THE ARASON INVARIANT AND MOD 2 ALGEBRAIC CYCLES

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Introduction

In topology, one associates to a complex quadratic vector bundle E over a topological space X its *Stiefel-Whitney classes*

$$w_i(E) \in H^i(X, \mathbb{Z}/2).$$

These classes are essentially the only characteristic classes attached to quadratic bundles: any such bundle is classified by the homotopy class of a map $X \to BO(n, \mathbb{C})$ where n is the rank of E. The classifying space $BO(n, \mathbb{C})$ has a tautological quadratic bundle $\mathcal{E}$ of rank n, and $H^*(BO(n, \mathbb{C}), \mathbb{Z}/2)$ is a polynomial algebra on the Stiefel-Whitney classes of $\mathcal{E}$.

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The same holds in algebraic geometry, where to any quadratic vector bundle E over a $\mathbb{Z}[1/2]$-scheme X (a vector bundle provided with an unimodular symmetric bilinear form) one can attach Stiefel-Whitney classes, living in mod 2 étale cohomology [15]

$$w_i(E) \in H^i_{\text{ét}}(X, \mathbb{Z}/2).$$

Here again, these classes can be defined as pull-backs of universal classes w_i in the cohomology of the simplicial scheme $BO(n)/\mathbb{Z}[1/2]$. Since the latter cohomology is a polynomial algebra on the w_i over the étale cohomology of $\operatorname{Spec}\mathbb{Z}[1/2]$ [25], the $w_i(E)$ are essentially the only characteristic classes with values in étale cohomology with $\mathbb{Z}/2$ coefficients attached to quadratic bundles in this context.

If we restrict to (say) virtual quadratic bundles E of rank 0 such that $w_1(E) = w_2(E) = 0$, no new mod 2 characteristic classes arise: such bundles are classified by the infinite spinor group Spin and one can show that $H^*(B\,\mathrm{Spin}, \mathbb{Z}/2)$ is a quotient of $H^*(BO, \mathbb{Z}/2)$, both in the topological and the étale context. In particular, the Wu formula

$$w_3 = w_1 w_2 + Sq^1 w_2$$

shows that $w_3(E) = 0$ if $w_1(E) = w_2(E) = 0$, so that there are no non-trivial degree 3 mod 2 characteristic classes for such bundles.

The situation is quite different if we restrict to quadratic bundles over schemes of the form $\operatorname{Spec} k$, where k is a field of characteristic $\neq 2$. To any k-quadratic form q, of dimension divisible by 8 and such that $w_1(q) = w_2(q) = 0$, Arason [1] has attached a non-trivial invariant

$$e^3(q) \in H^3_{\text{ét}}(k, \mathbb{Z}/2)$$

(see section 1 for less restrictive conditions on q). From the preceding discussion, we know that e^3 cannot be expected to extend to a 'global' invariant, i.e. one defined for quadratic bundles over arbitrary schemes. A question which arises naturally is to determine the *obstruction* to the existence of such a global extension. The aim of this paper is to answer this question in the case of quadratic bundles on smooth varieties over fields.

More specifically, let X be a smooth, irreducible variety over k (still assumed to be of characteristic $\neq 2$); let K be the function field of X and E a quadratic bundle over X. The generic fiber E_η corresponds to a quadratic form q over K. Assume its Arason invariant $e = e^3(q) \in H^3_{\text{ét}}(K, \mathbb{Z}/2)$ is defined; then one easily shows that e in fact lies in the subgroup $H^0_{\text{Zar}}(X, \mathcal{H}^3_{\text{ét}}(\mathbb{Z}/2))$. There is an exact sequence

$$H^3_{\text{ét}}(X, \mathbb{Z}/2) \longrightarrow H^0_{\text{Zar}}(X, \mathcal{H}^3_{\text{ét}}(\mathbb{Z}/2)) \xrightarrow{d_2} CH^2(X)/2 \xrightarrow{\mathrm{cl}^2} H^4_{\text{ét}}(X, \mathbb{Z}/2)$$

where $CH^2(X)$ is the second Chow group of X. This sequences stems from the Bloch-Ogus spectral sequence for X, with coefficients $\mathbb{Z}/2$ [2], and cl^2 is the cycle class map modulo 2. Our main result is the *computation of* $d_2(e) \in CH^2(X)/2$.

In order to explain this result, we recall that any quadratic bundle has a *Clifford invariant*

$$c(E) \in H^2_{\text{ét}}(X, \mathbb{Z}/2)$$

(a variant of $w_2(E)$, see Definition 2.3); in the case considered, we have

$$c(E) \in \operatorname{Ker}(H^2_{\text{ét}}(X, \mathbb{Z}/2) \to H^2_{\text{ét}}(K, \mathbb{Z}/2)) \simeq \operatorname{Pic}(X)/2.$$

On the other hand, the vector bundle underlying E has a *second Chern class* $c_2(E) \in CH^2(X)$. We then have:

Theorem 1. *Under the above assumptions,*

$$d_2(e) = c_2(E) + c(E)^2 \in CH^2(X)/2.$$

Corollary 1. $\operatorname{cl}^2(c_2(E) + c(E)^2) = 0$.

In fact, this corollary can be obtained by more elementary means than Theorem 1: generalizing the well-known relations between Chern and Stiefel-Whitney classes which exist in topology, e.g. [45, p. 181, prob. 15-A] yields the formula $c_2(E) = c(E)^2$ in $H^4_{\text{ét}}(X, \mathbb{Z}/2)$.

The proof of Theorem 1 can be sketched as follows. We show that the hypothesis on E implies that its class $[E] \in H^1_{\text{ét}}(X, O(n,n))$ lifts to a class $\widetilde{[E]} \in H^1_{\text{ét}}(X, \operatorname{Cliff}(n,n))$, where $\operatorname{Cliff}(n,n)$ is the *split special Clifford group*. Now we shall associate to any $\operatorname{Cliff}(n,n)$-torsor F on X two characteristic classes

$$\begin{aligned} \gamma_1(F) &\in \operatorname{Pic}(X) \\ \gamma_2(F) &\in \mathbb{H}^4_{\text{ét}}(X, \Gamma(2)) \end{aligned}$$

(see 6.7), where the right-hand-side group on the second line is Lichtenbaum's *étale weight-two motivic cohomology*. Recall the exact sequence ([37], [28, th. 1.1])

$$0 \to CH^2(X) \to \mathbb{H}^4_{\text{ét}}(X, \Gamma(2)) \to H^0_{\text{Zar}}(X, \mathcal{H}^3_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(2))) \to 0.$$

In the light of this sequence, we show in Theorem 6.9 that

$$2\gamma_2(F) = c_2(F) + \gamma_1(F)^2 \in CH^2(X) \tag{0.1}$$

where $c_2(F)$ is the second Chern class of the $SL(2n)$-torsor (a vector bundle) stemming from F. Theorem 1 follows from this identity and the identification of the map d_2 as a differential in a snake diagram.

This paper is organized as follows. In section 1 we review Arason's invariant, and section 2 the special Clifford group. The heart of the paper is sections 3 and 4, where we compute low-degree $\mathcal{K}$-cohomology of split reductive linear algebraic groups with simply connected derived subgroup and their classifying schemes. We collect the fruits of our labor in section 6, where we define the invariants $\gamma_1(F)$ and $\gamma_2(F)$ and prove identity (0.1). Theorem 1 is proven in section 8. In section 9 we give some applications to quadratic forms over a field.

There are 3 appendices. Appendix A shows how different models of the simplicial classifying scheme of a split torus yield the same $\mathcal{K}$-cohomology. Appendix B presents a construction and a characterization of the invariant announced by Rost for torsors under a simple, simply connected algebraic group H over a field (see the forthcoming paper [53]): in the case of Spin, this allows this paper to be self-contained. Let us point out that our method tackles the p-primary part of the Rost invariant as well, in case char $k = p > 0$. Finally, appendix C compares $\mathcal{K}$-cohomology of the simplicial scheme BH with that of an approximating variety B_rH: it turns out that they don't coincide. In this last appendix, we have to stay away from the characteristic of k if it is nonzero.

Acknowledgements. It will be clear to the reader that Rost's ideas permeate this article. In particular, his suggestion that one could use motivic cohomology to construct his invariants was a key insight for our proof. We also thank Burt Totaro and Jean-Pierre Serre for useful comments.

1. Review of the Arason invariant

Let k be a field of characteristic $\neq 2$. As is customary, we write

$$q =< a_1, \dots, a_r >$$

for the isomorphism class of the quadratic form $q(x) = a_1x_1^2 + \cdots + a_rx_r^2$ $(a_i \in k^*)$.

Let $W(k)$ be the Witt ring of k [31], [54]. The dimension of forms induces an augmentation

$$W(k) \xrightarrow{\dim} \mathbb{Z}/2$$

whose kernel, denoted by Ik, is the ideal of even-dimensional forms. Its n-th power is denoted by I^nk. Since Ik is additively generated by the forms $< 1, -a >$ $(a \in k^*)$, I^nk is generated by *n-fold Pfister forms*

$$\ll a_1, \dots, a_n \gg := < 1, -a_1 > \otimes \cdots \otimes < 1, -a_n > .$$

For $n \leq 4$, there are homomorphisms

$$e^n : I^nk/I^{n+1}k \to H^n(k, \mathbb{Z}/2)$$

characterized by $e^n(\ll a_1, \dots, a_n \gg) = (a_1, \dots, a_n) := (a_1) \cdot \dots \cdot (a_n)$, where, for $a \in k^*$, $(a) \in H^1(k, \mathbb{Z}/2)$ is the class of a via Kummer theory. For $n = 0, 1, 2$, the e^n come from elementary invariants $\dim, d_\pm, c$ defined over the whole Witt ring $W(k)$. They can be described as follows:

- $n = 0$: $e^0(q) = \dim q \pmod 2$.
- $n = 1$: $e^1(q) = d_\pm q := ((-1)^{\frac{r(r-1)}{2}} \operatorname{disc} q)$, where $r = \dim q$ and $\operatorname{disc} q = a_1 \dots a_r$ if $q =< a_1, \dots, a_r >$.
- $n = 2$: let $C(q)$ be the Clifford algebra of q and $C_0(q)$ the even part of $C(q)$. The algebra $C(q)$ (resp. $C_0(q)$) is a central simple algebra of exponent 2 over k if $\dim q$ is even (resp. odd). Then $e^2(q) = c(q) = \begin{cases} [C(q)] \in {}_2\operatorname{Br} k & \text{if } \dim q \text{ even} \\ [C_0(q)] \in {}_2\operatorname{Br} k & \text{if } \dim q \text{ odd.} \end{cases}$

  Note that ${}_2\operatorname{Br} k \simeq H^2(k, \mathbb{Z}/2)$ by Hilbert's Theorem 90.

The relationship of $d_\pm q$ and $c(q)$ with $w_1(q)$ and $w_2(q)$ is as follows:

- $d_\pm(q) = w_1(q) + \frac{r(r-1)}{2}(-1)$ (since $\operatorname{disc} q = w_1(q)$);
- $c(q) = w_2(q) + a(-1) \cdot w_1(q) + b(-1, -1)$, with $a = \frac{(r-1)(r-2)}{2}$ and $b = \frac{(r+1)r(r-1)(r-2)}{24}$ [31, prop. V.3.20]. In particular, if $r \equiv 0 \pmod 4$, $w_1(q) = w_2(q) = 0$ if and only if $e^1(q) = e^2(q) = 0$.

The existence of e^3 was proven by Arason in his thesis [1]: it cannot be extended to a function $W(k) \to H^3(k, \mathbb{Z}/2)$ which would be natural under change of base field [1, p. 491] (see Corollary 9.3 for an unstable refinement.) Similarly, Jacob-Rost [24] and independently Szyjewski [58] proved the existence of e^4. Merkurjev [39] proved that e^2 is an isomorphism, which shows with the above remarks that $e^3(q)$ is defined as soon as $w_1(q) = w_2(q) = 0$. Rost [51] and independently Merkurjev-Suslin [43] proved that e^3 is an isomorphism. Voevodsky has recently announced a proof that e^n exists and is an isomorphism for all n and all fields.

2. The special Clifford group

Recall [15, 1.9] that a quadratic bundle E over a scheme X has a Clifford algebra $C(E)$. If E has even rank, $C(E)$ is an Azumaya algebra with a canonical involution σ, restricting to the identity on $E \hookrightarrow C(E)$. Recall also the Clifford group $C^*(E)$ [15, 1.9], defined as the homogeneous stabilizer of E in $C(E)^*$ (acting by inner automorphisms). It is representable by a linear algebraic group scheme over X. When $E = \mathbb{H}(\mathbb{A}^n_X)$ is the split bundle associated with the affine n-space $\mathbb{A}^n_X$ [15, 5.5], we denote this algebraic group scheme by $C^*(n,n)$: it is defined over $\mathbb{Z}$.

There is a "spinor norm" homomorphism $C^*(E) \xrightarrow{\gamma_1} \mathbb{G}_m$, given by $\gamma_1(x) = x\sigma(x)$; as in [15, 1.9], we denote its kernel by $\widetilde{O}(E)$. The action of $C^*(E)$ on E by inner automorphisms is orthogonal, hence defines a homomorphism $C^*(E) \to O(E)$ with kernel the center of $C^*(E)$, which is nothing else than $\mathbb{G}_m$. The situation can be summarized by the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{O}(n,n) & \longrightarrow & O(n,n) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & C^*(n,n) & \longrightarrow & O(n,n) & \longrightarrow & 1 \\
 & & {\scriptstyle 2}\downarrow & & {\scriptstyle \gamma_1}\downarrow & & & & \\
 & & \mathbb{G}_m & \xrightarrow{=} & \mathbb{G}_m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
\tag{2.1}
$$

Let us denote by $\mathrm{Cliff}(E)$ the even part of $C^*(E)$: this is the *special Clifford group*. The group $\mathrm{Cliff}(E) \cap \widetilde{O}(E)$ is nothing else than the *spinor group* $\mathrm{Spin}(E)$. In case

$E = \mathbb{H}(\mathbb{A}^n_X)$, this is summarised by the following diagram, similar to (2.1):

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Spin}(n,n) & \longrightarrow & SO(n,n) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{Cliff}(n,n) & \longrightarrow & SO(n,n) & \longrightarrow & 1 \\
 & & {\scriptstyle 2}\downarrow & & {\scriptstyle \gamma_1}\downarrow & & & & \\
 & & \mathbb{G}_m & \xrightarrow{=} & \mathbb{G}_m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{2.2}
$$

This allows one to recover Cliff in terms of Spin, if one wishes:

$$\mathrm{Cliff}(n,n) = \mathrm{Spin}(n,n) \times \mathbb{G}_m/\mu_2$$

via the diagonal action $-(g,t) = (-g,-t)$.

Suppose $X = \operatorname{Spec} K$. Given a quadratic form q, the action of $O(q)$ on the vector space underlying q extends to an action of $O(q)$ on $C(q)$ by algebra automorphisms. When $q = n\mathbb{H}$ is split, $C(q) \simeq M_{2^n}(K)$; we denote by ρ the corresponding homomorphism $O(n,n) \to PGL(2^n)$. Recall the invariant $c(q)$ from section 1.

2.1. Lemma. *Let K be a field and let q be a quadratic form with even rank $2n$. Then*

$$c(q) = \partial[q] \in H^2(K,\mu_2) = {}_2\operatorname{Br}(K)$$

where $[q]$ is the class of q in $H^1(K,O(n,n))$ and ∂ is the boundary map in non-abelian cohomology coming from the exact sequence

$$1 \to \mu_2 \to \widetilde{O}(n,n) \to O(n,n) \to 1$$

of diagram (2.1).

Proof. This follows immediately from the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{O}(n,n) & \longrightarrow & O(n,n) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & {\scriptstyle \rho}\downarrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & GL(2^n) & \longrightarrow & PGL(2^n) & \longrightarrow & 1
\end{array}
$$

in which the middle vertical map is the natural embedding (from the definition of $C^*(n,n)$ and $\widetilde{O}(n,n)$). □

2.2. Remark. Hilbert's Theorem 90 implies that the map

$$H^1(K,\mathrm{Cliff}(n,n)) \to H^1(K,SO(n,n))$$

is injective. In other words, over a field any quadratic form in I^2 can be refined into a Clifford bundle in a *unique* way (contrary to the situation for Spin-bundles). Similarly for $O(n,n)$ and $C^*(n,n)$.

We now extend the invariants $d_\pm q$ and $c(q)$ to quadratic bundles of even rank over arbitrary schemes as follows.

2.3. Definition. If E is a quadratic bundle of rank $2n$ over X, then its signed discriminant $d_\pm E$ is the image of $[E] \in H^1_{\text{ét}}(X, O(n,n))$ into $H^1_{\text{ét}}(X, \mathbb{Z}/2)$ via the determinant map $\det : O(n,n) \to \mu_2 \simeq \mathbb{Z}/2$. Its Clifford invariant $c(E)$ is the image of $[E]$ in $H^2_{\text{ét}}(X, \mathbb{Z}/2)$ by the non-abelian boundary map associated with the exact sequence

$$1 \to \mu_2 \to \widetilde{O}(n,n) \to O(n,n) \to 1.$$

2.4. Remark. One can check that $d_\pm(E)$ and $c(E)$ coincide with the similar invariants defined by Parimala and Srinivas in [48, 2.2 and Lemma 6]. For the latter, one proceeds as in the proof of Lemma 2.1, replacing $GL(2^n)$ and $PGL(2^n)$ by the relevant $\pm$-orthogonal and projective orthogonal groups, corresponding to the canonical involution carried by $C(E)$.

3. $\mathcal{K}$-cohomology of split reductive algebraic groups

Let G be a split reductive algebraic group over k. In the next section, we shall partially compute the $\mathcal{K}$-cohomology of a classifying scheme BG; for this we have to partially compute the $\mathcal{K}$-cohomology groups $H^i_{\text{Zar}}(G^a, \mathcal{K}_j)$ for various a. Obviously we can assume $a = 1$. The method we use is the one of [34, § 2] (where it is applied to computing $K_*(G)$).

The $\mathcal{K}$-cohomology of G has been computed in full by Suslin in the cases $G = SL(N), GL(N)$ and $Sp(2n)$ [57].

3.1. To start the computation of the $\mathcal{K}$-cohomology of G, recall that since G is smooth this cohomology is given by the (co)homology of the corresponding Gersten complex. The computation in fact applies to a large extent to arbitrary "cycle modules" in the sense of Rost [52]. So we give ourselves a cycle module $K \mapsto M_*(K)$, for K running through finitely generated extensions of k. For any variety (smooth or not) V/k, we write

$$C_*(V, M_j)$$

for the Gersten complex

$$\dots \to \bigoplus_{x \in V_{(i+1)}} M_{i+j+1}(k(x)) \to \bigoplus_{x \in V_{(i)}} M_{i+j}(k(x)) \to \bigoplus_{x \in V_{(i-1)}} M_{i+j-1}(k(x)) \to \dots$$

where $V_{(i)}$ denotes the set of points of V of dimension i, and $A_i(V, M_j)$ for its homology. Since C_* is covariant for arbitrary morphisms, it can be extended to simplicial k-schemes $V_\bullet$ by taking the total complex associated with the bicomplex

$$\dots \to C_*(V_{n+1}, M_j) \to C_*(V_n, M_j) \to C_*(V_{n-1}, M_j) \to \dots$$

This allows us to define cycle homology of $V_\bullet$ as the homology of this total complex. There is a spectral sequence

$$E^1_{p,q} = A_q(V_p, M_j) \Rightarrow A_{p+q}(V_\bullet, M_j).$$

We can do the same with an augmented simplicial scheme.

3.2. The pairings $K_i^M \otimes_{\mathbb{Z}} M_j \to M_{i+j}$ give morphisms of complexes, for two varieties V and W [52, (14.1)]:

$$C_*(V, K_i^M) \otimes_{\mathbb{Z}} C_*(W, M_j) \to C_*(V \times_k W, M_{i+j}) \tag{3.1}$$

hence homomorphisms

$$A_m(V, K_i^M) \otimes_{\mathbb{Z}} A_n(W, M_j) \to A_{m+n}(V \times_k W, M_{i+j})$$

and, for $i = -m$:

$$CH_m(V) \otimes_{\mathbb{Z}} A_n(W, M_j) \to A_{m+n}(V \times_k W, M_{j-m})$$

where $CH_n(V)$ are *Chow homology groups* [16, § 1.3]. Putting all gradings together, we note that (3.1) refines into a morphism of complexes

$$C_*(V, K_*^M) \otimes_{K_*^M(k)} C_*(W, M_*) \to C_*(V \times_k W, M_*) \tag{3.2}$$

since the $C_m(V, K_*^M)$ and $C_n(W, M_*)$ are all modules over $K_*^M(k)$.

3.3. If Z is a closed subset of V, one has an exact sequence of complexes

$$0 \to C_*(Z, M_j) \to C_*(V, M_j) \to C_*(V - Z, M_j) \to 0 \tag{3.3}$$

which is canonically split as an exact sequence of graded abelian groups [52, (3.10.1)]. This yields a "localization" exact sequence [52, § 5]

$$\cdots \to A_i(Z, M_j) \to A_i(V, M_j) \to A_i(V - Z, M_j) \to A_{i-1}(Z, M_j) \to \dots \tag{3.4}$$

Putting all gradings together, we note that (3.3) gives an exact sequence of complexes

$$0 \to C_*(Z, M_*) \to C_*(V, M_*) \to C_*(V - Z, M_*) \to 0 \tag{3.5}$$

which is split as an exact sequence of graded $K_*^M(k)$-modules.

Suppose we have a finite closed covering $Z = \bigcup_i Z_i$ of some variety Z, and let $Z_\bullet$ be the associated simplicial scheme. By a well-known argument, (3.4) implies that the augmentation $Z_\bullet \to Z$ gives an isomorphism on A_*, yielding a Čech spectral sequence of homological type:

$$E^1_{p,q} = \bigoplus_{i_0 < \cdots < i_p} A_q(Z_{i_0} \cap \cdots \cap Z_{i_p}, M_j) \Rightarrow A_{p+q}(Z, M_j).$$

Suppose now that $U \subset V$ is a open subset of a variety V such that $Z = V - U$ is covered by the Z_i. Considering the augmented simplicial scheme $Z_\bullet \to V$, we get a spectral sequence analogous to [34, $(1.5)_G$]

$$E^1_{p,q} \Rightarrow A_{p+q}(U, M_j) \tag{3.6}$$

with

$$E^1_{p,q} = \begin{cases} A_q(V, M_j) & \text{if } p = 0 \\ \bigoplus_{i_1 < \cdots < i_p} A_q(Z_{i_1} \cap \cdots \cap Z_{i_p}, M_j) & \text{if } p > 0. \end{cases}$$

3.4. If V is purely of dimension d, define $A^i(V, M_j)$ as $A_{d-i}(V, M_{j-d})$. This cycle cohomology is contravariant for all maps to a smooth variety [52, § 12]. We have *homotopy invariance*:

$$A^i(V, M_j) \xrightarrow{\sim} A^i(W, M_j)$$

if V is equidimensional and $W \to V$ is an affine bundle [52, prop. 8.6].

3.5. Let us say that a variety X over k is *Künneth* if, for any k-variety Y and any cycle-module M, the pairing of complexes (3.2) is a quasi-isomorphism. The following lemma gives examples of Künneth varieties:

3.6. Lemma.

(i) $\operatorname{Spec} k$ *is Künneth.*
(ii) *If X and Y are Künneth, so is $X \times_k Y$.*
(iii) *Any affine bundle over a Künneth variety is Künneth.*
(iv) *Let X be a k-variety, Z a closed subset of X and U the complementary open subset. If among X, Z, U, two are Künneth varieties, then the third is.*

Proof. (i) and (ii) are trivial and (iii) follows from 3.4. To see (iv), we apply the exact sequence of complexes (3.3) to (X, Z, U) and $(X \times_k Y, Z \times_k Y, U \times_k Y)$. Since (3.5) is split as an exact sequence of graded $K_*^M(k)$-modules, it remains exact after tensorization over $K_*^M(k)$. So we get a commutative diagram of short exact sequences of complexes:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
C_*(Z, K_*^M) \otimes_{K_*^M(k)} C_*(W, M_*) & \longrightarrow & C_*(Z \times_k W, M_*) \\
\downarrow & & \downarrow \\
C_*(V, K_*^M) \otimes_{K_*^M(k)} C_*(W, M_*) & \longrightarrow & C_*(V \times_k W, M_*) \\
\downarrow & & \downarrow \\
C_*(U, K_*^M) \otimes_{K_*^M(k)} C_*(W, M_*) & \longrightarrow & C_*(U \times_k W, M_*) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The five lemma now shows that if two rows are quasi-isomorphisms, so is the third one.□

Recall that a k-variety X is *cellular* if X contains a closed subset $Z \neq X$ such that $X - Z \simeq \mathbb{A}_k^n$ for some n and Z is cellular (a recursive definition).

3.7. Proposition. *a) Any cellular variety is Künneth. Moreover, if X is cellular and Y is arbitrary, then the $CH_p(X)$ are finitely generated free abelian groups and the natural map*

$$\bigoplus_{p \geq 0} CH_p(X) \otimes_{\mathbb{Z}} A_{n-p}(Y, M_{i+p}) \xrightarrow{\sim} A_n(X \times Y, M_i)$$

is an isomorphism all M_*, n, i.
b) A split torus is a Künneth variety.

Proof. The fact that cellular varieties and tori are Künneth follows immediately from Lemma 3.6. The fact that Chow groups of a cellular variety are finitely generated free is well-known [16, ex. 1.9.1]. It remains to show the isomorphism. For this, it suffices to show that the $A_i(X, K_*^M)$ are *free* modules over $K_*^M(k)$. This follows from

3.8. Lemma. [40, proof of prop. 1] *Let X be a cellular variety over k. Then the natural maps from* 3.2

$$CH_i(X) \otimes K_j^M(k) \to A_i(X, K_{j-i}^M)$$

are isomorphisms. □

3.9. If V is smooth, one has

$$A^p(V, M_i) = H^p_{\mathrm{Zar}}(V, \mathcal{M}_i)$$

where $\mathcal{M}_i$ is the Zariski sheaf $U \mapsto A^0(U, M_i)$ (Gersten's conjecture, [52, cor. 6.5]). When M_i is given by a suitable cohomology theory with supports (defined on all smooth k-schemes) satisfying a purity theorem, $\mathcal{M}_i$ can further be identified with the Zariski sheafification of $U \mapsto M_i(U)$. This applies to algebraic K-theory (Quillen [50]) and to étale cohomology with coefficients in twisted roots of unity or singular cohomology with integer coefficients when $k = \mathbb{C}$ (Bloch-Ogus [2]).

3.10. Let G be our split reductive algebraic group. We let $L_G = \mathrm{Hom}(T, \mathbb{G}_m)$ be the character group of a maximal torus T of G. The choice of a $\mathbb{Z}$-basis of L_G gives a k-isomorphism

$$T \xrightarrow{\sim} \mathbb{G}_m^r$$

with $r = \mathrm{rank}\, G$.

3.11. Consider the projection $G \to G/T$, with fibers T. Letting $X := G/T$, one has

$$H^i(X, \mathcal{K}_j) \simeq H^i(G/B, \mathcal{K}_j)$$

where B is a Borel subgroup, because $X \to G/B$ is an affine bundle (i.e. a torsor under a vector bundle).

The isomorphism $T \xrightarrow{\sim} \mathbb{G}_m^r$ defines r rank one bundles $L_1, \ldots, L_r$ on X such that

$$G = L_1^\times \times_X \ldots \times_X L_r^\times$$

where $L_i^\times$ is the total space of the corresponding $\mathbb{G}_m$-bundle. We can then embed G into the affine bundle

$$\bar{G} := L_1 \times_X \ldots \times_X L_r.$$

One has the following properties

(i)
$$\bar{G} - G = \bigcup_{i=1}^{r} D_i$$
where D_i is the divisor $D_i = L_1 \times_X \ldots \times_X \{0\} \times_X \ldots \times_X L_r$ in which the zero section $\{0\}$ is taken on the i-th factor of $L_1 \times_X \ldots \times_X L_r$.

(ii) $[D_i] \in \mathrm{Pic}(\bar{G})$ corresponds to $c_1(L_i) \in \mathrm{Pic}(X)$ under the isomorphism $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(\bar{G})$.

3.12. We now apply the spectral sequence (3.6) to $V = \bar{G}$, $U = G$ and $Z_i = D_i$. This gives a spectral sequence

$$E^1_{p,q} = \begin{cases} A_q(\bar{G}, M_j) & \text{if } p = 0 \\ \bigoplus_{i_1<\cdots<i_p} A_q(D_{i_1} \cap \cdots \cap D_{i_p}, M_j) & \text{if } p > 0 \end{cases} \Rightarrow A_{p+q}(G, M_j).$$

Let $d = \dim G = \dim \bar{G}$; note that $\dim(\bar{G} - G) = \dim D_i = d - 1$ for all i and similarly $\dim(D_{i_1} \cap \cdots \cap D_{i_p}) = d - p$ for all $p > 0$. Hence the spectral sequence can be rewritten

$$E^1_{p,q} = \begin{cases} A^{d-q}(\bar{G}, M_{j+d}) & \text{if } p = 0 \\ \bigoplus_{i_1<\cdots<i_p} A^{d-p-q}(D_{i_1} \cap \cdots \cap D_{i_p}, M_{j+d-p}) & \text{if } p > 0 \end{cases} \Rightarrow A^{d-p-q}(G, M_{j+d}).$$

Since $\bar{G}$ and the $D_{i_1} \cap \cdots \cap D_{i_p}$ are all affine bundles over X, we can rewrite the E^1-term, using homotopy invariance 3.4

$$E^1_{p,q} = \textstyle\bigwedge^p L_G \otimes A^{d-p-q}(X, M_{j+d-p})$$

where L_G is the group of characters of the split maximal torus T.

Since X is an affine bundle over the cellular variety G/B (for this, e.g. [6]), Lemma 3.8 and homotopy invariance yield the final form of the E^1-term of the above spectral sequence (after a shift on j):

$$E^1_{p,q} = \textstyle\bigwedge^p L_G \otimes CH^{d-p-q}(X) \otimes M_{j+q-d}(k) \Rightarrow A^{d-p-q}(G, M_j) \tag{3.7}$$

compare [34, § 2]. This spectral sequence is contravariant in G (for group scheme homomorphisms).

It would be beyond the scope of this article to study this spectral sequence in detail, and in particular to show that it degenerates at E^2 like the similar one in [34]. We will content ourselves here with elementary remarks and low-degree computations.

3.13. From now on, we assume that $M_j = 0$ for $j < 0$. Note that this implies $A^n(X, M_j) = 0$ for $n > j$, any X. Hence we shall care about $A^n(X, M_j)$ only for $n \leq j$.

3.14. In view of the definition of the Čech d^1-differential, the complexes

$$(K.(G, d-q)) \quad \ldots \to \textstyle\bigwedge^{p+1} L_G \otimes CH^{d-p-q-1}(X) \to \bigwedge^p L_G \otimes CH^{d-p-q}(X) \to \bigwedge^{p-1} L_G \otimes CH^{d-p-q+1}(X) \to \ldots$$

of the E^1-terms of (3.7) can be described as follows. Let $c_1 : L_G \to \mathrm{Pic}(X)$ be the homomorphism given by the first Chern class. It gives rise to a Koszul complex [23, prop. 4.3.1.2]:

$$(Kos.(c_1, d-q)) \quad \ldots \to \textstyle\bigwedge^{p+1} L_G \otimes \mathbf{S}^{d-p-q-1}(\mathrm{Pic}(X)) \to \bigwedge^p L_G \otimes \mathbf{S}^{d-p-q}(\mathrm{Pic}(X)) \to \bigwedge^{p-1} L_G \otimes \mathbf{S}^{d-p-q+1}(\mathrm{Pic}(X)) \to \ldots$$

Then the natural maps $\mathbf{S}^r(\mathrm{Pic}(X)) \to CH^r(X)$ given by the intersection product provide a morphism of complexes from $Kos.(c_1, d-q)$ to $K.(G, d-q)$.

3.15. The first two terms of $K.(G, d-q)$ and $Kos.(c_1, d-q)$ coincide. In particular, this yields

$$E^2_{p,d-p} = \bigwedge^p(\operatorname{Ker} c_1).$$

3.16. Suppose $k = \mathbb{C}$ and $M_i(K) = \varinjlim H^i_{\rm an}(U, \mathbb{Z})$, where U runs through the open subsets of a model of $K/\mathbb{C}$. Then $M_i(\mathbb{C}) = 0$ for $i \neq 0$ and the spectral sequence (3.7) degenerates, yielding isomorphisms

$$H_p(K.(G, d-q)) \simeq H^{d-p-q}(G, \mathcal{H}^{d-q}_{\rm an}(\mathbb{Z})).$$

3.17. Suppose that G is a torus. Then $X = \operatorname{Spec} k$, hence $CH^i(X) = 0$ for $i > 0$ and (3.7) degenerates at E^1, yielding

- $A^p(G, M_j) = 0$ $(p > 0)$;
- There is a filtration on $A^0(G, M_j)$ with successive quotients $\bigwedge^p L_G \otimes M_{j-p}(k)$.

3.18. For $j = 0$, (3.7) gives an isomorphism $M_0(k) \xrightarrow{\sim} A^0(G, M_0)$. For $j = 1$, it gives an exact sequence

$$(3.10)\quad 0 \to M_1(k) \to A^0(G, M_1) \to L_G \otimes M_0(k) \xrightarrow{c_1 \otimes 1} \operatorname{Pic}(X) \otimes M_0(k) \to A^1(G, M_1) \to 0.$$

¿From now on, we make the following

3.19. Assumption. G is split reductive and its derived subgroup H is simply connected.

Therefore we have an exact sequence $1 \to H \to G \to S \to 1$ where

- S is a split torus
- H is semi-simple, simply connected and has a split torus T_H.

The unique maximal torus of G containing T_H is $T_G = Z(G)^0 T_H$, where $Z(G)^0$ is the connected component of 1 in the center of G [11, exposé XXII, p. 260, prop. 6.2.8]. We have an exact sequence

$$1 \to T_H \to T_G \to S \to 1$$

and the assumption that H is simply connected implies that $L_H \xrightarrow{c_1} \operatorname{Pic}(X)$ is an isomorphism [12]. We also have $X_G = X_H$ (and $X_S = \operatorname{Spec} k$). This gives a split short exact sequence

$$(3.11)\qquad 0 \to L_S \to L_G \xrightarrow{c_1} \operatorname{Pic}(X) \to 0.$$

3.20. Proposition. *Under assumption* 3.19*,*

(i) *For $j \geq 0$, the maps $A^0(S, M_j) \to A^0(G, M_j)$ are isomorphisms.*
If $S = \{1\}$, we have $M_j(k) \xrightarrow{\sim} A^0(G, M_j)$ for all $j > 0$.

(ii) *There is for $j = 1$ a short exact sequence*

$$0 \to M_1(k) \to A^0(G, M_1) \to L_S \otimes M_0(k) \to 0.$$

Moreover, $A^1(G, M_1) = 0$.

(iii) *For $j = 2$, we have*

- *an exact sequence*

$$0 \to A^1(G, M_2) \to \mathbf{S}^2(L_H) \otimes M_0(k) \xrightarrow{c\otimes 1} CH^2(X) \otimes M_0(k) \to 0$$

where c is the characteristic map $\mathbf{S}^2(L_H) \to CH^2(X)$;
- *isomorphisms $A^1(G, M_2) \xrightarrow{\sim} A^1(H, M_2)$ and equalities $A^2(G, M_2) = A^2(H, M_2) = 0$.*

Proof. To see the first claim of (i), note that 3.15 implies that the $E^2_{p,q}(S, M_j) \xrightarrow{\sim} E^2_{p,q}(G, M_j)$ for $p + q = d$ in the spectral sequence (3.7) attached to S and G. As observed in 3.17, $E^2_{p,q}(S, M_j) = 0$ for $p+q \neq d$, which implies that all differentials starting from $E^2_{p,q}(G, M_j)$ are 0. Since no differentials arrive at $E^2_{p,q}(G, M_j)$, this means that $E^2_{p,q}(G, M_j) = E^\infty_{p,q}(G, M_j)$. The map $A^0(S, M_j) \to A^0(G, M_j)$ respects the filtrations from (3.7) and is an isomorphism on the associated graded, so it is an isomorphism. The second claim of (i) follows immediately.

(ii) follows from (3.10) and the fact that $c_1 \otimes 1$ is surjective (3.11). We now look at the spectral sequence (3.7) for $j = 2$. It follows from (3.11) and the description of $(K.(G, d - q))$ as a Koszul-like complex that

(3.12)

$$\begin{array}{lll} E^2_{2,d-2} = \Lambda^2(L_S) \otimes M_0(k) & E^2_{1,d-2} = \operatorname{Ker} c \otimes M_0(k) & E^2_{0,d-2} = \operatorname{Coker} c \otimes M_0(k) \\ & E^2_{1,d-1} = L_S \otimes M_1(k) & \\ & & E^2_{0,d} = M_2(k) \\ E^2_{p,q} = 0 \text{ otherwise.} & & \end{array}$$

(iii) follows easily from this computation, except for the vanishing of $A^2(G, M_2)$. To see this, suppose first that $M = K^M$ (Milnor K-theory). Then $A^2(G, M_2) = CH^2(G)$ and this group is 0 by [38] for G semi-simple classical, [34, th. 2.1] in general. Indeed, [34, th. 2.1] implies that $K_0(G) \simeq \mathbb{Z}$ with trivial topological filtration, and it is well-known that for any smooth variety V, the natural map $CH^i(V) \to \operatorname{gr}^i K_0(V) = 0$ is surjective with kernel killed by $(i-1)!$. So the characteristic map c is surjective, which gives the result in general. □

Let $\mathcal{C}$ be a category with finite products. Recall that a contravariant functor $T : \mathcal{C}^o \to \{\text{abelian groups}\}$ is *additive* if $T(*) = 0$, where $*$ is the final object of $\mathcal{C}$, and $T(X) \oplus T(Y) \to T(X \times Y)$ is an isomorphism for all $X, Y \in \mathcal{C}$, where the map is given by the two projections.

3.21. Corollary. *Let $\mathcal{C}$ be the category of k-reductive groups satisfying assumption* 3.19*. Then $G \mapsto A^1(G, M_2)$ is additive.*

Proof. Recall that, for $G \in \mathcal{C}$, the first Chern class identifies L_H with $\operatorname{Pic}(X)$. Let $G_1, G_2 \in \mathcal{C}$, with split maximal tori T_1 and T_2, $G = G_1 \times G_2$ with split maximal torus $T = T_1 \times T_2$, and $X_1 = G_1/T_1$, $X_2 = G_2/T_2$, $X = G/T$. Then $X \simeq X_1 \times X_2$, hence we get decompositions (using Proposition 3.7):

$$\mathbf{S}^2(\operatorname{Pic}(X)) \simeq \mathbf{S}^2(\operatorname{Pic}(X_1)) \oplus \operatorname{Pic}(X_1) \otimes \operatorname{Pic}(X_2) \oplus \mathbf{S}^2(\operatorname{Pic}(X_2))$$

$$CH^2(X) \simeq CH^2(X_1) \oplus \operatorname{Pic}(X_1) \otimes \operatorname{Pic}(X_2) \oplus CH^2(X_2).$$

Moreover, the multiplication map $\mu : \mathbf{S}^2(\mathrm{Pic}(X)) \to CH^2(X)$ is diagonal with respect to these decompositions:

$$\mu = \mathrm{diag}(\mu_1, Id, \mu_2)$$

with obvious notation. □

We shall need the following corollary in appendices B and C:

3.22. Corollary. *Let A be a semi-local ring of a smooth variety over k. Then, for any split semi-simple simply connected algebraic group H over k and any cycle module M_*, there are isomorphisms:*

$$H^0_{\mathrm{Zar}}(H_A, \mathcal{M}_i) \simeq H^0(A, \mathcal{M}_i) \text{ for all } i \geq 0$$
$$H^1_{\mathrm{Zar}}(H_A, \mathcal{M}_2) \simeq \mathrm{Ker}\, c \otimes M_0(k)$$
$$H^q_{\mathrm{Zar}}(H_A, \mathcal{M}_2) = 0 \text{ for } q \geq 2$$

where c is the characteristic map of Proposition 3.20 (ii) *and $\mathcal{M}_i$ is the Zariski sheaf associated to M_i as in* 3.9.

Proof. Consider the cohomology theory with supports

$$(X, Z) \mapsto h^*_Z(X) := H^*_{H\times Z}(H \times X, \mathcal{M}_i)$$

(Zariski cohomology) for some $i \geq 0$. It satisfies étale excision (in the sense that $h^*_Z(X) \xrightarrow{\sim} h^*_Z(X')$ for an étale morphism $X' \xrightarrow{f} X$ such that $f^{-1}(Z) \xrightarrow{\sim} Z$) and is homotopy invariant; the first fact follows from the stronger localization property (3.4) for cycle cohomology, and the second is 3.4. By the arguments of [17] (see also [9]), this cohomology theory satisfies Gersten's conjecture. In particular, for A as in Corollary 3.22, with field of fractions K, we have exact sequences:

$$0 \to h^q(A) \to h^q(K) \to \bigoplus_{y\in Y^{(1)}} h^{q+1}_y(A)$$

where $Y = \mathrm{Spec}\, A$. Identifying $h^{q+1}_y(A)$ with $H^q(H_{k(y)}, \mathcal{M}_{i-1})$ via (3.4), this translates as

$$0 \to H^q(H_A, \mathcal{M}_i) \to H^q(H_K, \mathcal{M}_i) \to \bigoplus_{y\in Y^{(1)}} H^q(H_{k(y)}, \mathcal{M}_{i-1}).$$

Corollary 3.22 follows from this and the computations of Proposition 3.20 and Corollary 3.21. □

4. $\mathcal{K}$-cohomology of BG

In this section, we compute the groups $H^i_{\mathrm{Zar}}(BG, \mathcal{M}_j)$, where $\mathcal{M}_j$ is as in 3.9:

- in general when G is a split torus;
- for $j \leq 2$ when G is as in 3.19.

For simplicity, we sometimes drop the index $_{\mathrm{Zar}}$ from the groups $H^i_{\mathrm{Zar}}(BG, \mathcal{M}_j)$.

4.1. Let $X_\bullet$ be a simplicial k-scheme such that all X_n are smooth. Let $\mathfrak{T}$ be a Grothendieck topology over the category of schemes of finite type over k (for example the Zariski or the étale topology, or the analytic topology if $k = \mathbb{C}$). Recall the spectral sequence [10]

$$E_1^{pq}(\mathcal{F}_\bullet) = \mathbb{H}^q_{\mathfrak{T}}(X_p, \mathcal{F}_p) \Rightarrow \mathbb{H}^{p+q}_{\mathfrak{T}}(X_\bullet, \mathcal{F}_\bullet) \quad (4.1)$$

for any complex of simplicial sheaves $\mathcal{F}_\bullet$ over $X_\bullet$, with differential

$$d_1 : E_1^{pq} \longrightarrow E_1^{p+1,q}, \quad d_1 = \sum_{i=0}^{p+1} (-1)^i \delta_i^*.$$

4.2. We are especially interested in the case where $X_\bullet = BG$, where G is an algebraic group over k and $BG = EG/G$, where EG is defined by

$$(EG)_\ell = G^{\Delta_\ell}$$

with $\Delta_\ell = \{0, \dots, \ell\}$. Here G acts on EG diagonally on the right:

$$(g_0, \dots, g_\ell) \cdot h = (g_0 h, \dots, g_\ell h)$$

for $(g_0, \dots, g_\ell) \in (EG)_\ell$ and $h \in G$. The face map δ_i is just "forgetting i".

4.3. Lemma. *Suppose k is algebraically closed, let U be a unipotent subgroup of G, and take u in $U(k)$. Then conjugation by u acts by the identity on $H^*(BG, \mathcal{M}_*)$.*

Proof. Let $\mu\colon G \times_k BG \to BG$ be the morphism giving the action of conjugation. As a variety U is an affine space over k, there is a map $\varphi\colon \mathbb{A}^1_k \to G$ such that $u = \varphi(1)$, $1_G = \varphi(0)$. Pulling back μ by φ gives the morphism

$$\nu\colon \mathbb{A}^1 \times_k BG \longrightarrow BG$$

We have the sections

$$i_0, i_1\colon BG \longrightarrow \mathbb{A}^1 \times_k BG$$

with respective values 0 and 1. The projection p_2 gives a map

$$p_2^*\colon H^*(BG, \mathcal{M}_*) \longrightarrow H^*(\mathbb{A}^1 \times_k BG, \mathcal{M}_*).$$

This map is an isomorphism by homotopy invariance for the cohomology of G^p and a comparison of spectral sequences. It follows that

$$\mathrm{id} = i_0^* \circ \nu^* = i_1^* \circ \nu^* = \text{ conjugation by } u.$$ □

4.4. Proposition. *(compare* [20, lemme 1]*) Suppose k is algebraically closed. Then the natural action of $G(k)$ on the cohomology groups $H^i_{\mathrm{Zar}}(BG, M_j)$ via inner automorphisms is trivial.*

Proof. (compare *loc. cit.*) The group $G(k)$ is generated by the k-points of unipotent subgroups of G, together with the k-points of the center: if G is simple the subgroup of G generated by all unipotent subgroups is normal and not contained in the center, hence equal to G. The simple case implies the semi-simple case, and in general G is generated by its derived subgroup and its center. Since k-points of the center obviously act trivially, the conclusion follows from Lemma 4.3. □

4.5. Lemma. *Let $\mathcal{C}$ be a category with finite products and $T : \mathcal{C}^o \to \{$abelian groups$\}$ an additive functor. Let G be a group object of $\mathcal{C}$. Then the cohomotopy of the cosimplicial abelian group $T(BG)$ is $T(G)$ in degree 1 and 0 elsewhere.*

This is clear, since $T(BG)_n = T(G)^n$ and therefore $T(BG)$ is "dual" to $BT(G)$.

4.6. Theorem. *Let S be a split torus over k, with character group L_S. Then, for all $i, j \geq 0$, we have a canonical isomorphism*

$$\mathbf{S}^i(L_S) \otimes M_{j-i}(k) \xrightarrow{\sim} H^i_{\mathrm{Zar}}(BS, \mathcal{M}_j).$$

Proof. By 3.9, 3.17 and (4.1), $H^i_{\mathrm{Zar}}(BS, \mathcal{M}_j)$ is the i-th homotopy group of the simplicial abelian group

$$\cdots \vdots A^0(S^{n-1}, M_j) \vdots A^0(S^n, M_j) \vdots A^0(S^{n+1}, M_j) \vdots \cdots$$

and this simplicial abelian group has a filtration whose typical quotient is

$$(\cdots \vdots \Lambda^i(L_S^{n-1}) \vdots \Lambda^i(L_S^n) \vdots \Lambda^i(L_S^{n+1}) \vdots \cdots) \otimes M_{j-i}(k). \tag{4.2}$$

Consider the cosimplicial abelian group "BL_S". By Lemma 4.5, its homotopy is L_S in degree 1 and 0 in all other degrees. By [23, prop. 4.3.2.1], the homotopy of

$$\cdots \vdots \Lambda^i(L_S^{n-1}) \vdots \Lambda^i(L_S^n) \vdots \Lambda^i(L_S^{n+1}) \vdots \cdots$$

is therefore $\mathbf{S}^i(L_S)$ in degree i and 0 elsewhere; since this group is torsion-free, the homotopy of (4.2) is $\mathbf{S}^i(L_S) \otimes M_{j-i}(k)$ in degree i and 0 elsewhere. Since all quotients (4.2) have their homotopy concentrated in one degree and these degrees are all distinct, this yields Theorem 4.6. □

4.7. Theorem. *Under assumption* 3.19*,*

(i) *For all j, we have isomorphisms*

$$M_j(k) \xrightarrow{\sim} H^0(BG, \mathcal{M}_j)$$
$$H^1(BS, \mathcal{M}_j) \xrightarrow{\sim} H^1(BG, \mathcal{M}_j)$$

and an exact sequence

$$0 \to H^2(BS, \mathcal{M}_j) \to H^2(BG, \mathcal{M}_j) \to E_2^{1,1}(G, \mathcal{M}_j)$$
$$\to H^3(BS, \mathcal{M}_j) \to H^3(BG, \mathcal{M}_j)$$

where $E_2^{1,1}(G, \mathcal{M}_j)$ is a subgroup of $H^1(H, \mathcal{M}_j)$.

(ii) *We have*

$$H^0(BG, \mathcal{M}_1) \simeq M_1(k)$$
$$H^1(BG, \mathcal{M}_1) \simeq L_S \otimes M_0(k)$$
$$H^n(BG, \mathcal{M}_1) = 0 \text{ for } n \geq 2.$$

(iii) *We have*

$$H^0(BG, \mathcal{M}_2) \simeq M_2(k)$$
$$H^1(BG, \mathcal{M}_2) \simeq L_S \otimes M_1(k)$$
$$H^n(BG, \mathcal{M}_2) = 0 \text{ for } n \geq 3.$$

(iv) *The spectral sequence* (4.1) *yields an exact sequence*

$$0 \to E_2^{20}(G, \mathcal{M}_2) \to H^2_{\mathrm{Zar}}(BG, \mathcal{M}_2) \to E_2^{11}(G, \mathcal{M}_2) \to 0$$

which coincides canonically with the (exact) sequence

$$0 \to H^2_{\mathrm{Zar}}(BS, \mathcal{M}_2) \to H^2_{\mathrm{Zar}}(BG, \mathcal{M}_2) \to H^2_{\mathrm{Zar}}(BH, \mathcal{M}_2) \to 0.$$

Moreover, $E_2^{11}(G, \mathcal{M}_2) \xrightarrow{\sim} H^1(H, \mathcal{M}_2)$.

Proof. Note that $E_1^{0,q}(G, \mathcal{M}_j) = 0$ for $q > 0$ in the spectral sequence (4.1); (i) follows from this and Proposition 3.20 (i). On the other hand, $E_1^{p,q}(G, \mathcal{M}_j) = 0$ for $q > j$ (Gersten's conjecture). If $j = 1$, we have moreover $E_1^{p,1} = 0$ for all p by Proposition 3.20: this and Theorem 4.6 give (ii). Assume now $j = 2$ and let us simply write $E_r^{p,q}(G)$ for $E_r^{p,q}(G, \mathcal{M}_2)$. This time, Proposition 3.20 (iii) implies that $E_2^{p,2}(G) = 0$ for all $p \geq 0$. Note also that Corollary 3.21 and Lemma 4.5 give $E_2^{p,1}(G) = 0$ for $p > 1$ and $E_2^{1,1}(G) = H^1(G, \mathcal{M}_2)$. Moreover, by Proposition 3.20 (i) and Theorem 4.6, we have

$$\mathbf{S}^p(L_S) \otimes M_{2-p}(k) \xrightarrow{\sim} E_2^{p,0}(S) \xrightarrow{\sim} E_2^{p,0}(G).$$

Finally, the only nonzero E_2-terms are $E_2^{1,1}(G)$ and $E_2^{p,0}(G)$ $(0 \leq p \leq 2)$; in particular, $E_2 = E_\infty$. Theorem 4.7 follows easily from all these facts and Corollary 3.21. □

Let N_T be the normalizer of $T = T_G$ in G, which we let act on G by conjugation. The Weyl group $W(G)$ is by definition the quotient N_T/T. The actions of N_T on G and T extend to actions on EG and ET, giving an action of N_T on BG and BT.

Let k_s be a separable closure of k. The restriction map

$$H^i_{\mathrm{Zar}}(BG/k_s, K_i^M) \to H^i_{\mathrm{Zar}}(BT/k_s, K_i^M) \ (i \leq 2) \tag{4.3}$$

is $N_G(T)(k_s)$-equivariant; by Proposition 4.4, the action of the latter group on $H^i_{\mathrm{Zar}}(BG/\overline{k}, K_i^M)$ is trivial. On the other hand, since T is commutative, the action of T on ET by conjugation is trivial, hence the $N_T(k_s)$-action on $H^i_{\mathrm{Zar}}(BT/\overline{k}, K_i^M)$ descends to an action of $W(G)$. It follows that the image of (4.3) is contained in the Weyl invariants $H^i_{\mathrm{Zar}}(BT/\overline{k}, K_i^M)^{W(G)}$. By Theorem 4.7, $H^i_{\mathrm{Zar}}(BG/k, K_i^M) \to H^i_{\mathrm{Zar}}(BG/\overline{k}, K_i^M)$ and $H^i_{\mathrm{Zar}}(BT/k, K_i^M) \to H^i_{\mathrm{Zar}}(BT/\overline{k}, K_i^M)$ are isomorphisms, hence the image of $H^i_{\mathrm{Zar}}(BG/k, K_i^M) \to H^i_{\mathrm{Zar}}(BT/k, K_i^M)$ is also contained in the Weyl invariants. We are now all set to prove:

4.8. Theorem. *Under assumption* 3.19*, restriction to the maximal torus T of G yields a chain of isomorphisms*

$$H^i_{\mathrm{Zar}}(BG, \mathcal{M}_j) \xleftarrow{\sim} H^i_{\mathrm{Zar}}(BG, \mathcal{K}_i^M) \otimes M_{j-i}(k) \xrightarrow{\sim} H^i_{\mathrm{Zar}}(BT, \mathcal{K}_i^M)^{W(G)} \otimes M_{j-i}(k) \xleftarrow{\sim} \mathbf{S}^i(L_G)^{W(G)} \otimes M_{j-i}(k)$$

for $0 \leq i \leq j \leq 2$, where $W(G)$ is the Weyl group of G.

Proof. The left isomorphism follows from Theorem 4.7 and the right one from Theorem 4.6. It remains to prove that the middle map is an isomorphism. It suffices to do this for $M_* = K_*^M$ and $j = i$. We proceed in three steps:

Step 1. *G is simple.* The cases $i = 0, 1$ are trivial. We compute the $\mathcal{K}$-cohomology of BT via the spectral sequence associated to its simplicial model EG/T (see example A.6). The E_1-term of this spectral sequence is

$$E_1^{p,q} = H^q_{\mathrm{Zar}}(G/T \times G^p, \mathcal{K}_2).$$

We have $E_1^{p,q} = 0$ for $q > 2$. Since $X = G/T$ is an affine bundle over a cellular variety, Propositions 3.7 a) and 3.20 give isomorphisms:

$$\begin{aligned} E_1^{p,0} &= K_2(k) \\ E_1^{p,1} &= \mathrm{Pic}(X) \otimes K_1(k) \oplus H^1_{\mathrm{Zar}}(G^p, \mathcal{K}_2) \\ E_1^{p,2} &= CH^2(X). \end{aligned}$$

It follows that $E_2^{p,q} = 0$, except for

$$\begin{array}{ll} E_2^{0,2} = CH^2(X) & \\ E_2^{0,1} = \mathrm{Pic}(X) \otimes K_1(k) & E_2^{1,1} = H^1_{\mathrm{Zar}}(G, \mathcal{K}_2) \\ E_2^{0,0} = K_2(k) & \end{array}$$

We therefore get a short exact sequence

$$0 \to H^1_{\mathrm{Zar}}(G, \mathcal{K}_2) \to H^2_{\mathrm{Zar}}(BT, \mathcal{K}_2) \to CH^2(X) \to 0 \tag{4.4}$$

and comparing with the spectral sequence for BG, it is clear that the isomorphism of Theorem 4.7 identifies the first map with the restriction map

$$H^2_{\mathrm{Zar}}(BG, \mathcal{K}_2) \to H^2_{\mathrm{Zar}}(BT, \mathcal{K}_2).$$

Equation (4.4) shows that $\mathrm{Coker}(H^2_{\mathrm{Zar}}(BG, \mathcal{K}_2) \to H^2_{\mathrm{Zar}}(BT, \mathcal{K}_2))$ is torsion-free. Since G is simple, it is well-known that the Weyl invariants $H^2_{\mathrm{Zar}}(BT, \mathcal{K}_2)^W \simeq \mathbf{S}^2(L_G)^W$ (Theorem 4.6) have rank 1. It follows that the injection $H^2_{\mathrm{Zar}}(BG, \mathcal{K}_2) \to H^2_{\mathrm{Zar}}(BT, \mathcal{K}_2)^W$ must be *surjective*.

Step 2. *G is semi-simple.* Follows from step 1 by additivity.

Step 3. *The general case.* We need a lemma:

4.9. Lemma. *Let W be a finite group acting on a finitely generated free $\mathbb{Z}$-module A. Let $B \subseteq A$ be a subgroup such that W acts trivially on B and $C := A/B$ is free. Then:*

(i) *The sequence*

$$0 \to B \to A^W \to C^W \to 0$$

is exact.

(ii) *If $C^W = 0$, the sequence*

$$0 \to \mathbf{S}^2(B) \to \mathbf{S}^2(A)^W \to \mathbf{S}^2(C)^W$$

is exact.

Proof. The first claim follows from the cohomology exact sequence and the equality $H^1(W, B) = \mathrm{Hom}(W, B) = 0$. To see the second one, consider the complex of W-modules

$$0 \to \mathbf{S}^2(B) \to \mathbf{S}^2(A) \to \mathbf{S}^2(C) \to 0. \tag{K}$$

This complex is acyclic, except at $\mathbf{S}^2(A)$ where its cohomology is $B \otimes C$. We have two hypercohomology spectral sequences

$$I_1^{p,q} = H^q(W, K^p) \Rightarrow \mathbb{H}^{p+q}(W, K) \Leftarrow II_2^{p,q} = H^p(W, H^q(K)).$$

The spectral sequence II degenerates, yielding a spectral sequence

$$I_1^{p,q} = H^q(W, K^p) \Rightarrow H^{p+q-1}(W, B \otimes C).$$

Since $C^W = 0$ and B is free, $(B \otimes C)^W = 0$ too. So we get $I_2^{0,0} = I_2^{1,0} = 0$ and the claim follows. □

End of proof of Theorem 4.8. We check it case by case with the help of Theorems 4.6 and 4.7. The case $i = 0$ is trivial. Note that $W(G) = W(H)$ [22, p. 181, Lemma 29.5] and $(L_H)^{W(H)} = 0$ since H is semi-simple. This yields immediately the case $i = 1$. As for $i = 2$, it follows from the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{S}^2(L_S) \otimes M_0(k) & \longrightarrow & H^2_{\text{Zar}}(BG, \mathcal{M}_2) & \longrightarrow & \mathbf{S}^2(L_H)^W \otimes M_0(k) & \longrightarrow & 0 \\
 & & || & & \downarrow & & || & & \\
0 & \longrightarrow & \mathbf{S}^2(L_S) \otimes M_0(k) & \longrightarrow & \mathbf{S}^2(L_G)^W \otimes M_0(k) & \longrightarrow & \mathbf{S}^2(L_H)^W \otimes M_0(k) & &
\end{array}$$

where the top row is exact by Theorem 4.7 and the bottom row is exact by Lemma 4.9 (note that $\mathbf{S}^2(L_S)$, $\mathbf{S}^2(L_G)^W$ and $\mathbf{S}^2(L_H)^W$ are torsion-free). □

4.10. Remark. One could check that the exact sequence (4.4) coincides with the one from Proposition 3.20 (ii), modulo the identification of $H^2_{\text{Zar}}(BT, \mathcal{M}_2)$ with $\mathbf{S}^2(L_G) \otimes M_0(k)$ (Theorem 4.6). Also, we incidentally recover a special case of a theorem of Demazure [12, cor. 2 to prop. 3] and [13], completed by [56]).

We now compare K-cohomology with analytic cohomology. This will be used in appendices B and C.

4.11. Theorem. *Under assumption* 3.19*, there are isomorphisms*

$$H^1(BG, \mathcal{K}_1) \xrightarrow{\sim} H^2_{\text{an}}(BG(\mathbb{C}), \mathbb{Z})$$
$$H^2(BG, \mathcal{K}_2) \xrightarrow{\sim} H^4_{\text{an}}(BG(\mathbb{C}), \mathbb{Z})$$

which are natural with respect to algebraic group homomorphisms.

The proof will be in four steps.

Step 1. *For $i = 1, 2$, $H^i(BG, \mathcal{K}_i)$ is invariant under base change.*

This is clear from Theorem 4.7 or Theorem 4.8.

This shows that we may assume $k = \mathbb{C}$ in Theorem 4.11 (passing through $\mathbb{Z}$ via Corollary 3.22 if k has nonzero characteristic).

Step 2. *For $i = 1, 2$, there is a natural map of Zariski sheaves $\mathcal{K}_i \to \mathcal{H}^i_{\mathrm{an}}(\mathbb{Z})$, and this map induces isomorphisms $H^i(BG, \mathcal{K}_i) \xrightarrow{\sim} H^i(BG, \mathcal{H}^i_{\mathrm{an}}(\mathbb{Z}))$.*

For $i = 1$, the map is given by the composite map of presheaves

$$\Gamma(U, \mathcal{O}^*_U) \to H^0_{\mathrm{an}}(U, \mathbb{G}_m) \to H^1_{\mathrm{an}}(U, \mathbb{Z})$$

where the second map comes from the exponential exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathbb{G}_a \xrightarrow{\exp} \mathbb{G}_m \to 1.$$

This induces a composite

$$\mathcal{K}_1 \otimes \mathcal{K}_1 \to \mathcal{H}^1_{\mathrm{an}}(\mathbb{Z}) \otimes \mathcal{H}^1_{\mathrm{an}}(\mathbb{Z}) \to \mathcal{H}^2_{\mathrm{an}}(\mathbb{Z})$$

in which the last map is cup-product. Since $H^2_{\mathrm{an}}(\mathbb{A}^1_{\mathbb{C}} - \{0, 1\}, \mathbb{Z}) = 0$, this composite factors through $\mathcal{K}_2$. Now that we have comparison maps, the claim follows once again from Theorem 4.7.

Step 3. *For $i = 1, 2$, there is a natural map $H^i(BG, \mathcal{H}^i_{\mathrm{an}}(\mathbb{Z})) \to H^{2i}_{\mathrm{an}}(BG, \mathbb{Z})$.*

Indeed, for $p + q = 2$ or 4, we have $H^p(BG, \mathcal{H}^q_{\mathrm{an}}(\mathbb{Z})) = 0$ for $p > q$ by Theorem 4.7. The Bloch-Ogus spectral sequence then yields the desired homomorphism.

Step 4. *The map of step* 3 *is an isomorphism.* Indeed, by Theorem 4.7 (iv), we have $H^p(BG, \mathcal{H}^{4-p}_{\mathrm{an}}(\mathbb{Z})) = 0$ for $p = 0, 1$. □

4.12. Remark. Theorem 4.7 shows that, for any cycle module M,

$$H^i(BG, \mathcal{K}_i) \otimes M_{j-i}(k) \to H^i(BG, \mathcal{M}_j)$$

is an isomorphism for $0 \le j \le 2$. Together with Theorem 4.11, this yields canonical isomorphisms

$$H^i(BG, \mathcal{M}_j) \simeq H^{2i}_{\mathrm{an}}(BG(\mathbb{C}), \mathbb{Z}) \otimes M_{j-i}(k)$$

for $0 \le j \le 2$.

5. $GL(N)$ and $\mathrm{Cliff}(n, n)$

In this section, we use Theorem 4.8 to compute explicitly the lower $\mathcal{K}$-cohomology of BG, where $G = GL(N), SL(N), \mathrm{Cliff}(n, n)$ and $\mathrm{Spin}(n, n)$.

5.1. SL(N) and GL(N). We take as maximal split torus for $GL(N)$ the group T of diagonal matrices and for $SL(N)$ diagonal matrices T_0 with determinant 1. If $x_1, \ldots, x_N$ is the corresponding basis of characters of $L_{GL(N)}$, then $L_{SL(N)} = L_{GL(N)}/ < \sum x_i >$. The Weyl group $W = \mathfrak{S}_N$ acts by permutation of the x_i; it follows that $\mathbf{S}(L_{GL(N)})^W$ is the free polynomial algebra on the elementary symmetric functions c_r of the x_i and that $\mathbf{S}(L_{SL(N)})^W$ is its quotient by the ideal generated by $c_1 = \sum x_i$. In particular:

$$L^W_{GL(N)} = \mathbb{Z}c_1 \qquad L^W_{SL(N)} = 0$$
$$\mathbf{S}^2(L_{GL(N)})^W = \mathbb{Z}c_1^2 \oplus \mathbb{Z}c_2 \quad \mathbf{S}^2(L_{SL(N)})^W = \mathbb{Z}c_2.$$

The restriction map induces homomorphisms $H^i(BGL(N),\mathcal{K}_j) \to H^i(BT,\mathcal{K}_j)^W$ and $H^i(BSL(N),\mathcal{K}_j) \to H^i(BT_0,\mathcal{K}_j)^W$, which are seen to be isomorphisms by Theorem 4.8. By Theorem 4.6 and the above computation, we get:

5.2. Theorem. *For $N \geq 1$, restriction to the maximal torus yields isomorphisms:*

$$\begin{array}{ll} H^1(BGL(N),\mathcal{K}_1) = \mathbb{Z}c_1 & H^1(BSL(N),\mathcal{K}_1) = 0 \\ H^2(BGL(N),\mathcal{K}_2) = \mathbb{Z}c_1^2 \oplus \mathbb{Z}c_2 & H^2(BSL(N),\mathcal{K}_2) = \mathbb{Z}c_2. \end{array}$$

5.3. Proposition. *The Whitney formula holds for c_1 and c_2: for $M, N \geq 1$ one has*

$$\rho^* c_1 = c_1 \times 1 + 1 \times c_1$$
$$\rho^* c_2 = c_2 \times 1 + c_1 \times c_1 + 1 \times c_2$$

where ρ is the embedding $GL(M) \times GL(N) \hookrightarrow GL(M+N)$. In particular, c_1 and c_2 are stable.

Proof. This can be proven by restriction to the maximal torus (Theorem 4.8) or by reduction to topology (Theorem 4.11). □

5.4. Remark. This shows that the classes c_1, c_2 of Theorem 5.2 coincide with the Chern classes defined by Gillet in [18]. For c_1, reduce by stability to the tautological case of $GL(1)$. For c_2, reduce by Theorem 4.8 and the Whitney formula for the Gillet classes to the case of c_1.

5.5. $\mathbf{Spin(n,n)}$ and $\mathbf{Cliff(n,n)}$. We have the following

5.6. Proposition. *Let $q = n\mathbb{H}$, where $\mathbb{H}$ is the quadratic form xy. Let $(e_1, f_1, \dots, e_n, f_n)$ be the corresponding basis of the space underlying q. Then, in $\mathrm{Cliff}(n\mathbb{H}) = \mathrm{Cliff}(n,n)$, the assignment*

$$(t_0, t_1, \dots, t_n) \mapsto t_0(t_1e_1 + f_1)(e_1 + f_1) \dots (t_ne_n + f_n)(e_n + f_n)$$

defines an isomorphism $\mathbb{G}_m^{n+1} \xrightarrow[\sim]{\tau} T$ of $\mathbb{G}_m^{n+1}$ onto a split maximal torus T. We have:

(i) *$\gamma_1 \circ \tau(t_0, \dots, t_n) = t_0^2 t_1 \dots t_n$, where γ_1 is the spinor norm of section 2 (see diagram (2.2)).*
(ii) *$\psi \circ \tau(t_0, \dots, t_n) = \mathrm{diag}(t_1, t_1^{-1}, \dots, t_n, t_n^{-1})$, where $\psi : \mathrm{Cliff}(n,n) \to SL(2n)$ is the natural map.*
(iii) *For $n \geq 2$, the Weyl group $W(\mathrm{Cliff}(n,n))$ is isomorphic to the subgroup of the wreath product $\mathfrak{S}_n \wr \mu_2 = \mathfrak{S}_n \ltimes \{\pm 1\}^n$ consisting of elements $(\sigma, \varepsilon_1, \dots, \varepsilon_n)$ such that $\varepsilon_1 \dots \varepsilon_n = 1$. For $n = 1$, it is trivial.*
(iv) *Suppose $n \geq 2$. Via the isomorphism τ, $W(\mathrm{Cliff}(n,n))$ acts on $\mathbb{G}_m^{n+1}$ as follows:*

$$\sigma(t_0, t_1, \dots, t_n) = (t_0, t_{\sigma(1)}, \dots, t_{\sigma(n)})$$
$$\varepsilon(t_0, t_1, \dots, t_n) = (t_0 \prod_{\varepsilon_i = -1} t_i, t_1^{\varepsilon_1}, \dots, t_n^{\varepsilon_n})$$

where $\sigma \in \mathfrak{S}_n$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ (with $\prod \varepsilon_i = 1$).

Proof. Let us record the identities in the Clifford algebra of $n\mathbb{H}$:

$$e_i^2 = f_i^2 = 0$$
$$e_ie_j = -e_je_i; f_if_j = -f_jf_i$$
$$e_if_j = -f_je_i \, (i \neq j)$$
$$e_if_i + f_ie_i = 1$$
$$e_if_ie_i = e_i;\ f_ie_if_i = f_i$$

We first show that τ is a homomorphism. Since the e_i and f_i with different indices anticommute, the $(t_ie_i + f_i)(e_i + f_i)$ mutually commute and we may assume $n = 1$ and obviously $t_0 = 1$. Let us drop the indices $_1$ for simplicity. We have

$$(te + f)(e + f) = tef + fe$$
$$(sef + fe)(tef + fe) = st(ef)^2 + (fe)^2 = stef + fe.$$

Similarly, to compute $\gamma_1 \circ \tau$ and $\psi \circ \tau$, we may assume $n = 1$ and $t_0 = 1$. We have:

$$\gamma_1((te + f)(e + f)) = (te + f)(e + f)(e + f)(te + f) = (te + f)^2 = t.$$

On the other hand,

$$(tef + fe)e(tef + fe)^{-1} = (tef + fe)e(t^{-1}ef + fe) = tefefe = te;$$
$$(tef + fe)f(tef + fe)^{-1} = (tef + fe)f(t^{-1}ef + fe) = t^{-1}fefef = t^{-1}f;$$

and, for v orthogonal to $< e, f >$:

$$(tef + fe)v(tef + fe)^{-1} = (tef + fe)v(t^{-1}ef + fe) = v(tef + fe)(t^{-1}ef + fe) = v.$$

The composition $\mathbb{G}_m^{n+1} \xrightarrow{\tau} T \xrightarrow{(\psi,\gamma_1)} T_{SL(2n)} \times \mathbb{G}_m$ obviously has kernel $\mu_2 \times (1, \dots, 1)$, and scalar multiplication acts faithfully on $C(n\mathbb{H})$, so that τ is injective. Since the dimension of a maximal torus of $\mathrm{Spin}(n, n)$ or $SO(n, n)$ is n, τ is also surjective.

For $n \geq 2$, the Weyl group of $\mathrm{Cliff}(n, n)$ is the same as that of its derived subgroup $\mathrm{Spin}(n, n)$. This Weyl group is classically known [5, ch. VI, planche IV, (X)]. For $n = 1$, $\mathrm{Cliff}(n, n) \simeq \mathbb{G}_m \times \mathbb{G}_m$, so $W = 1$. Finally, let us prove the last claim. It suffices to observe that σ is represented by an element of $\mathrm{Cliff}(n, n)$ that maps (e_i, f_i) to $\pm(e_{\sigma(i)}, f_{\sigma(i)})$ by conjugation (for $\sigma = (1, 2)$, we may choose for such an element $(e_1 + e_2)(f_1 + f_2) - 1$) and that ε is represented by an element of $\mathrm{Cliff}(n, n)$ that exchanges e_i and f_i exactly for those i such that $\varepsilon_i = -1$ (we may choose for such an element $\prod_{\varepsilon_i=-1}(e_i + f_i)$). □

We translate the action of $W(\mathrm{Cliff}(n, n))$ on the group of characters $L_{\mathrm{Cliff}(n,n)}$, provided with the basis $x_0, \dots, x_n$ given by τ. We get:

- For $\sigma \in \mathfrak{S}_n$:
$$\sigma x_0 = x_0, \quad \sigma x_i = x_{\sigma^{-1}(i)} \ (i > 0)$$
- For $\varepsilon \in \{\pm 1\}^n$:
$$\varepsilon x_i = \begin{cases} \varepsilon_i x_i & \text{if } i > 0 \\ x_0 + \sum_{j>0} \frac{1-\varepsilon_j}{2} x_j & \text{if } i = 0. \end{cases}$$

On the other hand, a maximal torus of $\mathrm{Spin}(n,n)$ is given by $\mathrm{Ker}((\gamma_1)_{|T})$, hence $L_{\mathrm{Spin}(n,n)}$ is the quotient of $L_{\mathrm{Cliff}(n,n)}$ by the subgroup generated by

$$\gamma_1 = 2x_0 + \sum_{i>0} x_i. \tag{5.1}$$

Let us also define

$$\gamma_2 = 2x_0^2 + 2x_0 \sum_{i>0} x_i + \sum_{0<i<j} x_i x_j. \tag{5.2}$$

The following proposition follows from elementary computations.

5.7. Proposition. *We have*

(i) $(L_{\mathrm{Cliff}(n,n)})^W = \begin{cases} \mathbb{Z}\gamma_1 & \text{for } n \geq 2 \\ \mathbb{Z}\gamma_1 \oplus \mathbb{Z}x_0 & \text{for } n = 1. \end{cases}$

(ii) $(L_{\mathrm{Spin}(n,n)})^W = \begin{cases} 0 & \text{for } n \geq 2 \\ \mathbb{Z}x_0 & \text{for } n = 1. \end{cases}$

(iii) $\mathbf{S}^2(L_{\mathrm{Cliff}(n,n)})^W = \begin{cases} \mathbb{Z}\gamma_1^2 \oplus \mathbb{Z}\gamma_2 & \text{for } n \geq 3 \\ \mathbb{Z}\gamma_1^2 \oplus \mathbb{Z}\gamma_2 \oplus \mathbb{Z}x_0(\gamma_1 - x_0) & \text{for } n = 1, 2. \end{cases}$

(iv) $\mathbf{S}^2(L_{\mathrm{Spin}(n,n)})^W = \begin{cases} \mathbb{Z}\gamma_2 & \text{for } n \geq 3 \\ \mathbb{Z}\gamma_2 \oplus \mathbb{Z}x_0^2 & \text{for } n = 2 \\ \mathbb{Z}x_0^2 & \text{for } n = 1. \end{cases}$

(v) $\psi^* c_2 = 2\gamma_2 - \gamma_1^2 \in \mathbf{S}^2(L_{\mathrm{Cliff}(n,n)})$. □

5.8. Theorem. *For $n \geq 3$, restriction to the maximal torus yields isomorphisms:*

$$H^1(B\,\mathrm{Cliff}(n,n), \mathcal{K}_1) = \mathbb{Z}\gamma_1 \qquad H^1(B\,\mathrm{Spin}(n,n), \mathcal{K}_1) = 0$$
$$H^2(B\,\mathrm{Cliff}(n,n), \mathcal{K}_2) = \mathbb{Z}\gamma_1^2 \oplus \mathbb{Z}\gamma_2 \quad H^2(B\,\mathrm{Spin}(n,n), \mathcal{K}_2) = \mathbb{Z}\gamma_2.$$

We have the identity, valid for all $n \geq 1$:

$$\psi^* c_2 = 2\gamma_2 - \gamma_1^2 \in H^2_{\mathrm{Zar}}(B\,\mathrm{Cliff}(n,n), \mathcal{K}_2) \tag{5.3}$$

where $\psi : \mathrm{Cliff}(n,n) \to SL(2n)$ is the natural map. □

5.9. Proposition. *The Whitney formula holds for γ_1 and γ_2: for $m, n \geq 1$ one has*

$$\rho^*\gamma_1 = \gamma_1 \times 1 + 1 \times \gamma_1$$
$$\rho^*\gamma_2 = \gamma_2 \times 1 + \gamma_1 \times \gamma_1 + 1 \times \gamma_2$$

where ρ is the embedding $\mathrm{Cliff}(m,m) \times \mathrm{Cliff}(n,n) \hookrightarrow \mathrm{Cliff}(m+n, m+n)$. In particular, γ_1 and γ_2 are stable.

Proof. This is clear for γ_1. For γ_2, the easiest is to use formula (5.3) of Theorem 5.8. Note that c_2 is additive since it comes from $SL(2n)$. So (5.3) gives the Whitney formula for γ_2 multiplied by 2, and we can then divide by 2 since

$$H^2_{\mathrm{Zar}}(B(\mathrm{Cliff}(m,m) \times \mathrm{Cliff}(n,n)), \mathcal{K}_2)$$

is torsion-free. □

6. Two invariants for Clifford bundles

In this section we associate to a torsor F under $\mathrm{Cliff}(n,n)$ on a scheme X two invariants with values in étale motivic cohomology of X, which are related to the second Chern class of the vector bundle underlying F. When $X = \operatorname{Spec} K$, K a field, we relate these to the Arason invariant.

6.1. Recall Lichtenbaum's complexes $\Gamma(i)(i \leq 2)$ ([35], [36], [37]). One has $\Gamma(0) = \mathbb{Z}$ placed in degree 0, $\Gamma(1) = \mathbb{G}_m$ placed in degree 1 and $\Gamma(2)$ is constructed in [36]. There are products $\Gamma(i) \overset{L}{\otimes} \Gamma(j) \to \Gamma(i+j)$ for $i + j \leq 2$. If X is a smooth variety defined over a field k, one has ([37], [28, th. 1.1]):

$$\mathbb{H}^i_{\text{ét}}(X, \Gamma(2)) = \begin{cases} 0 & i \leq 0 \\ K_3(k(X))_{\text{ind}} & i = 1 \\ H^0_{\text{Zar}}(X, \mathcal{K}_2) & i = 2 \\ H^1_{\text{Zar}}(X, \mathcal{K}_2) & i = 3, \end{cases} \tag{6.1}$$

and an exact sequence

$$0 \to CH^2(X) \to \mathbb{H}^4_{\text{ét}}(X, \Gamma(2)) \to H^0_{\text{Zar}}(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0 \tag{6.2}$$

when $\mathcal{H}^j(\mathcal{F})$ is the Zariski sheaf associated to the presheaf $U \mapsto H^j_{\text{ét}}(U, \mathcal{F})$.

This computation is done via the Leray spectral sequence for the map $\alpha : X_{\text{ét}} \to X_{\text{Zar}}$, together with the following computation of the Zariski sheaves

$$R^q\alpha_*\Gamma(2) = \begin{cases} 0 & \text{for } q \leq 0 \\ \text{the constant sheaf } K_3(k(X))_{\text{ind}} & \text{for } q = 1 \\ \mathcal{K}_2 & \text{for } q = 2 \\ 0 & \text{for } q = 3 \\ \mathcal{H}^{q-1}(\mathbb{Q}/\mathbb{Z}(2)) & \text{for } q \geq 4. \end{cases} \tag{6.3}$$

Here the étale sheaf $\mathbb{Q}/\mathbb{Z}(2)$ is defined as $\varinjlim \mu_n^{\otimes 2}$ if $\operatorname{char} k = 0$ and

$$\varinjlim_{(n, \operatorname{char} F)=1} \mu_n^{\otimes 2} \oplus \varinjlim_r W_r\Omega^2_{\log}[-2],$$

where $W_r\Omega^2_{\log}$ is the sheaf of logarithmic de Rham-Witt differentials over the big étale site of $\operatorname{Spec} k$ and the transition maps are given by the Verlagerung (compare [28]).

On the other hand, one has ([37], [28, th. 1.2])

$$\mathbb{H}^i_{\text{Zar}}(X, \Gamma(2)) = \begin{cases} K_3(k(X))_{\text{ind}} & i = 1 \\ H^{i-2}_{\text{Zar}}(X, \mathcal{K}_2) & 2 \leq i \leq 4 \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

6.2. Lemma. *Let X/k be a smooth, geometrically connected rational variety. (This means that $k_s(X)/k_s$ is a purely transcendental extension, where k_s is a separable closure of k.) Then the map $K_3(k)_{\text{ind}} \to K_3(k(X))_{\text{ind}}$ is an isomorphism.*

Proof. If $k(X)/k$ is purely transcendental, this follows from [33, pp. 327–328] or [44, lemma 4.2]. In general this follows from the commutative diagram

$$\begin{array}{ccc}
(K_3(k_s)_{\mathrm{ind}})^{G_k} & \xrightarrow{\sim} & (K_3(k_s(X))_{\mathrm{ind}})^{G_k}\\
\wr\uparrow & & \wr\uparrow\\
K_3(k)_{\mathrm{ind}} & \longrightarrow & K_3(k(X))_{\mathrm{ind}}
\end{array}$$

in which the vertical isomorphisms follow from [33, th. 4.13] or [44, prop. 11.4]. □

6.3. Remark. One could weaken the assumption "rational" into "unirational". It is in fact conjectured that the result holds for *any* geometrically connected X (unirational or not) [44, Conj. 11.7].

Hypercohomology with coefficients in $\Gamma(2)$ extends to simplicial schemes. We have:

6.4. Lemma. *a) Let $X_\bullet$ be a simplicial k-scheme, with all X_n smooth, rational and geometrically connected over k. Then $\mathbb{H}^1_{\text{ét}}(X_\bullet,\Gamma(2)) \simeq K_3(k)_{\mathrm{ind}}$ and the other formulæin* (6.1) *and* (6.2) *hold for étale and Zariski cohomology of $X_\bullet$ (replacing $CH^2(X)$ by $\mathbb{H}^2_{\mathrm{Zar}}(X_\bullet,\mathcal{K}_2)$ in* (6.2)*).*
b) Assume further that $X_0 = \operatorname{Spec} k$. Then the exact sequence (6.2) *degenerates into a canonical isomorphism*

$$\mathbb{H}^4_{\text{ét}}(X_\bullet,\Gamma(2)) \simeq H^3_{\text{ét}}(k,\mathbb{Q}/\mathbb{Z}(2)) \oplus \mathbb{H}^2_{\mathrm{Zar}}(X_\bullet,\mathcal{K}_2).$$

This applies in particular to $X_\bullet = BG/k$, where G is a connected linear algebraic group over k.

Proof. a) To compare $\mathbb{H}^*_{\text{ét}}(X_\bullet,\Gamma(2))$ with $H^*_{\mathrm{Zar}}(X_\bullet,\mathcal{K}_2)$, we use the "Leray" spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X_\bullet, R^q\alpha_*\Gamma(2)) \Rightarrow \mathbb{H}^{p+q}_{\text{ét}}(X_\bullet,\Gamma(2))$$

where α is the natural map from the big étale site of $\operatorname{Spec} k$ to its big Zariski site. The simplicial Zariski sheaves $R^q\alpha_*\Gamma(2)$ are given by (6.3); moreover the assumption on $X_\bullet$ and lemma 6.2 imply that $R^1\alpha_*\Gamma(2)$ is the constant simplicial sheaf with value $K_3(k)_{\mathrm{ind}}$. Therefore,

$$E_2^{p,1} = H^p_{\mathrm{Zar}}(X_\bullet, K_3(k)_{\mathrm{ind}}) = \begin{cases} K_3(k)_{\mathrm{ind}} & p=0\\ 0 & p>0,\end{cases}$$

and the computations of [37], [28] apply *mutatis mutandis*. For b), it suffices to see that the map $H^3(k,\mathbb{Q}/\mathbb{Z}(2)) \to H^0_{\mathrm{Zar}}(X_\bullet,\mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$ is surjective. This is obvious from the spectral sequence (4.1). The last claim follows from the fact that G, hence all G^p, are geometrically connected rational varieties over k. □

6.5. Let X be a scheme. We shall give parallel definitions of Chern classes $c_1(V)$, $c_2(V)$ for a vector bundle V on X (with values in Zariski motivic cohomology) and classes $\gamma_1(F)$, $\gamma_2(F)$ for a Clifford bundle F on X (with values in étale motivic cohomology). The classifying simplicial schemes $BGL(N)$ and $B\operatorname{Cliff}(n,n)$ used below will be considered over $\mathbb{Z}$.

6.6. Vector bundles. Let V be a vector bundle of rank N on X. Then V is locally trivial for the Zariski topology. Let (U_i) be a Zariski cover of X trivializing V, and let $U_\bullet$ be the associated simplicial scheme. We have a diagram

$$\begin{array}{ccc} & & BGL(N) \\ & \overset{[V]}{\nearrow} & \\ U_\bullet & & \\ & \searrow & \\ & & X \end{array} \tag{6.5}$$

in which X is considered as a constant simplicial scheme. The top map $[V]$ is induced by transition functions between given trivializations of V on the U_i's. Since the bottom map induces an isomorphism on Zariski cohomology, (6.5) yields homomorphisms

$$\mathbb{H}^{2i}_{\mathrm{Zar}}(BGL(N),\Gamma(i)) \xrightarrow{[V]^*} \mathbb{H}^{2i}_{\mathrm{Zar}}(X,\Gamma(i))$$

which only depend on the isomorphism class of V. We define

$$c_1(V) = [V]^*c_1 \in \mathbb{H}^{2}_{\mathrm{Zar}}(X,\Gamma(1))$$
$$c_2(V) = [V]^*c_2 \in \mathbb{H}^{4}_{\mathrm{Zar}}(X,\Gamma(2)).$$

Note that $\mathbb{H}^{2i}_{\mathrm{Zar}}(X,\Gamma(i)) = H^{i}_{\mathrm{Zar}}(X,\mathcal{K}_i)$ $(i \le 2)$. If X is a smooth variety over a field, then the Bloch-Quillen isomorphism

$$H^{i}_{\mathrm{Zar}}(X,\mathcal{K}_i) \simeq CH^i(X)$$

together with remark 5.4 identifies $c_1(V)$ and $c_2(V)$ with the classical Chern classes with values in the Chow ring of X.

6.7. Clifford bundles. Let F be a torsor on X under $\mathrm{Cliff}(n,n)$ (briefly, a $\mathrm{Cliff}(n,n)$-bundle). Then F is locally trivial for the étale topology. Let (U_i) be an étale cover of X trivializing F, and let $U_\bullet$ be the associated simplicial scheme. We have a diagram

$$\begin{array}{ccc} & & B\,\mathrm{Cliff}(n,n) \\ & \overset{[F]}{\nearrow} & \\ U_\bullet & & \\ & \searrow & \\ & & X \end{array} \tag{6.6}$$

in which X is considered as a constant simplicial scheme. The top map $[F]$ is induced by transition functions between given trivializations of F on the U_i's. Since the bottom map induces an isomorphism on étale cohomology, (6.6) yields homomorphisms

$$\mathbb{H}^{2i}_{\text{ét}}(B\,\mathrm{Cliff}(n,n),\Gamma(i)) \xrightarrow{[F]^*} \mathbb{H}^{2i}_{\text{ét}}(X,\Gamma(i))$$

which only depend on the isomorphism class of F. We define

$$\gamma_1(F) = [F]^*\gamma_1 \in \mathbb{H}^2_{\text{ét}}(X, \Gamma(1))$$
$$\gamma_2(F) = [F]^*\gamma_2 \in \mathbb{H}^4_{\text{ét}}(X, \Gamma(2)).$$

Note that, even though γ_1 and γ_2 are classes in $H^i_{\text{Zar}}(B\,\text{Cliff}(n,n), \mathcal{K}_i)$ $(i = 1, 2)$, $[F]$ is only defined in the étale topology, so $\gamma_i(F)$ is not a priori a Zariski cohomology class. The class $\gamma_1(F)$ certainly is, since $\mathbb{H}^2_{\text{Zar}}(X, \Gamma(1)) \to \mathbb{H}^2_{\text{ét}}(X, \Gamma(1))$ is an isomorphism. But, when X is smooth over a field, the map

$$CH^2(X) \simeq \mathbb{H}^4_{\text{Zar}}(X, \Gamma(2)) \to \mathbb{H}^4_{\text{ét}}(X, \Gamma(2))$$

(cf (6.4)) coincides with the map of (6.2). The main point of this paper is that in general $\gamma_2(F) \notin CH^2(X)$, i.e. is *not algebraic*.

6.8. By pushout, the map

$$\text{Cliff}(n,n) \to SO(n,n)$$

associates to F a $SO(n,n)$-torsor E on X, that we shall call the *underlying quadratic bundle* of F. Similarly, the composite

$$\text{Cliff}(n,n) \to SO(n,n) \to SL(2n)$$

associates to F an $SL(2n)$-torsor V, the *underlying vector bundle* of F.

The vector bundle V has a second Chern class $c_2(V) \in \mathbb{H}^4_{\text{Zar}}(X, \Gamma(2))$, that we may map to $\mathbb{H}^4_{\text{ét}}(X, \Gamma(2))$. If 2 is invertible on X, the quadratic bundle E has a Clifford invariant $c(E) \in H^2_{\text{ét}}(X, \mathbb{Z}/2)$ (Definition 2.3). We define $c_2(F)$ as $c_2(V) \in \mathbb{H}^4_{\text{ét}}(X, \Gamma(2))$ and $c(F)$ as $c(E)$.

6.9. Theorem. *The characteristic classes* $\gamma_1(F) \in \mathbb{H}^2_{\text{ét}}(X, \Gamma(1)), \gamma_2(F) \in \mathbb{H}^4_{\text{ét}}(X, \Gamma(2))$ *have the following properties and relations with* $c_2(F) \in \mathbb{H}^4_{\text{ét}}(X, \Gamma(2))$ *and* $c(F) \in H^2_{\text{ét}}(X, \mathbb{Z}/2)$ $(1/2 \in \mathcal{O}_X)$*:*

(i) *naturality for all morphisms;*

(ii) *additivity:* $\begin{cases} \gamma_1(F \perp F') = \gamma_1(F) + \gamma_1(F') \\ \gamma_2(F \perp F') = \gamma_2(F) + \gamma_1(F) \cdot \gamma_1(F') + \gamma_2(F'), \end{cases}$
where the product corresponds to the pairing $\Gamma(1) \times \Gamma(1) \to \Gamma(2)$*;*

(iii) *relation with* c_2*:* $c_2(F) = 2\gamma_2(F) - \gamma_1(F)^2$.

(iv) *relation with* c $(1/2 \in \mathcal{O}_X)$*:* $\delta_1(\gamma_1(F)) = c(F)$*, where* $\delta_1 : \mathbb{H}^2_{\text{ét}}(X, \Gamma(1)) = H^1_{\text{ét}}(X, \mathbb{G}_m) \to H^2_{\text{ét}}(X, \mathbb{Z}/2)$ *is the boundary map from the Kummer exact sequence.*

Note that $\gamma_1(F)$ has an elementary description as the image of $[F] \in H^1_{\text{ét}}(X, \text{Cliff}(n,n))$ into $H^1_{\text{ét}}(X, \mathbb{G}_m) = \mathbb{H}^2_{\text{ét}}(X, \Gamma(1))$ via the spinor norm γ_1 of (2.2).

Proof. (i) is trivial; (ii) and (iii) follow from Theorem 5.8. It remains to prove (iv). We have to show that the diagram

$$\begin{array}{ccc} H^1(X,\mathrm{Cliff}(n,n)) & \longrightarrow & H^1(X,SO(n,n)) \\ {\scriptstyle (\gamma_1)_*}\downarrow & & \downarrow \\ H^1(X,\mathbb{G}_m) & \xrightarrow{\delta_1} & H^2(X,\mathbb{Z}/2) \end{array}$$

commutes, where for simplicity we drop the indices $_{\text{ét}}$ from the cohomology groups.

We use Čech cohomology, choosing suitable étale covers $\mathcal{U}=(U_i)$ of X. Let $C_{ij}\in \check{Z}^1(\mathcal{U},\mathrm{Cliff}(n,n))$ be a 1-cocycle representing our Clifford bundle: we have $\delta C_{ij}=0$, where δ is the Čech boundary. Let c_{ij} be its image in $\check{Z}^1(\mathcal{U},\mathbb{G}_m)$ and e_{ij} its image in $\check{Z}^1(\mathcal{U},SO(n,n))$. The image of c_{ij} in $\check{H}^2(\mathcal{U},\mathbb{Z}/2)$ is $\delta\sqrt{c_{ij}}$, and $\sqrt{c_{ij}}$ is a lift of c_{ij} in $\check{C}^1(\mathcal{U},\mathbb{G}_m)$ by $2:\mathbb{G}_m\to\mathbb{G}_m$ (up to refining $\mathcal{U}$). The image of e_{ij} into $\check{H}^2(\mathcal{U},\mathbb{Z}/2)$ is $\delta\tilde{e}_{ij}$, where $\tilde{e}_{ij}$ is a lift of e_{ij} in $\check{C}^1(\mathcal{U},\mathrm{Spin}(n,n))$ (up to refining $\mathcal{U}$). But one has

$$C_{ij}-\tilde{e}_{ij}=\sqrt{c_{ij}}+n_{ij}$$

where $n_{ij}\in\mathbb{Z}/2$. Applying δ, this implies $-\operatorname{Im}e=\operatorname{Im}c+$ coboundary. □

Suppose $X=\operatorname{Spec}K$ where K is a field of characteristic $\neq 2$. Then $\mathbb{H}^2(K,\Gamma(1))=H^1(K,\mathbb{G}_m)=0$ (Hilbert 90) and $c_2(F)=0$ since any vector bundle over $\operatorname{Spec}K$ is trivial. So $\gamma_1(F)=0$ and formula (ii) in Theorem 6.9 says that γ_2 is *additive*, while formula (iii) reduces to

$$2\gamma_2(E)=0. \tag{6.7}$$

6.10. Lemma. *Let F be a $\mathrm{Cliff}(n,n)$-bundle over a field K, and let q be its underlying quadratic form. Then $q\in I^3K$.*

Proof. By Merkurjev's theorem [39] it suffices to see that $c(q)=0$: this follows immediately from Lemma 2.1 (or Theorem 6.9 (iv)).

6.11. Theorem. *Let F be a $\mathrm{Cliff}(n,n)$-bundle over K and q the underlying quadratic form. Then*

$$\delta_2(e^3(q))=\gamma_2(F)$$

where $\delta_2: H^3(K,\mathbb{Z}/2)\to\mathbb{H}^4(K,\Gamma(2))$ is the boundary map coming from the Kummer triangle [37], [28, (9)]

$$\mathbb{Z}/2[-1]\to\Gamma(2)\xrightarrow{2}\Gamma(2)\to\mathbb{Z}/2. \tag{6.8}$$

Proof. Write $q+\sum\varphi_i=0$ in $W(K)$, where the φ_i are multiples of 3-fold Pfister forms. So we have an isomorphism $q\perp\perp_i\varphi_i\cong m\mathbb{H}$ for some m. Letting F_i and H denote Cliff-bundles representing the φ_i and $m\mathbb{H}$, we have (with obvious notation)

$$F\perp\perp_i F_i\simeq H$$

in view of remark 2.2, hence $\gamma_2(F\perp\perp_i F_i)=\gamma_2(H)=0$, and by Theorem 6.9 (ii):

$$\gamma_2(F)+\sum\gamma_2(F_i)=0$$

(note that $\gamma_1 \equiv 0$ on $\operatorname{Spec} k$). Since $e^3(q) = \sum e^3(\psi_i)$ too, we are reduced to the case in which q is a 3-fold Pfister form.

Recall that $\mathbb{H}^3(K, \Gamma(2)) = 0$ (see 6.1), so that δ_2 is an isomorphism onto ${}_2\mathbb{H}^4(K, \Gamma(2))$. By diagram (6.7), $\gamma_2(F)$ is 2-torsion: let $\tilde{\gamma}_2(F)$ denote $\delta_2^{-1}(\gamma_2(F)) \in H^3(K, \mathbb{Z}/2)$.

Let $\widetilde{F}$ be a Spin-bundle lifting F, whose existence is assured by diagram (2.2) and Hilbert's Theorem 90. Then, by construction, $\tilde{\gamma}_2(F)$ is nothing else than the Rost invariant associated with $\widetilde{F}$ (see appendix B). Let $K_1 = K(q)$ be the function field of the quadric defined by q and $K_2 = K(\widetilde{F})$ the function field of the torsor $\widetilde{F}$. Since q is a Pfister form, q_{K_1} is hyperbolic [31, cor. X.1.6], hence $\widetilde{F}_{K_1}$ is trivial as the map $H^1(K_1, \operatorname{Spin}(n,n)) \to H^1(K_1, SO(n,n))$ has trivial kernel (this follows from the surjectivity of the spinor norm for isotropic forms). Conversely, $\widetilde{F}_{K_2}$ is trivial, hence q_{K_2} is hyperbolic. It follows that

$$e^3(q), \tilde{\gamma}_2(F) \in \operatorname{Ker}(H^3(K, \mathbb{Z}/2) \to H^3(K_1, \mathbb{Z}/2)) \cap \operatorname{Ker}(H^3(K, \mathbb{Z}/2) \to H^3(K_2, \mathbb{Z}/2)).$$

By Arason's theorem [1, th. 5.6], the first kernel is generated by $e^3(q)$. By Rost's theorem (Theorem B.11), the second kernel is generated by $\tilde{\gamma}_2(F)$. Therefore $\tilde{\gamma}_2(F) = e^3(q)$, as we wanted. □

7. Snaking a Bloch-Ogus differential

Let X be a smooth variety over k and n be prime to $\operatorname{char} k$. Consider the commutative diagram with exact rows and columns

$$(7.1)\quad \begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & & & & & H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2})) \\
 & & & & & & \downarrow \\
0 \to & CH^2(X) & \longrightarrow & \mathbb{H}^4_{\text{ét}}(X, \Gamma(2)) & \longrightarrow & & H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0 \\
 & \downarrow n & & \downarrow n & & & \downarrow n \\
0 \to & CH^2(X) & \longrightarrow & \mathbb{H}^4_{\text{ét}}(X, \Gamma(2)) & \longrightarrow & & H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0 \\
 & \downarrow & & & & & \\
 & CH^2(X)/n & & & & & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
\end{array}$$

The snake lemma defines a map

$$\mathcal{S} : H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2})) \to CH^2(X)/n$$
$$\mathcal{S}(x) = \operatorname{Im} n\widetilde{x} \in CH^2(X)/n$$

where $\widetilde{x}$ is a lift of x in $\mathbb{H}^4_{\text{ét}}(X, \Gamma(2))$.

7.1. Theorem. *Let*

$$d_2 : H^0(X, \mathcal{H}^3(\mu_n^{\otimes 2})) \to H^2(X, \mathcal{H}^2(\mu_n^{\otimes 2})) \simeq CH^2(X)/n$$

be the d_2-differential from the Bloch-Ogus spectral sequence. Then $d_2 = \mathcal{S}$.

Proof. Let I^* be a torsion free acyclic complex quasi-isomorphic to $\Gamma(2)$. Multiplication by n gives an injective map of complexes:

$$\begin{array}{ccccccccc} \alpha_* I^0 & \xrightarrow{d} & \alpha_* I^1 & \longrightarrow & \dots & \longrightarrow & \alpha_* I^i & \longrightarrow & \dots \\ \downarrow n & & \downarrow n & & & & \downarrow n & & \\ \alpha_* I^0 & \xrightarrow{d} & \alpha_* I^1 & \longrightarrow & \dots & \longrightarrow & \alpha_* I^i & \longrightarrow & \dots \end{array} \tag{7.2}$$

with cokernel $\alpha_* I^i / n\alpha_* I^i = \alpha_*(I^i/nI^i)$ quasi-isomorphic to $\mu_n^{\otimes 2}$.

Any n-torsion class $e \in H^0(X, R^4\alpha_*\Gamma(2)) = H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))$ is represented on a suitable Zariski covering $X_\bullet$ of X by $x_i \in \Gamma(X_i, \alpha_* I^4)_{d \text{ closed}}$, with $dx_i = 0$, $nx_i = dy_i$ for some $y_i \in \Gamma(X_i, \alpha_* I^3)$. To obtain $\mathcal{S}(e)$, one first lifts e as a class in $\mathbb{H}^4_{\text{ét}}(X, \Gamma(2))$, with Čech cocycle

$$x = (x_{i_0 \dots i_4}, x_{i_0 \dots i_3}, \dots, x_{i_0}) \in (\mathcal{C}^4(\alpha_* I^0) \times \dots \times \mathcal{C}^0(\alpha_* I^4))_{d-\delta},$$

where δ is the Čech differential. Thus

$$\begin{aligned} nx &= (nx_{i_0 \dots i_4}, \dots, nx_{i_0}) \\ &\equiv (nx_{i_0 \dots i_4}, \dots, nx_{i_0 i_1} - (\delta y)_{i_0, i_1}, 0). \end{aligned}$$

As $H^1_{\text{Zar}}(X, R^3\alpha_*\Gamma(2)) = 0$, there are (after refining $X_\bullet$) elements $z_{i_0 i_1} \in \mathcal{C}^1(\alpha_* I^2)$ verifying: $dz_{i_0 i_1} = nx_{i_0 i_1} - (\delta y)_{i_0 i_1}$. Thus

$$nx \equiv (nx_{i_0 \dots i_4}, nx_{i_0 i_1 i_2 i_3}, nx_{i_0 i_1 i_2} - (\delta z)_{i_0 i_1 i_2}, 0, 0)$$

and $\mathcal{S}(e)$ is the class of

$$nx_{i_0 i_1 i_2} - (\delta z)_{i_0 i_1 i_2} \ \text{ in } \ CH^2(X) = H^2_{\text{Zar}}(X, R^2\alpha_*\Gamma(2)).$$

On the other hand, $d_2(e)$ is obtained as follows:

$$e \ \text{ as a class in } \ H^0_{\text{Zar}}(X, \frac{\operatorname{Ker}\{\alpha_*(I^3/n) \to \alpha_*(I^4/n)\}}{\operatorname{Im} \alpha_*(I^2/n)})$$

is given by $y_i \pmod n$.

One takes $y_{i_0 i_1} \in \Gamma(X_{i_0 i_1}, \alpha_* I^2/n)$ verifying $dy_{i_0 i_1} = (\delta y)_{i_0 i_1} \bmod n$.

Then $d_2(e) = \delta y \in H^2_{\text{Zar}}(X, \mathcal{H}^2(\mu_n^{\otimes 2}))$. But we can take $y_{i_0 i_1} \equiv -z_{i_0 i_1} \pmod n$. Thus

$$\delta y \equiv -\delta z \pmod n \equiv +nx - \delta z \pmod n.$$

This is $\mathcal{S}(e)$. □

8. Proof of Theorem 1

First we remark that $e = e^3(q)$ lies in $H^0(X, \mathcal{H}^3(\mathbb{Z}/2))$: this is obvious from Theorem 6.11 and the fact that $H^3_{\text{ét}}(\mathcal{O}, \mathbb{Z}/2) \xrightarrow{\delta_2} {}_2\mathbb{H}^4_{\text{ét}}(\mathcal{O}, \Gamma(2))$ is bijective for the local rings $\mathcal{O}$ of X (Hilbert 90 for K_2, see (6.3) and (6.8)).

We now consider the signed discriminant and Clifford invariant $d_\pm E$, $c(E)$ of Definition 2.3. The Bloch-Ogus spectral sequence gives an exact sequence

$$0 \to \operatorname{Pic}(X)/2 \to H^2_{\text{ét}}(X, \mathbb{Z}/2) \to H^2(K, \mathbb{Z}/2) \tag{8.1}$$

8.1. Lemma. *We have*

$$d_\pm E = 0$$
$$[E] \in \operatorname{Im}(H^1_{\text{ét}}(X, SO(n,n)) \to H^1_{\text{ét}}(X, O(n,n))^1$$

and

$$c(E) \in \operatorname{Pic}(X)/2.$$

Proof. By assumption we have

$$(d_\pm E)_\eta = 0 = c(E)_\eta$$

where η is the generaic point of X. Since $H^1_{\text{ét}}(X, \mathbb{Z}/2) \to H^1(K, \mathbb{Z}/2)$ is injective, this gives the first two claims, and the third follows from (8.1). □

8.2. Lemma. *The class* $[E] \in H^1_{\text{ét}}(X, O(n,n))$ *is in the image of*

$$H^1_{\text{ét}}(X, \operatorname{Cliff}(n,n)) \to H^1_{\text{ét}}(X, O(n,n)).$$

Proof. By Lemma 8.1, $[E]$ can be lifted to $H^1_{\text{ét}}(X, SO(n,n))$. Diagram (2.2) gives a commutative diagram of pointed sets

$$\begin{array}{ccccc}
H^1_{\text{ét}}(X, \operatorname{Spin}(n,n)) & \longrightarrow & H^1_{\text{ét}}(X, SO(n,n)) & \xrightarrow{\partial_{\mu_2}} & H^2_{\text{ét}}(X, \mu_2) \\
\downarrow & & \| & & \downarrow \theta \\
H^1_{\text{ét}}(X, \operatorname{Cliff}(n,n)) & \longrightarrow & H^1_{\text{ét}}(X, SO(n,n)) & \xrightarrow{\partial_{\mathbb{G}_m}} & H^2_{\text{ét}}(X, \mathbb{G}_m)
\end{array}$$

But by the Kummer exact sequence, $\operatorname{Pic}(X)/2 = \operatorname{Ker}\theta$, so $\partial_{\mathbb{G}_m}([E]) = \theta(\partial_{\mu_2}([E])) = \theta(c(E)) = 0$ by Lemma 8.1. □

Let F be a Cliff-bundle refining E (Lemma 8.2). By Theorem 6.11, we have

$$\gamma_2(F)_\eta = \delta(e^3(q)) \tag{8.2}$$

---

[1]Although $H^1_{\text{ét}}(X, SO(n,n)) \to H^1_{\text{ét}}(X, O(n,n))$ is injective if X is the spectrum of a field, it need not be in general: we are indebted to Serre and Parimala for pointing this out.

where δ is the "Kummer" boundary for weight-two étale motivic cohomology. By Theorem 7.1, we have

$$d_2(e^3(q)) = \mathcal{S}(e^3(q))$$

where $\mathcal{S}$ is the snake map of section 7. Equation (8.2) and the commutative diagram

$$\begin{array}{ccccc} & & H^0(X, \mathcal{H}^3(\mathbb{Z}/2)) & \hookrightarrow & H^3(K, \mathbb{Z}/2) \\ & & \downarrow & & \downarrow \delta \\ \mathbb{H}^4_{\text{ét}}(X, \Gamma(2)) & \longrightarrow & H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) & \hookrightarrow & \mathbb{H}^4(K, \Gamma(2)) \end{array}$$

shows that $\gamma_2(E)$ lifts the image of $e^3(q)$ in $H^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$. Therefore, the image of $e^3(q)$ by $\mathcal{S}$ is the projection of $2\gamma_2(F) \in CH^2(X)$ in $CH^2(X)/2$. By Theorem 6.9 (iii), this is $c_2(F) + \gamma_1(F)^2$. But by Theorem 6.9 (iv) we have $\delta_1(\gamma_1(F)) = c(F)$, hence (with an obvious abuse of notation) $\gamma_1(F) = c(E) \in \mathrm{Pic}(X)/2$. Theorem 1 is proven.

9. Application to quadratic forms

Let q be a quadratic form over k. We assume $q \in I^2k$, *i.e.* $\dim q$ even and $d_\pm q = 1$. The Clifford algebra $C(q)$ is central simple over k: let X be its Severi-Brauer variety and $K = k(X)$. Over K, $C(q)$ is split, hence (by Merkurjev's and Arason's theorems) $q_K \in I^3K$ and $e^3(q)$ is defined.

9.1. Theorem. *If* $\operatorname{ind} C(q) \geq 8$, *then* $d_2(e^3(q_K)) \neq 0$.

9.2. Corollary. *Under the conditions of Theorem* 9.1, $e^3(q_K) \neq 0$, *hence* $q_K \notin I^4K$. □

9.3. Corollary. *Let* $n, i \geq 0$ *and let* $Q(k, 2n, i)$ *be the set of isomorphism classes of quadratic forms* q *over* k *such that* $\dim q = 2n$, $d_\pm q = 1$ *and* $\operatorname{ind} C(q) \leq i$. *Then, if* $i \geq 8$, *there exists no cohomological invariant* $e_F : Q(F, 2n, i) \to H^3(F, \mathbb{Z}/2)$ $(F \supseteq k)$ *commuting with change of base field and such that* $e(q) = e^3(q)$ *if* $q \in I^3F$.

Corollary 9.2 is wrong if $\operatorname{ind} C(q) \leq 4$. Let φ be a quaternion form or an Albert form, $\psi \in I^4k$ and let q be the anisotropic part of $\varphi \perp \psi$. Then $\varphi_K \sim 0$, hence $q_K \sim \psi_K \in I^4K$. One could for example take for φ (resp. ψ) a generic quaternion or Albert form (resp. a generic 4-fold Pfister form).

Similarly, Corollary 9.3 is wrong for $i = 2$: one can then define $e(q) = e^3(q \perp a\tau)$, where τ is the quaternion form such that $c(q) = c(\tau)$ and a is an arbitrary fixed scalar. On the other hand, it seems likely that Corollary 9.3 still holds if $i = 4$, provided $n \geq 4$.

Proof of Theorem 9.1. Let E be the quadratic bundle $q \otimes_k X$, with generic fiber q_K: we are in the situation of Theorem 1. Since E is extended from k, its underlying vector bundle is trivial, hence $c_2(E) = 0$ and Theorem 1 reduces to

$$d_2 e^3(q_K) = c(E)^2.$$

To prove Theorem 9.1, it therefore suffices to show that $c(E)^2 \neq 0 \in CH^2(X)/2$. Let $\overline{X} = X \otimes_k k_s$, where k_s is a separable closure of k. Recall that $\overline{X} \simeq \mathbb{P}^n_{k_s}$, with $n = \dim X$. This gives:

- $H^2(k, \mathbb{Z}/2) \to H^2(X, \mathbb{Z}/2)$ is injective by the Hochschild-Serre spectral sequence; in particular $0 \neq c(E) \in \mathrm{Pic}(X)/2 \subseteq H^2(X, \mathbb{Z}/2)$.

- $CH^i\mathbb{P}^n_{k_s} = h^i\mathbb{Z}$, where h is the class of a hyperplane section. Panin [46] has shown that $\mathrm{Pic}(X) = 2h\mathbb{Z} \subset \mathrm{Pic}(\mathbb{P}^n_{k_s})$; on the other hand it is easy to show that $\mathrm{Im}(CH^2(X) \to CH^2(\mathbb{P}^n_{k_s})) = 4h^2\mathbb{Z}$ when $\mathrm{ind}\, C(q) \geq 8$ (compare [29, § 4]); in particular, if $H = 2h$ is the generator of $\mathrm{Pic}(X)$, then $H^2 \notin 2CH^2(X)$.

It follows from these facts that $\mathrm{Pic}(X)/2 \simeq \mathbb{Z}/2$ and $c(E)$ *generates* $\mathrm{Pic}(X)/2$, hence $c(E) \equiv H \pmod{2\,\mathrm{Pic}(X)}$. Therefore $c(E)^2 \equiv H^2 \not\equiv 0 \pmod{2CH^2(X)}$. □

Proof of Corollary 9.3. Suppose e exists. Let $q \in Q(k, n, i)$ with $\mathrm{ind}\, C(q) \geq 8$. Let K be the function field of the Severi-Brauer variety of $C(q)$. By naturality of e, $e(q)_K = e(q_K) = e^3(q_K)$. Then $e^3(q_K)$ is defined over k, hence $d_2(e^3(q_K)) = 0$, which contradicts Theorem 9.1. □

9.4. Example. $\dim q = 8$. Then q_K is similar to a 3-fold Pfister form. If $\mathrm{ind}\, C(q) = 8$, Corollary 9.2 implies that q_K is not hyperbolic, hence anisotropic. Laghribi [30] has shown that this still holds if $\mathrm{ind}\, C(q) < 8$, but the reason is entirely different: it relies on the Arason-Pfister Hauptsatz.

Appendix A. Toral descent

A.1. Let $\pi : X \to Y$ be a morphism of schemes, and let X^n_Y denote the n-fold fiber product of X over Y. Form the simplicial scheme $E_Y X$ with n-simplices X^{n+1}_Y, where the map

$$E_Y X(g) : X^{m+1}_Y \longrightarrow X^{n+1}_Y$$

coming from $g : \Delta_n \to \Delta_m$ in Δ is given on ring-valued points by

$$E_Y X(g)(x_0, \dots , x_m) = (x_{g(0)}, \dots , x_{g(n)}).$$

If we are working in the category of schemes over a fixed base B, we write EX for $E_B X$.

The map π induces a natural augmentation $\varepsilon_{X/Y} : E_Y X \to Y$.

The construction of $E_Y X$ is functorial in the map $X \to Y$; in particular, if $X \to Y$ is a map of simplicial schemes, we have the bi-simplicial scheme $E_Y X$, with (n, m)-simplices given by

$$(E_Y X)_{(n,m)} = (E_{Y_m} X_m)_n$$

and with augmentation $\varepsilon_{X/Y} : E_X Y \to Y$.

Let $\mathcal{F}$ be a sheaf over the big Zariski site of k. The augmentation $\varepsilon_{X/Y}$ gives a natural map

$$\varepsilon^*_{X/Y} : H^*(Y, \mathcal{F}) \to H^*(E_Y X, \mathcal{F}). \tag{A.1}$$

A.2. Lemma. *Let $X \xrightarrow{\pi} Y$ be a map of (simplicial) schemes. Suppose π has a section σ. Then the augmentation map $E_Y X \xrightarrow{\varepsilon_{X/Y}} Y$ is a homotopy equivalence, where we consider Y as a constant (bi)-simplicial scheme, and* (A.1) *is an isomorphism. In particular, if $\mathcal{C}$ is a subcategory of the category of k-schemes, closed under finite products over k, and if $\mathcal{F}$ is a Zariski sheaf on $\mathcal{C}$, then the map*

$$\varepsilon^*_{X/k} : H^0(\mathrm{Spec}(k), \mathcal{F}) = H^*(\mathrm{Spec}(k), \mathcal{F}) \longrightarrow H^*(EX, \mathcal{F})$$

is an isomorphism for all X in $\mathcal{C}$ having a k-point.

Proof. For notational simplicity, we give the proof supposing that X and Y are schemes. The section σ induces a map $E\sigma\colon E_YY = Y \to E_YX$ splitting $\varepsilon_{X/Y}$.

The simplicial set $[0,1]$ is the nerve of the category associated to the partially ordered set $0 < 1$, hence $[0,1]$ has n-simplices given as the set of length $n+1$ non-decreasing sequences of 0's and 1's. Given such a sequence $s\colon \{0,\ldots,n\} \to \{0,1\}$, define

$$p_s\colon X_Y^{n+1} \to X_Y^{n+1}$$

by

$$p_s(x_0,\ldots,x_n) = (y_0,\ldots,y_n)$$

where $y_i = \sigma(\pi(x_i))$ if $s(i) = 0$ and $y_i = x_i$ if $s(i) = 1$. Letting $E_YX \times [0,1]$ be the diagonal simplicial scheme

$$(E_YX \times [0,1])_n := (E_YX)_n \times [0,1]_n$$

the maps p_s define the map of simplicial schemes

$$p\colon E_YX \times [0,1] \to E_YX$$

with

$$p_{|E_YX\times 0} = E\sigma \circ \varepsilon_{X/Y}; \qquad p_{|E_YX\times 1} = \mathrm{id}_{E_YX}.$$

□

Let X be a simplicial scheme, $T \cong \mathbb{G}_m^r$ a split torus of rank r, and $\mu\colon X \times T \to X$ an action of T on X. We call the T-action *free* if the action on the n-simplices $X_n \times T \to X_n$ is free for each n. Assuming that the quotients X_n/T exist for each n, we may form the simplicial scheme Y with n-simplices $Y_n := X_n/T$ and canonical morphism $\pi\colon X \to Y$.

A.3. Proposition. *Let $\mu\colon X \times T \to X$ be a free T-action on a smooth (simplicial) k-scheme X such that the quotient $X \to Y := X/T$ exists. Then the map* (A.1) *is an isomorphism.*

Proof. Suppose that X is a smooth simplicial k-scheme. We have the spectral sequences

$$E_1^{p,q}(E_YX) = H^q(E_{Y_p}X_p, \mathcal{F}) \Longrightarrow H^{p+q}(E_YX, \mathcal{F})$$
$$E_1^{p,q}(Y) = H^q(Y_p, \mathcal{F}) \Longrightarrow H^{p+q}(Y, \mathcal{F})$$

and the augmentation induces a map of spectral sequences. This reduces us to considering the case of a smooth k-scheme X with T-action.

Since the T-action is free, the map $X \to Y := X/T$ makes X into a T-torsor for the étale topology. By Hilbert's Theorem 90, $X \to Y$ is a Zariski locally trivial T-bundle. Let

$$\mathcal{U} := \{U_0,\ldots,U_s\}$$

be a Zariski open cover of Y trivializing $\pi\colon X \to Y$, let $V_i = \pi^{-1}(U_i)$, and let $\mathcal{V}$ be the open cover $\{V_0,\ldots,V_s\}$ of X. We form the simplicial schemes $\mathcal{N}(\mathcal{U})$ and $\mathcal{N}(\mathcal{V})$ (where

$\mathcal{N}$ stands for "nerve"), giving the map of augmented simplicial schemes

$$\begin{array}{ccc} \mathcal{N}(\mathcal{V}) & \xrightarrow{\Pi} & \mathcal{N}(\mathcal{U}) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & Y. \end{array}$$

This induces the augmentation

$$\varepsilon_\Pi \colon E_{\mathcal{N}(\mathcal{U})}\mathcal{N}(\mathcal{V}) \to \mathcal{N}(\mathcal{U})$$

and the commutative diagram

$$\begin{array}{ccc} E_{\mathcal{N}(\mathcal{U})}\mathcal{N}(\mathcal{V}) & \xrightarrow{\varepsilon_\Pi} & \mathcal{N}(\mathcal{U}) \\ \downarrow & & \downarrow \\ E_Y X & \xrightarrow{\varepsilon_\pi} & Y. \end{array} \tag{A.2}$$

As Zariski cohomology of a Zariski sheaf satisfies Mayer-Vietoris for Zariski open covers, the right-hand vertical arrow in (A.2) induces an isomorphism

$$H^*(Y, \mathcal{F}) \to H^*(\mathcal{N}(\mathcal{U}), \mathcal{F}).$$

The cover $\mathcal{V}$ induces a cover $\mathcal{V}_n$ of $(E_Y X)_n$ by the open subsets $\{(E_{U_0}V_0)_n, \ldots, (E_{U_s}V_s)_n\}$. We have the canonical identification

$$[E_{\mathcal{N}(\mathcal{U})}\mathcal{N}(\mathcal{V})]_{n,*} \cong \mathcal{N}(\mathcal{V}_n).$$

By a spectral sequence argument as above, this implies that left-hand vertical arrow in (A.2) induces an isomorphism

$$H^*(E_Y X, \mathcal{F}) \to H^*(E_{\mathcal{N}(\mathcal{U})}\mathcal{N}(\mathcal{V}), \mathcal{F}).$$

Using the other spectral sequence for the cohomology of $E_{\mathcal{N}(\mathcal{U})}\mathcal{N}(\mathcal{V})$ and $\mathcal{N}(\mathcal{U})$, we thus reduce to the case $X = Y \times_k T$, with T acting by multiplication on the factor T.

In this case, the projection $X \to Y$ has the section $\sigma : Y \to X$ given by $\sigma(y) = (y, 1)$. We then apply Lemma A.2. □

A.4. Remark. The proof works just as well for $X \to Y$ a Zariski-locally trivial family with fiber F, such that F has a k-point, and similarly for $X \to Y$ an étale-locally trivial family with fiber F, such that F has a k-point, provided we use étale cohomology instead of Zariski cohomology.

A.5. Proposition. *Let $f \colon X \to X'$ be a T-equivariant map of smooth (simplicial) k-schemes with free T action, such that the quotients $Y := X/T$ and $Y' := X'/T$ are defined, and let $g \colon Y \to Y'$ be the induced map. Suppose that f induces an isomorphism*

$$f^* \colon H^*(X', \mathcal{F}) \to H^*(X, \mathcal{F}).$$

for all sheaves $\mathcal{F}$ on the big Zariski site over k. Then g induces an isomorphism

$$g^* \colon H^*(Y', \mathcal{F}) \to H^*(Y, \mathcal{F}).$$

for all sheaves $\mathcal{F}$.

Proof. The commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & Y' \end{array}$$

defines a commutative diagram

$$\begin{array}{ccc} E_Y X & \xrightarrow{h} & E_{Y'} X' \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & Y' \end{array}$$

By Proposition A.3, we need only show that the map

$$h^* \colon H^*(E_{Y'}X', \mathcal{F}) \to H^*(E_Y X, \mathcal{F})$$

is an isomorphism. The map h induces a map of spectral sequences, given on the E_1-terms by

$$E_1^{p,q}(E_{Y'}X') = H^q((E_{Y'}X')_{p,*}, \mathcal{F}) \xrightarrow{h_p^*} E_1^{p,q}(E_Y X) = H^q((E_Y X)_{p,*}, \mathcal{F}).$$

We have natural isomorphisms

$$(E_Y X)_{p,*} \cong X_* \times_k T^p; \qquad (E_{Y'}X')_{p,*} \cong X'_* \times_k T^p$$

which identify the map h_p on p-simplices with $f \times \mathrm{id}_{T^p}$. The cohomology of each of these spaces is the abutment of Leray spectral sequences

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathcal{F}) \Rightarrow H^{p+q}(X \times_k T^p, \mathcal{F})$$

(where $\pi : X \times_k T^p \to X$ is the first projection) and similarly for X'. By our assumption on f and a spectral sequence comparison argument, each h_p^* is an isomorphism. □

A.6. Example. Let T be a split torus in a reductive algebraic group scheme G. Then the diagonal action of T on G^n and on T^n is free, giving the T-equivariant morphism $ET \to EG$ induced by the inclusion of T into G. It is easy to see that the quotient EG/T exists; the quotient ET/T is by definition BT, giving the commutative diagram

$$\begin{array}{ccc} ET & \longrightarrow & EG \\ \big\downarrow & & \big\downarrow \\ BT & \xrightarrow{i} & EG/T \end{array}$$

Now T and G have the k-point 1; it thus follows from Lemma A.2 and Proposition A.5 that the map

$$i^* \colon H^*(EG/T, \mathcal{M}_*) \to H^*(BT, \mathcal{M}_*)$$

is an isomorphism. This holds more generally when replacing T by a reductive subgroup whose torsors are locally trivial for the Zariski topology (e.g. a product of $\mathbb{G}_m, SL(n)$ and $Sp(2n)$), or by any reductive subgroup if we replace Zariski cohomology by étale cohomology.

Appendix B. The Rost invariant

Let H be a semi-simple, simply connected linear algebraic group over k. If H is split, we have an isomorphism $H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2) \xrightarrow{\sim} \mathbf{S}^2(L_H)^W$ by Theorem 4.8, where T is a split maximal torus, $L_H = \mathrm{Hom}(T, \mathbb{G}_m)$ and where W is the Weyl group of H. By Lemma 6.4, we therefore have a canonical isomorphism:

$$\mathbb{H}^4_{\text{ét}}(BH, \Gamma(2)) \simeq H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \oplus \mathbf{S}^2(L_H)^W.$$

If H is simple, the group $\mathbf{S}^2(L_H)^W$ is known to be free of rank 1. We show that this situation extends to the non-split case in a straightforward way.

B.1. Proposition. *Let H be a (not necessarily split) semi-simple, simply connected linear algebraic group over k. Let $X_\bullet$ be a smooth simplicial scheme over k and $E_\bullet \to X_\bullet$ be an H-torsor (this means that, for each n, $E_n \to X_n$ is an H-torsor, and that all faces and degeneracies preserve the torsor structures). Then there are isomorphisms*

$$\mathbb{H}^i_{\text{ét}}(X_\bullet, \Gamma(2)) \xrightarrow{\sim} \mathbb{H}^i_{\text{ét}}(E_\bullet, \Gamma(2)) \ (i \leq 2)$$

and an exact sequence

$$0 \to \mathbb{H}^3_{\text{ét}}(X_\bullet, \Gamma(2)) \to \mathbb{H}^3_{\text{ét}}(E_\bullet, \Gamma(2)) \to H^0_{\text{ét}}(X_\bullet, \mathbf{H}^1(E_\bullet, \mathcal{K}_2)) \\ \to \mathbb{H}^4_{\text{ét}}(X_\bullet, \Gamma(2)) \to \mathbb{H}^4_{\text{ét}}(E_\bullet, \Gamma(2))$$

where $\mathbf{H}^1(E_\bullet, \mathcal{K}_2)$ is the simplicial sheaf defined as follows: its component over X_n is the étale sheaf associated to the presheaf $U \mapsto H^1_{\mathrm{Zar}}(E_n \times_{X_n} U, \mathcal{K}_2)$. This simplicial sheaf is locally constant.

To prove this proposition, we use the following lemma:

B.2. Lemma. *Let X be a smooth variety over k and E be an H-torsor on X. Let*

$$\Gamma_{\text{ét}}(\pi, 2) = cone(\Gamma_{\text{ét}}(X, 2) \to R\pi_* \Gamma_{\text{ét}}(E, 2))$$

where $\pi : E \to X$ is the projection. Then the cohomology sheaves $\mathcal{H}^i(\Gamma_{\text{ét}}(\pi, 2))$ are:

- 0 *for $i \leq 2$;*
- *the (locally constant) sheaf $\mathbf{H}^1(E, \mathcal{K}_2)$ defined as in Proposition* B.1 *for $i = 3$.*

Proof. We have an exact sequence of étale sheaves

$$0 \to \mathcal{H}^0(\Gamma_{\text{ét}}(\pi, 2)) \to \mathcal{H}^1(\Gamma_{\text{ét}}(X, 2)) \to R^1\pi_*\Gamma_{\text{ét}}(E, 2) \\ \to \mathcal{H}^1(\Gamma_{\text{ét}}(\pi, 2)) \to \mathcal{H}^2(\Gamma_{\text{ét}}(X, 2)) \to R^2\pi_*\Gamma_{\text{ét}}(E, 2) \to \dots$$

By [36] (and [28, lemma 1.4 (ii)]), the étale sheaf $\mathcal{H}^q(\Gamma_{\text{ét}}(X, 2))$ is 0 for $q \neq 1, 2$, and its stalk at a geometric point $x \in X$ is $K_3(K_x^{sh})_{\text{ind}}$ for $n = 1$ (resp. $K_2(\mathcal{O}^{sh}_{X,x})$ for $n = 2$), where $\mathcal{O}^{sh}_{X,x}$ is the strict henselization of $\mathcal{O}_X$ at x and K_x^{sh} is its field of fractions. Since H is locally split and E is locally trivial for the étale topology, the stalks of $R^q\pi_*\Gamma_{\text{ét}}(E, 2)$ for $q \leq 4$ are given by (6.1), (6.2) and Corollary 3.22: for a geometric point x of X, we have

- $R^1\pi_*\Gamma_{\text{ét}}(E, 2)_x = K_3(K_x^{sh})_{\text{ind}}$;
- $R^2\pi_*\Gamma_{\text{ét}}(E, 2)_x = K_2(\mathcal{O}^{sh}_{X,x})$;
- $R^3\pi_*\Gamma_{\text{ét}}(E, 2)_x = H^1_{\mathrm{Zar}}(\mathcal{O}^{sh}_{X,x} \times_X E, \mathcal{K}_2) = H^1_{\mathrm{Zar}}(K_x^{sh} \times_X E, \mathcal{K}_2)$.

To get the first isomorphism, note that by lemma 6.2, $K_3(K)_{\mathrm{ind}} \xrightarrow{\sim} K_3(K(H))_{\mathrm{ind}}$ for all K, since H is a geometrically connected rational variety. To get the second isomorphism, use Corollary 3.22 for $M_* = K_*^M$ and $i = 2$ plus Gersten's conjecture for K_2. Similarly for the third isomorphisms. Note that, if H_{split} is the split semi-simple group associated to H, $\mathbf{H}^1(E, \mathcal{K}_2)$ is locally isomorphic to the constant sheaf $H^1_{\mathrm{Zar}}(H_{\mathrm{split}}, \mathcal{K}_2)$.

All this gives $\mathcal{H}^0(\Gamma_{\mathrm{\acute{e}t}}(\pi, 2)) = 0$ and the rest of the sequence as

$$0 \to \mathcal{H}^1(\Gamma_{\mathrm{\acute{e}t}}(\pi, 2)) \to \mathcal{K}_2 \to \mathcal{K}_2 \to \mathcal{H}^2(\Gamma_{\mathrm{\acute{e}t}}(\pi, 2)) \to 0 \to \mathbf{H}^1(H, \mathcal{K}_2) \to \mathcal{H}^3(\Gamma_{\mathrm{\acute{e}t}}(\pi, 2)) \to 0.$$

□

Proposition B.1 now follows from Lemma B.2 by noting that

$$\mathbb{H}^3_{\mathrm{\acute{e}t}}(X_\bullet, \Gamma(\pi_\bullet, 2)) \simeq H^0_{\mathrm{\acute{e}t}}(X_\bullet, \mathbf{H}^1(E_\bullet, \mathcal{K}_2))$$

where $\pi_\bullet : E_\bullet \to X_\bullet$ is the projection and $\Gamma(\pi_\bullet, 2)$ is the simplicial complex of sheaves with components $\Gamma(\pi_n, 2)$. □

B.3. Corollary. *Let H be as in Proposition* B.1 *and $\overline{H} = H \times_k k_s$, where k_s is a separable closure of k. Then $K_2(k) \xrightarrow{\sim} H^0_{\mathrm{Zar}}(H, \mathcal{K}_2)$ and there is a commutative square of isomorphisms*

$$\begin{array}{ccc} H^2_{\mathrm{Zar}}(B\overline{H}, \mathcal{K}_2)^{G_k} & \longrightarrow & H^1_{\mathrm{Zar}}(\overline{H}, \mathcal{K}_2)^{G_k} \\ \uparrow & & \uparrow \\ H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2) & \longrightarrow & H^1_{\mathrm{Zar}}(H, \mathcal{K}_2). \end{array}$$

Proof. The vertical homomorphisms in the diagram are induced by extension of scalars. We have two torsors $H \to \operatorname{Spec} k$ and $EH \to BH$. Applying Proposition B.1 to the first one gives the first claim, by taking (6.1) into account (this fact is well-known anyway). This in turn allows us to define the bottom horizontal homomorphism of the diagram via the spectral sequence (4.1) for BH, as in the proof of Theorem 4.7. Proposition B.1 and (6.1) also give an isomorphism

$$H^1_{\mathrm{Zar}}(H, \mathcal{K}_2) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\overline{H}, \mathcal{K}_2)^{G_k}$$

noting that the unit section splits the map $H \to \operatorname{Spec} k$. So the right vertical map in the diagram is bijective, and so is the top horizontal one by Theorem 4.7. On the other hand, the second torsor gives an exact sequence

$$0 \to H^1_{\mathrm{Zar}}(\overline{H}, \mathcal{K}_2)^{G_k} \to \mathbb{H}^4_{\mathrm{\acute{e}t}}(BH, \Gamma(2)) \to \mathbb{H}^4_{\mathrm{\acute{e}t}}(EH, \Gamma(2))$$

via Proposition B.1 and Lemma 6.4. Noting that $H^3_{\mathrm{\acute{e}t}}(k, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{\sim} \mathbb{H}^4_{\mathrm{\acute{e}t}}(k, \Gamma(2)) \xrightarrow{\sim} \mathbb{H}^4_{\mathrm{\acute{e}t}}(EH, \Gamma(2))$ (for the second isomorphism, see Lemma A.2), this gives via Lemma 6.4 again an isomorphism $H^1_{\mathrm{Zar}}(\overline{H}, \mathcal{K}_2)^{G_k} \xrightarrow{\sim} H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2)$. Still using Lemma 6.4, one checks that the composition

$$H^1_{\mathrm{Zar}}(H, \mathcal{K}_2) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\overline{H}, \mathcal{K}_2)^{G_k} \xrightarrow{\sim} H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2)$$

is inverse to the map defined above. Therefore the bottom horizontal homomorphism is also bijective, and so is the left vertical one by commutativity of the diagram. □

B.4. Lemma. *Suppose that in Proposition* B.1*, H is absolutely simple. Then the sheaf $\mathbf{H}^1(E_\bullet,\mathcal{K}_2)$ is constant, with value $\mathbb{Z}$.*

Proof. We first deal with the case $H=SL(N)$. Then, for each n, $E_n\to X_n$ is locally trivial for the Zariski topology. It follows that the *Zariski* sheaf associated to the presheaf $U\mapsto H^1_{\text{Zar}}((E_n\times_{X_n}U,\mathcal{K}_2)$ is locally isomorphic to the constant sheaf with value $H^1(SL(N),\mathcal{K}_2)\simeq\mathbb{Z}$, hence is itself constant with value $\mathbb{Z}$. The same is a fortiori true for the corresponding étale sheaf.

In general, let $\rho:H\to SL(N)$ be a nontrivial representation defined over k, and let $\rho_*E_\bullet$ be the induced torsor over $X_\bullet$. The map $H^1_{\text{Zar}}(SL(N)\times_k k_s,\mathcal{K}_2)\to H^1_{\text{Zar}}(\overline{H},\mathcal{K}_2)$ is nontrivial (see below). Since both groups are infinite cyclic, it is injective. It follows that the natural map of étale sheaves $\mathbf{H}^1(\rho_*E_\bullet,\mathcal{K}_2)\to\mathbf{H}^1(E_\bullet,\mathcal{K}_2)$ is a monomorphism. Since both sheaves are locally isomorphic to $\mathbb{Z}$ and the first one is constant, the second one must be constant too. □

B.5. Theorem. *Let H be a simple simply connected algebraic group over k. Then:*
a) There is an isomorphism

$$\mathbb{H}^4_{\text{ét}}(BH,\Gamma(2))\simeq H^3_{\text{ét}}(k,\mathbb{Q}/\mathbb{Z}(2))\oplus\mathbb{Z}.$$

b) For $X_\bullet,E_\bullet$ as in Proposition B.1*, with X_0 geometrically connected, the exact sequence of Proposition* B.1 *simplifies to*

$$0\to\mathbb{H}^3_{\text{ét}}(X_\bullet,\Gamma(2))\to\mathbb{H}^3_{\text{ét}}(E_\bullet,\Gamma(2))\to\mathbb{Z}\xrightarrow{\alpha}\mathbb{H}^4_{\text{ét}}(X_\bullet,\Gamma(2))\to\mathbb{H}^4_{\text{ét}}(E_\bullet,\Gamma(2)).$$

Moreover, if $Y_\bullet\xrightarrow{f}X_\bullet$ is a map of smooth simplicial k-schemes, with Y_0 geometrically connected, and $F_\bullet=f^*E_\bullet$, then the map

$$H^0_{\text{ét}}(X_\bullet,\mathbf{H}^1(E,\mathcal{K}_2))\to H^0_{\text{ét}}(Y_\bullet,\mathbf{H}^1(F,\mathcal{K}_2))$$

is an isomorphism.
c) If $X_\bullet$ is either constant or satisfies the assumptions of Lemma 6.4 *a), then this exact sequence can be rewritten*

$$0\to H^1_{\text{Zar}}(X_\bullet,\mathcal{K}_2)\to H^1_{\text{Zar}}(E_\bullet,\mathcal{K}_2)\to\mathbb{Z}\xrightarrow{\alpha}\mathbb{H}^4_{\text{ét}}(X_\bullet,\Gamma(2))\to\mathbb{H}^4_{\text{ét}}(E_\bullet,\Gamma(2)).$$

Proof. The hypothesis on H implies that $H\simeq R_{\ell/k}H'$ where H' is absolutely simple, ℓ/k is a finite separable extension and $R_{\ell/k}$ denotes Weil's restriction of scalars [59, p. 46]. By Corollary 3.21, we have

$$H^1_{\text{Zar}}(\overline{H},\mathcal{K}_2)\simeq Ind^{G_k}_{G_\ell}H^1_{\text{Zar}}(\overline{H}',\mathcal{K}_2)$$

as Galois modules, and Lemma B.4 shows that G_ℓ acts trivially on $H^1_{\text{Zar}}(\overline{H}',\mathcal{K}_2)$. Therefore, $H^1_{\text{Zar}}(\overline{H},\mathcal{K}_2)$ is a permutation module under G_k and $H^1_{\text{Zar}}(\overline{H},\mathcal{K}_2)^{G_k}\simeq H^1_{\text{Zar}}(H',\mathcal{K}_2)$ $\simeq\mathbb{Z}$. a) follows from this, together with Corollary B.3 and Lemma 6.4.

To prove b), we observe that there exists up to isomorphism a unique H'-torsor $E'_\bullet$ over $X'_\bullet:=X_\bullet\otimes_k\ell$ such that $E_\bullet\simeq f_*E'_\bullet$, where $f:X'_\bullet\to X_\bullet$ is the projection. Then $\mathbf{H}^1(E_\bullet,\mathcal{K}_2)\simeq f_*\mathbf{H}^1(E'_\bullet,\mathcal{K}_2)\simeq f_*\mathbb{Z}$, hence

$$H^0_{\text{ét}}(X_\bullet,\mathbf{H}^1(E_\bullet,\mathcal{K}_2))\simeq H^0_{\text{ét}}(X'_\bullet,\mathbb{Z})=H^0_{\text{ét}}(X'_0,\mathbb{Z})=\mathbb{Z}$$

since $X_0' = X_0 \times_k l$ is connected. The last claim of b) is obvious, by a similar argument. Finally, c) follows from b) and (6.1) or lemma 6.4. (The computation $H^1_{\mathrm{Zar}}(H, \mathcal{K}_2) \simeq \mathbb{Z}$ is due to Rost [53].) □

B.6. Let H be simple simply connected and let $\rho : H \to SL(N)$ be some nontrivial representation as above. If $k = \mathbb{C}$, the map $H^2_{\mathrm{Zar}}(BSL(N), \mathcal{K}_2) \xrightarrow{\rho^*} H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2)$ is nontrivial. An easy way to see this is to use Theorem 4.11 to reduce to topology, in which case the result is well-known. If H is split, the same holds by reduction to the complex case (passing through $\mathbb{Z}$ if $\operatorname{char} k > 0$, via Corollary 3.22). This is still the case in general, as one sees by passing to the separable closure of k. If ρ' is another such representation of H, then $\rho^* c_2$ and $\rho'^* c_2$ differ in $H^2(BH, \mathcal{K}_2) \simeq \mathbb{Z}$ by a *positive* rational number. To see this, embed ρ and ρ' into $\rho + \rho'$. This reduces us to the case in which $\rho' = \lambda \circ \rho$ for some $\lambda : SL(N) \to SL(N + N')$; then it can be checked that $\lambda^* c_2$ is a positive multiple of c_2 by reducing to the fundamental representations of $SL(N)$ (alternatively, reduce to topology). It follows that there is a unique generator γ_H of $H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2)$ such that, for $\rho : H \to SL(N)$ a homomorphism of algebraic groups,

$$\rho^* c_2 = d_\rho \gamma_H$$

where d_ρ is a *positive integer*. Note that the string of isomorphisms

$$H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(H, \mathcal{K}_2) \xrightarrow{\sim} H^1_{\mathrm{Zar}}(\overline{H}, \mathcal{K}_2)^{G_k} = H^0_{\text{ét}}(BH, \mathbf{H}^1(EH, \mathcal{K}_2))$$

(compare Corollary B.3) yields a canonical generator of the three other groups, that we still denote by γ_H.

Let d_H be the greatest common divisor of the integers d_ρ. If H is split, these multipliers are clearly independent of k; they were computed explicitly by Dynkin [14] in the case $k = \mathbb{C}$ for analytic cohomology (with a few mistakes for $H = E_8$, see [32, proof of prop. 2.6]). It turns out that, at least in the split case, d_H is always realized by a certain fundamental SL-representation ψ of H.

For the reader's convenience, we recall the list of Dynkin indices of split simple groups, together with the weight of a fundamental representation ψ such that $d_\psi = d_H$ (compare [32, prop. 2.6]):

H	A_r	$B_r, r \geq 3$	C_r	$D_r, r \geq 4$	E_6	E_7	E_8	F_4	G_2
d_H	1	2	1	2	6	12	60	6	2
weight of ψ	ϖ_1	ϖ_1	ϖ_1	ϖ_1	ϖ_6	ϖ_7	ϖ_8	ϖ_4	ϖ_1

The reader should take special care with D_3, corresponding to $\mathrm{Spin}(3,3)$, which is not in this table. In fact, D_3 is isomorphic to A_3 and $\mathrm{Spin}(3,3)$ is accordingly isomorphic to $SL(4)$, so its Dynkin index is 1. However the index associated to the representation ψ of Theorem 5.8 is 2 even for $n = 3$ and Theorem 5.8 is correct as stated.

Let X be a k-scheme and E an H-torsor on X, as in proposition B.1. Then E is locally trivial for the étale topology of X and γ_H yields a characteristic class

$$\gamma_H(E) = [E]^* \gamma_H \in \mathbb{H}^4_{\text{ét}}(X, \Gamma(2))$$

where $[E] \in [X, BH]_{\text{ét}}$ is the homotopy class associated to E. We have:

B.7. Lemma. *Let $\rho : H \to SL(N)$ be a linear representation of H, and let $V = \rho_* E$ be the associated vector bundle. Then*

$$\rho^* c_2(V) = d_\rho \gamma_H(E)$$

where d_ρ is the multiplier described in B.6. □

Suppose now that $X = \operatorname{Spec} k$. Then $H^3_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{H}^4_{\text{ét}}(k, \Gamma(2))$ is an isomorphism. Denote by $e(E)$ the inverse image of $\gamma_H(E)$ in $H^3_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(2))$: this is the *Rost invariant* of E.

B.8. Proposition. *For any E over* Spec k*, we have*

$$d_H e(E) = 0$$

where d_H is the Dynkin index of H.

Proof. This is obvious from Lemma B.7.

As in [28, end of introduction], let

$$\mathbb{Z}/d_H(2) = \begin{cases} \mu_{d_H}^{\otimes 2} & \text{if char } k = 0 \\ \mu_{d'_H}^{\otimes 2} \oplus W_r\Omega^2_{\log}[-2] & \text{if char } k = p > 0 \end{cases}$$

where (if char $k = p > 0$) d'_H is the prime-to-p part of d_H and $W_r\Omega^2_{\log}$ is the weight-two logarithmic part of the de Rham-Witt complex at length r, where $p^r \| d_H$. From the Merkurjev-Suslin theorem [42] (and the Bloch-Gabber-Kato theorem at the characteristic [3, Corollary 2.8]), the sequence

$$0 \to H^3_{\text{ét}}(k, \mathbb{Z}/d_H(2)) \to H^3_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(2)) \xrightarrow{d_H} H^3_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(2))$$

is exact. So the Rost invariant refines into an invariant in $H^3_{\text{ét}}(k, \mathbb{Z}/d_H(2))$.

Let X be a smooth variety over k and E an H-torsor over X. One sees as in section 8 that the component $e(E_\eta)$ in $H^3_{\text{ét}}(k(X), \mathbb{Z}/d'_H(2))$ is unramified over X. We can then show the following analogue to Theorem 6.11, exactly in the same way as above:

B.9. Theorem. *Let $\rho : H \to SL(N)$ be a representation of H, and let $e'(E_\eta) \in H^0(X, \mathcal{H}^3(\mu_{d'_\rho}^{\otimes 2}))$ be the prime-to-the-characteristic part of $e(E_\eta)$, viewed in the group $H^0(X, \mathcal{H}^3(\mu_{d'_\rho}^{\otimes 2}))$ where d'_ρ is the prime-to-p part of d_ρ. Then*

$$d_2(e'(E_\eta)) = c_2(E) \in CH^2(X)/d'_\rho$$

where $c_2(E)$ is the second Chern class of the vector bundle deduced from E via the representation ρ. □

If it happens that $d_\rho = d_H$, this theorem gives a computation of $d_2(e'(E_\eta)) \in CH^2(X)/d_H$, viewing $e'(E_\eta)$ as an element of $H^0(X, \mathcal{H}^3(\mu_{d'_H}^{\otimes 2}))$.

We conclude this section with a proof of Rost's announced theorem. When $H = \text{Spin}$, this allows this paper to be self-contained.

B.10. Proposition. *Let H be simple, simply connected and let $E \to X$ be an H-torsor on a smooth k-scheme X. Then, with notation as in Theorem* B.5 *b), we have $\alpha(1) = \gamma_H(E)$. In particular,*

$$\operatorname{Ker}(\mathbb{H}^4_{\text{ét}}(X,\Gamma(2)) \to \mathbb{H}^4_{\text{ét}}(E,\Gamma(2))) = <\gamma_H(E)> .$$

Proof. This follows from the commutative diagram, coming from Theorem B.5 c):

$$\begin{array}{ccccccc} H^1_{\mathrm{Zar}}(E,\mathcal{K}_2) & \longrightarrow & \mathbb{Z}\gamma_H & \xrightarrow{\alpha} & \mathbb{H}^4_{\text{ét}}(X,\Gamma(2)) & \longrightarrow & \mathbb{H}^4_{\text{ét}}(E,\Gamma(2)) \\ & & {\scriptstyle [E]^*}\uparrow{\scriptstyle =} & & {\scriptstyle [E]^*}\uparrow & & {\scriptstyle [E]^*}\uparrow \\ 0 = H^1(EH,\mathcal{K}_2) & \longrightarrow & \mathbb{Z}\gamma_H & \xrightarrow{\alpha} & \mathbb{H}^4_{\text{ét}}(BH,\Gamma(2)) & \longrightarrow & H^3_{\text{ét}}(k,\mathbb{Q}/\mathbb{Z}(2)) \end{array} \tag{B.1}$$

(note that, by Lemma A.2, $\mathbb{H}^1_{\mathrm{Zar}}(EH,\mathcal{K}_2) = 0$ and the composite $\mathbb{H}^4_{\text{ét}}(k,\Gamma(2)) \to \mathbb{H}^4_{\text{ét}}(BH,\Gamma(2)) \to \mathbb{H}^4_{\text{ét}}(EH,\Gamma(2))$ is an isomorphism). □

B.11. Theorem. (Rost) *Let H be a simple, simply connected algebraic group over k. Let E be an H-torsor on k and let $K = k(E)$ be its function field. Let η be the map $H^3_{\text{ét}}(k,\mathbb{Q}/\mathbb{Z}(2)) \to H^3_{\text{ét}}(K,\mathbb{Q}/\mathbb{Z}(2))$ given by extension of scalars. Then*

$$\operatorname{Ker}\eta = <e(E)> .$$

Proof. Note that $\overline{E} \simeq \overline{H}$ and that $K_2(\overline{k}) \xrightarrow{\sim} H^0(\overline{H},\mathcal{K}_2)$, as follows from Proposition 3.20. We therefore have an exact sequence extending that of Theorem B.5 c) (for $X_\bullet = \operatorname{Spec} k$):

$$H^1(E,\mathcal{K}_2) \to (H^1(\overline{E},\mathcal{K}_2)^{G_F} \simeq \mathbb{Z}) \xrightarrow{\alpha} \operatorname{Ker}\eta \to \operatorname{Ker}(CH^2(E) \to CH^2(\overline{E})).$$

(This exact sequence follows from [8, prop. 3.6] and [27, th. 3.1], see [49] or [28, th. 1].) We have $CH^2(E) = CH^2(\overline{E}) = 0$: as has already been pointed out, the case of $\overline{E} \simeq \overline{H}$ follows from [34, th. 2.1], and the general case of E is [47, cor. 5.2 (4)]. Finally, the equality $\alpha(\gamma_H) = e(E)$ is a special case of Theorem B.5 c) (for $X_\bullet = \operatorname{Spec} k$). □

Appendix C. An amusing example

Let H be as above. We apply Proposition B.10 to the following "generic" case: let $\rho : H \to SL(N)$ be a faithful linear representation of H. To ρ and $r \geq 1$ we associate the k-variety

$$B_r H = \frac{SL(N+r)}{H \times SL(r)}$$

where H is identified with its image in $SL(N)$. We also associate the vector bundle $H^{\Delta_l} \times k^N/\rho$ on BH, and its class $\rho^* c_2 \in \mathbb{H}^4_{\text{ét}}(BH,\Gamma(2))$.

The variety $B_r H$ is smooth and carries a tautological H-torsor $E = SL(N+r)/SL(r)$. As before, E determines a homotopy class of map

$$[E] \in [B_r H, BH/k]_{\text{ét}}.$$

C.1. Theorem. *a) For $r \geq 2$, the map*

$$\mathbb{H}^4_{\text{ét}}(BH, \Gamma(2)) \xrightarrow{[E]^*} \mathbb{H}^4_{\text{ét}}(B_rH, \Gamma(2)) \tag{C.1}$$

is injective, with p-primary torsion cokernel, where p is the characteristic exponent of k (so, in characteristic 0, it is an isomorphism).
b) For $r = 1$, there is a split exact sequence, up to p-primary torsion groups

$$0 \to H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to \mathbb{H}^4(B_1H, \Gamma(2)) \to \mathbb{Z}/d_\rho \to 0$$

and (C.1) *has p-primary torsion cokernel. Its kernel is generated by $\rho^* c_2$.*

Proof. We first compute $H^*_{\text{Zar}}(E, \mathcal{K}_2)$ for $r \geq 2$. We could go via the Leray spectral sequence of the fibration $\pi : SL(N+r) \to E$ (for the Zariski cohomology), using the fact that any $SL(r)$-bundle is locally trivial for the Zariski topology and using Corollary 3.22 as above. It is perhaps more elegant to go back to cycle cohomology and use Rost's spectral sequence [52, cor. 8.2]:

$$E_2^{p,q} = A^p(E, A^q[\pi, K_2]) \Rightarrow A^{p+q}(SL(N+r), K_2)$$

where $\pi : SL(N+r) \to E$ is the projection (the two arguments are essentially the same anyway). $A^q[\pi, K_j]$ is defined by $A^q[\pi, K_j](K) = A^q(SL(N+r) \times_E \operatorname{Spec} K, K_j)$ for any point $\operatorname{Spec} K \to E$. It is a cycle module, since the fibration π is a $SL(r)$-torsor, its fiber is trivial at all such points and in particular $A^q[\pi, K_j](K) \simeq A^q(SL(r)/K, K_j)$ for any K. In view of Proposition 3.20, we get:

$$A^q[\pi, K_2] \simeq \begin{cases} K_2 & q = 0 \\ \mathbb{Z}c_2 \text{ (constant)} & q = 1 \\ 0 & q \geq 2 \end{cases}$$

hence an exact sequence

$$0 \to A^1(E, K_2) \to \mathbb{Z}c_2 \to A^0(E, \mathbb{Z}c_2) \to A^2(E, K_2) \to 0$$

whence $A^1(E, K_2) = A^2(E, K_2) = 0$. By (6.2), it follows that $\mathbb{H}^4_{\text{ét}}(E, \Gamma(2)) \xrightarrow{\sim} H^0_{\text{Zar}}(E, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$. On the other hand, $H^3(k, \mathbb{Q}_l/\mathbb{Z}_l(2)) \xrightarrow{\sim} H^0_{\text{Zar}}(E, \mathcal{H}^3(\mathbb{Q}_l/\mathbb{Z}_l(2)))$ for all $l \neq p$. To see this, apply the Rost spectral sequence to the cycle module $K \mapsto H^*(K, \mathbb{Q}_l/\mathbb{Z}_l(*-1))$ and use Proposition 3.20 (i). Hence the map

$$\mathbb{H}^4_{\text{ét}}(k, \Gamma(2)) \to \mathbb{H}^4_{\text{ét}}(E, \Gamma(2))$$

has p-primary torsion cokernel; and this map is injective since E has a rational point.

For $r \geq 2$, Theorem C.1 now follows from diagram (B.1). Finally, in the case $r = 1$, we have $E = SL(N+1)$ and Proposition 3.20 shows that $A^i(E, K_2) = K_2(k), \mathbb{Z}c_2$ or 0 according as $i = 0, 1$ or 2, and the conclusion again follows from diagram (B.1). □

In contrast to Theorem C.1, the Zariski cohomology groups of BH and B_rH are in general "different", as the following corollary shows.

C.2. Corollary. *For $r \geq 2$,*
a) There is an exact sequence

$$0 \to CH^2(B_rH) \otimes \mathbb{Z}[1/p] \xrightarrow{cl^2} H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2) \otimes \mathbb{Z}[1/p] \to \frac{H^0_{\text{ét}}(B_rH, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))}{H^3_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(2))} \otimes \mathbb{Z}[1/p] \to 0$$

where, as before, p is the characteristic exponent of k. This exact sequence realizes $CH^2(B_rH) \otimes \mathbb{Z}[1/p]$ as a subgroup of index d_H of $H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2) \otimes \mathbb{Z}[1/p] = \mathbb{Z}[1/p]\gamma_H$.
b) We have:

$$H^0_{\text{ét}}(B_rH, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \otimes \mathbb{Z}[1/p] \simeq H^3_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(2)) \otimes \mathbb{Z}[1/p] \oplus \mathbb{Z}[1/p]e(E_\eta)$$

where $e(E_\eta)$ is the Rost invariant of the generic fiber of the H-torsor E; this invariant has order d_H.

Proof. We assume in the sequel that everything has been tensored by $\mathbb{Z}[1/p]$. The first claim of a) follows easily from Theorem C.1, the diagram with exact rows

$$\begin{array}{ccccccc} 0 \to H^2_{\mathrm{Zar}}(BH, \mathcal{K}_2) & \longrightarrow & \mathbb{H}^4_{\text{ét}}(BH, \Gamma(2)) & \longrightarrow & H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to 0 \\ & & \wr\downarrow & & \\ 0 \to CH^2(B_rH) & \longrightarrow & \mathbb{H}^4_{\text{ét}}(B_rH, \Gamma(2)) & \longrightarrow & H^0_{\mathrm{Zar}}(B_rH, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0 \end{array}$$

and the fact that $H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to H^0_{\mathrm{Zar}}(B_rH, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$ is split injective, since B_rH has a rational point. (The top row is a split exact sequence by Lemma 6.4.) On the other hand, b) follows from a).

It remains therefore to prove the second claim of a). This is equivalent to a statement proved by Rost (see [55, p. 783-15, exemple]). We shall give a simple proof of it, based on a recent result of Merkurjev [41]. We first need a well-known lemma:

C.3. Lemma. *Let X be a smooth variety over k. Then $CH^2(X)$ is generated by the $c_2(E)$, where E runs through the algebraic vector bundles over X of determinant* 1.

Proof. Consider the composition (for all $i \geq 0$)

$$CH^i(X) \xrightarrow{Cl^i} K_0(X) \xrightarrow{c_i} CH^i(X)$$

in which Cl^i is the K-theoretic cycle class map and c_i is the i-th Chern class with values in the Chow group. It follows from Riemann-Roch without denominators [26] that $c_i \circ Cl^i = (-1)^{i-1}(i-1)! Id_{CH^i(X)}$ [21, formula (4.5) and further comments]. In particular, for $i = 2$, this composition is minus the identity. To obtain bundles of determinant 1, one replaces E by $E \oplus \det(E)^{-1}$. □

By [41, cor. 6.6], the natural map

$$R(H) \to K_0(B_1H)$$

given by the "Borel construction" is surjective. Here $R(H)$ is the representation ring of H. Together with Lemma C.3, this shows that $CH^2(B_1H)$ is generated by the

$\psi^* c_2$, where ψ runs through the special linear representations of H. Consider now the commutative diagram

$$\begin{array}{ccccc} CH^2(B_1H) & \longrightarrow & \mathbb{H}^4_{\text{ét}}(B_1H,\Gamma(2)) & \longrightarrow & \mathbb{H}^4_{\text{ét}}(B_1H,\Gamma(2))/\mathbb{H}^4_{\text{ét}}(k,\Gamma(2)) \\ \uparrow & & \alpha\uparrow & & \beta\uparrow \\ CH^2(B_rH) & \longrightarrow & \mathbb{H}^4_{\text{ét}}(B_rH,\Gamma(2)) & \longrightarrow & \mathbb{H}^4_{\text{ét}}(B_rH,\Gamma(2))/\mathbb{H}^4_{\text{ét}}(k,\Gamma(2)) \end{array}$$

where $r \geq 2$. By Theorem C.1 a), the bottom composition coincides with the map cl^2 of Corollary C.2. By the above remark and the surjectivity of the left vertical map (Theorem C.1), the image of $\beta \circ cl^2$ is the subgroup generated by the $\psi^* c_2$. But, by Theorem C.1 b), α is surjective with kernel generated by $\rho^* c_2$. So the same conclusion holds for the image of cl^2, which therefore has index d_H, by definition of the Dynkin index of H. □

C.4. Question. The class $e(E_\eta)$ is unramified over B_rH. Consider a smooth compactification X of B_rH. Is $e(E_\eta)$ unramified over the whole of X?

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