the full embedding $i : \text{Cor}(k) \rightarrow \text{DM}^H(k)$. The extension to $\text{Mot}(k)$ follows since $\text{DM}^H(k)$ is pseudo-abelian. \square

2.4. The homological motive. Let $\mathbb{Z}\mathbf{SmProj}/k$ be the additive category generated by \mathbf{SmProj}/k : for X, Y integral define

$$\text{Hom}_{\mathbb{Z}\mathbf{SmProj}/k}(X, Y) := \mathbb{Z}[\text{Hom}_{\mathbf{SmProj}/k}(X, Y)]$$

and extend to arbitrary X and Y by taking direct sums over the irreducible components. The composition law in $\mathbb{Z}\mathbf{SmProj}/k$ is induced from \mathbf{SmProj}/k .

Form the category of bounded complexes $C^b(\mathbb{Z}\mathbf{SmProj}/k)$ and the homotopy category $K^b(\mathbb{Z}\mathbf{SmProj}/k)$. We denote the complex concentrated in degree 0 associated to an $X \in \mathbf{SmProj}/k$ by $[X]$. Sending X to $[X]$ defines the functor

$$[-] : \mathbf{SmProj}/k \rightarrow C^b(\mathbb{Z}\mathbf{SmProj}/k)$$

Let $i : Z \rightarrow X$ be a closed immersion in \mathbf{SmProj}/k , $\mu : X_Z \rightarrow X$ the blow-up of X along Z and $i_E : E \rightarrow X_Z$ the exceptional divisor with structure morphism $q : E \rightarrow Z$. Let $C(\mu)$ be the complex

$$[E] \xrightarrow{(i_E, -q)} [X_Z] \oplus [Z] \xrightarrow{\mu+i} [X]$$

with $[X]$ in degree 0.

Definition 2.7. The category $\mathcal{D}_{\text{hom}}(k)$ is the localization of the triangulated category $K^b(\mathbb{Z}\mathbf{SmProj}/k)$ with respect to the thick subcategory generated by the complexes $C(\mu)$.

Let

$$m_{\text{hom}} : \mathbf{SmProj}/k \rightarrow \mathcal{D}_{\text{hom}}(k)$$

be the composition

$$\mathbf{SmProj}/k \xrightarrow{[-]} C^b(\mathbf{SmProj}/k) \rightarrow K^b(\mathbf{SmProj}/k) \rightarrow \mathcal{D}_{\text{hom}}(k).$$

Lemma 2.8. $\mathcal{D}_{\text{hom}}(k)$ with functor $m_{\text{hom}} : \mathbf{SmProj}/k \rightarrow \mathcal{D}_{\text{hom}}(k)$ is a category of homological descent, in the sense of Guillén and Navarro Aznar [16].

Proof. We use the notations of [16]. We first need to extend m_{hom} to a functor from the category of cubical objects of \mathbf{SmProj}/k to $\mathcal{D}_{\text{hom}}(k)$. In fact, we can use the evident total complex functor to extend $[-]$ to a functor from cubical objects of \mathbf{SmProj}/k to $C^b(\mathbb{Z}\mathbf{SmProj}/k)$. The remaining conditions are direct consequences of the definition of $\mathcal{D}_{\text{hom}}(k)$. \square

We let \mathbf{Sch}'_k denote the subcategory of proper morphisms in \mathbf{Sch}_k .

Theorem 2.9. *Suppose that k admits resolution of singularities. Then the functor m_{hom} extends to a functor*

$$M_{\text{hom}} : \mathbf{Sch}'_k \rightarrow \mathcal{D}_{\text{hom}}(k)$$

such that

1. Let $\mu : Y \rightarrow X$ be a proper morphism in \mathbf{Sch}_k , $i : Z \rightarrow X$ a closed immersion. Suppose that $\mu : \mu^{-1}(X \setminus Z) \rightarrow X \setminus Z$ is an isomorphism. There is a canonical extension of the commutative square

$$\begin{array}{ccc} M_{\text{hom}}(\mu^{-1}(Z)) & \longrightarrow & M_{\text{hom}}(Y) \\ \downarrow & & \downarrow \\ M_{\text{hom}}(Z) & \longrightarrow & M_{\text{hom}}(X) \end{array}$$

to a map of distinguished triangles in $\mathcal{D}_{\text{hom}}(k)$

$$\begin{array}{ccccccc} M_{\text{hom}}(\mu^{-1}(Z)) & \longrightarrow & M_{\text{hom}}(Y) & \longrightarrow & C_1 & \longrightarrow & M_{\text{hom}}(\mu^{-1}(Z))[1] \\ \downarrow & & \downarrow & & \downarrow \alpha & & \downarrow \\ M_{\text{hom}}(Z) & \longrightarrow & M_{\text{hom}}(X) & \longrightarrow & C_2 & \longrightarrow & M_{\text{hom}}(Z)[1] \end{array}$$

such that α is an isomorphism.

2. Let $j : U \rightarrow X$ be an open immersion in \mathbf{Sch}_k with closed complement $i : Z \rightarrow X$. Then there is a canonical distinguished triangle

$$M_{\text{hom}}(Z) \xrightarrow{i_*} M_{\text{hom}}(X) \xrightarrow{j^*} M_{\text{hom}}(U) \rightarrow M_{\text{hom}}(Z)[1],$$

natural with respect to proper morphisms of pairs $f : (X, U) \rightarrow (X', U')$.

Proof. It follows from lemma 2.8 that the category $\mathcal{D}_{\text{hom}}(k)^{\text{op}}$ with functor

$$m_{\text{hom}}^{\text{op}} : \mathbf{SmProj}/k^{\text{op}} \rightarrow \mathcal{D}_{\text{hom}}(k)^{\text{op}}$$

is a category of cohomological descent, in the sense of [16]. By [16], théorème 2.2.2. $m_{\text{hom}}^{\text{op}}$ extends to a functor (“cohomology with compact

supports”)

$$M_{\text{hom}}^c : \mathbf{Sch}_k^{\text{op}} \rightarrow \mathcal{D}_{\text{hom}}(k)^{\text{op}}$$

satisfying (1) and (2) with all arrows reversed. We take $M_{\text{hom}} := (M_{\text{hom}}^c)^{\text{op}}$. \square

Fix an integer r . We have the functor

$$m(r) : \mathbf{SmProj}/k \rightarrow \text{DM}^H(k)$$

Since the composition (as correspondences) of the graphs of composable morphisms is always defined, m extends canonically to the exact functor

$$K^b(m(r)) : K^b(\mathbb{Z}\mathbf{SmProj}/k) \rightarrow \text{DM}^H(k)$$

Lemma 2.10. *$K^b(m)$ extends canonically to an exact functor*

$$\mathcal{D}_{\text{hom}}(m(r)) : \mathcal{D}_{\text{hom}}(k) \rightarrow \text{DM}^H(k)$$

Proof. We have already seen that the functors $m(r)$ extends to a functor

$$i : \text{Mot}_{\text{CH}}(k) \rightarrow \text{DM}^H(k).$$

Let

$$\begin{array}{ccc} E & \xrightarrow{i_E} & X_Z \\ q \downarrow & & \downarrow \mu \\ Z & \xrightarrow{i} & X \end{array}$$

be a blow-up square in \mathbf{SmProj}/k . It is well-known that

$$m(E)(r) \xrightarrow{(i_{E*}, -q^*)} m(X_Z)(r) \oplus m(Z)(r) \xrightarrow{\mu_* + i_*} m(X)(r)$$

is a split exact sequence in $\text{Mot}_{\text{CH}}(k)$. Thus $K^b(m(r))(C(\mu)) \cong 0$ in $\text{DM}^H(k)$, giving the desired extension to $\mathcal{D}_{\text{hom}}(k)$. \square

Definition 2.11. Let $bm(r) : \mathbf{Sch}'_k \rightarrow \text{DM}^H(k)$ be the composition

$$\mathbf{Sch}'_k \xrightarrow{M_{\text{hom}}} \mathcal{D}_{\text{hom}}(k) \xrightarrow{\mathcal{D}(m(r))} \text{DM}^H(k).$$

We write $bm(X)(r)$ for $bm(r)(X)$, and call $bm(X)(r)$ the *twisted Borel-Moore motive of X* .

2.5. The category of extended motives.

Definition 2.12. For $r \in \mathbb{Z}$, we have the full subcategory $\mathcal{C}(r) \subset \mathrm{DM}^H(k)$ with objects of the form $\mathcal{D}_{\mathrm{hom}}(m(r))(X)$, with $X \in \mathcal{D}(k)$ and $r \in \mathbb{Z}$. Clearly the pseudo-abelian hull of $\mathcal{C}(r)$ defines a full additive subcategory $\hat{\mathrm{Mot}}(k)_r$ of $\mathrm{DM}^H(k)$. Let $\hat{\mathrm{Mot}}(k)$ be the full additive subcategory of $\mathrm{DM}^H(k)$ generated by the $\hat{\mathrm{Mot}}(k)_r$, $r \in \mathbb{Z}$, and let $\hat{\mathrm{Mot}}(k)^*$ denote the smallest full subcategory of $\mathrm{DM}^H(k)$ containing $\hat{\mathrm{Mot}}(k)$ and closed under translation. We call $\hat{\mathrm{Mot}}(k)^*$ the category of *extended motives over k* .

Clearly the full embedding $i : \mathrm{Mot}(k) \rightarrow \mathrm{DM}^H(k)$ factors through $\hat{\mathrm{Mot}}(k)$, and the functor $bm(r) : \mathbf{Sch}'_k \rightarrow \mathrm{DM}^H(k)$ factors through the inclusion $\hat{\mathrm{Mot}}(k)_r \rightarrow \mathrm{DM}^H(k)$.

2.6. Contravariant functoriality. For $X \in \mathbf{Sm}/k$ equi-dimensional over k , set $h(X) := bm(X)(\dim X)$. Extend to arbitrary $X \in \mathbf{Sm}/k$ by taking the direct sum over the components of X .

We have the functor

$$h_{\mathbf{SmProj}} : \mathbf{SmProj}/k^{\mathrm{op}} \rightarrow \mathrm{Mot}(k)$$

with $h(X)$ defined as above and with $h(f : Y \rightarrow X) = [{}^t\Gamma_f]_*$.

Let $f : X \rightarrow Y$ be a morphism in \mathbf{Sm}/k . We proceed to define a morphism

$$f^* : h(Y) \rightarrow h(X).$$

By resolution of singularities, we may find projective completions $j : X \rightarrow \bar{X}$, $j' : Y \rightarrow \bar{Y}$ such that

1. the complements $D := \bar{X} \setminus X$, $E := \bar{Y} \setminus Y$ are strict normal crossing divisors.

Let $\bar{\Gamma}_f \subset \bar{X} \times \bar{Y}$ be the closure of the graph of f . Write $E = \sum_{j=1}^m E_j$ with each E_j irreducible, and let $E_I = \cap_{i \in I} E_i$ for each subset $I \subset \{1, \dots, m\}$. By [4], applied to the projection $\bar{\Gamma}_f \rightarrow \bar{Y}$, we assume that

2. $\bar{\Gamma}_f$ intersects $\bar{X} \times E_I$ properly for each I .

We say that a pair of projective completions $j : X \rightarrow \bar{X}$, $j' : Y \rightarrow \bar{Y}$ satisfying (1) and (2) are *good for f* .

Write $D = \sum_{i=1}^n D_i$ with each D_i smooth, and let $D_I := \cap_{i \in I} D_i$ for each subset $I \subset \{1, \dots, n\}$. Then $m_{\mathrm{hom}}(X)$ is represented by the

complex D_* :

$$D_{1,\dots,n} \xrightarrow{d_{n-1}} \bigoplus_{|I|=n-1} D_I \xrightarrow{d_{n-2}} \dots \xrightarrow{d_1} \bigoplus_i D_i \xrightarrow{d_0} \bar{X}$$

with \bar{X} in degree 0, and where the differential is the signed sum of the inclusion maps $D_I \rightarrow D_{I \setminus i}$, with sign $(-1)^{j-1}$ if $I = i_1 < \dots < i_s$ and $i = i_j$. We have a similar description of a complex E_* representing $m_{\text{hom}}(Y)$. We write $D_*(d_X)$ for the finite diagram

$$(\bigoplus_{|I|=j} (D_I, d_X), f^{j+1,j}),$$

where $f^{j+1,j}$ is the signed sum of the graphs of the corresponding inclusion maps. We define the finite diagram $E_*(d_Y)$ similarly. By definition of $h(X)$ and $h(Y)$, we have canonical isomorphisms in $\text{DM}^H(k)$

$$D_*(d_X) \cong h(X), \quad E_*(d_Y) \cong h(Y).$$

We have the highly distinguished subcomplexes

$$C^*((E_J, d_Y), (D_I, d_X))' \subset C^*((E_J, d_Y), (D_I, d_X))$$

defined by

$$C^*((E_J, d_Y), (D_I, d_X))' := z_{d_X - |J|}(E_J \times D_I, -)_{\bar{\mathcal{C}}_{J,I}},$$

where $\bar{\mathcal{C}}_{J,I}$ is the set of closed subsets $E_{J'} \times D_I$, $J \subset J'$; we include the cases $J = \emptyset$ or $I = \emptyset$, $E_\emptyset = \bar{Y}$, $D_\emptyset = \bar{X}$. The $C^*((E_J, d_Y), (D_I, d_X))'$ form a triple complex: the first differential is given by the differential in $C^*((E_J, d_Y), (D_I, d_X))'$, the second is the alternating sum of the push-forward maps

$$C^*((E_J, d_Y), (D_I, d_X))' \xrightarrow{i_{J, I \setminus \{i\}, i \in I^*}} C^*((E_J, d_Y), (D_{I \setminus \{i\}}, d_X))'$$

and the third is the alternating sum of the restriction maps

$$C^*((E_J, d_Y), (D_I, d_X))' \xrightarrow{i_{J \cup \{j\}, I \setminus \{i\} \subset I}} C^*((E_{J \cup \{j\}}, d_Y), (D_I, d_X))'$$

The total complex of this triple complex is the internal Hom complex $\mathcal{H}om(E_*(d_Y), D_*(d_X))'$ with respect to our choices of highly distinguished subcomplexes.

For each $j = 1, \dots, m$, the cycle $\bar{\Gamma}_f \cdot (\bar{X} \times E_j)$ is supported in $D \times E_j$, so we may write

$$\bar{\Gamma}_f \cdot (\bar{X} \times E_j) = \sum_{i=1}^n \bar{\Gamma}_f(i, j).$$

with $\bar{\Gamma}_f(i, j)$ supported in $D_i \times E_j$. This decomposition is not in general unique, as some components of $\bar{\Gamma}_f \cdot \bar{X} \times E_j$ may lie in $D_{i' \setminus i} \times E_j$.

It follows from our definitions that

$$\begin{aligned} {}^t\bar{\Gamma}_f &\text{ is in } C^0((E_\emptyset, d_Y), (D_\emptyset, d_X)), \\ {}^t\bar{\Gamma}_f(i, j) &\text{ is in } C^0((E_j, d_Y), (D_i, d_X)), \\ {}^t\bar{\Gamma}_f + \sum_{i,j} {}^t\bar{\Gamma}_f(i, j) &\text{ is in } Z^0(\mathcal{H}om(E_*(d_Y), D_*(d_X))). \end{aligned}$$

We let $f_{\bar{X}, \bar{Y}}^*$ denote the class

$$\left[{}^t\bar{\Gamma}_f + \sum_{i,j} {}^t\bar{\Gamma}_f(i, j) \right] \in H^0(\mathcal{H}om(E_*(d_Y), D_*(d_X))).$$

Having fixed \bar{X} and \bar{Y} , the only choice we have made is the decomposition of the $\bar{\Gamma}_f \cdot (\bar{X} \times E_j)$ into pieces $\bar{\Gamma}_f(i, j)$; as the difference of two such choices comes from the image of the differential

$$\begin{aligned} z_{d_X-1}(E_j \times D_{ii'}, -*)\bar{c}_{j, \{ii'\}} \\ \rightarrow z_{d_X-1}(E_j \times D_i, -*)\bar{c}_{j,i} \oplus z_{d_X-1}(E_j \times D_i, -*)\bar{c}_{j, i'} \end{aligned}$$

the cohomology class $f_{\bar{X}, \bar{Y}}^*$ depends only on the choice of compactifications \bar{X}, \bar{Y} .

By resolution of singularities, the collection of projective completions of X and Y that are good for f form a filtered system. Suppose we have morphisms $a : \bar{X}' \rightarrow \bar{X}$, $b : \bar{Y}' \rightarrow \bar{Y}$ of completions. Let $\bar{\Gamma}'_f \subset \bar{X}' \times \bar{Y}'$ denote the closure of Γ_f . Then

$$(a \times \text{id})_*(\bar{\Gamma}'_f) = (\text{id} \times b)^*(\bar{\Gamma}_f)$$

as cycles on $\bar{X} \times \bar{Y}'$; this implies that, via the canonical isomorphisms

$$D_*(d_X) \cong h(X) \cong D'_*(d_X), \quad E_*(d_Y) \cong h(Y) \cong E'_*(d_Y),$$

the morphisms $f_{\bar{X}, \bar{Y}}^*$ and $f_{\bar{X}', \bar{Y}'}^*$ induce the same morphism $h(Y) \rightarrow h(X)$.

Thus we have for each morphism $f : X \rightarrow Y$ in \mathbf{Sm}/k a well-defined morphism

$$f^* : h(Y) \rightarrow h(X)$$

in $\text{DM}^H(k)$. Clearly f^* is the morphism defined by ${}^t[\Gamma_f]$ in case X and Y are in \mathbf{SmProj}/k .

Thus, it remains to prove the functoriality $(fg)^* = g^*f^*$. This does not follow by a direct computation, because the composition of the representatives we have used to define f^* and g^* may not be defined. We therefore proceed in stages. We require first a result from resolution of singularities:

Lemma 2.13. *Let D be a reduced strict normal crossing divisor on some $X \in \mathbf{Sm}/k$ and let $i : W \rightarrow X$ be a closed immersion. Suppose that W is irreducible, not contained in D and that $W \setminus D$ is smooth. Then there is a sequence of blow-ups with irreducible smooth centers lying over $W \cap D$,*

$$X_N \rightarrow \dots \rightarrow X_1 \rightarrow X_0 = X,$$

such that

- (1) *the reduced pull-back D_i of D to X_i is a strict normal crossing divisor for each i .*
- (2) *Let $Z_i \subset X_i$ be the smooth irreducible closed subscheme blown up to form X_{i+1} , and let $F \subset D_i$ be the minimal face of D_i containing Z_i . Then for each face F' of D_i , Z_i intersects $F \cap F'$ properly and transversely on F .*
- (3) *letting $W_N \subset X_N$ be the proper transform of W and $D_N \subset X_N$ the reduced inverse image of D , W_N intersects each face of D_N properly and transversely.*

We can now handle the case of the composition of closed immersions.

Lemma 2.14. *Let $i_1 : X \rightarrow Y$, $i_2 : Y \rightarrow Z$ be closed immersions. Then $(i_2 i_1)^* = i_1^* i_2^*$.*

Proof. We use the previous lemma to find projective completions $j' : Y \rightarrow \bar{Y}$, $j'' : Z \rightarrow \bar{Z}$ with strict normal crossing complements $D' \subset \bar{Y}$, $D'' \subset \bar{Z}$ such that

- (1) i_2 extends to a closed immersion $\bar{i}_2 : \bar{Y} \rightarrow \bar{Z}$
- (2) \bar{Y} intersects each face D''_f properly and transversely

Now let $X' \subset \bar{Y}$ be the closure of X . Then $X = X' \setminus D'$, and we may apply the previous lemma to the closed subscheme X' of \bar{Y} . Let $T_i \subset \bar{Y}_i$ be the center of the i th blow-up $\bar{Y}_{i+1} \rightarrow \bar{Y}_i$ used to achieve the conclusion of the lemma. We blow-up \bar{Z} , starting with the blow-up along T_0 . Since each face of D'' intersects \bar{Y} transversely, it follows that $T_0 \subset D''$ satisfies the same transversality in lemma 2.13(2) that T_0 satisfies with respect to D' . Thus D'' pulls back to a strict normal crossing divisor D''_1 on $\bar{Z}_1 := \bar{Z}_{T_0}$, intersecting the proper transform $\bar{Y}_1 \subset \bar{Z}_1$ transversely. Thus we may continue the process, eventually replacing the closed immersion $\bar{Y} \rightarrow \bar{Z}$ with the closed immersion $\bar{Y}_n \rightarrow \bar{Z}_n$.

Replacing \bar{Z} and \bar{Y} with \bar{Z}_n and \bar{Y}_n , we may thus assume that we have a projective completion $j : X \rightarrow \bar{X}$ such that i_1 extends to $\bar{i}_1 : \bar{X} \rightarrow \bar{Y}$, with \bar{X} transverse to each face of D' . Let $D = \bar{X} \setminus X$.

We have the graphs $\Gamma_{\bar{i}_1} \subset \bar{X} \times \bar{Y}$, $\Gamma_{\bar{i}_2} \subset \bar{Y} \times \bar{Z}$ and $\Gamma_{\bar{i}_2 \circ \bar{i}_1}$. By our conditions on \bar{i}_1 and \bar{i}_2 , it follows that $\Gamma_{\bar{i}_1} \cap \bar{X} \times D'_J$ is a disjoint union of components D_I with $|I| = |J|$, and similarly for $\Gamma_{\bar{i}_2} \subset \bar{Y} \times \bar{Z}$ and $\Gamma_{\bar{i}_2 \circ \bar{i}_1}$. This easily implies that the composition of correspondences $i_{1\bar{X},\bar{Y}}^* \circ i_{2\bar{Y},\bar{Z}}^*$ is defined and

$$i_{1\bar{X},\bar{Y}}^* \circ i_{2\bar{Y},\bar{Z}}^* = (i_2 \circ i_1)_{\bar{X},\bar{Z}}^*$$

As we are free to choose our projective completions to define the maps i_1^* , i_2^* and $(i_2 \circ i_1)^*$, this gives $i_1^* i_2^* = (i_2 i_1)^*$. \square

Lemma 2.15. *Let $g : U \rightarrow X$ be an open immersion with complement $X \setminus U$ a strict normal crossing divisor, $f : X \rightarrow Y$, $h : V \rightarrow U$ morphisms in \mathbf{Sm}/k . Then $(fg)^* = f^* g^*$ and $(gh)^* = h^* g^*$.*

Proof. Let $j : X \rightarrow \bar{X}$ be a completion such that $\bar{X} \setminus U$ is a strict normal crossing divisor, and let $j' : Y \rightarrow \bar{Y}$ be a completion such that $f_{\bar{X},\bar{Y}}^*$ is defined. Taking $\bar{U} = \bar{X}$, $(fg)_{\bar{X},\bar{Y}}^*$ is defined and $g_{\bar{U},\bar{X}}^*$ is defined. As $\bar{\Gamma}_g$ is the diagonal in \bar{X} , the composition $g_{\bar{U},\bar{X}}^* \circ f_{\bar{X},\bar{Y}}^*$ is defined and

$$g_{\bar{U},\bar{X}}^* \circ f_{\bar{X},\bar{Y}}^* = (fg)_{\bar{X},\bar{Y}}^*.$$

Similarly, let $V \rightarrow \bar{V}$ and $U \rightarrow \bar{U}$ be projective completions that are good for h . Set $\bar{X} = \bar{U}$. Then $\bar{\Gamma}_g$ is the diagonal in \bar{X} and (\bar{V}, \bar{X}) is good for gh , so $h_{\bar{V},\bar{U}}^*$, $(gh)_{\bar{V},\bar{X}}^*$ and $g_{\bar{U},\bar{X}}^*$ are all defined. Since $\bar{g} = \text{id}_{\bar{U}}$, the composition $g_{\bar{U},\bar{X}}^* \circ h_{\bar{V},\bar{U}}^*$ is defined and

$$g_{\bar{U},\bar{X}}^* \circ h_{\bar{V},\bar{U}}^* = (gh)_{\bar{V},\bar{X}}^*.$$

\square

Lemma 2.16. *Given a morphism $f : X \rightarrow Y$ in \mathbf{Sm}/k , factor f as $p_2 \circ i$, with $p_2 : X \times Y \rightarrow Y$ the projection and $i : X \rightarrow X \times Y$ the closed immersion (id_X, f) . Then $f^* = i^* \circ p_2^*$.*

Proof. First choose completions $X \rightarrow \bar{X}$, $Y \rightarrow \bar{Y}$ such that f extends to a morphism $\bar{f} : \bar{X} \rightarrow \bar{Y}$. Let $X' := \bar{f}^{-1}(Y)$. Then the restriction of \bar{f} to $f' : X' \rightarrow Y$ is a proper morphism, and we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ \downarrow i & & \downarrow i' \\ X \times Y & \xrightarrow{j \times \text{id}} & X' \times Y \\ \downarrow p_2 & & \downarrow p_2 \\ Y & \xlongequal{\quad} & Y \end{array} \begin{array}{l} \\ \\ f \\ \\ \end{array} \begin{array}{l} \\ \\ f' \\ \\ \end{array}$$

Suppose we have proven the result for $f' : X' \rightarrow Y$. Using lemma 2.15 we have

$$\begin{aligned}
i^* \circ p_2^* &= i^* \circ (j \times \text{id})^* \circ p_2^* \\
&= (j \times \text{id} \circ i)^* \circ p_2^* \\
&= (i' \circ j)^* \circ p_2^* \\
&= j^* \circ i'^* \circ p_2^* \\
&= j^* \circ f'^* \\
&= f^*
\end{aligned}$$

Thus we may assume that f is proper.

Choose completions $j : X \rightarrow \bar{X}$, $j' : Y \rightarrow \bar{Y}$ for which the morphism $f_{\bar{X}, \bar{Y}}^*$ is defined. Let $\bar{i} : X \rightarrow \bar{X} \times Y$ be the composition $(j \times \text{id}) \circ i$ and let $\bar{p}_2 : \bar{X} \times Y \rightarrow Y$ be the projection. Since f is proper, \bar{i} is a closed immersion. Using lemma 2.15 again, it suffices to show

$$f^* = \bar{i}^* \circ \bar{p}_2^*.$$

The fact that $f_{\bar{X}, \bar{Y}}^*$ is defined implies that $\bar{i}_{\bar{X}, \bar{X} \times \bar{Y}}^*$ is also defined. Since \bar{X} is smooth and projective, $\bar{p}_{2, \bar{X} \times \bar{Y}, \bar{Y}}^*$ is also defined, and the graph closure $\bar{\Gamma}_{\bar{p}_2} \subset \bar{X} \times \bar{Y} \times \bar{Y}$ is $\bar{X} \times \Delta_{\bar{Y}}$. The intersection

$$\bar{\Gamma}_{\bar{i}} \times \bar{Y} \cap \bar{X} \times \bar{\Gamma}_{\bar{p}_2} \subset \bar{X} \times \bar{X} \times \bar{Y} \times \bar{Y}$$

is $\delta_{\bar{X} \times \bar{Y}}(\bar{\Gamma}_f)$ (after rearranging the factors), hence the composition $\bar{i}_{\bar{X}, \bar{X} \times \bar{Y}}^* \circ \bar{p}_{2, \bar{X} \times \bar{Y}, \bar{Y}}^*$ is defined and

$$\bar{i}_{\bar{X}, \bar{X} \times \bar{Y}}^* \circ \bar{p}_{2, \bar{X} \times \bar{Y}, \bar{Y}}^* = f_{\bar{X}, \bar{Y}}^*.$$

□

A slight variation:

Lemma 2.17. *Let $i : X \rightarrow Y \times T$ be a closed immersion, with X , Y and T in \mathbf{Sm}/k . Suppose that $p_2 \circ i : X \rightarrow Y$ is a closed immersion. Then $(p_2 \circ i)^* = i^* p_2^*$.*

Proof. Using lemma 2.15, we may replace T with a projective completion, i.e., we may assume T is smooth and projective. If we take projective completions \bar{X} , \bar{Y} good for $p_2 \circ i$, then \bar{X} , $\bar{Y} \times T$ is good for i and $\bar{Y} \times T$, \bar{Y} is good for p_2 . Also, the composition $i_{\bar{X}, \bar{Y} \times T}^* \circ p_{2, \bar{Y} \times T, \bar{Y}}^*$ is defined and

$$i_{\bar{X}, \bar{Y} \times T}^* \circ p_{2, \bar{Y} \times T, \bar{Y}}^* = (p_2 \circ i)_{\bar{X}, \bar{Y}}^*.$$

□

Next, we consider the case of a projection.

Lemma 2.18. *For $X, Y \in \mathbf{Sm}/k$, let $p_2 : X \times Y \rightarrow Y$ be the projection. Let $f : Y \rightarrow Z$ be a morphism in \mathbf{Sm}/k . Then $(fp_2)^* = p_2^*f^*$.*

Proof. Let $j : X \rightarrow \bar{X}$ be a projective completion with strict normal crossing complement $\bar{X} \setminus X$. This gives us the commutative diagram

$$\begin{array}{ccc} X \times Y & & \\ p_2 \downarrow & \searrow^{j \times \text{id}} & \\ \bar{X} \times Y & \xrightarrow{\bar{p}_2} & Y \xrightarrow{f} Z \end{array}$$

Using lemma 2.15, it suffices to show that $(f\bar{p}_2)^* = \bar{p}_2^*f^*$, so we may assume that X is smooth and projective.

Take projective completions $Y \rightarrow \bar{Y}$ and $Z \rightarrow \bar{Z}$ good for f . Since p_2 is smooth and projective, $X \times Y \rightarrow X \times \bar{Y}$, $Y \rightarrow \bar{Y}$ is good for p_2 , and $X \times Y \rightarrow X \times \bar{Y}$, $Z \rightarrow \bar{Z}$ is good for fp_2 . Similarly, the composition $p_{2X \times \bar{Y}, \bar{Y}}^* \circ f_{\bar{Y}, \bar{Z}}^*$ is defined and

$$p_{2X \times \bar{Y}, \bar{Y}}^* \circ f_{\bar{Y}, \bar{Z}}^* = (fp_2)_{X \times \bar{Y}, \bar{Z}}^*.$$

□

Given an $X \in \mathbf{Sm}/k$ and a morphism $g : Y \rightarrow Z$ in \mathbf{Sm}/k , we have the closed immersions

$$i_1 := \text{id}_X \times (\text{id}_Y, g) : X \times Y \rightarrow X \times Y \times Z$$

and

$$i_2 := (\text{id}_Y, g) : Y \rightarrow Y \times Z$$

giving the commutative diagram

$$\begin{array}{ccccc} X \times Y & \xrightarrow{i_1} & X \times Y \times Z & & \\ p_2 \downarrow & & p_{23} \downarrow & \searrow^{p_3} & \\ Y & \xrightarrow{i_2} & Y \times Z & \xrightarrow{p_2} & Z \end{array}$$

Lemma 2.19. $p_2^* \circ i_2^* = i_1^* \circ p_{23}^*$ and $p_3^* = p_{23}^* \circ p_2^*$.

Proof. We first show that $p_2^* \circ i_2^* = i_1^* \circ p_{23}^*$. Let $h : X \times Y \rightarrow Y \times Z$ be the composition $p_{23} \circ i_1$. By lemma 2.16 we have

$$h^* = i_1^* \circ p_{23}^*,$$

so we need only show that $h^* = p_2^* \circ i_2^*$. This follows from lemma 2.18.

The identity $p_3^* = p_{23}^* \circ p_2^*$ also follows from lemma 2.18. □

Theorem 2.20. *The assignment sending X to $h(X)$ and $f : X \rightarrow Y$ to $f^* : h(Y) \rightarrow h(X)$ defines a functor*

$$h : \mathbf{Sm}/k^{\text{op}} \rightarrow \text{DM}^H(k)$$

with $h|_{\mathbf{SmProj}} = h_{\mathbf{SmProj}}$.

Proof. We need to show that $(gf)^* = g^*f^*$ for morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ in \mathbf{Sm}/k . We have the commutative diagram

$$\begin{array}{ccccc}
 & & & & i_3 \\
 & & & & \curvearrowright \\
 X & \xrightarrow{i_1} & X \times Y & \xrightarrow{\text{id} \times i_2} & X \times Y \times Z \\
 & \searrow f & \downarrow p_2 & & \downarrow p_{23} \\
 & & Y & \xrightarrow{i'} & Y \times Z & \xrightarrow{p_3} \\
 & & & \searrow g & \downarrow p_2 \\
 & & & & Z
 \end{array}$$

with $i_1 = (\text{id}_X, f)$, $i_2 = (\text{id}_Y, g)$, $i_3 := \text{id}_X \times i_2 \circ i_1$. Using lemmas 2.14, 2.16, 2.18 and 2.19, we have

$$f^*g^* = i_3^* \circ p_3^*$$

Factoring p_3 as

$$X \times Y \times Z \xrightarrow{p_{31}} X \times Z \xrightarrow{p_2} Z$$

and using lemma 2.18 gives $p_3^* = p_{13}^* \circ p_2^*$. Thus

$$f^*g^* = i_3^* \circ p_{13}^* \circ p_2^*.$$

We have the commutative diagram

$$\begin{array}{ccc}
 & X \times Y \times Z & \\
 & \nearrow i_3 & \downarrow p_{13} \\
 X & \xrightarrow{i_4} & X \times Z \\
 & \searrow gf & \downarrow p_2 \\
 & & Z
 \end{array}$$

with $i_4 := (\text{id}_X, gf)$. By lemma 2.17 $i_4^* = i_3^*p_{13}^*$, so by lemma 2.16 again

$$f^*g^* = i_4^*p_2^* = (gf)^*.$$

□

3. ADDITIVE CHOW GROUPS

3.1. Normal schemes. We prove some elementary results on normal varieties over a field.

Lemma 3.1. *Let k be a perfect field. and let X and Y be reduced normal finite type k -schemes. Then $X \times_k Y$ is normal.*

Proof. Since k is perfect, the regular locus of X or Y is smooth over k , hence the singular locus of $X \times_k Y$ is

$$(X \times_k Y)_{\text{sing}} = X_{\text{sing}} \times_k Y \cup X \times_k Y_{\text{sing}}.$$

In particular, $X \times_k Y$ is smooth in codimension one.

Take a point $z \in (X \times_k Y)_{\text{sing}}$ and let $x = p_1(z)$, $y = p_2(z)$. Then either x has codimension at least two on X or y has codimension at least two on Y ; we may suppose the former.

We recall that a local domain \mathcal{O} is normal if and only if $\text{Spec } \mathcal{O}$ is regular in codimension one and \mathcal{O} has depth at least two (*cf.* [25]).

Since X is normal, the local ring $\mathcal{O}_{X,x}$ has depth at least two. Let s_1, s_2 be a regular sequence in the maximal ideal of $\mathcal{O}_{X,x}$. Since $Y \rightarrow \text{Spec } k$ is flat, $p_1^*(s_1), p_2^*(s_2)$ form a regular sequence in the maximal ideal of $\mathcal{O}_{X \times_k Y, z}$, hence $\mathcal{O}_{X \times_k Y, z}$ has depth at least two. Thus $X \times_k Y$ is normal. \square

Lemma 3.2. *Let k be a field, $f : Y \rightarrow X$ a projective, surjective map of normal k -schemes of finite type. Let D be a Cartier divisor on X such that $f^*D \geq 0$ on X . Then $D \geq 0$ on Y .*

Proof. Localizing at the set of generic points of $\text{supp}(D)$, we may assume that $X = \text{Spec } \mathcal{O}$ with \mathcal{O} a DVR. Thus $Y \rightarrow X$ factors as a closed embedding $Y \rightarrow \mathbb{P}_{\mathcal{O}}^N$ followed by the projection $\mathbb{P}_{\mathcal{O}}^N \rightarrow X$.

By repeatedly intersecting Y with a hypersurface $H \subset \mathbb{P}_{\mathcal{O}}^N$ of large degree and normalizing, we may assume that $f : Y \rightarrow X$ is generically finite.

Let $Y \rightarrow Y' \rightarrow X$ be the Stein factorization of f . As Y' is normal and $Y' \rightarrow X$ is finite and surjective, we must have $Y' = \text{Spec } \mathcal{O}'$, with \mathcal{O}' a semi-local PID. But $Y \rightarrow Y'$ is projective and birational, hence an isomorphism: $Y = \text{Spec } \mathcal{O}'$.

Let $t \in \mathcal{O}$ be a generator of the maximal ideal m of \mathcal{O} , let $m' \subset \mathcal{O}'$ be a maximal ideal and let s be a generator of m' . Since $\mathcal{O} \rightarrow \mathcal{O}'$ is finite, $m\mathcal{O}' \subset m'$, hence $t = us^n$ for some unit u in $\mathcal{O}'_{m'}$ and some integer $n > 0$.

But D is defined by t^a for some $a \in \mathbb{Z}$, hence the restriction of f^*D to $\text{Spec } \mathcal{O}'_{m'}$ is defined by s^{an} . Since f^*D is effective, $an \geq 0$ hence $a \geq 0$ and thus $D \geq 0$. \square

Let X be a normal variety over k . Recall that a Weil divisor on D on X is an element of the free abelian group $Div(X)$ on the set $PDiv(X)$ of prime divisors on X . Thus D is uniquely written as a finite sum $\sum a_Y[Y]$, where Y 's are the prime divisors on X . Then the assignment $D \mapsto a_Y$ gives a homomorphism $ord_Y : Div(X) \rightarrow \mathbb{Z}$, called the order function of Y . A Weil divisor is called *effective* if $ord_Y \geq 0$ for all prime divisors Y and one writes $D \geq 0$. The support of D is the set of all prime divisors Y such that $ord_Y(D) \neq 0$. This is clearly a finite set.

Let Y_1, \dots, Y_n be a set of Weil divisors on X . The *supremum* of these divisors, denoted by $sup_{1 \leq i \leq n} Y_i$ is the Weil divisor defined to be

$$(3.1) \quad sup_{1 \leq i \leq n} Y_i := \sum_{Y \in PDiv(X)} (\max_{1 \leq i \leq n} ord_Y(Y_i)) [Y].$$

3.2. The additive cycle complex. We recall the definition of the additive cycle complexes with modulus from Park [27]. We fix a field k .

Set $\mathbb{A}^1 := \text{Spec } k[t]$, $\mathbb{G}_m := \text{Spec } k[t, t^{-1}]$, $\mathbb{P}^1 := \text{Proj } k[Y_0, Y_1]$, and let $y := Y_1/Y_0$ be the standard coordinate function on \mathbb{P}^1 . We set $\square^n := (\mathbb{P}^1 \setminus \{1\})^n$.

For $n \geq 1$, let $B_n = \mathbb{A}^1 \times \square^{n-1}$, $\overline{B}_n = \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1} \supset B_n$ and $\widehat{B}_n = \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \supset \overline{B}_n$. We use the coordinate system (t, y_1, \dots, y_{n-1}) on \overline{B}_n , with $y_i := y \circ p_i$.

Let $F_{n,i}^1$, $i = 1, \dots, n-1$ be the Cartier divisor on \overline{B}_n defined by $y_i = 1$ and $F_{n,0} \subset \overline{B}_n$ the Cartier divisor defined by $t = 0$.

A *face* of B_n is a subscheme F defined by equations of the form

$$y_{i_1} = \epsilon_1, \dots, y_{i_s} = \epsilon_s; \quad \epsilon_j \in \{0, \infty\}.$$

For $\epsilon = 0, \infty$, $i = 1, \dots, n-1$ let

$$\iota_{n,i,\epsilon} : B_{n-1} \rightarrow B_n$$

be the inclusion

$$\iota_{n,i,\epsilon}(t, y_1, \dots, y_{n-2}) = (t, y_1, \dots, y_{i-1}, \epsilon, y_i, \dots, y_{n-2}),$$

Definition 3.3 (Park[27]). Let X be a finite type k -scheme, r, m non-negative integers $m \geq 1$.

(0) $\underline{Tz}_r(X, 1; m)$ is the free abelian group on integral closed subschemes Z of $X \times \mathbb{A}^1$ of dimension r such that $Z \cap (X \times \{0\}) = \emptyset$.

For $n > 1$, $\underline{Tz}_r(X, n; m)$ is the free abelian group on integral closed subschemes Z of $X \times B_n$ of dimension $r + n - 1$ such that:

(1) (Good position) For each face F of B_n , Z intersects $X \times F$ properly:

$$\dim_{X \times_k F}(Z \cap (X \times_k F)) \leq r + \dim_k(F),$$

(2) (Modulus condition) Let \bar{Z} be the closure of Z in $X \times \bar{B}_n$, $\bar{p} : \bar{Z}^N \rightarrow \bar{Z}$ the normalization, and $p : \bar{Z}^N \rightarrow X \times \bar{B}_n$ the evident map. Then

$$(m+1) \cdot p^*(X \times F_{n,0}) \leq \sup_{i=1, \dots, n-1} p^*(X \times F_{n,i}^1)$$

(3) (Induction) For each $i = 1, \dots, n-1$, $\epsilon = 0, \infty$, $(\text{id}_X \times \iota_{n,i\epsilon})^*(Z)$ is in $\underline{\text{Tz}}_r(X, n-1; m)$.

Note that the good position condition on Z implies that the cycle $(\text{id}_X \times \iota_{n,i\epsilon})^*(Z)$ is well-defined and each component satisfies the good position condition. If X is equi-dimensional of dimension d over k , set

$$\underline{\text{Tz}}^r(X, n; m) = \underline{\text{Tz}}_{d+1-r}(X, n; m).$$

We thus have the cubical abelian group $\underline{n} \mapsto \underline{\text{Tz}}_r(X, n; m)$, and if X is equi-dimensional over k , the cubical abelian group $\underline{n} \mapsto \underline{\text{Tz}}^r(X, n; m)$.

Remark 3.4. We have changed Park's notation a bit; our m corresponds to his $m+1$. This fits with the convention in [29]. \square

Definition 3.5. $\text{Tz}_r(X, *; m)$ is the *additive cycle complex* of X in dimension r and with modulus m which in degree n :

$$\text{Tz}_r(X, n; m) := \frac{\underline{\text{Tz}}_r(X, n; m)}{\underline{\text{Tz}}_r(X, n; m)_{\text{degn}}}$$

$$\text{TCH}_r(X, n; m) := H_n(\text{Tz}_r(X, *; m)); \quad n \geq 1$$

is the *additive higher Chow group* of X with modulus m .

3.3. Projective push-forward. For a morphism $f : X \rightarrow Y$, we have the induced morphism

$$f_n := f \times \text{id}_{B_n} : X \times B_n \rightarrow Y \times B_n.$$

We write \bar{f}_n for the extension $f \times \text{id}_{\bar{B}_n}$.

Lemma 3.6. *Let $f : X \rightarrow Y$ be a projective morphism of finite type k -schemes. Then*

$$Z \in \underline{\text{Tz}}_r(X, n; m) \implies f_{n*}(Z) \in \underline{\text{Tz}}_r(Y, n; m).$$

Proof. We may assume that Z is an integral cycle and that $Z \rightarrow f_n(Z)$ is generically finite. Since f_n is proper, we have

$$f_n(Z) \cap Y \times F = f_n(Z \cap X \times F)$$

(as closed subsets) for each face F . Thus $f_{n*}(Z)$ is in good position.

We now show that $f_{n*}(Z)$ satisfies the modulus condition. Let \bar{Z} be the closure of Z in $X \times \bar{B}_n$ and let $p : \bar{Z}^N \rightarrow \bar{Z} \hookrightarrow X \times \bar{B}_n$ be the normalization map for \bar{Z} . Put $V = f_n(W)$. Let \bar{V} be the closure of V in $Y \times \bar{B}_n$ and let $q : \bar{V}^N \rightarrow \bar{V} \hookrightarrow Y \times \bar{B}_n$ be the normalization map. We have a commutative diagram

$$\begin{array}{ccc} \bar{Z}^N & \xrightarrow{p} & X \times \bar{B}_n \\ h \downarrow & & \downarrow f \times id \\ \bar{V}^N & \xrightarrow{q} & Y \times \bar{B}_n \end{array}$$

Thus

$$\begin{aligned} h^*q^*(Y \times F_{n,0}) &= p^*(X \times F_{n,0}) \\ h^*q^*(Y \times F_{n,i}^1) &= p^*(X \times F_{n,i}^1) \end{aligned}$$

Therefore the modulus condition for Z and Lemma 3.2 imply the modulus condition for V .

Suppose $n = 0$. Since f is proper, we have $f(X \times 0 \cap \bar{Z}) = \bar{f}_0(\bar{Z}) \cap (Y \times 0)$. Thus the condition (0) for Z implies the same for $f_{0*}(Z)$.

For $n > 0$, we have

$$\partial_i^\epsilon(f_{n*}(Z)) = f_{n*}(\partial_i^\epsilon(Z)).$$

Thus the inductive condition for Z (and induction) implies the same for $f_{n*}(Z)$ \square

Coming back to the construction of push-forward map, Lemma 3.6 shows that the maps f_{n*} gives a well defined map $f_{n*} : \mathrm{Tz}_r(X, n; m) \rightarrow \mathrm{Tz}_r(Y, n; m)$. These maps clearly commute with the boundary maps, as in the case of higher Chow cycles. Thus we get a map of complexes

$$f_* : \mathrm{Tz}_r(X, *, m) \rightarrow \mathrm{Tz}_r(Y, *, m)$$

and hence maps $f_* : \mathrm{TCH}_r(X, n; m) \rightarrow \mathrm{TCH}_r(Y, n; m)$.

The functoriality

$$(fg)_* = f_*g_*$$

follows from the functoriality on the level of cycles.

3.4. Flat pull-back. Let $g : T \rightarrow S$ be a flat morphism of schemes. Recall that $f^* : z^i(S) \rightarrow z^i(T)$ is defined on generators by

$$f^*(1 \cdot Z) := [f^{-1}(Z)],$$

where $f^{-1}(Z) := T \times_S Z$, considered as a closed subscheme of T and for a closed subscheme $W \subset T$ of pure codimension i , $[W] \in z^i(T)$ is

the associated cycle:

$$[W] := \sum_{w \in W_{gen}} \ell_{\mathcal{O}_{T,w}}(\mathcal{O}_{W,w})[w].$$

Let $f : X \rightarrow Y$ be a flat morphism of finite type k -schemes, of relative dimension d . The maps $f_n : X \times B_n \rightarrow Y \times B_n$ are therefore flat as well, hence we have a well-defined pull-back map

$$f_n^* : z_r(Y \times B_n) \rightarrow z_{r+d}(X \times B_n).$$

Lemma 3.7. *For $V \in \underline{\mathbb{T}}_{Z_r}(Y, n; m)$, $f_n^*(V)$ is in $\underline{\mathbb{T}}_{Z_{r+d}}(X, n; m)$.*

Proof. The good position condition follows immediately from the flatness of f_n .

To check the modulus condition, let $V \in \underline{\mathbb{T}}_{Z_r}(Y, n; m)$ be an integral cycle, \bar{V} the closure in $Y \times \bar{B}_n$, and let $Z = f_n^{-1}(V)_{red}$. We may assume that Z is non-empty.

Since \bar{f}_n is flat, we have (as closed subsets)

$$\bar{Z} = \bar{f}_n^{-1}(\bar{V}).$$

In particular, the morphism $\bar{Z} \rightarrow \bar{V}$ induced by \bar{f}_n is equi-dimensional and dominant. Thus the map on the normalizations $\bar{Z}^N \rightarrow \bar{V}^N$ is also equi-dimensional and for each irreducible component \bar{Z}_α^N , the restriction $q_\alpha : \bar{Z}_\alpha^N \rightarrow \bar{V}^N$ is dominant.

We thus have the commutative diagram, where all the vertical maps are equi-dimensional and dominant.

$$\begin{array}{ccc} \bar{Z}_\alpha^N & \xrightarrow{p_{Z_\alpha}} & X \times \bar{B}_n \\ q_\alpha \downarrow & & \downarrow \bar{f}_n \\ \bar{V}^N & \xrightarrow{p_V} & Y \times \bar{B}_n \end{array}$$

This gives the identities

$$\begin{aligned} p_{Z_\alpha}^*(X \times F_{n,0}) &= q_\alpha^*(p_V^*(Y \times F_{n,0})) \\ p_{Z_\alpha}^*(X \times F_{n,i}^1) &= q_\alpha^*(p_V^*(Y \times F_{n,i}^1)) \end{aligned}$$

which show that the modulus condition for V implies the same for Z .

If $n = 0$, the identity $\bar{Z} = \bar{f}_0^{-1}(\bar{V})$ shows that the condition (0) for V implies the condition (0) for Z .

The flatness of f_n implies

$$\partial_{n,i}^\epsilon(f_n^*V) = f_{n-1}^*(\partial_{n,i}^\epsilon(V))$$

which yields the inductive condition. \square

The maps f_n^* thus gives us the map of complexes

$$f^* : \mathrm{Tz}^i(Y, *, m) \rightarrow \mathrm{Tz}^i(X, *, m).$$

The functoriality of flat pull-back for cycles yields the functoriality

$$(fg)^* = g^* f^*$$

as maps of complexes, for composable flat morphisms f and g .

The compatibility of projective push-forward and flat pull-back of cycles in cartesian squares yields:

Lemma 3.8. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of maps of finite type k -schemes, where f is flat and g is projective. Then

$$f^* g_*(\alpha) = g'_* f'^*(\alpha)$$

as maps of complexes $\mathrm{Tz}^i(Y', *, m) \rightarrow \mathrm{Tz}^i(X, *, m)$.

4. HIGHER CHOW GROUPS AND ADDITIVE CHOW GROUPS

Our aim in this section is to show that the higher Chow groups of smooth quasi-projective varieties act on the degenerate Chow groups in a natural way. We suppose in this section that k is a perfect field.

4.1. External product. We begin with the construction of an external product.

Definition 4.1. Let X and Y be finite type k -schemes. Define

$$\boxtimes_{a,b} : \underline{z}_r(X, a) \otimes \underline{\mathrm{Tz}}_s(Y, b; m) \rightarrow \underline{z}_{r+s}(X \times_k Y \times B_{a+b})$$

by

$$Z \boxtimes W := \tau_*(Z \times W),$$

where

$$\tau : X \times \square^a \times Y \times \mathbb{A}^1 \times \square^b \rightarrow X \times Y \times \mathbb{A}^1 \times \square^a \times \square^b$$

is the exchange of factors.

Lemma 4.2. *For $Z \in \underline{z}_r(X, a)$ and $W \in \underline{\mathrm{Tz}}_s(Y, b; m)$, $Z \boxtimes W$ is in $\underline{\mathrm{Tz}}_{r+s}(X \times_k Y, a + b; m)$.*

Proof. We may assume that Z and W are integral closed subschemes of $X \times \square^a$ and $Y \times B_b$, respectively.

Since

$$Z \times W \cap X \times F \times Y \times F' = (Z \cap X \times F) \times (W \cap Y \times F'),$$

for faces F of \square^a and F' of B_b , the good position condition is clear.

The condition (0) in case $a = b = 0$ is checked the same way.

For the modulus condition, let $\bar{Z} \subset X \times \mathbb{P}^{1a}$, $\bar{W} \subset Y \times \bar{B}_n$ be the respective closures of Z and W , and let

$$\begin{aligned} p_Z : \bar{Z}^N &\rightarrow X \times \mathbb{P}^{1a} \\ p_W : \bar{W}^N &\rightarrow Y \times \bar{B}_n \end{aligned}$$

be the normalizations of \bar{Z} and \bar{W} . By lemma 3.1

$$p_{Z \times W} : \bar{Z}^N \times_k \bar{W}^N \rightarrow X \times \mathbb{P}^{1a} \times Y \times \bar{B}_n$$

is the normalization of $\bar{Z} \times \bar{W} = \overline{Z \times W}$.

Suppose that j is an index with

$$m \cdot p_W^*(X \times F_{b,0}) \leq p_W^*(X \times F_{b,j}^1).$$

Then

$$(m+1) \cdot (\bar{Z}^N \times p_W^*(X \times F_{b,0})) \leq \bar{Z}^N \times p_W^*(X \times F_{b,j}^1);$$

since

$$\begin{aligned} p_{Z \times W}^*(\tau^*(X \times Y \times F_{a+b,0})) &= \bar{Z}^N \times p_W^*(X \times F_{b,0}) \\ p_{Z \times W}^*(\tau^*(X \times Y \times F_{a+b,j}^1)) &= \bar{Z}^N \times p_W^*(X \times F_{b,j}^1), \end{aligned}$$

the modulus condition for $Z \boxtimes W$ is satisfied.

We finish by checking the inductive condition. If $b = 0$, then $\overline{Z \boxtimes W} \cap X \times F_{a,0} = \emptyset$, so the inductive condition is trivially satisfied.

Suppose that $a + b > 0$. For $1 \leq i \leq a$, we have

$$\partial_{a+b,i}^\epsilon(Z \boxtimes W) = \partial_{a,i}^\epsilon(Z) \boxtimes W$$

and for $a + 1 \leq i \leq a + b$, we have

$$\partial_{a+b,i}^\epsilon(Z \boxtimes W) = Z \boxtimes \partial_{b,i-a}^\epsilon(W).$$

In the second case, the inductive condition for W , plus induction on b , shows that $Z \boxtimes \partial_{b,i-a}^\epsilon(W)$ is in $\underline{\mathbb{T}}_{Z_{r+s}}(X \times_k Y, a+b-1; m)$. In particular, if $a = 0$, then $Z \boxtimes W$ is in $\underline{\mathbb{T}}_{Z_{r+s}}(X \times_k Y, a+b; m)$. By induction on a , $\partial_{a,i}^\epsilon(Z) \boxtimes W$ is in $\underline{\mathbb{T}}_{Z_{r+s}}(X \times_k Y, a+b-1; m)$ as well, which completes the proof. \square

Theorem 4.3. *Let X and Y be finite type k -schemes. Then the maps*

$$\boxtimes_{a,b} : z_r(X, a) \otimes \underline{\mathrm{Tz}}_s(Y, b; m) \rightarrow \underline{\mathrm{Tz}}_{r+s}(X \times_k Y, *; m)$$

define a map of complexes

$$\boxtimes : z_r(X, *) \otimes \underline{\mathrm{Tz}}_s(Y, *; m) \rightarrow \underline{\mathrm{Tz}}_{r+s}(X \times_k Y, *; m)$$

Proof. A formal computation shows that the product maps $\boxtimes_{a,b}$ are compatible with boundaries, and that

$$\begin{aligned} \boxtimes_{a,b} (z_r(X, a)_{\mathrm{degn}} \otimes \underline{\mathrm{Tz}}_s(Y, b; m) + z_r(X, a) \otimes \underline{\mathrm{Tz}}_s(Y, b; m)_{\mathrm{degn}}) \\ \subset \underline{\mathrm{Tz}}_{r+s}(X \times_k Y, *; m)_{\mathrm{degn}}. \end{aligned}$$

□

Remark 4.4. It is easy to check that \boxtimes is compatible with flat pull-back: for $f : X' \rightarrow X$, $g : Y' \rightarrow Y$ flat, we have

$$f^*(Z) \boxtimes g^*(W) = (f \times g)^*(Z \boxtimes W)$$

and projective push-forward: if $f : X' \rightarrow X$ and $g : Y' \rightarrow Y$ are projective then

$$f_*(Z) \boxtimes g_*(W) = (f \times g)_*(Z \boxtimes W).$$

Furthermore we have the associativity

$$Z \boxtimes (W \boxtimes V) = (Z \boxtimes W) \boxtimes V$$

□

Passing to homology, we thus have the external product

$$\boxtimes : \mathrm{CH}_r(X, a) \otimes \mathrm{TCH}_s(Y, b) \rightarrow \mathrm{TCH}_{r+s}(X \times_k Y, a + b)$$

4.2. Cap product. We refine the external product to a ‘‘cap product’’

$$\cap_X : z^r(X, *) \otimes \underline{\mathrm{Tz}}^s(X, *; m) \rightarrow \underline{\mathrm{Tz}}^{s+r}(X, *; m)$$

for $X \in \mathbf{Sm}/k$. Let $\delta_X : X \rightarrow X \times_X X$ be the diagonal, and set

$$\delta_{X,n} := \delta_X \times \mathrm{id} : X \times B_n \rightarrow X \times_k X \times B_n.$$

Definition 4.5. For $X \in \mathbf{Sm}/k$, let

$$\underline{\mathrm{Tz}}^r(X \times_k X, n; m)_\Delta \subset \underline{\mathrm{Tz}}^r(X \times_k X, n; m)$$

be the subgroup generated by integral closed subschemes $Z \subset X \times X \times B_n$ such that

1. Z is in $\underline{\mathrm{Tz}}^r(X \times_k X, n; m)$
2. $\mathrm{codim}_{X \times F}(\delta_{X,n}^{-1}(Z) \cap X \times F) \geq r$ for all faces F of B_n .
3. $\delta_{X,n}^*(Z)$ is in $\underline{\mathrm{Tz}}^r(X, n; m)$.

The subgroups $\underline{\mathrm{Tz}}^r(X \times_k X, n; m)_\Delta$ form a cubical subgroup

$$\underline{\mathrm{Tz}}^r(X \times_k X, -; m)_\Delta \subset \underline{\mathrm{Tz}}^r(X \times_k X, -; m)$$

and the maps $\delta_{X,n}^*$ give a well-defined map of complexes

$$\delta_X^* : \mathrm{Tz}^r(X \times_k X, *; m)_\Delta \rightarrow \mathrm{Tz}^r(X, *; m).$$

Let \mathcal{W} be a finite set of closed subsets $W_n \subset X \times B_n$, $n = 1, \dots, N$. \mathcal{W} generates a ‘‘cubical closed subset’’ \mathcal{W}^c by

$$\mathcal{W}^c(\underline{n}) := \bigcup_{1 \leq m \leq N} g_{\underline{n} \rightarrow m}^{-1}(W_m) \subset \square^n.$$

Definition 4.6. Let \mathcal{W} be a finite set of closed subsets $W_n \subset X \times B_n$, such that each W_n is the support of a cycle in $\underline{\mathrm{Tz}}_s(X, n; m)$. Let $\underline{\mathrm{Tz}}_s^{\mathcal{W}}(X, n; m) \subset \underline{\mathrm{Tz}}_s(X, n; m)$ be the subgroup of cycles Z with

$$\mathrm{supp}(Z) \subset \mathcal{W}^c(\underline{n}).$$

Similarly, $z_{\mathcal{W}}^r(X, n) \subset z^r(X, n)$ is the subgroup of cycles Z such that $\mathrm{supp}(Z) \subset \mathcal{W}^c(\underline{n})$.

From the construction of \mathcal{W}^c , it is immediate that the $\underline{\mathrm{Tz}}_s^{\mathcal{W}}(X, n; m)$ form a cubical subgroup $\underline{\mathrm{Tz}}_s^{\mathcal{W}}(X, -; m)$ of $\underline{\mathrm{Tz}}_s(X, -; m)$, giving us the subcomplex

$$\mathrm{Tz}_s^{\mathcal{W}}(X, *; m) := \frac{\underline{\mathrm{Tz}}_{\mathcal{W}}^r(X, n; m)}{\underline{\mathrm{Tz}}_{\mathcal{W}}^r(X, n; m)_{\mathrm{degn}}} \subset \mathrm{Tz}_s(X, *; m).$$

Similarly, we have the subcomplex $z_{\mathcal{W}}^r(X, *)$ of $z^r(X, *)$.

The construction of the cap product is based on the following result:

Lemma 4.7. *Fix an integer s . Let \mathcal{W} be a finite set of closed subsets $W_n \subset X \times B_n$, $n = 1, \dots, N$, such that each W_n is the support of a cycle in $\mathrm{Tz}_s(X, n; m)$. Let $f : X \rightarrow Y$ be a smooth morphism. Then there is a finite set of locally closed subsets \mathcal{C} such that $(A, B) \mapsto f^*(A) \boxtimes B$ restricts to a well-defined map of complexes*

$$f^*(-) \boxtimes : z^r(Y, *)_{\overline{\mathcal{C}}} \otimes \mathrm{Tz}_s^{\mathcal{W}}(X, *; m) \rightarrow \mathrm{Tz}_{\dim_k X + s - r}(X \times_k X, *; m)_\Delta.$$

for all r .

Proof. It is obvious that $\underline{\mathrm{Tz}}_s^{\mathcal{W}}(X, n; m) = \underline{\mathrm{Tz}}_s^{\mathcal{W}}(X, n; m)_{\mathrm{degn}}$ for all $n > N$, hence $\mathrm{Tz}_s^{\mathcal{W}}(X, n; m) = 0$ for $n > N$.

If F is a closed subset of some $T \in \mathbf{Sch}_k$, we write $\dim_k F \geq d$ if there is an irreducible component F_α of F with $\dim_k F_\alpha \geq d$.

For each $n = 0, \dots, N$, define the constructible subsets $C_{n,d}$ of Y by

$$C_{n,d} = \{y \in Y \mid \dim_k f^{-1}(y) \times B_n \cap \mathcal{W}^c(\underline{n}) \geq d\}$$

Define the constructible subsets $C_{n,d}^0$ of Y by

$$C_{n,d}^0 = \{y \in Y \mid \dim_k f^{-1}(y) \times F_{n,0} \cap \overline{\mathcal{W}^c(\underline{n})} \geq d\}$$

and $C_{n,j,d}^1$ by

$$C_{n,j,d}^1 = \{y \in Y \mid \dim_k f^{-1}(y) \times F_{n,j}^1 \cap \overline{\mathcal{W}^c(\underline{n})} \geq d\}$$

Write $C_{n,d} \setminus C_{n,d-1}$ and $C_{n,d}^0 \setminus C_{n,d-1}^0$ as finite unions of locally closed subsets of X :

$$\begin{aligned} C_{n,d} \setminus C_{n,d-1} &= \cup_i C_{n,d,i} \\ C_{n,d}^0 \setminus C_{n,d-1}^0 &= \cup_i C_{n,d,i}^0 \\ C_{n,j,d}^1 \setminus C_{n,j,d-1}^1 &= \cup_i C_{n,j,d,i}^1 \end{aligned}$$

Let

$$\mathcal{C} = \{C_{n,d,i}, C_{n,d,i}^0, C_{n,j,d,i}^1 \mid n = 1, \dots, N, d = 0, \dots, n + s, \\ j = 1, \dots, n, i = 1, 2, \dots\}.$$

For subschemes $Z \subset Y \times \square^a$, $W \subset X \times \mathbb{A}^1 \times \square^b$, define $f^*(Z) \cap_X W$ as the image of $f_a^*(Z) \times_X W$ under the exchange of factors isomorphism

$$(X \times \square^a) \times_X (X \times \mathbb{A}^1 \times \square^b) \xrightarrow{\tau'} X \times B_{a+b}.$$

Note that $f_a^*(Z)$ is well-defined since f is smooth. Define $\overline{f^*(Z)} \cap_X \overline{W} \subset X \times \overline{B_{a+b}}$ similarly. Showing that $f^*(Z) \boxtimes W$ is in $\mathrm{Tz}_{\dim_k X + s - r}(X \times_k X, *, m)_\Delta$ is equivalent to showing that $f^*(Z) \cap_X W$ is in $\mathrm{Tz}_{s-r}(X, *, m)$, which we proceed to show for $Z \in \underline{z}^r(Y, *)_{\overline{c}}$, $W \in \underline{\mathrm{Tz}}_s^{\mathcal{W}}(X, *, m)$.

For Z, W integral cycles in $\underline{z}^r(Y, *)_{\overline{c}}$, $\underline{\mathrm{Tz}}_s^{\mathcal{W}}(X, b; m)$ respectively, one easily computes:

$$\dim_k f_a^{-1}(Z) \cap_X W \cap X \times F \leq s - r + \dim_k F$$

for each face F of B_{a+b} . If moreover Z is in $\underline{z}^r(Y, a)_{\overline{c}} \subset \underline{z}^r(Y, a)_{\overline{c}}$, we have

$$\begin{aligned} \dim_k \overline{f_a^{-1}(Z)} \cap_X (\overline{W} \cap X \times F_{b,0}) &\leq s - r + a + b - 1 \\ \dim_k \overline{f_a^{-1}(Z)} \cap_X (\overline{W} \cap X \times F_{b,j}^1) &\leq s - r + a + b - 1; \quad j = 1, \dots, b. \end{aligned}$$

The first condition shows that each component of $f^{-1}(Z) \cap_X W$ has the correct dimension and is in good position; in particular, the cycle $f^*(Z) \cap_X W$ is defined. The second shows that $\overline{f^*(Z)} \times \overline{B_b}$ intersects $\tau'_*(\overline{\square^a} \times (\overline{W} \cap X \times F_{b,0}))$ and the $\tau'_*(\overline{\square^a} \times (\overline{W} \cap X \times F_{b,j}^1))$ properly on $X \times \overline{B_{a+b}}$. This in turn shows that the modulus condition for W implies the modulus condition for $f^*(Z) \cap_X W$.

The condition (0) in case $b = 0$ trivially implies the induction condition for $f^*(Z) \cap_X W$; in general, the inductive condition follows by induction on b . \square

The same proof, ignoring the modulus and inductive conditions, yields

Lemma 4.8. *Fix an integer $r \geq 0$. Let \mathcal{W} be a finite set of closed subsets $W_n \subset X \times B_n$, $n = 1, \dots, N$, such that each W_n is the support of a cycle in $\underline{z}^s(X, n)$. Let $f : X \rightarrow Y$ be a smooth morphism. Then there is a finite set of locally closed subsets \mathcal{C} such that the partially defined map $(A, B) \mapsto f^*(A) \cup_X B$ restricts to a well-defined map of complexes*

$$f^*(-) \cup_X (-) : z^r(Y, *)_{\mathcal{C}} \otimes z_{\mathcal{W}}^s(X, *) \rightarrow \underline{z}^{r+s}(X, *).$$

Lemma 4.9. *Let s, \mathcal{W} and $f : X \rightarrow Y$ be as in lemma 4.7, and let \mathcal{C} be a finite set of locally closed subsets of X given by lemma 4.7. Let \mathcal{C}_1 be a finite set of locally closed subsets of X containing \mathcal{C} .*

*Fix an integer $r \geq 0$, and let \mathcal{T} be a finite set of closed subsets of $Y \times B_n$ of the form $\text{supp}(T_n)$, where T_n is in $\underline{z}^r(Y, n)_{\overline{\mathcal{C}}_1}$. Let $g : Y \rightarrow Z$ be a smooth morphism in \mathbf{Sm}/k . Then there is a finite set \mathcal{C}_2 of locally closed subsets of Z such that for each $W \in \text{Tz}_s^{\mathcal{W}}(X, *, m)$, $Z \in \underline{z}_{\mathcal{T}}^r(Y, *)$ and $V \in \underline{z}^q(Z, *)_{\overline{\mathcal{C}}_2}$:*

(1) *The cycles $g^*(V) \cup_Y Z$, $(gf)^*(V) \cap_X (f^*(Z) \cap_X W)$ and $f^*(g^*(V) \cup_Y Z) \cap_X W$ are defined.*

(2) *$g^*(V) \cup_Y Z$ is in $\underline{z}^{q+r}(Y, *)$*

(3) *$(gf)^*(V) \cap_X (f^*(Z) \cap_X W)$ and $f^*(g^*(V) \cup_Y Z) \cap_X W$ are in $\text{Tz}_*(X, *, m)$ and*

$$(gf)^*(V) \cap_X (f^*(Z) \cap_X W) = f^*(g^*(V) \cup_Y Z) \cap_X W.$$

Proof. Since the groups $\text{Tz}_s^{\mathcal{W}}(X, *, m)$ and $\underline{z}_{\mathcal{T}}^r(Y, *)$ is finitely generated, it suffices to exhibit a finite set \mathcal{C}_2 satisfying (1)-(3) for each integral $W \in \text{Tz}_s^{\mathcal{W}}(X, *, m)$ and integral $Z \in \underline{z}_{\mathcal{T}}^r(Y, *)$.

Given $W \in \text{Tz}_s^{\mathcal{W}}(X, *, m)$ and integral $Z \in \underline{z}_{\mathcal{T}}^r(Y, *)$, the cycle $f^*(Z) \cap_X W$ is in $\text{Tz}_{s-r}(X, *, m)$. Letting \mathcal{U} be the set of intersections of $\text{supp}(f^*(Z) \cap_X W)$ with faces $X \times F$, we may apply lemma 4.7 to $\text{Tz}_{s-r}^{\mathcal{U}}(X, *, m)$ and morphism $gf : X \rightarrow Z$, to yield the set $\mathcal{C}_2(fg)$, which is “good” for the intersection $(gf)^*(V) \cap_X (f^*(Z) \cap_X W)$. Similarly, we may apply lemma 4.9 to the set of intersections \mathcal{U}' of $\text{supp}(Z)$

with faces $Y \times F$ to yield the set $\mathcal{C}_2(g)$, which is “good” for the intersection $g^*(V) \cup_Y Z$.

Let $\mathcal{C}_2 = \mathcal{C}_2(g) \cup \mathcal{C}_2(fg)$. We claim that all $V \in \underline{z}^q(Z, *)_{\overline{\mathcal{C}}_2}$ satisfy (1)-(3). For this, it suffices to show that for integral $V \in \underline{z}^q(Z, *)_{\overline{\mathcal{C}}_2}$, the cycle $f^*(g^*(V) \cup_Y Z) \cap_X W$ is well-defined (only as a cycle on $X \times B_n$ for appropriate n) and that this cycle is equal to $(gf)^*(V) \cap_X (f^*(Z) \cap_X W)$. But since f is smooth and the cycle $g^*(V) \cup_Y Z$ is defined, so is the cycle

$$f^*(g^*(V) \cup_Y Z) = (gf)^*(V) \cup_X f^*(Z).$$

Since the cycle $(gf)^*(V) \cap_X (f^*(Z) \cap_X W)$ is also defined and all cycles involved are effective, this implies that $((gf)^*(V) \cup_X f^*(Z)) \cap f^*(Z)$ is defined. The associativity of cycle intersection implies that

$$(gf)^*(V) \cap_X (f^*(Z) \cap_X W) = ((gf)^*(V) \cup_X f^*(Z)) \cap f^*(Z),$$

completing the proof. \square

Theorem 4.10. *Let X be a smooth quasi-projective variety over k .*

(1) *If X is projective, there is a product*

$$\cap_X : \mathrm{CH}^r(X, p) \otimes \mathrm{TCH}_s(X, q; m) \rightarrow \mathrm{TCH}_{s-r}(X, p+q; m),$$

natural with respect to flat pull-back, and satisfying the projection formula

$$f_*(f^*(a) \cap_X b) = a \cap_Y f_*(b)$$

for $f : X \rightarrow Y$ a morphism of smooth projective varieties over k . If f is a flat morphism, we have in addition the projection formula

$$f_*(a \cap_X f^*(b)) = f_*(a) \cap_Y b$$

(2) *In general, there is a product*

$$\cap_X : \mathrm{CH}^r(X) \otimes \mathrm{TCH}_s(X, q; m) \rightarrow \mathrm{TCH}_{s-r}(X, q; m),$$

natural with respect to flat pull-back, and satisfying the projection formula

$$f_*(f^*(a) \cap_X b) = a \cap_Y f_*(b)$$

for $f : X \rightarrow Y$ a projective morphism of smooth quasi-projective varieties over k , and the projection formula

$$f_*(a \cap_X f^*(b)) = f_*(a) \cap_Y b$$

if f is flat.

In addition, all products are associative.

Proof. The existence of the product in (1) follows from lemma 4.7: the map

$$\operatorname{colim}_{\mathcal{W}} \mathbb{T}z_s^{\mathcal{W}}(X, *, m) \rightarrow \mathbb{T}z_s(X, *, m)$$

is an isomorphism and by lemma 1.15(1), the inclusion $z^r(X, *)_{\bar{\mathcal{C}}} \rightarrow z^r(X, *)$ is a quasi-isomorphism.

To prove the first projection formula, let $\Gamma \subset X \times_k Y$ be the graph of f . Let \mathcal{W}_X be a finite set of closed subsets $W_n \subset X \times B_n$ as in the statement of lemma 4.7 and let \mathcal{W} be the finite set of subsets $(\operatorname{id}_X, f) \times \operatorname{id}_{B_n}(W_n)$ of $X \times_k Y \times B_n$. We apply lemma 4.7 to the smooth morphism $p_2 : X \times_k Y \rightarrow Y$, the set \mathcal{W} and the integer s . This gives us the finite set of locally closed subsets \mathcal{C} of Y and a well-defined map of complexes

$$(-) \cap_{X \times_k Y} p_2^*(-) : z^r(Y, *)_{\bar{\mathcal{C}}} \otimes \mathbb{T}z_s^{\mathcal{W}}(X \times_k Y, *, m) \rightarrow \mathbb{T}z_{s-r}(X \times_k Y, *, m)$$

Applying lemma 4.8 with $s = \dim_k Y$ and $\mathcal{W} = \{\Gamma\}$, we may add elements to \mathcal{C} so that we have a well-defined map of complexes

$$p_2^*(-) \cap_{X \times_k Y} (-) : z^r(Y, *)_{\bar{\mathcal{C}}} \otimes z_{\Gamma}^{\dim_k Y}(X \times_k Y, *) \rightarrow z^{s+r}(X \times_k Y, *)$$

For $Z \in z^r(Y, *)_{\bar{\mathcal{C}}}$, the cycle $f^*(Z)$ is just $p_{1*}(\Gamma \cap p_2^*(Z))$, it follows that f^* gives a well-defined map of complexes

$$f^* : z^r(Y, *)_{\bar{\mathcal{C}}} \rightarrow z^r(X, *).$$

Similarly, for $W \in \underline{\mathbb{T}}z_s(X, *, m)$,

$$f^*(Z) \cap_X W = p_{1*}(p_2^*(Z) \cap_{X \times_k Y} (\operatorname{id}, f)_*(W)),$$

so we have a well-defined map of complexes

$$f^*(-) \cap_X (-) : z^r(Y, *)_{\bar{\mathcal{C}}} \otimes \underline{\mathbb{T}}z_s^{\mathcal{W}_X}(X, *, m) \rightarrow \underline{\mathbb{T}}z_{s-r}(X, *, m).$$

The proof of the second projection formula for flat f is similar, and is left to the reader.

Finally, if Z is an integral cycle such that $f^*(Z)$ is defined and W is an integral cycle such that $f^*(Z) \cap_X W$ is defined, then $Z \cap_Y f_*(W)$ is defined and

$$f_*(f^*(Z) \cap_X W) = Z \cap_Y f_*(W).$$

Thus sending (Z, W) to $Z \cap_Y f_*(W)$ gives a well-defined map of complexes

$$(-) \cap_Y f_*(-) : z^r(Y, *)_{\bar{\mathcal{C}}} \otimes \underline{\mathbb{T}}z_s^{\mathcal{W}_X}(X, *, m) \rightarrow \underline{\mathbb{T}}z_{s-r}(Y, *, m).$$

and $f_* \circ (f^*(-) \cap_X (-)) = (-) \cap_Y f_*(-)$. The projection formula thus follows from lemma 1.15(1).

The proof of the associativity $(z_1 \cup_X z_2) \cap_X w = z_1 \cap_X (z_2 \cap_X w)$ is proven similarly: Applying lemma 4.9 to the projections $X^3 \xrightarrow{p_{23}}$

$X^2 \xrightarrow{p_2} X$ and then using lemma 1.15(1), we see that given classes $z_1 \in \text{CH}^r(X, a)$, $z_2 \in \text{CH}^t(X, b)$ and $w \in \text{TCH}_s(X, q; m)$, we can find representatives Z_1, Z_2, W such that all intersections $Z_1 \cup_X Z_2$, $Z_2 \cap_X W$, $Z_1 \cap_X (Z_2 \cap_X W)$ and $(Z_1 \cup_X Z_2) \cap_X W$ are defined, are all in the appropriate complexes and satisfying

$$(Z_1 \cup_X Z_2) \cap_X W = Z_1 \cap_X (Z_2 \cap_X W)$$

as cycles.

The proof of (2) is the same, we just use lemma 1.15(2) instead of lemma 1.15(1). \square

5. THE ADDITIVE CHOW GROUPS OF AN S -MOTIVE

5.1. The action of correspondences. Theorem 4.10 gives us an action of correspondences on the groups $\text{Tz}_s(X, *, m)$.

Definition 5.1. Take $X, Y \in \mathbf{Sm}/S$ with Y projective and take $\alpha \in \text{Cor}^r(X \times_S Y) = \text{CH}_{r+\dim_S X}(X \times Y)$. Define

$$\alpha_* : \text{Tz}_s(X, *, m) \rightarrow \text{Tz}_{s+r}(X, *, m)$$

to be the composition

$$\begin{aligned} \text{Tz}_s(X, *, m) &\xrightarrow{p_1^*} \text{Tz}_{s+\dim Y}(X \times_S Y, *, m) \\ &\xrightarrow{\alpha \cap -} \text{Tz}_{s+r}(X \times_S Y, *, m) \xrightarrow{p_{1*}} \text{Tz}_{s+r}(X, *, m) \end{aligned}$$

Proposition 5.2. *Let S be in \mathbf{Sm}/k . For $X, Y, Z \in \mathbf{Sm}/S$ with Y and Z projective, and for $\alpha \in \text{Cor}^r(X, Y)$, $\beta \in \text{Cor}^{r'}(Y, Z)$, we have*

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$$

as maps from $\text{Tz}_s(Z, *, m)$ to $\text{Tz}_{s+r+r'}(X, *, m)$.

Proof. Relying on our previous results, especially theorem 4.10, the proof is standard, using repeated application of the functoriality of flat pull-back, projective pushforward, associativity, compatibility of projective push-forward and flat pull-back in transverse cartesian squares

and the projection formula for a smooth projective morphism:

$$\begin{aligned}
(\beta\alpha)_*(w) &:= p_{X^*}^{XZ} \left(p_{XZ^*}^{XYZ} \left(p_{YZ^*}^{XYZ^*}(\beta) \cup_{XYZ} p_{XY^*}^{XYZ^*}(\alpha) \right) \cap_{XZ} p_X^{XZ^*}(w) \right) \\
&= p_{Z^*}^{XZ} \left(p_{XZ^*}^{XYZ} \left[p_{YZ^*}^{XYZ^*}(\beta) \cap_{XYZ} \left(p_{XY^*}^{XYZ^*}(\alpha) \cap_{XYZ} p_X^{XZ^*}(w) \right) \right] \right) \\
&= p_{Z^*}^{XYZ} \left(p_{YZ^*}^{XYZ^*}(\beta) \cap_{XYZ} \left(p_{XY^*}^{XYZ^*}(\alpha) \cap_{XYZ} p_X^{XZ^*}(w) \right) \right) \\
&= p_{Z^*}^{YZ} \left(p_{YZ^*}^{XYZ} \left[p_{YZ^*}^{XYZ^*}(\beta) \cap_{XYZ} \left(p_{XY^*}^{XYZ^*}(\alpha) \cap_{XYZ} p_X^{XZ^*}(w) \right) \right] \right) \\
&= p_{Z^*}^{YZ} \left(\beta \cap_{YZ} \left[p_{YZ^*}^{XYZ} \left(p_{XY^*}^{XYZ^*}(\alpha \cap_{XY} p_X^{XY^*}(w)) \right) \right] \right) \\
&= p_{X^*}^{XY} \left(\beta \cap_{YZ} \left[p_Y^{YZ^*} \left(p_{Y^*}^{XY}(\alpha \cap_{XY} p_X^{XY^*}(w)) \right) \right] \right) \\
&= \beta_*(\alpha_*(w)).
\end{aligned}$$

□

The proposition immediately implies

Theorem 5.3. *Let \mathbf{GrAb} denote the category of graded abelian groups. For each integer $p \geq 1$ assignment*

$$(X, n) \mapsto \mathrm{TCH}_n(X, p; m)$$

extends to an additive functor

$$\mathrm{TCH}_*(p; m) : \mathrm{Mot}(S) \rightarrow \mathbf{GrAb}$$

with

$$\mathrm{TCH}_*(p; m)(m(X)(n), \alpha) = \alpha_*(\mathrm{TCH}_n(X, p; m)) \subset \mathrm{TCH}_n(X, p; m).$$

Corollary 5.4. *For each $f : X \rightarrow Y$ in \mathbf{SmProj}/S , there is a well-defined pull-back map*

$$f^* : \mathrm{TCH}^s(Y, *, m) \rightarrow \mathrm{TCH}^s(X, *, m)$$

with $(gf)^ = f^*g^*$. If f is flat, then f^* is the same as the flat pull-back. Finally, the projection formula*

$$f_*(a \cap_X f^*(b)) = f_*(a) \cap_Y b$$

*is satisfied for $a \in \mathrm{CH}^r(X)$, $b \in \mathrm{TCH}_s(Y, *, m)$.*

Proof. Let $\Gamma_f \subset X \times_S Y$ be the graph of f , and let ${}^t\Gamma_f \subset Y \times_S X$ be the transpose, giving the element $[{}^t\Gamma_f] \in \mathrm{Cor}_S^{\dim_S X - \dim_S Y}(Y, X)$. Define $f^* = [{}^t\Gamma_f]_*$. The functoriality follows from the identity

$$[{}^t\Gamma_f] \circ [{}^t\Gamma_g] = [{}^t\Gamma_{gf}]$$

in Cor_S and proposition 5.2.

The fact that the new definition of f^* agrees with the old one for flat f follows from the identity

$$(\mathrm{id}_X, f)_*(f_{old}^*(w)) = [\Gamma_f] \cap p_2^*(w).$$

The operations $a \cap_X (-)$, and $f_*(a) \cap_Y (-)$ can be also written as the action of correspondences, namely $\delta_{X*}(a)_*$ and $\delta_{Y*}(f_*(a))_*$, where $\delta_X : X \rightarrow X \times_S X$ and $\delta_Y : Y \rightarrow Y \times_S Y$ are the diagonals. Similarly, $f_* = [\Gamma_f]_*$.

The projection formula follows from proposition 5.2 and the identity of correspondences

$$[\Gamma_f] \circ \delta_{X*}(a) \circ {}^t[\Gamma_f] = \delta_{Y*}(f_*(a)).$$

□

Remark 5.5. Let $X \rightarrow S$ be a smooth projective morphism in \mathbf{Sm}/k and let $i : Z \rightarrow X$ be a closed immersion in \mathbf{Sm}/k . Then we have the identity of operators on $\mathrm{TCH}^*(X, *, m)$:

$$i_* \circ i^* = [Z] \cap_X (-),$$

where $[Z] \in \mathrm{CH}^*(X)$ denotes the class of Z . This follows from the projection formula:

$$\begin{aligned} i_*(i^*(w)) &= i_*(1_Z \cap_Z i^*(w)) \\ &= i_*(1_Z) \cap_X w \\ &= [Z] \cap_X w. \end{aligned}$$

□

5.2. Projective bundle formula. In this section, we compute the additive higher Chow groups of the projective bundles over smooth varieties.

Theorem 5.6. *Let X be a smooth quasi-projective variety and let E be a vector bundle on X of rank $r + 1$. Let $p : \mathbb{P}(E) \rightarrow X$ be the associated projective bundle over X . Let $\eta \in \mathrm{CH}^1(\mathbb{P}(E))$ be the class of the tautological line bundle $\mathcal{O}(1)$. Then for any $q, n \geq 1$ and $m \geq 2$, the map*

$$\theta : \bigoplus_{i=0}^r \mathrm{TCH}^{q-i}(X, n; m) \rightarrow \mathrm{TCH}^q(\mathbb{P}(E), n; m)$$

given by

$$(a_0, \dots, a_r) \mapsto \sum_{0 \leq i \leq r} \eta^i \cap_{\mathbb{P}(E)} p^*(a_i)$$

is an isomorphism.

Proof. Write $\mathbb{P} := \mathbb{P}(E)$. As is well-known, the correspondences

$$\alpha_i := (\mathrm{id}_{\mathbb{P}}, p)_*(\eta^i) \in \mathrm{Cor}_S^i(\mathbb{P}, X)$$

give an isomorphism

$$\sum_{i=0}^r \alpha_i : m(\mathbb{P}) \cong \bigoplus_{i=0}^r m(X)(i)$$

in $\text{Mot}(S)$.

It is easy to see that

$$\text{TCH}^*(*; m)(\alpha_i)(a) = \eta^i \cap_{\mathbb{P}} p^*(a),$$

whence the result. \square

5.3. Blow-up formula. Let $i : Z \rightarrow X$ be a closed immersion in \mathbf{SmProj}/S . Denote the blow-up of X along Z by $\mu : X_Z \rightarrow X$. Let $E := \mu^{-1}(Z)$ be the exceptional divisor, giving us the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{i_E} & X_Z \\ q \downarrow & & \downarrow \mu \\ Z & \xrightarrow{i} & X \end{array}$$

$E \rightarrow Z$ is the projective space bundle $p : \mathbb{P}(N_{Z/X}) \rightarrow Z$, where $N_{Z/X}$ is the normal bundle. Applying the functor $m : \mathbf{SmProj}/S \rightarrow \text{Mot}(S)$ to this square gives us the sequence in $\text{Mot}(S)$

$$(5.1) \quad m(E) \xrightarrow{(m(i_E), -m(q))} m(X_Z) \oplus m(Z) \xrightarrow{m(\mu) + m(i)} m(X)$$

Lemma 5.7. *The maps $({}^t[\Gamma_\mu], 0) : m(X) \rightarrow m(X_Z) \oplus m(Z)$ and $(m(i_E), -m(q)) : m(E) \rightarrow m(X_Z) \oplus m(Z)$ gives an isomorphism*

$$\phi : m(E) \oplus m(X) \rightarrow m(X_Z) \oplus m(Z)$$

exhibiting the sequence (5.1) as a split exact sequence in $\text{Mot}(S)$.

Proof. One computes that $m(\mu) \circ ({}^t[\Gamma_\mu]) = \text{id}_{m(X)}$, hence $({}^t[\Gamma_\mu], 0)$ gives a splitting to $m(\mu) + m(i)$. Thus, it suffices to check that ϕ is an isomorphism.

By Manin's identity principle, it suffices to check that the map

$$\text{id}_T \otimes \phi : m(T) \otimes [m(E) \oplus m(X)] \rightarrow m(T) \otimes [m(X_Z) \oplus m(Z)]$$

induces an isomorphism

$$(\text{id}_T \otimes \phi)^* : \text{CH}^*(T \times X_Z) \oplus \text{CH}^*(T \times Z) \rightarrow \text{CH}^*(T \times E) \oplus \text{CH}^*(T \times X)$$

for all smooth projective $T \rightarrow S$. This follows easily from the known blow-up formula for the Chow groups. \square

Theorem 5.8. *Let $i : Z \rightarrow X$ be a closed immersion in \mathbf{SmProj}/S and let $\mu : X_Z \rightarrow X$ be the blow-up of X along Z , $i_E : E \rightarrow X_Z$ the exceptional divisor with morphism $q : E \rightarrow Z$. Then the sequences*

$$0 \rightarrow \mathrm{TCH}_s(E, n; m) \xrightarrow{(q_*, -i_{E*})} \mathrm{TCH}_s(Z, n; m) \oplus \mathrm{TCH}_s(X_Z, n; m) \xrightarrow{i_* + \mu_*} \mathrm{TCH}_s(X, n; m) \rightarrow 0$$

and

$$0 \rightarrow \mathrm{TCH}^s(X, n; m) \xrightarrow{(i^*, \mu^*)} \mathrm{TCH}^s(Z, n; m) \oplus \mathrm{TCH}^s(X_Z, ; m) \xrightarrow{q^* - i_E^*} \mathrm{TCH}^s(E, n; m) \rightarrow 0$$

are split exact.

Proof. The result for the second sequence follows by applying the functor $\mathrm{TCH}(n; m)$ to the sequence (5.1) and using lemma 5.7. For the first sequence, we take the dual of the sequence (5.1) and use the same argument. \square

5.4. Additive Chow groups of varieties with group actions. Let k be a perfect field which may not be algebraically closed. As an application of our extension of the additive chow groups to a functor on the category of motives $\mathrm{Mot}(S)$ (theorem 5.3), we compute the additive Chow groups of smooth varieties with group actions, in particular, for projective homogeneous spaces and Grassmann bundles over smooth projective varieties, generalising theorem 5.6.

Theorem 5.9. *Let X be a smooth projective variety of dimension d over k equipped with an action of the multiplicative group \mathbb{G}_m . Let $\{Z_i, 0 \leq i \leq r\}$ be the connected components of the fixed point locus. Then Z_i 's are all smooth projective and one has for $n \geq 0, p \geq 1$ and $m \geq 1$ the formula*

$$(5.2) \quad \mathrm{TCH}_n(X, p; m) \cong \bigoplus_{i=0}^r \mathrm{TCH}_{a_i}(Z_i, p; m)$$

where a_i is the dimension of the positive eigenspace of the action of \mathbb{G}_m on the tangent space of X at an arbitrary point $z \in Z_i$.

Proof. By a theorem of Bialynicki-Birula [3], generalized by Hesselink [18] to case of non-algebraically closed fields, the fixed point locus $X^{\mathbb{G}_m}$ is smooth, closed subscheme of X with the decomposition into connected components as stated. Furthermore, X has a filtration

$$X = X_r \supset X_{r-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

and $X_i \setminus X_{i-1}$ admits the structure of an \mathbb{A}^{a_i} -bundle $\phi_i : X_i \setminus X_{i-1} \rightarrow Z_i$.

By a result of Karpenko and Chernousov-Gille-Merkurjev [11], theorem 7.1, the motive of X has the following decomposition in the category $\text{Mot}(k)$: $m(X) = \bigoplus_{i=0}^r m(Z_i)(a_i)$, hence

$$(5.3) \quad m(X)(n) = \bigoplus_{i=0}^r m(Z_i)(n + a_i)$$

The theorem now follows by applying theorem 5.3 (with $S = \text{Spec}(k)$) to the above decomposition. \square

Remark 5.10. Theorem 5.9 applies to a smooth projective variety X which has an action of a k -split reductive group G , by restricting the action to a given \mathbb{G}_m in G . Furthermore, if X is a quasi-homogeneous space of such a reductive group, then Brosnan [10] has shown that the components Z_i 's are also quasi-homogeneous spaces of much smaller dimensions and hence their additive Chow groups should be easier to compute.

Corollary 5.11. *Let G be a k -split reductive group and let P be a parabolic subgroup of G . Then one has for $n \geq 0, p \geq 1$ and $m \geq 1$,*

$$\text{TCH}_n(G/P, p; m) \cong \bigoplus_w \text{TCH}_{n-l(w)}(k, p; m)$$

where w runs through the set of cells of G/P in its Bruhat decomposition and $l(w)$ is the dimension of the corresponding cell.

Proof. Köck has shown in [21] that in this case one can choose a $\mathbb{G}_m \subset G$ so that the varieties Z_i 's are just rational points. The result thus follows directly from theorem 5.9. \square

Remark 5.12. Corollary 5.11 shows in particular that the additive Chow groups of a Grassmann variety has a decomposition in terms of the additive Chow groups of the ground field. One can write down the similar formula for the additive Chow groups of any Grassmann bundle over a smooth projective variety X in terms of the additive Chow groups of X because one has the corresponding decomposition in $\text{Mot}(k)$ (see [21]).

6. LOGARITHMIC ADDITIVE CHOW GROUPS

The additive Chow groups quite clearly do not satisfy the homotopy invariance property; for this reason it is difficult to apply the existing technology to prove either a localization property with respect to a closed immersion, or a Mayer-Vietoris property for a Zariski open cover. However, since we have a blow-up exact sequence, we can use the machinery of Guillén and Navarro Aznar to define “additive Chow groups with log poles”, at least if we assume k admits resolution of singularities.

6.1. The extension theorem.

Theorem 6.1. *The functor*

$$\mathrm{Tz}_r(-; m) : \mathbf{SmProj}/k \rightarrow C^-(\mathbf{Ab})$$

extends canonically to a functor

$$\mathrm{Tz}_r(-, m) : \mathcal{D}_{\mathrm{hom}}(k) \rightarrow D^-(\mathbf{Ab}).$$

Proof. Since $\mathrm{Tz}_r(-; m)$ is additive, there is a canonical extension to an additive functor

$$\mathrm{Tz}_r(-; m) : \mathbb{Z}\mathbf{SmProj}/k \rightarrow C^-(\mathbf{Ab})$$

Taking the total complex and passing to the homotopy categories gives the exact functor

$$\mathrm{Tz}_r(-; m) : K^b(\mathbb{Z}\mathbf{SmProj}/k) \rightarrow K^-(\mathbf{Ab})$$

For each blow-up $\mu : X_Z \rightarrow X$ in \mathbf{SmProj}/k , theorem 5.8 shows that $\mathrm{Tz}_r(C(\mu); m)$ is acyclic. This gives us the desired extension

$$\mathrm{TCH}_r(-, m) : \mathcal{D}_{\mathrm{hom}}(k) \rightarrow D^-(\mathbf{Ab}).$$

□

Corollary 6.2. *The functor $\mathrm{Tz}_r(-; m) : \mathbf{SmProj}/k \rightarrow C^-(\mathbf{Ab})$ extends to a functor*

$$\mathrm{Tz}_r^{\mathrm{log}}(-; m) : \mathbf{Sch}'_k \rightarrow D^-(\mathbf{Ab})$$

satisfying:

1. *Let $\mu : Y \rightarrow X$ be a proper morphism in \mathbf{Sch}_k , $i : Z \rightarrow X$ a closed immersion. Suppose that $\mu : \mu^{-1}(X \setminus Z) \rightarrow X \setminus Z$ is an isomorphism. Set $E := \mu^{-1}(X \setminus Z)$ with maps $i_E : E \rightarrow Y$, $q : E \rightarrow Z$. There is a canonical extension of the sequence in $D^-(\mathbf{Ab})$*

$$\mathrm{Tz}_r^{\mathrm{log}}(E; m) \xrightarrow{(i_{E*}, -q^*)} \mathrm{Tz}_r^{\mathrm{log}}(Y; m) \oplus \mathrm{Tz}_r^{\mathrm{log}}(Z; m) \xrightarrow{\mu_* + i_*} \mathrm{Tz}_r^{\mathrm{log}}(X; m)$$

to a distinguished triangle in $D^-(\mathbf{Ab})$.

2. *Let $i : Z \rightarrow X$ be a closed immersion in \mathbf{Sch}_k , $j : U \rightarrow X$ the open complement. Then there is a canonical distinguished triangle in $D^-(\mathbf{Ab})$:*

$$\mathrm{Tz}_r^{\mathrm{log}}(Z; m) \xrightarrow{i_*} \mathrm{Tz}_r^{\mathrm{log}}(X; m) \xrightarrow{j^*} \mathrm{Tz}_r^{\mathrm{log}}(U; m) \rightarrow \mathrm{Tz}_r^{\mathrm{log}}(Z; m)[1],$$

which is natural with respect to proper morphisms of pairs $(X, U) \rightarrow (X', U')$.

Proof. This follows from theorems 2.9 and 6.1. □

6.2. The logarithmic additive Chow groups of an extended motive. For the Borel-Moore motive $bm(X)(r)$ of a finite-type k -scheme X , we have the associated object $\mathrm{Tz}_r^{\mathrm{log}}(X; m)$ in $D^-(\mathbf{Ab})$; sending (X, r) to $\mathrm{Tz}_r^{\mathrm{log}}(X; m)$ in $D^-(\mathbf{Ab})$ defines a functor

$$\mathrm{Tz}^{\mathrm{log}}(-; m) : \mathbf{Sch}'_k \times \mathbb{Z} \rightarrow D^-(\mathbf{Ab}).$$

We have as well the functor

$$bm : \mathbf{Sch}'_k \times \mathbb{Z} \rightarrow \hat{\mathrm{Mot}}(k) \subset \mathrm{DM}^H(k)$$

which is $bm(r)$ on $\mathbf{Sch}'_k \times r$.

For each $n \in \mathbb{Z}$, let $\mathrm{TCH}^{\mathrm{log}}(-, n; m)$ be the functor $H_n(\mathrm{Tz}^{\mathrm{log}}(-; m))$.

Theorem 6.3. *For each $n \in \mathbb{Z}$, the functor $\mathrm{TCH}^{\mathrm{log}}(-, n; m)$ extends via bm to*

$$\mathrm{TCH}_*^{\mathrm{log}}(-, n; m) : \hat{\mathrm{Mot}}(k)^* \rightarrow \mathbf{Ab}$$

such that the restriction of $\mathrm{TCH}^{\mathrm{log}}(-, n; m)$ to $\mathrm{Mot}(k) \subset \hat{\mathrm{Mot}}(k)^*$ is the functor $\mathrm{TCH}(-, n; m)$

Proof. Since $\hat{\mathrm{Mot}}(k)$ is the full pseudo-abelian subcategory of $\mathrm{DM}^H(k)$ generated by the objects $bm(X)(r)$ with $X \in \mathbf{Sch}_k$, it suffices to define an action of higher correspondences $\alpha \in \mathrm{Hom}(bm(X)(r), bm(Y)(s)[a])$,

$$\alpha_* : \mathrm{TCH}_r^{\mathrm{log}}(X, n; m) \rightarrow \mathrm{TCH}_s^{\mathrm{log}}(Y, n + a; m)$$

agreeing with the action already defined for $X, Y \in \mathbf{SmProj}/k$, $a = 0$, and satisfying the functoriality

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_*.$$

The construction is essentially the same as for $\mathrm{Mot}(k)$. Take $X \in \mathbf{Sch}_k$. Then there is a cubical object \tilde{X}_* of \mathbf{SmProj}/k such that the image under $C^b(m(r))$ of the total complex of \tilde{X}_* in $C^b(\mathbb{Z}\mathbf{SmProj}/k)$ is a representative for $bm(X)(r) \in \mathrm{DM}^H(k)$. In other words, the finite diagram $(K^m, f^{m+1, m})$ with

$$K^m = \bigoplus_{|I|=1-m} (\tilde{X}_I, r)$$

and $f^{m+1, m} : K^m \rightarrow K^{m+1}$ the map induced by the graphs of the morphisms $f_{I, J} : \tilde{X}_I \rightarrow \tilde{X}_J$ with $|I| = |J| + 1$ represents $bm(X)(r)$. Note that

$$f_{I, J_*} : \mathrm{Tz}_r(\tilde{X}_I, *, m) \rightarrow \mathrm{Tz}_r(\tilde{X}_J, *, m)$$

is well-defined for every I, J , yielding the double complex $\mathrm{Tz}(K^*, *, m)$ with second differential given by the maps f_{I, J_*} . The total complex $\mathrm{Tot}(\mathrm{Tz}(K^*, *, m))$ is thus by definition a representative in $C^-(\mathbf{Ab})$ for $\mathrm{Tz}_r^{\mathrm{log}}(X, *, m)$.

Taking a second scheme $Y \in \mathbf{Sch}_k$, we have a choice of cubical object \tilde{Y}_* , giving us the finite diagram $L = (L^m, g^{m+1,m})$ and the identification of $\mathrm{Tz}_r^{\mathrm{log}}(X, *, m)$ with $\mathrm{Tot}(\mathrm{Tz}(L^*, *, m))$.

Choose an element $[x] \in H_n(\mathrm{Tot}(\mathrm{Tz}(K^*, *, m)))$. We may represent $[x]$ by some $x \in Z_n(\mathrm{Tot}(\mathrm{Tz}(K^*, *, m)))$, i.e., a collection of cycles $x_I \in \mathrm{Tz}_r(\tilde{X}_I, n - |I|; m)$. Letting W_I be the set of all irreducible components of all cycles involved in x_I , we have for each I the cubical subset \tilde{W}_I generated by W_I . We enlarge this by adding all the images of the \tilde{W}_J for all compositions of maps $X_J \rightarrow \dots \rightarrow X_I$ involved in the diagram K^* , giving the finitely generated sub double complex $\mathrm{Tz}^{\mathcal{W}}(K^*, *, m)$ of $\mathrm{Tz}(K^*, *, m)$.

Applying lemma 4.9, we may construct highly distinguished subcomplexes $C^*(K^m, L^n)' \subset C^*(K^m, L^n)$ such that the associated internal Hom complex $\mathcal{H}om(K, L)'$ is defined, and that the action of correspondences for the individual schemes $\tilde{X}_I \times \tilde{Y}_J$ give rise to a well-defined action

$$(-)_* (-) : \mathcal{H}om(K, L)' \otimes \mathrm{Tot}(\mathrm{Tz}^{\mathcal{W}}(K^*, *, m)) \rightarrow \mathrm{Tot}(\mathrm{Tz}(L^*, *, m)).$$

For a given $\alpha \in \mathrm{Hom}(K, L[n]) = \mathrm{Hom}(bm(X)(r), bm(Y)(s)[n])$, we define $\alpha_*([x])$ by

$$\alpha_*([x]) := \tilde{\alpha}_*(x)$$

where $\tilde{\alpha} \in Z^a(\mathcal{H}om(K, L))$ is a representative for α .

The argument we used to show that the composition law in $\mathrm{DM}^H(k)$ is well-defined and associative also shows that $\alpha_*([x])$ is well-defined, independent of the choices we have made for representing elements and highly distinguished subcomplexes, and that we have the functoriality

$$\beta_*(\alpha_*([x])) = (\beta\alpha)_*([x]),$$

for $\beta \in \mathrm{Hom}(bm(Y)(s), bm(Z)(t)[b])$, once we are given a cubical object \tilde{Z}_* of \mathbf{SmProj}/k for Z , yielding the resulting finite diagram (L, h^{**}) representing $bm(Z)(t)$.

Since different choices of cubical objects \tilde{X}_* , \tilde{Y}_* representing X and Y give rise to isomorphic objects in $\mathrm{DM}^H(k)$, the functoriality we have already proved shows that the map α_* is also independent of these choices.

In case X and Y are in \mathbf{SmProj}/k , we may take the $\tilde{X}_* = X$, $\tilde{Y}_* = Y$, so that $K = (X, r)$, $L = (Y, s)$ and we are back to our original definition of the action of the correspondence $\alpha \in \mathrm{Cor}^{s-r}(X, Y)$. \square

For $X \in \mathbf{Sch}_k$ equi-dimensional over k , set $\mathrm{TCH}_{\log}^s(X, n; m) := \mathrm{TCH}_{\dim X - s}^{\log}(X, n; m)$ and extend this notation to X locally equi-dimensional by taking the direct sum over the connected components of X .

Corollary 6.4. *Sending $X \in \mathbf{Sm}/k$ to $\mathrm{TCH}_{\log}^s(X, n; m)$ extends to a functor*

$$\mathrm{TCH}_{\log}^s(-, n; m) : \mathbf{Sm}/k^{\mathrm{op}} \rightarrow \mathbf{Ab}.$$

In addition, if $j : U \rightarrow X$ is an open immersion in \mathbf{Sm}/k , the map

$$\mathrm{TCH}_{\log}^s(j, n; m) : \mathrm{TCH}_{\log}^s(X, n; m) \rightarrow \mathrm{TCH}_{\log}^s(U, n; m)$$

agrees with the map j^ from corollary 6.2.*

Proof. The existence of the functor $\mathrm{TCH}_{\log}^s(-, n; m)$ follows from theorems 2.20 and 6.3.

For the assertion on the open immersion $j : U \rightarrow X$, the result follows essentially from the definitions in case the complement of U in X is a strict normal crossing divisor. In general, temporarily write $[j]^*$ for the map in the distinguished triangle of corollary 6.2. Let $\mu : X' \rightarrow X$ be a projective birational morphism which is an isomorphism over U such that the induced open immersion $j' : U \rightarrow X'$ case strict normal crossing complement. μ_* induces the map of distinguished triangles

$$\begin{array}{ccccccc} bm(X' \setminus U) & \xrightarrow{i'_*} & bm(X') & \xrightarrow{j'^*} & bm(U) & \longrightarrow & bm(X' \setminus U)[1] \\ \mu_* \downarrow & & \mu_* \downarrow & & \parallel & & \mu_*[1] \downarrow \\ bm(X \setminus U) & \xrightarrow{i_*} & bm(X) & \xrightarrow{[j]^*} & bm(U) & \longrightarrow & bm(X \setminus U)[1] \end{array}$$

i.e $j'^* = [j]^* \circ \mu_*$. One easily calculates that $\mu_* \mu^* = \mathrm{id}_{bm(X)}$ and $j'^* = j^* \circ \mu_*$, hence $[j]^* = j^*$. \square

6.3. Codimension one. We conclude this section with a computation of $\mathrm{TCH}_{\log}^1(X, 1; m)$ for X smooth and quasi-projective over k ; we thank K. Rülling for suggesting that we make this computation. We continue to assume that k admits resolution of singularities.

Proposition 6.5. *Suppose that $p : X \rightarrow \mathrm{Spec} k$ is geometrically irreducible of dimension d over k . Then the pull-back map*

$$p^* : \mathrm{TCH}_0^{\log}(\mathrm{Spec} k, n; m) = \mathrm{TCH}_0(\mathrm{Spec} k, n; m) \rightarrow \mathrm{TCH}_d^{\log}(X, n; m)$$

is an isomorphism for all $n, m \geq 1$.

Proof. Let $j : X \rightarrow \bar{X}$ be an open immersion to a smooth projective geometrically irreducible \bar{X} over k , with complement $i : Z \rightarrow \bar{X}$. We have the localization distinguished triangle

$$\mathrm{Tz}_d^{\mathrm{log}}(Z; m) \xrightarrow{i^*} \mathrm{Tz}_d^{\mathrm{log}}(\bar{X}; m) \xrightarrow{j^*} \mathrm{Tz}_d^{\mathrm{log}}(X; m) \rightarrow \mathrm{Tz}_d^{\mathrm{log}}(Z; m)[1]$$

from corollary 6.2. Since \bar{X} is projective, we have

$$\mathrm{Tz}_d^{\mathrm{log}}(\bar{X}; m) = \mathrm{Tz}_d(\bar{X}; m)$$

and similarly for Z . Since $d > \dim_k Z$, the complex $\mathrm{Tz}_d(Z; m)$ is the 0 complex and thus the map j^* is a quasi-isomorphism. This reduces us to the case of projective X .

Let $W \subset X \times B_n$ be a generator in $\underline{\mathrm{Tz}}_d(X, n; m)$. In particular, W is a codimension one integral closed subscheme of $X \times \mathbb{A}^1 \times \square^{n-1}$, with

$$W \cap X \times 0 \times \square^{n-1} = \emptyset$$

Since X is projective, the projection \bar{W} of W to B_n is closed and disjoint from $0 \times \square^{n-1}$. Also, \bar{W} is irreducible; since X is geometrically irreducible over k , $X \times \bar{W}$ is irreducible. By reason of codimension, we therefore have

$$W = p_{B_n}^{-1}(\bar{W}),$$

and thus

$$p^* : \mathrm{Tz}_0(\mathrm{Spec} k, *, m) \rightarrow \mathrm{Tz}_d(X, *, m)$$

is an isomorphism of complexes. Since

$$\mathrm{Tz}_d^{\mathrm{log}}(X, *, m) = \mathrm{Tz}_d(X, *, m), \quad \mathrm{Tz}_0^{\mathrm{log}}(\mathrm{Spec} k, *, m) = \mathrm{Tz}_0(\mathrm{Spec} k, *, m)$$

the proof is complete. \square

Remark 6.6. Using localization as above, if X has irreducible components X_1, \dots, X_r , such that X_1, \dots, X_n all have maximal dimension d and X_{n+1}, \dots, X_r have dimension $< d$, and if k_i is the field of constants in $k(X_i)$, then

$$\mathrm{TCH}_d^{\mathrm{log}}(X, n; m) \cong \bigoplus_{i=1}^n \mathrm{TCH}_0(\mathrm{Spec} k_i, n; m).$$

Combining this with Rülling's result [29] that $\mathrm{TCH}^n(\mathrm{Spec} k, n; m) \cong \mathbb{W}_m \Omega_k^{n-1}$ gives the isomorphism

$$\mathrm{TCH}_d^{\mathrm{log}}(X, 1; m) \cong \bigoplus_{i=1}^n \mathbb{W}_m(k_i).$$

In particular, if X is smooth and geometrically irreducible over k , then

$$\mathrm{TCH}_{\mathrm{log}}^1(X, 1; m) \cong \mathbb{W}_m(k).$$

Park has made an analog of the Beilinson-Soulé vanishing conjectures for the groups $\mathrm{TCH}^q(\mathrm{Spec} k, p; m)$; for $q = 1$, these say that $\mathrm{TCH}^1(\mathrm{Spec} k, p; m) = 0$ for $p \neq 1$. \square

7. ADDITIVE CHOW GROUPS OF 1-CYCLES ON FIELDS

The additive Chow groups of 0-cycles on a field k were studied by Bloch-Esnault [8] (for modulus $m = 1$) and Rülling [29] (for general modulus m). Their main results are the construction of natural regulator maps

$$(7.1) \quad R_{0,m}^n : \mathrm{TCH}^n(k, n; m) \rightarrow \mathbb{W}_m \Omega_k^{n-1}.$$

for $n \geq 1$ and $m \geq 1$ (the case $m = 1$ is due to Bloch-Esnault), which they show are isomorphisms. Here the groups on the right are the homology of the generalized deRham-Witt complex of Hasselholt-Madsen and $\Omega_k^{n-1} := \Omega_{k/\mathbb{Z}}^{n-1}$ are absolute differentials of k .

In particular, for $n \geq 1$ and $m \geq 1$, there is a natural surjection

$$(7.2) \quad \mathrm{TCH}^n(k, n; m) \twoheadrightarrow \Omega_k^{n-1}.$$

These results motivate one to look for existence of such nontrivial regulator maps on the additive Chow groups of higher dimensional cycles for a field. If the field k has characteristic zero, Park [27], main theorem, has constructed regulator maps for $n \geq 1$ and $m \geq 1$,

$$(7.3) \quad \mathrm{Tz}^{n-1}(k, n; m) \rightarrow \Omega_k^{n-3}$$

which induces the natural maps

$$(7.4) \quad R_{1,m}^n : \mathrm{TCH}^{n-1}(k, n; m) \rightarrow \Omega_k^{n-3}.$$

He has shown in [28], theorem 1.12, that this regulator map is nontrivial for $n = 3$.

As an application of theorem 4.10, we prove in this section the following strengthening of the result of Park:

Theorem 7.1. *Let k be an algebraically closed field of characteristic zero. Then for $n \geq 1$ and $m \geq 1$, the regulator map*

$$R_{1,m}^n : \mathrm{TCH}^{n-1}(k, n; m) \rightarrow \Omega_k^{n-3}$$

is surjective. In particular, all the additive Chow groups of 1-cycles on k are nontrivial for $n \geq 3$.

Remark 7.2. In view of the known calculations of relative K -theory of nilpotent ideals and the expectation that the additive Chow groups would compute these relative K -groups, it is conjectured that the regulator maps $R_{1,2}^n$ should be isomorphisms. In that context, the above result gives some evidence for the conjecture. \square

The proof of this theorem will be given after we establish some other preliminary results. In this section, we will assume our field k to have characteristic zero. Recall the notation from section 3.2: for $n \geq 1$, we

have $B_n = \mathbb{A}^1 \times \square^{n-1}$, $\overline{B}_n = \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1} \supset B_n$ and $\widehat{B}_n = \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \supset \overline{B}_n$.

The group of units k^* acts on B_n by

$$a * (x, t_1, \dots, t_{n-1}) = \left(\frac{x}{a}, t_1, \dots, t_{n-1} \right)$$

This action clearly extends to an action of k^* on \overline{B}_n and \widehat{B}_n by the same formula, where now x could take value ∞ . It is easy to check that k^* acts on the groups $\mathrm{Tz}_r(k, n; m)$ such that all the boundary maps are k^* -equivariant. In particular, the additive Chow groups are equipped with k^* -action.

For an irreducible curve $C \subset B_n$ such that $[C] \in \mathrm{Tz}^{n-1}(k, n; m)$, let \overline{C} and \widehat{C} be the closures of C in \overline{B}_n and \widehat{B}_n respectively. Let

$$p : \widehat{B}_n \rightarrow \mathbb{P}^1$$

be the projection map. For $a \in k^*$, let σ_a denote the multiplication by a on \mathbb{P}^1 . We denote the induced action on \widehat{B}_n also by σ_a . We let k^* act on k by multiplication.

Lemma 7.3. *For any $a \in k^*$, one has $\widehat{\sigma_a(C)} = \sigma_a(\widehat{C})$ and $p^{-1}(0) \cap \widehat{C} = p^{-1}(0) \cap \sigma_a(\widehat{C})$.*

Proof. Let $\sigma_a : \widehat{B}_n \rightarrow \widehat{B}_n$ be the multiplication map. This map is clearly closed which gives $\widehat{\sigma_a(C)} \subset \sigma_a(\widehat{C})$. To show the other inclusion, let $z \in \sigma_a(\widehat{C})$ and let $y \in \widehat{C}$ such that $z = \sigma_a(y)$. Let U be an open neighborhood of z and put $V = \sigma_a^{-1}(U)$. Then $V \cap C \neq \emptyset$ which gives $\sigma_a(C) \cap U \neq \emptyset$. This proves the first part. The second part is immediate from the description of the action. \square

Fix integers $n \geq 1, m \geq 1$ and let $C \subset B_n$ be an irreducible curve such that $[C] \in \mathrm{Tz}^{n-1}(k, n; m)$. Recall that the modulus condition for additive cycle $[C]$ (see section 3) is equivalent to the following: for each closed point $P \in p^{-1}(t=0)$ on the normalization \widehat{C}^N , there is an index $i, 1 \leq i \leq n-1$ such that

$$(7.5) \quad \mathrm{ord}_P(p^*(F_{n,i}^1) - (m+1)p^*(F_{n,0})) \geq 0$$

In such a case, we write $(C, P) \in \mathcal{M}^m(t_i)$.

Fix $a \in k^*$ and let C_a denote $\sigma_a(C)$. We similarly denote $\sigma_a(\widehat{C})$ by \widehat{C}_a . Consider the diagram

$$\begin{array}{ccccc} \widehat{C}^N & \xrightarrow{\pi} & \widehat{C} & \longrightarrow & \widehat{B}_n \\ \sigma_a^N \downarrow & & \sigma_a \downarrow & & \downarrow \sigma_a \\ \widehat{C}_a^N & \xrightarrow{\pi_a} & \widehat{C}_a & \longrightarrow & \widehat{B}_n \end{array}$$

where $\sigma_a^N : \widehat{C}^N \rightarrow \widehat{C}_a^N$ is the map induced by σ_a on the normalizations. Let

$$p_a : \widehat{C}_a^N \rightarrow \widehat{B}_n$$

be the composition of the bottom arrows in the above diagram.

Let $\{P_1, \dots, P_r\} := p^{-1}(t=0)$. Then by lemma 7.3, $p_a^{-1}(t=0) = \{Q_1, \dots, Q_r\}$, where $Q_i = \sigma_a^N(P_i)$.

Lemma 7.4. *For each $i = 1, \dots, r$, and $j = 1, \dots, n-1$, $(C, P_i) \in \mathcal{M}^m(t_j)$ if and only if $(C_a, Q_i) \in \mathcal{M}^m(t_j)$.*

Proof. Let $\iota : F_{n,j}^1 \rightarrow \widehat{B}_n$ be the inclusion. We have the cartesian diagram

$$\begin{array}{ccc} F_{n,j}^1 & \xrightarrow{\iota} & \widehat{B}_n \\ \sigma_a \downarrow & & \downarrow \sigma_a \\ F_{n,j}^1 & \xrightarrow{\iota} & \widehat{B}_n \end{array}$$

from which it follows that

$$\sigma_a^N : p^*(F_{n,j}^1) \rightarrow p_a^*(F_{n,j}^1)$$

is an isomorphism. Similarly,

$$\sigma_a^N : p^*(F_{n,0}) \rightarrow p_a^*(F_{n,0})$$

is an isomorphism, from which the lemma follows directly. \square

We recall from [27] the rational absolute Kähler differential $(n-1)$ -forms $\omega_{l,m}^n \in \Gamma\left(\widehat{B}_{n+1}, \Omega_{\widehat{B}_{n+1}/\mathbb{Z}}^{n-1}(\log F_{n+1})(*\{x=0\})\right)$:

$$\begin{aligned} \omega_{1,m}^n &= \frac{1-t_1}{x^{m+1}} \frac{dt_2}{t_2} \wedge \dots \wedge \frac{dt_n}{t_n} \\ \omega_{l,m}^n &= \frac{1-t_l}{x^{m+1}} \frac{dt_{l+1}}{t_{l+1}} \wedge \dots \wedge \frac{dt_n}{t_n} \wedge \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_{l-1}}{t_{l-1}} \quad (1 < l < n) \\ \omega_{n,m}^n &= \frac{1-t_n}{x^{m+1}} \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_{n-1}}{t_{n-1}} \end{aligned}$$

We shall denote these forms simply by ω_l when n and m are fixed. For the definition of residue of a meromorphic form at a closed point of a nonsingular curve used here, we refer the reader to [27].

We also recall here the definition of the regulator maps $R_{1,m}^n$; we refer the reader to [27], section 3, for details. For cycle of the form $[C] \in \text{Tz}^{n-1}(k, n; m)$ where $C \subset B_n$ is an irreducible curve, we have

$$(7.6) \quad R_{1,m}^n([C]) = \sum_{P \in \text{Supp}(p^*\{x=0\})} R_{1,m}^n(C, P)$$

where

$$(7.7) \quad R_{1,m}^n(C, P) := (-1)^{l-1} \text{res}_P(p^*(\omega_l)) \quad \text{if } (C, P) \in \mathcal{M}^m(t_l)$$

for $1 \leq l \leq n-1$; this definition extends to $\text{Tz}^{n-1}(k, n; m)$ by \mathbb{Z} -linearity.

The following result, which explains how the residues behave with respect to the action of k^* on a smooth curve, will be one of our main tools to prove theorem 7.1.

Proposition 7.5. *Assume $n = 3$. Then for any $i = 1, \dots, r$, we have for $(C, P_i) \in \mathcal{M}^m(t_j)$,*

$$\text{res}_{P_i}(p^*(\omega_l)) = \frac{1}{a^{m+1}} \text{res}_{Q_i}(p_a^*(\omega_l)) \quad \text{for } l = 1, 2.$$

Proof. We give the proof for ω_1 ; the other cases are identical. We can work locally in order to compute the residue at a given point; take $\mathbb{A}^1 := \mathbb{P}^1 \setminus \{\infty\}$, with $0 \in \mathbb{A}^1$ corresponding to $1 \in \mathbb{P}^1$. Let $A = k[x, t_1, t_2]$ $X = \mathbb{A}^1 \times (\mathbb{A}^1)^2 = \text{Spec}(A)$. Then the action of $a \in k^*$ translates to the k -algebra automorphism $f : A \rightarrow A$ given by $f(x) = a^{-1}x, f(t_1) = t_1, f(t_2) = t_2$.

Let $P = p(P_i) = (0, \alpha_1, \alpha_2)$. Then $\sigma_a(P) = P$. Let $\mathcal{O}_{X,P}$ be the local ring of X at P . We obtain a commutative diagram of complete local rings.

$$\begin{array}{ccc} \widehat{\mathcal{O}_{X,P}} & \xrightarrow{h_a} & \widehat{\mathcal{O}_{\widehat{C}_a^N, Q_i}} \\ \downarrow \hat{f} & & \downarrow \phi \\ \widehat{\mathcal{O}_{X,P}} & \xrightarrow{h} & \widehat{\mathcal{O}_{\widehat{C}^N, P_i}} \end{array}$$

Observe that the vertical maps are isomorphisms. Then

$$(7.8) \quad \begin{aligned} \text{res}_{Q_i}(p_a^*\omega_1) &= \text{res}_{Q_i}(h_a(\omega_1)) \\ &= \text{res}_{P_i}(\phi \circ h_a(\omega_1)) = \text{res}_{P_i}(h \circ \hat{f}(\omega_1)). \end{aligned}$$

Choose a parameter $t := t_{P_i}$ of the local ring at P_i and use the parameter $t_{Q_i} := \phi^{-1}(t)$ in $\widehat{\mathcal{O}_{\widehat{C}_a^N, Q_i}}$. There are power series

$$x(t), t_1(t), t_2(t) \in \widehat{\mathcal{O}_{\widehat{C}^N, P_i}} \cong k(P_i)[[t]]$$

with $h(x) = x(t)$, etc. Since $f(t_i) = t_i$ and $f(x) = a^{-1}x$, we have

$$h \circ \hat{f}(x) = a^{-1}x(t), \quad h \circ \hat{f}(t_1) = t_1(t), \quad h \circ \hat{f}(t_2) = t_2(t).$$

Thus, using the explicit description of ω_1 , we have

$$p^*(\omega_1) = p^* \left(\frac{1 - t_1}{x^{m+1}} \frac{dt_2}{t_2} \right) = \frac{1 - t_1(t)}{x(t)^{m+1}} \frac{d(t_2(t))}{t_2(t)}.$$

Also

$$\begin{aligned} h \circ \hat{f}(\omega_1) &= h \circ \hat{f} \left(\frac{1 - t_1}{x^{m+1}} \frac{dt_2}{t_2} \right) \\ &= \frac{1 - h \circ \hat{f}(t_1)}{(h \circ \hat{f}(x))^{m+1}} \frac{d(h \circ \hat{f}(t_2))}{h \circ \hat{f}(t_2)} \\ &= \frac{1 - t_1(t)}{(a^{-1}x)^{m+1}} \frac{d(t_2(t))}{t_2(t)} \\ &= a^{m+1} \cdot p^*(\omega_1) \end{aligned}$$

It follows directly from the definition of residue that res_{P_i} is k -linear. Thus

$$\text{res}_{P_i}(h \circ \hat{f}(\omega_1)) = a^{m+1} \text{res}_{P_i}(p^*(\omega_1)).$$

This together with (7.8) completes the proof. \square

Corollary 7.6. *Consider the regulator map*

$$R_{1,m}^3 : \text{TCH}^2(k, 3; m) \rightarrow k$$

Then for each $\alpha \in \text{TCH}^2(k, 3; m)$ and $a \in K^$*

$$R_{1,m}^3(a * \alpha) = \frac{1}{a^{m+1}} R_{1,m}^3(\alpha)$$

Proof. Since the regulator map is \mathbb{Z} -linear, it suffices to show that the above equality holds when α represents the class of an irreducible curve $C \subset B_3$. But in that case we have from (7.6)

$$R_{1,m}^3(a * [C]) = \sum_{i=1}^r R_{1,m}^3(a * C, Q_i) = \sum_{i=1}^r R_{1,m}^3(C_{a^{-1}}, Q_i)$$

Now for a given $i, 1 \leq i \leq r$, we have by lemma 7.4, $(C_{a^{-1}}, Q_i) \in \mathcal{M}^m(t_l) \Leftrightarrow (C, P_i) \in \mathcal{M}^m(t_l)$. Applying proposition 7.5 in (7.7), we get

$$\begin{aligned} R_{1,m}^3(C_{a^{-1}}, Q_i) &= (-1)^{l-1} \text{res}_{Q_i}(p_{a^{-1}}^*(\omega_l)) \\ &= (a^{-1})^{m+1} (-1)^{l-1} \text{res}_{P_i}(p^*(\omega_l)) \\ &= a^{-m-1} R_{1,m}^3(C, P_i) \end{aligned}$$

Summing over i gives the result. \square

The following is the case $n = 3$ of theorem 7.1.

Corollary 7.7. *If k is algebraically closed, then $R_{1,m}^3$ is surjective.*

Proof. By Park [28], there exists $\alpha \in \text{TCH}^2(k, 3; m)$ such that

$$R_{1,m}^3(\alpha) = \beta \neq 0$$

in k . Fix any $0 \neq y \in k$. $k = \bar{k}$ implies there are $z, \beta' \in k$ such that $z^{m+1} = y$ and $(\beta')^{m+1} = \beta^{-1}$. Put $a = z\beta'$. Then $a \in k^*$ and by corollary 7.6,

$$R_{1,m}^3(a^{-1} * \alpha) = a^{m+1} \beta = (z\beta')^{m+1} \beta = y\beta^{-1} \beta = y.$$

Since 0 is clearly in the image of $R_{1,m}^3$, we see that $R_{1,m}^3$ is surjective. \square

Let k be as before an arbitrary field of characteristic zero. Recall that by theorem 4.3 we have the product

$$\boxtimes : z^r(k, *) \otimes \text{Tz}^s(k, *, m) \rightarrow \text{Tz}^{r+s}(k, *, m)$$

induced by the external product. By theorem 4.10, the map \boxtimes induces the natural map

$$(7.9) \quad \mu : \text{CH}^n(k, n) \otimes \text{TCH}^2(k, n; m) \rightarrow \text{TCH}^{n+2}(k, n+3; m)$$

For any $n \geq 1$, a cycle in $z^n(k, n)$ is a \mathbb{Z} -linear combination of closed points $b = (b_1, \dots, b_n) \in \square^n$, with each b_i in $\square - \{0, \infty\}$. For a curve $C \in B_3$, we denote by C^b the curve in $B_{n+3} \cong B_3 \times \square^n$ given by the product $C \times b$. We thus have

$$(7.10) \quad \mu([C] \otimes [b]) = [C^b]$$

Lemma 7.8. *Let $[C] \in \text{Tz}^2(k, 3; m)$ be the cycle defined by an irreducible curve $C \subset B_3$ and let $b = (b_1, \dots, b_n) \in \square^n(k)$ be a k -point defining a cycle in $z^n(k, n)$. Then*

$$\begin{aligned} R_{1,m}^{n+3}([C^b]) = & \left(\sum_{(C,P) \in \mathcal{M}^m(t_1)} R_{1,m}^3(C, P) + (-1)^n \sum_{(C,P) \in \mathcal{M}^m(t_2)} R_{1,m}^3(C, P) \right) \cdot \prod_{i=1}^n \frac{db_i}{b_i}. \end{aligned}$$

Proof. It is immediate that $\widehat{C^b} = \widehat{C} \times b$ and $(\widehat{C^b})^N = \widehat{C}^N \times b$. Let

$$p : \widehat{C}^N \rightarrow \widehat{B}_3; p^b : (\widehat{C^b})^N \rightarrow \widehat{B}_{n+3}$$

denote the maps on the normalizations, and for $P \in p^{-1}(\{x = 0\})$, set $P^b := P \times b$. Then

- (i) $(p^b)^{-1}(\{x = 0\}) = p^{-1}(\{x = 0\}) \times b$.
- (ii) for $l = 1, 2, \forall P \in p^{-1}(\{x = 0\})$:

$$(C, P) \in \mathcal{M}^m(t_l) \Leftrightarrow (C^b, P^b) \in \mathcal{M}^m(t_l).$$

- (iii) for $l \geq 3, \forall P \in p^{-1}(\{x = 0\})$: $p(P^b) \notin F_{n+3,l}^1$.

Thus

$$\begin{aligned} R_{1,m}^{n+3}([C^b]) &= \sum_{Q \in (p^b)^{-1}(\{x=0\})} R_{1,m}^{n+3}(C^b, Q) \\ &= \sum_{(C,P) \in \mathcal{M}^m(t_1)} R_{1,m}^{n+3}(C^b, Q) + \sum_{(C^b,Q) \in \mathcal{M}^m(t_2)} R_{1,m}^{n+3}(C^b, Q) \\ &\quad + \sum_{l \geq 3} \sum_{(C^b,Q) \in \mathcal{M}^m(t_l)} R_{1,m}^{n+3}(C^b, Q) \end{aligned}$$

Note that the second equality holds because if

$$(C^b, Q) \in \mathcal{M}^m(t_l) \cap \mathcal{M}^m(t_{l'})$$

for some $Q \in (p^b)^{-1}(\{x = 0\})$, and for some $l \neq l'$, then $R_{1,m}^{n+3}(C^b, Q) = 0$ (see [27], lemma 3.2).

Using (i), (ii) and (iii) above, this gives

$$\begin{aligned}
R_{1,m}^{n+3}([C^b]) &= \sum_{(C^b, Q) \in \mathcal{M}^m(t_1)} R_{1,m}^{n+3}(C^b, Q) + \sum_{(C^b, Q) \in \mathcal{M}^m(t_2)} R_{1,m}^{n+3}(C^b, Q) \\
&= \sum_{(C, P) \in \mathcal{M}^m(t_1)} R_{1,m}^{n+3}(C^b, P^b) + \sum_{(C, P) \in \mathcal{M}^m(t_2)} R_{1,m}^{n+3}(C^b, P^b) \\
&= \sum_{(C, P) \in \mathcal{M}^m(t_1)} \text{res}_{P^b} \left(\frac{1-t_1}{x^{m+1}} \cdot \bigwedge_{j=2}^{n+2} \frac{dt_j}{t_j} \right) \\
&\quad - \sum_{(C, P) \in \mathcal{M}^m(t_2)} \text{res}_{P^b} \left(\frac{1-t_2}{x^{m+1}} \cdot \bigwedge_{j=3}^{n+2} \frac{dt_j}{t_j} \wedge \frac{dt_1}{t_1} \right) \\
&= \sum_{(C, P) \in \mathcal{M}^m(t_1)} \left(\text{res}_P \frac{1-t_1}{x^{m+1}} \cdot \frac{dt_2}{t_2} \right) \cdot \bigwedge_{j=1}^n \frac{db_j}{b_j} \\
&\quad + (-1)^{n+1} \sum_{(C, P) \in \mathcal{M}^m(t_2)} \left(\text{res}_P \frac{1-t_2}{x^{m+1}} \cdot \frac{dt_1}{t_1} \right) \cdot \bigwedge_{j=1}^n \frac{db_j}{b_j} \\
&= \sum_{(C, P) \in \mathcal{M}^m(t_1)} R_{1,m}^3(C, P) \cdot \bigwedge_{j=1}^n \frac{db_j}{b_j} \\
&\quad + (-1)^n \sum_{(C, P) \in \mathcal{M}^m(t_2)} R_{1,m}^3(C, P) \cdot \bigwedge_{j=1}^n \frac{db_j}{b_j}
\end{aligned}$$

This completes the proof of the lemma. \square

Corollary 7.9. *If n is even, then*

$$R_{1,m}^{n+3}(\mu(\alpha \otimes [b])) = R_{1,m}^3(\alpha) \cdot \bigwedge_{j=1}^n \frac{db_j}{b_j}.$$

Next we recall some explicit 1-cycles in $\text{Tz}^2(k, 3; m)$ constructed by Park [28]. Let $a, a_1, a_2, \in k, b, b_1, b_2 \in k^*$. Let $\Gamma_1, \Gamma_2, C_1^{(a_1, a_2), b}, C_2^{a, (b_1, b_2)}$

be parametrized 1-cycles in $\mathrm{Tz}^2(k, 3; m)$ defined as follows:

$$C_1^{(a_1, a_2), b} = \begin{cases} \left\{ \left(t, \frac{(1-a_1t)(1-a_2t)}{1-(a_1+a_2)t}, b \right) \mid t \in k \right\} & \text{if } a_1 a_2 (a_1 + a_2) \neq 0 \\ \left\{ (t, 1 - a^2 t^2, b) \mid t \in \mathbb{A}_k^1 \right\} & \text{if } a := a_1 = -a_2 \neq 0 \\ 0 & \text{if } a_1 a_2 = 0 \end{cases}$$

$$C_2^{a, (b_1, b_2)} = \begin{cases} \left\{ \left(\frac{1}{a}, t, \frac{b_1 t - b_1 b_2}{t - b_1 b_2} \right) \mid t \in \mathbb{A}_k^1 \right\} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

$$\Gamma_1 = \left\{ \left(t, t, \frac{(1 - (1/2)t)^2}{1 - t} \right) \mid t \in \mathbb{A}_k^1 \right\}$$

$$\Gamma_2 = \left\{ \left(t, 1 + \frac{t}{6}, 1 - \frac{t^2}{6} \right) \mid t \in \mathbb{A}_k^1 \right\}$$

Let $\bar{\Gamma}_1 = \Gamma_1 + C_1^{(1/2, 1/2), 2}$ and

$$\begin{aligned} \bar{\Gamma}_2 = & \Gamma_2 + C_1^{(-1/2, 1/2), 2/3} + 3C_1^{(-1/3, -1/3), -1} - C_1^{(-1/6, -1/6), 2} \\ & - C_1^{(-1/6, -1/3), 2} - C_1^{(-1/2, 1/2), 2} + C_1^{(1/2, 1/2), 2} \\ & + C_2^{-1/6, (4, -2)} - C_2^{-1/6, (-2, -2)} + C_2^{1/2, (2/3, 3/2)} \\ & - 3C_2^{-1/6, (2, -1)} - 3C_2^{-1/3, (-1, -1)} + C_2^{-1/2, (4/3, 3/2)} \end{aligned}$$

Define the cycle Γ by

$$(7.11) \quad \Gamma := \bar{\Gamma}_1 - \bar{\Gamma}_2$$

Lemma 7.10. *For each $b = (b_1, \dots, b_n) \in (\square \setminus \{0, \infty\})^n(k)$ and $a \in k^*$, we have*

$$R_{1,m}^{n+3}(\mu(a * \Gamma \otimes [b])) = (-1)^n R_{1,m}^3(a * \Gamma) \cdot \bigwedge_{j=1}^n \frac{db_j}{b_j} \in \Omega_k^n.$$

Proof. It is enough to show that the above formula holds for a cycle of the form $[C] \in \left\{ \Gamma_1, \Gamma_2, C_1^{(a_1, a_2), b}, C_2^{a, (b_1, b_2)} \right\}$. We do it separately for each cycle in this set.

(i) $[C] = \Gamma_1$: By Park [28], lemma 1.9, for each $p \in p^{-1}(\{x = 0\})$, (C, P) is in $\mathcal{M}^m(t_2)$ and hence by lemma 7.4 and lemma 7.8, we see immediately that the formula holds for Γ_1 .

(ii) $[C] = \Gamma_2$: In this case again by Park [28], lemma 1.10, for each $p \in p^{-1}(\{x = 0\})$, (C, P) is in $\mathcal{M}^m(t_2)$ and hence the formula follows.

(iii) $[C] = C_1^{(a_1, a_2), b}$: In this case, we apply lemma 1.7 of Park to see that for each $p \in p^{-1}(\{x = 0\})$, (C, P) is in $\mathcal{M}^m(t_1)$. Hence the second sum in lemma 7.8 is zero which in particular means that the first sum is same as $R_{1,m}^3(a * [C])$. Thus applying lemma 7.8 again, we get

$$R_{1,m}^{n+3}(\mu(a * [C] \otimes [b])) = R_{1,m}^3(a * [C]) \cdot \bigwedge_{j=1}^n \frac{db_j}{b_j}.$$

So enough to show that the right hand term is zero. However, in this case we have $R_{1,m}^3(a * [C]) = \frac{1}{a^{m+1}} R_{1,m}^3([C])$ by corollary 7.6. But it is easy to see that $R_{1,m}^3([C]) = 0$ since $p^*(dt_2) = db = 0$.

(iv) $[C] = C_2^{a, (b_1, b_2)}$: In this case again we have by lemma 1.7 of Park, for each $p \in p^{-1}(\{x = 0\})$, (C, P) is in $\mathcal{M}^m(t_2)$. We apply lemma 7.4 and lemma 7.8 again to conclude that the formula holds for in this case. \square

Lemma 7.11. *Let k be a field of characteristic different from 2. Then there are surjective k -linear maps*

$$k \otimes_{\mathbb{Z}} T(k^*) \rightarrow k \otimes_{\mathbb{Z}} \bigwedge(k^*) \rightarrow k \otimes_{\mathbb{Z}} K_*^M(k) \rightarrow \Omega_k^*$$

where the $T(k^*)$ and $\bigwedge(k^*)$ denote the tensor and exterior algebras of k^* and $K_*^M(k)$ is the Milnor k -theory ring of k . The composite map is given in degree n by

$$a \otimes (b_1 \otimes \cdots \otimes b_n) \mapsto a \bigwedge_{j=1}^n \frac{db_j}{b_j} \in \Omega_k^n.$$

Proof. The exterior algebra of k^* is the quotient of the tensor algebra by the homogeneous two-sided ideal I generated by degree two elements $\{b \otimes b | b \in k^*\}$. Let I' be the homogeneous two-sided ideal generated by degree two elements $\{b_1 \otimes b_2 + b_2 \otimes b_1 | b_1, b_2 \in k^*\}$. It is clear that $I' \subset I$. However, since the characteristic of k is different from 2, it is easy to check that $k \otimes_{\mathbb{Z}} (I/I') = 0$. This shows that $k \otimes_{\mathbb{Z}} \bigwedge(k^*)$ is the quotient of the k -algebra $k \otimes_{\mathbb{Z}} T(k^*)$ by the homogeneous ideal I .

It is known [26] that the Milnor K -theory ring of k is the quotient of the tensor algebra of k^* by the homogeneous two-sided ideal J generated by degree two elements $\{b \otimes (1 - b) | b, 1 - b \in k^*\}$. Thus to get the middle map of the lemma, it suffices to show that $I \subset J$. For this,

it is enough to show that the images of the degree two generators of I under the quotient map $\tau : T(k^*) \rightarrow K_*^M(k)$ are all zero. But we have

$$\tau(b_1 \otimes b_2 + b_2 \otimes b_1) = \{b_1, b_2\} + \{b_2, b_1\} = 0$$

in $K_*^M(k)$ by [26], lemma 3.2(b).

By [7], lemma 4.1, there is a surjective map of k -vector spaces

$$\eta : k \otimes_{\mathbb{Z}} T(k^*) \rightarrow \Omega_k^*$$

which is given in degree n by

$$\eta(a \otimes (b_1 \otimes \cdots \otimes b_n)) = a \bigwedge_{j=1}^n \frac{db_j}{b_j}.$$

To show that this factors through $k \otimes K_*^M(k)$, it suffices to show that $\eta(k \otimes J) = 0$. A generator of $k \otimes J$ is of the form $a \otimes (b \otimes (1 - b))$ and one has

$$\begin{aligned} \eta(a \otimes (b \otimes (1 - b))) &= \frac{a}{b(1-b)} db \wedge d(1-b) \\ &= -\frac{a}{b(1-b)} (db \wedge db) = 0. \end{aligned}$$

□

We shall also denote the induced map $k \otimes K_*^M(k) \rightarrow \Omega_k^*$ by η . Now we go back to the situation of k being of characteristic zero. Let $\Gamma \in \text{TCH}^2(k, 3; m)$ be the class of the cycle defined in (7.11).

Proposition 7.12. *For $n \geq 0$, there are maps*

$$\begin{array}{ccc} CH^n(k, n) \otimes_{\mathbb{Z}} \text{TCH}^2(k, 3; m) & \xrightarrow{\mu} & \text{TCH}^{n+2}(k, n+3; m) \\ \downarrow S_{1,m}^n & & \downarrow R_{1,m}^{n+3} \\ K_n^M(k) \otimes_{\mathbb{Z}} k & \xrightarrow{\eta} & \Omega_k^n \end{array}$$

such that for $a \in k^*$ and $\delta \in CH^n(k, n)$, one has

$$R_{1,m}^{n+3} \circ \mu(a * \Gamma \otimes \delta) = (-1)^n \eta \circ S_{1,m}^n(a * \Gamma \otimes \delta).$$

Proof. The map μ comes from theorem 4.10 (part 1). $S_{1,m}^n$ is defined by $S_{1,m}^n = \phi^n \otimes R_{1,m}^3$ where $\phi^n : CH^n(k, n) \rightarrow K_n^M(k)$ is the Nesterenko-Suslin-Totaro isomorphism [26]. We can write $\delta = \sum_{i=1}^r b^i - \sum_{j=1}^s c^j$

where $b^i, c^j \in (\square \setminus \{0, \infty\})^n$. Then

$$\begin{aligned}
R_{1,m}^{n+3} \circ \mu(a * \Gamma \otimes \delta) &= \sum R_{1,m}^{n+3} \circ \mu(a * \Gamma \otimes b^i) \\
&\quad - \sum R_{1,m}^{n+3} \circ \mu(a * \Gamma \otimes c^j) \\
&= (-1)^n \sum R_{1,m}^3(a * \Gamma) \bigwedge_{l=1}^n \frac{db_l^i}{b_l^i} \\
&\quad - (-1)^n \sum R_{1,m}^3(a * \Gamma) \bigwedge_{l=1}^n \frac{dc_l^j}{c_l^j} \\
&\quad \text{(by lemma 7.10)} \\
&= (-1)^n \sum R_{1,m}^3(a * \Gamma) \phi^n([b^i]) \\
&\quad - (-1)^n \sum R_{1,m}^3(a * \Gamma) \phi^n([c^j]) \\
&= (-1)^n R_{1,m}^3(a * \Gamma) \left\{ \sum \phi^n([b^i]) - \sum \phi^n([c^j]) \right\} \\
&= (-1)^n R_{1,m}^3(a * \Gamma) (\eta(1 \otimes \phi^n(\delta))) \\
&= (-1)^n \eta \{ R_{1,m}^3(a * \Gamma) \otimes \phi^n(\delta) \} \\
&= (-1)^n \eta \circ S_{1,m}^n(a * \Gamma \otimes \delta)
\end{aligned}$$

□

Corollary 7.13. *Let $\widetilde{\text{TCH}}^2(k, 3; m) \hookrightarrow \text{TCH}^2(k, 3, m)$ be the $\mathbb{Z}[k^*]$ -submodule generated by Γ . Then for each element α in $CH^n(k, n) \otimes \widetilde{\text{TCH}}^2(k, 3; m)$, one has*

$$R_{1,m}^{n+3} \circ \mu(\alpha) = (-1)^n \eta \circ S_{1,m}^n(\alpha).$$

Proof. This follows from proposition 7.12 since elements of $CH^n(k, n) \otimes \widetilde{\text{TCH}}^2(k, 3; m)$ are \mathbb{Z} -linear combinations of elements of the type $a * \Gamma \otimes [b]$ with $a \in k^*$. □

Proof of theorem 7.1. For $n \leq 2$, the theorem is obvious as the term on the right is zero. Thus it is enough to show that $R_{1,m}^{n+3}$ is surjective for $n \geq 0$.

Let

$$\iota : CH^n(k, n) \otimes \widetilde{\text{TCH}}^2(k, 3; m) \rightarrow CH^n(k, n) \otimes \text{TCH}^2(k, 3; m)$$

be the natural map. It is enough to show that $R_{1,m}^{n+3} \circ \mu \circ \iota$ is surjective. η is surjective by lemma 7.11. ϕ^n is isomorphism for all $n \geq 0$ by [26]. By [28], proposition 1.11, $R_{1,m}^3(\Gamma) \neq 0$. Now since $k = \bar{k}$, we repeat

the argument of the proof of corollary 7.7 to see that the map

$$R_{1,m}^3 : \widetilde{\mathrm{TCH}}^2(k, 3; m) \rightarrow k$$

is surjective. Combining these implies that $\eta \circ S_{1,m}^n \circ \iota$ is surjective. We now apply corollary 7.13 to show $(-1)^n R_{1,m}^{n+3} \circ \mu \circ \iota$ is surjective. In particular,

$$\Omega_k^n \subset \mathrm{Image}(R_{1,m}^{n+3} \circ \mu \circ \iota).$$

□

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