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# Algebraic Cobordism

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**Abstract**

Together with F. Morel, we have constructed in [6, 7, 8] a theory of *algebraic cobordism*, an algebro-geometric version of the topological theory of complex cobordism. In this paper, we give a survey of the construction and main results of this theory; in the final section, we propose a candidate for a theory of higher algebraic cobordism, which hopefully agrees with the cohomology theory represented by the $\mathbb{P}^1$-spectrum $MGL$ in the Morel-Voevodsky stable homotopy category.

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## 1. Oriented cohomology theories

Fix a field $k$ and let $\mathbf{Sch}_k$ denote the category of separated finite-type $k$-schemes. We let $\mathbf{Sm}_k$ be the full subcategory of smooth quasi-projective $k$-schemes.

We have described in [7] the notion of an *oriented cohomology theory* on $\mathbf{Sm}_k$. Roughly speaking, such a theory $A^*$ consists of a contravariant functor from $\mathbf{Sm}_k$ to graded rings (commutative), which is also covariantly functorial for projective equi-dimensional morphisms $f : Y \to X$ (with a shift in the grading):

$$f_* : \mathbb{A}^*(Y) \to A^{*-\dim_X Y}(X).$$

The pull-back $g^*$ and push-forward $f_*$ satisfy a projection formula and commute in transverse cartesian squares. If $L \to X$ is a line bundle with zero-section $s : X \to L$, we have the *first Chern class* of $L$, defined by

$$c_1(L) := s^*(s_*(1_X)) \in A^1(X),$$

where $1_X \in A^0(X)$ is the unit. $A^*$ satisfies the *projective bundle formula*:

(PB) Let $\mathcal{E}$ be a rank $r + 1$ locally free coherent sheaf on $X$, with projective bundle $q : \mathbb{P}(\mathcal{E}) \to X$ and tautological quotient invertible sheaf $q^*\mathcal{E} \to \mathcal{O}(1)$. Let $\xi = c_1(\mathcal{O}(1))$. Then $A^*(\mathbb{P}(\mathcal{E}))$ is a free $A^*(X)$-module with basis $1, \xi, \ldots, \xi^r$.

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Finally, $A^*$ satisfies a homotopy property: if $p : V \to X$ is an affine-space bundle (i.e., a torsor for a vector bundle over $X$), then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

**Examples 1.1.** (1) The theories $\mathrm{CH}^*$ and $H^{2*}_{\text{ét}}(-, \mathbb{Z}/n(*))$ on $\mathbf{Sm}_k$ (also with $\mathbb{Z}_l(*)$ or $\mathbb{Q}_l(*)$ coefficients).
(2) The theory $K_0[\beta, \beta^{-1}]$ on $\mathbf{Sm}_k$. Here $\beta$ is an indeterminant of degree $-1$, used to keep track of the relative dimension when taking projective push-forward.

**Remarks 1.2.** (1) In [8], we consider a more general (dual) notion, that of an *oriented Borel-Moore homology theory* $A_*$. Roughly, this is a functor from a full subcategory of $\mathbf{Sch}_k$ to graded abelian groups, covariant for projective maps, and contravariant (with a shift in the grading) for local complete intersection morphisms. In addition, one has external products, and a degree -1 Chern class endomorphism $\tilde{c}_1(L) : A_*(X) \to A_{*-1}(X)$ for each line bundle $L$ on $X$, defined by $\tilde{c}_1(L)(\eta) = s^*(s_*(\eta))$, $s : X \to L$ the zero-section. As for an oriented cohomology theory, there are various compatibilities of push-forward and pull-back, and $A_*$ satisfies a projective bundle formula and a homotopy property.

This allows for a more general category of definition for $A_*$, e.g., the category $\mathbf{Sch}_k$. As we shall see, the setting of Borel-Moore homology is often more natural than cohomology. On $\mathbf{Sm}_k$, the two notions are equivalent: to pass from Borel-Moore homology to cohomology, one re-grades by setting $A^n(X) := A_{n-\dim_k X}(X)$ and uses the l.c.i. pull-back for $A_*$ to give the contravariant functoriality of $A^*$, noting that every morphism of smooth $k$-schemes is an l.c.i. morphism. We will state most of our results for cohomology theories on $\mathbf{Sm}_k$, but they extend to the setting of Borel-Moore homology on $\mathbf{Sch}_k$ (see [8] for details).
(2) Our notion of oriented cohomology is related to that of Panin [10], but is not the same.

## 2. The formal group law

Let $A_*$ be an oriented cohomology theory on $\mathbf{Sm}_k$. As noticed by Quillen [11], a double application of the projective bundle formula (PB) yields the isomorphism of rings

$$A^*(k)[[u,v]] \cong \varprojlim_{n,m} A^*(\mathbb{P}^n \times \mathbb{P}^m),$$

the isomorphism sending $u$ to $c_1(p_1^*\mathcal{O}(1))$ and $v$ to $c_1(p_2^*\mathcal{O}(1))$. The class of $c_1(p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1))$ thus gives a power series $F_A(u,v) \in A^*(k)[[u,v]]$ with

$$c_1(p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)) = F_A(c_1(p_1^*\mathcal{O}(1)), c_1(p_2^*\mathcal{O}(1))).$$

By the naturality of $c_1$, we have the identity for $X \in \mathbf{Sm}_k$ with line bundles $L$, $M$,

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M)).$$

In addition, $F_A(u,v) = u + v \mod uv$, $F_A(u,v) = F_A(v,u)$, and $F_A(F_A(u,v), w) = F_A(u, F_A(v,w))$. Thus, $F_A$ gives a formal group law with coefficients in $A^*(k)$.

**Remark 2.3.** Note that $c_1 : \mathrm{Pic}(X) \to A^1(X)$ is a group homomorphism if and only if $F_A(u,v) = u + v$. If this is the case, we call $A^*$ *ordinary*, if not, $A^*$ is *extraordinary*. If $F_A(u,v) = u + v - \alpha uv$ with $\alpha$ a unit in $A^*(k)$, we call $A^*$ *multiplicative and periodic*.

**Examples 2.4.** For $A^* = \mathrm{CH}^*$ or $H^{2*}$, $F_A = u + v$, giving examples of ordinary theories. For the theory $A = K_0[\beta, \beta^{-1}]$, $c_1(L) = (1 - L^\vee)\beta^{-1}$, and $F_A(u,v) = u + v - \beta uv$, giving an example of a multiplicative and periodic theory.

**Remark 2.5.** Let $\tilde{\mathbb{L}}^* = \mathbb{Z}[a_{ij} \mid i,j \geq 1]$, where we give $a_{ij}$ degree $-i-j+1$, and let $F \in \tilde{\mathbb{L}}^*[[u,v]]$ be the power series $F = u + v + \sum_{ij} a_{ij} u^i v^j$. Let

$$\mathbb{L}^* = \tilde{\mathbb{L}}^*/F(u,v) = F(v,u), F(F(u,v),w) = F(u, F(v,w)),$$

and let $F_{\mathbb{L}} \in \mathbb{L}^*[[u,v]]$ be the image of $F$. Then $(F_{\mathbb{L}}, \mathbb{L}^*)$ is the universal commutative dimension 1 formal group; $\mathbb{L}^*$ is called the *Lazard ring* (*cf.* [5]).

Thus, if $A^*$ is an oriented cohomology theory on $\mathbf{Sm}_k$, there is a canonical graded ring homomorphism $\phi_A : \mathbb{L}^* \to A^*(k)$ with $\phi_A(F_{\mathbb{L}}) = F_A$.

## 3. Algebraic cobordism

The main result of [7, 8] is

**Theorem 3.6.** *Let $k$ be a field of characteristic zero.*

1. *There is a universal oriented Borel-Moore homology theory $\Omega_*$ on $\mathbf{Sch}_k$. The restriction of $\Omega_*$ to $\mathbf{Sm}_k$ yields the universal oriented cohomology theory $\Omega^*$ on $\mathbf{Sm}_k$.*
2. *The homomorphism $\phi_\Omega : \mathbb{L}^* \to \Omega^*(k)$ is an isomorphism.*
3. *Let $i : Z \to X$ be a closed imbedding with open complement $j : U \to X$. Then the sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0.$$

   *is exact.*

*Idea of construction:* We construct $\Omega_*(X)$ in steps; the construction is inspired by Quillen's approach to complex cobordism [11].

1. Start with *cobordism cycles* $(f : Y \to X, L_1, \ldots, L_r)$, with $Y \in \mathbf{Sm}_k$ irreducible, $f : Y \to X$ projective and $L_1, \ldots, L_r$ line bundles on $Y$ (we allow $r = 0$). We identify two cobordism cycles if there is an isomorphism $\phi : Y \to Y'$, a permutation $\sigma$ and isomorphisms $L_j \cong \phi^* L'_{\sigma(j)}$. Let $\mathcal{Z}_*(X)$ be the free abelian group on the cobordism cycles, graded by giving $(f : Y \to X, L_1, \ldots, L_r)$ degree $\dim_k Y - r$.
2. Let $\mathcal{R}^{dim}(X)$ be the subgroup of $\mathcal{Z}_*(X)$ generated by cobordism cycles of the form $(f : Y \to X, \pi^* L_1, \ldots, \pi^* L_r, M_1, \ldots, M_s)$, where $\pi : Y \to Z$ is a smooth morphism in $\mathbf{Sm}_k$, the $L_i$ are line bundles on $Z$, and $r > \dim_k Z$. Let $\underline{\mathcal{Z}}_*(X) = \mathcal{Z}_*(X)/\mathcal{R}^{dim}(X)$.

3. Add the Gysin isomorphism: If $L \to Y$ is a line bundle and $s : Y \to L$ is a section transverse to the zero-section with divisor $i : D \to Y$, identify $(f : Y \to X, L_1, \ldots, L_r, L)$ with $(f \circ i : D \to X, i^*L_1, \ldots, i^*L_r)$. We let $\underline{\Omega}_*(X)$ denote the resulting quotient of $\underline{\mathcal{Z}}_*(X)$. Note that on $\underline{\Omega}_*(X)$ we have, for each line bundle $L \to X$, the *Chern class operator*

$$\begin{aligned}&\tilde{c}_1(L) : \underline{\Omega}_*(X) \to \underline{\Omega}_{*-1}(X)\\&(f : Y \to X, L_1, \ldots, L_r) \mapsto (f : Y \to X, L_1, \ldots, L_r, f^*L)\end{aligned}$$

as well as push-forward maps $f_* : \underline{\Omega}_*(X) \to \underline{\Omega}_*(X')$ for $f : X \to X'$ projective.
4. Impose the formal group law: Regrade $\mathbb{L}$ by setting $\mathbb{L}_n := \mathbb{L}^{-n}$. Let $\Omega_*(X)$ be the quotient of $\mathbb{L}_* \otimes \underline{\Omega}_*(X)$ by the imposing the identity of maps $\mathbb{L}_* \otimes \underline{\Omega}_*(Y) \to \mathbb{L}_* \otimes \underline{\Omega}_*(X)$

$$(\mathrm{id} \otimes f_*) \circ F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M)) = \mathrm{id} \otimes (f_* \circ \tilde{c}_1(L \otimes M))$$

for $f : Y \to X$ projective, and $L, M$ line bundles on $Y$. Note that, having imposed the relations in $\mathcal{R}^{dim}$, the operators $\tilde{c}_1(L)$, $\tilde{c}_1(M)$ are locally nilpotent, so the infinite series $F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))$ makes sense.

As the notation suggests, the most natural construction of $\Omega$ is as an oriented Borel-Moore homology theory rather than an oriented cohomology theory; the tranlation to an oriented cohomology theory on $\mathbf{Sm}_k$ is given as in remark 1.2(1). The proof of theorem 3.6 uses resolution of singularities [4] and the weak factorization theorem [1] in an essential way.

**Remark 3.7.** In addition to the properties of $\Omega_*$ listed in theorem 3.6, $\Omega_*(X)$ is generated by the classes of "elementary" cobordism cycles $(f : Y \to X)$.

# 4. Degree formulas

In the paper [12], Rost made a number of conjectures based on the theory of algebraic cobordism in the Morel-Voevodsky stable homotopy category. Many of Rost's conjectures have been proved by homotopy-theoretic means (see [3]); our construction of algebraic cobordism gives an alternate proof of these results, and settles many of the remaining open questions as well. We give a sampling of some of these results.

## 4.1. The generalized degree formula

All the degree formulas follow from the "generalized degree formula". We first define the degree map $\Omega^*(X) \to \Omega^*(k)$.

**Definition 4.8.** Let $k$ be a field of characteristic zero and let $X$ be an irreducible finite type $k$-scheme with generic point $i : x \to X$. For an element $\eta$ of $\Omega^*(X)$, define $\deg \eta \in \Omega^*(k)$ to be the element mapping to $i^*\eta$ in $\Omega^*(k(x))$ under the isomorphisms $\Omega^*(k) \cong \mathbb{L}^* \cong \Omega^*(k(x))$ given by theorem 3.6(2).

**Theorem 4.9(generalized degree formula).** *Let $k$ be a field of characteristic zero. Let $X$ be an irreducible finite type $k$-scheme, and let $\eta$ be in $\Omega_*(X)$. Let $f_0 : B_0 \to X$ be a resolution of singularities of $X$, with $B_0$ quasi-projective over $k$. Then there are $a_i \in \Omega_*(k)$, and projective morphisms $f_i : B_i \to X$ such that*

1. *Each $B_i$ is in $\mathbf{Sm}_k$, $f_i : B_i \to f(B_i)$ is birational and $f(B_i)$ is a proper closed subset of $X$ (for $i > 0$).*
2. *$\eta - (\deg \eta)[f_0 : B_0 \to X] = \sum_{i=1}^r a_i[f_i : B_i \to X]$ in $\Omega_*(X)$.*

*Proof.* It follows from the definitions of $\Omega^*$ that we have

$$\Omega^*(k(x)) = \lim_{\substack{\to \\ U}} \Omega^*(U),$$

where the limit is over smooth dense open subschemes $U$ of $X$, and $\Omega^*(k(x))$ is the value at $\operatorname{Spec} k(x)$ of the functor $\Omega^*$ on finite type $k(x)$-schemes. Thus, there is a smooth open subscheme $j : U \to X$ of $X$ such that $j^*\eta = (\deg \eta)[\mathrm{id}_U]$ in $\Omega^*(U)$. Since $U \times_X B_0 \cong U$, it follows that $j^*(\eta - (\deg \eta)[f_0]) = 0$ in $\Omega^*(U)$.

Let $W = X \setminus U$. From the localization sequence

$$\Omega_*(W) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0$$

we find an element $\eta_1 \in \Omega_*(W)$ with $i_*(\eta_1) = \eta - (\deg \eta)[f_0]$, and noetherian induction completes the proof. □

**Remark 4.10.** Applying theorem 4.9 to the class of a projective morphism $f : Y \to X$, with $X, Y \in \mathbf{Sm}_k$, we have the formula

$$[f : Y \to X] - (\deg f)[\mathrm{id}_X] = \sum_{i=1}^r a_i[f_i : B_i \to X]$$

in $\Omega^*(X)$. Also, if $\dim_k X = \dim_k Y$, $\deg f$ is the usual degree, i.e., the field extension degree $[k(Y) : k(X)]$ if $f$ is dominant, or zero if $f$ is not.

## 4.2. Complex cobordism

For a differentiable manifold $M$, one has the complex cobordism ring $MU^*(M)$. Given an embedding $\sigma : k \to \mathbb{C}$ and an $X \in \mathbf{Sm}_k$, we let $X^\sigma(\mathbb{C})$ denote the complex manifold associated to the smooth $\mathbb{C}$-scheme $X \times_k \mathbb{C}$. Sending $X$ to $MU^{2*}(X^\sigma(\mathbb{C}))$ defines an oriented cohomology theory on $\mathbf{Sm}_k$; by the universality of $\Omega^*$, we have a natural homomorphism

$$\Re_\sigma : \Omega^*(X) \to MU^{2*}(X^\sigma(\mathbb{C})).$$

Now, if $P = P(c_1, \dots, c_d)$ is a degree $d$ (weighted) homogeneous polynomial, it is known that the operation of sending a smooth compact $d$-dimensional complex manifold $M$ to the Chern number $\deg(P(c_1, \dots, c_d)(\Theta_M))$ (where $\Theta_M$ is the complex tangent bundle) descends to a homomorphism $MU^{-2d} \to \mathbb{Z}$. Composing with

$\Re_\sigma$, we have the homomorphism $P : \Omega^{-d}(k) \to \mathbb{Z}$. If $X$ is smooth and projective of dimension $d$ over $k$, we have $P([X]) = \deg(P(c_1, \ldots, c_d)(\Theta_{X^\sigma(\mathbb{C})}))$; $P([X])$ is in fact independent of the choice of embedding $\sigma$.

Let $s_d(c_1, \ldots, c_d)$ be the polynomial which corresponds to $\sum_i \xi_i^d$, where $\xi_1, \ldots$ are the Chern roots. The following divisibility is known (see [2]): if $d = p^n - 1$ for some prime $p$, and $\dim X = d$, then $s_d(X)$ is divisible by $p$.

In addition, for integers $d = p^n - 1$ and $r \geq 1$, there are mod $p$ characteristic classes $t_{d,r}$, with $t_{d,1} = s_d/p \mod p$. The $s_d$ and the $t_{d,r}$ have the following properties:

(4.1)

1. $s_d(X) \in p\mathbb{Z}$ is defined for $X$ smooth and projective of dimension $d = p^n - 1$. $t_{d,r}(X) \in \mathbb{Z}/p$ is defined for $X$ smooth and projective of dimension $rd = r(p^n - 1)$.
2. $s_d$ and $t_{d,r}$ extend to homomorphisms $s_d : \Omega^{-d}(k) \to p\mathbb{Z}$, $t_{d,r} : \Omega^{-rd}(k) \to \mathbb{Z}/p$.
3. If $X$ and $Y$ are smooth projective varieties with $\dim X, \dim Y > 0$, $\dim X + \dim Y = d$, then $s_d(X \times Y) = 0$.
4. If $X_1, \ldots, X_s$ are smooth projective varieties with $\sum_i \dim X_i = rd$, then $t_{d,r}(\prod_i X_i) = 0$ unless $d | \dim X_i$ for each $i$.

We can now state Rost's degree formula and the higher degree formula:

**Theorem 4.11(Rost's degree formula).** *Let $f : Y \to X$ be a morphism of smooth projective $k$-schemes of dimension $d$, $d = p^n - 1$ for some prime $p$. Then there is a zero-cycle $\eta$ on $X$ such that*

$$s_d(Y) - (\deg f)s_d(X) = p \cdot \deg(\eta).$$

**Theorem 4.12(Rost's higher degree formula).** *Let $f : Y \to X$ be a morphism of smooth projective $k$-schemes of dimension $rd$, $d = p^n - 1$ for some prime $p$. Suppose that $X$ admits a sequence of surjective morphisms*

$$X = X_0 \to X_1 \to \ldots \to X_{r-1} \to X_r = \operatorname{Spec} k$$

*such that:*

1. $\dim X_i = d(r - i)$.
2. *Let $\eta$ be a zero-cycle on $X_i \times_{X_{i+1}} \operatorname{Spec} k(X_{i+1})$. Then $p | \deg(\eta)$.*

*Then*

$$t_{d,r}(Y) = \deg(f)t_{d,r}(X).$$

*Proof.* These two theorems follow easily from the generalized degree formula. Indeed, for theorem 4.11, take the identity of remark 4.10 and push forward to $\Omega^*(k)$. Using remark 3.7, this gives the identity

$$[Y] - (\deg f)[X] = \sum_{i=1}^{r} m_i[A_i \times B_i].$$

in $\Omega^*(X)$, for smooth, projective $k$-schemes $A_j$, $B_j$, and integers $m_j$, where each $B_i$ admits a projective morphism $f_i : B_i \to X$ which is birational to its image and not dominant. Since $s_d$ vanishes on non-trivial products, the only relevant part of the sum involves those $B_j$ of dimension zero; such a $B_j$ is identified with the closed point $b_j := f_j(B_j)$ of $X$. Applying $s_d$, we have

$$s_d(Y) - \deg(f)s_d(X) = \sum_j m_j s_d(A_j)\deg_k(b_j).$$

Since $s_d(A_j) = pn_j$ for suitable integers $n_j$, we have

$$s_d(Y) - \deg(f)s_d(X) = p\deg(\sum_j m_j n_j b_j).$$

Taking $\eta = \sum_j m_j n_j b_j$ proves theorem 4.11.

The proof of theorem 4.12 is similar: Start with the decomposition of $[f : Y \to X] - (\deg f)[\mathrm{id}_X]$ given by remark 4.10. One then decomposes the maps $B_i \to X = X_0$ further by pushing forward to $X_1$ and using theorem 4.9. Iterating down the tower gives the identity in $\Omega_*(k)$

$$[Y] - (\deg f)[X] = \sum_i m_i[B_0^i \times \ldots \times B_r^i];$$

the condition (2) implies that, if $d|\dim_k B_j^i$ for all $j = 0, \ldots, r$, then $p|m_j$. Applying $t_{d,r}$ and using the property (4.1)(4) yields the formula. □

## 5. Comparison results

Suppose we have a formal group $(f, R)$, giving the canonical homomorphism $\phi_f : \mathbb{L}^* \to R$. Let $\Omega^*_{(f,R)}$ be the functor

$$\Omega^*_{(f,R)}(X) = \Omega^*(X) \otimes_{\mathbb{L}^*} R,$$

where $\Omega^*(X)$ is an $\mathbb{L}^*$-algebra via the homomorphism $\phi_\Omega : \mathbb{L}^* \to \Omega^*(k)$. The universal property of $\Omega^*$ gives the analogous universal property for $\Omega^*_{(f,R)}$.

In particular, let $\Omega^*_+$ be the theory with $(f(u,v), R) = (u+v, \mathbb{Z})$, and let $\Omega^*_\times$ be the theory with $(f(u,v), R) = (u+v-\beta uv, \mathbb{Z}[\beta, \beta^{-1}])$. We thus have the canonical natural transformations of oriented theories on $\mathbf{Sm}_k$

$$\Omega^*_+ \to \mathrm{CH}^*; \quad \Omega^*_\times \to K_0[\beta, \beta^{-1}]. \tag{5.2}$$

**Theorem 5.13.** *Let $k$ be a field of characteristic zero. The natural transformations* (5.2) *are isomorphisms, i.e.,* $\mathrm{CH}^*$ *is the universal ordinary oriented cohomology theory and* $K_0[\beta, \beta^{-1}]$ *is the universal multiplicative and periodic theory.*

*Proof.* For $\mathrm{CH}^*$, this uses localization, theorem 4.9 and resolution of singularities. For $K_0$, one writes down an integral Chern character, which gives the inverse isomorphism by the Grothendieck-Riemann-Roch theorem. □

# 6. Higher algebraic cobordism

The cohomology theory represented by the $\mathbb{P}^1$-spectrum $MGL$ in the Morel-Voevodsky $\mathbb{A}^1$-stable homotopy category [9, 13] gives perhaps the most natural algebraic analogue of complex cobordism. By universality, $\Omega^n(X)$ maps to $MGL^{2n,n}(X)$; to show that this map is an isomorphism, one would like to give a map in the other direction. For this, the most direct method would be to extend $\Omega^*$ to a theory of higher algebraic cobordism; we give one possible approach to this construction here.

The idea is to repeat the construction of $\Omega_*$, replacing abelian groups with symmetric monoidal categories throughout. Comparing with the $Q$-construction, one sees that the cobordism cycles in $\mathcal{R}^{dim}(X)$ should be homotopic to zero, but not canonically so. Thus, we cannot impose this relation directly, forcing us to modify the group law by taking a limit.

Start with the category $\widetilde{\mathcal{Z}}(X)_0$, with objects $(f : Y \to X, L_1, \ldots, L_r)$, where $Y$ is irreducible in $\mathbf{Sm}_k$, $f$ is projective, and the $L_i$ are line bundles on $Y$. A morphism $(f : Y \to X, L_1, \ldots, L_r) \to (f' : Y' \to X, L'_1, \ldots, L'_r)$ in $\widetilde{\mathcal{Z}}(X)_0$ consist of a tuple $(\phi, \psi_1, \ldots, \psi_r, \sigma)$, with $\phi : Y \to Y'$ an isomorphism over $X$, $\sigma$ a permutation, and $\psi_j : L_j \to \phi^* L'_{\sigma(j)}$ an isomorphism of line bundles on $Y$. Form the category $\widetilde{\mathcal{Z}}(X)$ as the symmetric monoidal category freely generated by $\widetilde{\mathcal{Z}}(X)_0$; grade $\widetilde{\mathcal{Z}}(X)$ by letting $\widetilde{\mathcal{Z}}_n(X)$ be the full symmetric monoidal subcategory generated by the $(f : Y \to X, L_1, \ldots, L_r)$ with $n = \dim_k Y - r$.

Next, form $\underline{\widetilde{\Omega}}(X)$ by adjoining (as a symmetric monoidal category) an isomorphism $\gamma_{L,s} : (f \circ i : D \to X, i^* L_1, \ldots, i^* L_r) \to (f : Y \to X, L_1, \ldots, L_r, L)$ for each section $s : Y \to L$ transverse to the zero-section with divisor $i : D \to X$. Given a morphism $\tilde{\phi} := (\phi, \ldots) : (f : Y \to X, L_1, \ldots, L_r, L) \to (f' : Y' \to X, L'_1, \ldots, L'_r, L')$ (with $L \cong \phi^* L'$ via $\tilde{\phi}$), let $i' : D' \to Y'$ be the map induced by $\phi$, $s' : Y' \to L'$ the section induced by $s$, and

$$\psi^D : (f \circ i : D \to X, i^* L_1, \ldots, i^* L_r) \to (f' \circ i' : D' \to X, i'^* L'_1, \ldots, i'^* L'_r)$$

the morphism induced by $\psi$. We impose the relation $\psi \circ \gamma_{L,s} = \gamma_{L',s'} \circ \psi^D$. Finally, for line bundles $L, M$ with smooth transverse divisors $i_D : D \to Y$, $i_E : E \to Y$ defined by sections $s : Y \to L$, $t : Y \to M$, respectively, we impose the relation $\gamma_{L,s} \circ \gamma_{i_D^* M, i_D^* t} = \gamma_{M,t} \circ \gamma_{i_E^* L, i_E^* s}$. The grading on $\widetilde{\mathcal{Z}}(X)$ extends to one on $\underline{\widetilde{\Omega}}(X)$.

Given $g : X \to X'$ projective, we have the functor $g_* : \underline{\widetilde{\Omega}}(X) \to \underline{\widetilde{\Omega}}(X')$, similarly, given a smooth morphism $h : X \to X'$, we have the functor $h^* : \underline{\widetilde{\Omega}}(X') \to \underline{\widetilde{\Omega}}(X)$. Given a line bundle $L$ on $X$, we have the natural transformation $\tilde{c}_1(L)$ sending $(f : Y \to X, L_1, \ldots, L_r)$ to $(f : Y \to X, L_1, \ldots, L_r, f^* L)$.

Now let $\mathcal{C}$ be a symmetric monoidal category such that all morphisms are isomorphisms, and let $R$ be a ring, free as a $\mathbb{Z}$-module. One can define a symmetric monoidal category $R \otimes_{\mathbb{N}} \mathcal{C}$ with a symmetric monoical functor $\mathcal{C} \to R \otimes_{\mathbb{N}} \mathcal{C}$ which is universal for symmetric monoidal functors $\mathcal{C} \to \mathcal{C}'$ such that $\mathcal{C}'$ admits an action of $R$ via natural transformations. In case $R = \mathbb{Z}$, $\mathbb{Z} \otimes_{\mathbb{N}} \mathcal{C}$ is the standard group completion $\mathcal{C}^{-1}\mathcal{C}$. In general, if $\{e_\alpha \mid \alpha \in A\}$ is a $\mathbb{Z}$-basis for $R$, then

$$R \otimes_{\mathbb{N}} \mathcal{C} = \coprod_{\alpha} \mathcal{C}^{-1}\mathcal{C},$$

with the $R$-action given by expressing $\times x : R \to R$ in terms of the basis $\{e_\alpha\}$.

For each integer $n \geq 0$, let $\mathbb{L}_*^{(n)}$ be the quotient of $\mathbb{L}_*$ by the ideal of elements of degree $> n$. We thus have the formal group $(F_{\mathbb{L}^{(n)}}, \mathbb{L}_*^{(n)})$.

We form the category $\mathbb{L}^{(n)} \otimes_{\mathbb{N}} \underline{\widetilde{\Omega}}(X)$, which we grade by total degree. For each $f : Y \to X$ projective, with $Y \in \mathbf{Sm}_k$, and line bundles $L$, $M$, $L_1, \ldots, L_r$ on $Y$, we adjoin an isomophism $\rho_{L,M}$

$$f_*(F_{\mathbb{L}^{(n)}}(\tilde{c}_1(L), \tilde{c}_1(M))(\mathrm{id}_Y, L_1, \ldots, L_r)) \xrightarrow{\sim} f_*(\mathrm{id} \otimes \tilde{c}_1(L \otimes M)(\mathrm{id}_Y, L_1, \ldots, L_r)).$$

We impose the condition of naturality with respect to the maps in $\mathbb{L}^{(n)} \otimes_{\mathbb{N}} \underline{\widetilde{\Omega}}_n(Y)$, in the evident sense; the Chern class transformations extend in the obvious manner.

We impose the following commutativity condition: We have the evident isomorphism $t_{L,M} : F_{\mathbb{L}^{(n)}}(\tilde{c}_1(L), \tilde{c}_1(M)) \to F_{\mathbb{L}^{(n)}}(\tilde{c}_1(M), \tilde{c}_1(L))$ of natural transformations, as well as $\tau_{L,M} : \tilde{c}_1(L \otimes M) \to \tilde{c}_1(M \otimes L)$, the isomorphism induced by the symmetry $L \otimes M \cong M \otimes L$. Then we impose the identity $\tau_{L,M} \circ \rho_{L,M} = \rho_{M,L} \circ t_{L,M}$. We impose a similar identity between the associativity of the formal group law and the associativity of the tensor product of line bundles.

We also adjoin $a \cdot \tau_{L,M}$ for all $a \in \mathbb{L}^{(n)}$, with similar compatibilities as above, respecting the $\mathbb{L}^{(n)}$-action and sum. This forms the symmetric monoidal category $\widetilde{\Omega}^{(n)}(X)$, which inherits a grading from $\underline{\widetilde{\Omega}}(X)$. We have the inverse system of graded symmetric monoidal categories:

$$\ldots \to \widetilde{\Omega}^{(n+1)}(X) \to \widetilde{\Omega}^{(n)}(X) \to \ldots \,.$$

**Definition 5.14.** Set $\Omega_{m,r}^{(n)}(X) := \pi_r(B\widetilde{\Omega}_m^{(n)}(X))$ and $\Omega_{m,r}(X) := \varprojlim_n \Omega_{m,r}^{(n)}(X)$.

At present, we can only verify the following:

**Theorem 5.15.** *There is a natural isomorphism* $\Omega_{m,0}(X) \cong \Omega_m(X)$.

*Proof.* First note that $\pi_0(\widetilde{\mathcal{Z}}_m(X))$ is a commutative monoid with group completion $\mathcal{Z}_m(X)$. Next, the natural map $\pi_0(\underline{\widetilde{\Omega}}_*(X))^+ \to \underline{\Omega}_*(X)$ is surjective with kernel generated by the classes generating $\mathcal{R}^{dim}(X)$. Given such an element $\psi := (f : Y \to X, \pi^*L_1, \ldots, \pi^*L_r, M_1, \ldots, M_s)$, with $\pi : Y \to Z$ smooth, and $r > \dim_k Z$, suppose that the $L_i$ are very ample. We may then choose sections $s_i : Z \to L_i$ with divisors $D_i$ all intersecting transversely. Iterating the isomorphisms $\gamma_{L_i,s_i}$ gives a path from $\psi$ to 0 in $B\underline{\widetilde{\Omega}}_*(X)$. Passing to $B\widetilde{\Omega}_m^{(n)}(X)$, the group law allows us to replace an arbitrary line bundle witha difference of very ample ones, so all the classes of this form go to zero in $\Omega_{m,0}^{(n)}(X)$. This shows that the natural map

$$\Omega_{m,0}^{(n)}(X) \to (\mathbb{L}^{(n)} \otimes_{\mathbb{L}} \Omega_*(X))_m$$

is an isomorphism. Since $(\mathbb{L}^{(n)} \otimes_{\mathbb{L}} \Omega_*(X))_m = \Omega_m(X)$ for $m \geq n$, we are done. □

The categories $\widetilde{\Omega}_m^{(n)}(X)$ are covariantly functorial for projective maps, contravariant for smooth maps (with a shift in the grading) and have first Chern class natural transformations $\tilde{c}_1(L) : \widetilde{\Omega}_m^{(n)}(X) \to \widetilde{\Omega}_{m-1}^{(n)}(X)$ for $L \to X$ a line bundle.

We conjecture that the inverse system used to define $\Omega_{m,r}(X)$ is eventually constant for all $r$, not just for $r = 0$. If this is true, it is reasonable to define the space $B\widetilde{\Omega}_m(X)$ as the homotopy limit

$$B\widetilde{\Omega}_m(X) := \operatorname*{holim}_n B\widetilde{\Omega}_m^{(n)}(X).$$

One would then have $\Omega_{m,r}(X) = \pi_r(B\widetilde{\Omega}_m(X), 0)$ for all $m, r$; hopefully the properties of $\Omega_*$ listed in theorem 3.6 would then generalize into properties of the spaces $B\widetilde{\Omega}_m(X)$.

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