INVERTING THE MOTIVIC BOTT ELEMENT

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ABSTRACT. We prove a version for motivic cohomology of Thomason's theorem on Bott-periodic K-theory, namely, that for a field k containing the nth roots of unity, the mod n motivic cohomology of a smooth k-scheme agrees with mod n étale cohomology, after inverting the element in $H^0(k, \mathbb{Z}/n(1))$ corresponding to a primitive nth root of unity.

1. INTRODUCTION

In his landmark paper [13], Thomason showed that the mod n étale K-theory of a scheme X could be described purely from the mod n algebraic K-theory of X, by simply inverting a canonical element β_n of $K_2(X;\mathbb{Z}/n)$ (this is assuming that the ring of functions on X contains a primitive nth root of unity, and that n is either odd, divisible by 4, or that ring of functions on X contains a square root of -1). In this paper we prove an analogous result relating the étale cohomology and the *motivic cohomology* of X. The main idea of the proof is the same as Thomason's, but the argument becomes much simpler: We avoid some of the technical difficulties by passing from spectra to complexes, and make additional simplifications by using the graded nature of motivic cohomology to sort out the mixing of the different levels in the gamma filtration inherent in K-theory. We also work in a less general setting than Thomason, in that we restrict ourselves to schemes smooth and essentially of finite type over a field.

The main player in this story is the mod n motivic cohomology of a scheme X, smooth and essentially of finite type over a field k, defined via Bloch's cycle complex $z^q(X,*)$ (see [1]) by

$$H^p(X, \mathbb{Z}(q)) := H_{2q-p}(z^q(X,*)),$$
$$H^p(X, \mathbb{Z}/n(q)) := H_{2q-p}(z^q(X,*) \otimes \mathbb{Z}/n).$$

For the field k, we have

$$H^p(\operatorname{Spec} k, \mathbb{Z}(1)) = \begin{cases} 0 & \text{for } p \neq 1 \\ k^\times & \text{for } p = 1. \end{cases}$$

The universal coefficient sequence for $H^0(\operatorname{Spec} k, \mathbb{Z}/n(1))$ thus gives the isomorphism

$$H^0(\operatorname{Spec} k, \mathbb{Z}/n(1)) \cong \mu_n(k).$$

If k contains a primitive nth root of unity ζ, we therefore have the corresponding *motivic Bott element* $\beta_n \in H^0(\operatorname{Spec} k, \mathbb{Z}/n(1))$, and we may invert β_n in the mod

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n motivic cohomology ring of X, forming the bi-graded ring

$$H^*(X,\mathbb{Z}/n(*))[\beta_n^{-1}] := (\bigoplus_{p,q} H^p(X,\mathbb{Z}/n(q)))[\beta_n^{-1}].$$

Natural *cycle class maps* $\mathrm{cl}^{p,q} : H^p(X,\mathbb{Z}/n(q)) \to H^p_{\text{ét}}(X,\mu_n^{\otimes q})$ are defined in [5, §3]. The cycle class maps give a ring homomorphism

$$H^*(X,\mathbb{Z}/n(*)) \xrightarrow{\mathrm{cl}^{*,*}} H^*_{\text{ét}}(X,\mu_n^{\otimes *})$$

with $\mathrm{cl}^{0,1}(\beta_n) = \zeta$. Since k contains ζ, cup product with ζ is an isomorphism on étale cohomology, so $\mathrm{cl}^{*,*}$ induces the map

$$H^*(X,\mathbb{Z}/n(*))[\beta_n^{-1}] \xrightarrow{\mathrm{cl}^{*,*}} H^*_{\text{ét}}(X,\mu_n^{\otimes *}) \tag{1.1}$$

Our main result is

Theorem 1.1. *Let k be a field, $n \geq 2$ an integer prime to* $\mathrm{char}\, k$, *and let X be a localization of a smooth k-scheme of finite type. Suppose that k contains a primitive nth root of unity. In case* $\mathrm{char}\, k = 0$, *we suppose further that either n is odd or that k contains a square root of -1. Then the map* (1.1) *is an isomorphism.*

The proof of Thereom 1.1 is, as mentioned above, essentially that used by Thomason. We use the Tate isomorphism (see §3.9) to show that the Bott element admits a family of inductors for a field extension of cohomological dimension one. The heart of the argument is Lemma 4.3, which shows how the family of inductors for the Bott element leads to étale descent for the Bott-inverted motivic cohomology for field extensions of cohomological dimension one. By an elementary tower construction, this allows us to reduce to the case of an algebraically closed field, where the result follows from the theorem of Suslin-Voevodsky [12] (extended in characteristic $p > 0$ by work of de Jong [6]), which computes the mod n motivic cohomology in terms of étale cohomology for smooth schemes over an algebraically closed field.

We actually prove a stronger version of Theorem 1.1 (Theorem 4.5) which asserts an isomorphism of objects in the derived category of sheaves on X. We have as well a version of Theorem 1.1 in case k does not contain a primitive nth root of unity (but still with the requirement that k contains a square root of -1 in case $\mathrm{char}\, k = 0$) where we invert a "power" of the Bott element. For details on this extension of Theorem 1.1, see §6.

2. Motivic cohomology

2.1. **Higher Chow groups.** We use Bloch's higher Chow groups to define motivic cohomology.

Let k be a field. We have the cosimplicial scheme Δ^*, with

$$\Delta^n := \mathrm{Spec}\, k[t_0, \dots, t_n]/\sum_i t_i - 1.$$

A *face* of Δ^n is a subscheme defined by equations of the form $t_{i_1} = \dots = t_{i_r} = 0$.

Let X be a k-scheme of finite type, locally equi-dimensional over k. Bloch [1] has defined complexes $z^q(X,*)$ with $z^q(X,p)$ the subgroup of the group of codimension q algebraic cycles on $X \times_k \Delta^p$ generated by those irreducible subschemes W such that $W \cap X \times F$ has codimenison $\geq q$ on $X \times F$ for all faces F of Δ^p. The differential

$z^q(X,p) \to z^q(X,p-1)$ is the alternating sum of the restriction to the faces $t_i = 0$. The *higher Chow group* $\mathrm{CH}^q(X,p)$ is defined by

$$\mathrm{CH}^q(X,p) := H_p(z^q(X,*)).$$

Actually, Bloch restricts the definition to quasi-projective X, but this is not necessary; see e.g. [8] and [9] for proofs of the essential properties of the higher Chow groups in the general setting.

We call a k-scheme X *essentially smooth* if X is the localization of a smooth, finite type k-scheme.

We reindex the homological complex $z^q(X,*)$, forming the cohomological complex $\mathcal{Z}^q(X,*)$ with $\mathcal{Z}^q(X,p) := z^q(X, 2q-p)$. For X essentially smooth over k, the *motivic cohomology* of X is defined by

$$H^p(X, \mathbb{Z}(q)) := H^p(\mathcal{Z}^q(X,*)).$$

The mod n motivic cohomology is similarly defined by

$$H^p(X, \mathbb{Z}/n(q)) := H^p(\mathcal{Z}^q(X,*)/n),$$

where we write M/n for $M \otimes \mathbb{Z}/n$. There are several alternative definitions of motivic cohomology and mod n motivic cohomology via categorical means; those of [7] agree with the ones given above, while those of [14] agree with the above in case $\mathrm{char}\, k = 0$.

Let $\mathbf{Ess}/k$ be the category of essentially smooth k-schemes. As explained in [5, §2.8], sending X to $\mathcal{Z}^q(X,*)$ extends to a functor

$$\mathcal{Z}^q(-,*) : \mathbf{Ess}/k^{\mathrm{op}} \to \mathbf{D}^-(\mathbf{Ab}).$$

2.2. **Products.** Let X and Y be k-schemes, essentially of finite type and locally equi-dimensional over k. There is a natural external product

$$\mathcal{Z}^q(X,*) \otimes^L \mathcal{Z}^{q'}(Y,*) \xrightarrow{\boxtimes_{X,Y}} \mathcal{Z}^{q+q'}(X \times_k Y, *) \tag{2.1}$$

in $\mathbf{D}^-(\mathbf{Ab})$, which, for $X = Y$ essentially smooth, gives the cup product in motivic cohomology by taking the pull-back via the diagonal $X \to X \times_k X$.

The product (2.1) is given via a *triangulation* of the bicosimplicial scheme $\Delta^* \times \Delta^*$. More precisely, let $[n]$ denote the ordered set $\{0, \ldots, n\}$, with the standard order. Each $i \in [n]$ gives us the corresponding vertex $v_i^n \in \Delta^n$ defined by $t_i = 1$, $t_j = 0$ for $j \neq i$. We give the product $[n] \times [m]$ the product partial order

$$(a,b) \leq (a',b') \iff a \leq a' \text{ and } b \leq b'.$$

Let $g := (g_1, g_2) : [n+m] \to [n] \times [m]$ be an injective order-preserving map, and let

$$T(g) : \Delta^{n+m} \to \Delta^n \times \Delta^m$$

be the affine-linear map with $T(g)(v_i^{n+m}) = (v^n_{g_1(i)}, v^m_{g_2(i)})$.

We have the permutation $\sigma(g) \in S_{n+m}$ determined by

$$\sigma(i) = \begin{cases} g_1(i) & \text{if } g_1(i-1) < g_1(i), \\ g_2(i) + n & \text{if } g_2(i-1) < g_2(i), \end{cases}$$

for $i = 1, \ldots, n+m$; this is well-defined since g is injective.

We let

$$T_{n,m} := \sum_{g:[n+m]\to[n]\times[m]} \mathrm{sgn}(\sigma(g))T(g)$$

where the sum is over injective, order-preserving maps g. The maps $T_{n,m}$ define the "standard" triangulation T of $\Delta^* \times \Delta^*$.

Let $z^q(X \times_k Y, (n,m))$ be the subgroup of $z^q(X \times Y \times \Delta^n \times \Delta^m)$ generated by irreducible codimension q subschemes W such that $W \cap X \times Y \times F_1 \times F_2$ has codimension q on $X \times Y \times F_1 \times F_2$ for all faces F_1 of Δ^n and F_2 of Δ^m. Sending (n,m) to $z^q(X \times Y, (n,m))$ defines a bisimplicial abelian group. Taking the product of cycles defines the map of complexes

$$\boxtimes_1 : z^q(X,*) \otimes z^{q'}(Y,*) \to \mathrm{Tot}\, z^{q+q'}(X \times_k Y, *, *).$$

Let $z^q(X \times_k Y, (n,m))_{\mathcal{T}}$ be the subgroup of $z^q(X \times_k Y, (n,m))$ generated by irreducible codimension q subschemes W in $z^q(X \times_k Y, (n,m))$ such that $W \cap X \times Y \times T(g)(F)$ has codimension q on $X \times_k Y \times T(g)(F)$ for all faces F of Δ^{n+m} and all injective g as above. One easily checks that the $z^q(X \times_k Y, (n,m))_{\mathcal{T}}$ form a sub-bicomplex $z^q(X \times_k Y, *, *)_{\mathcal{T}}$ of $z^q(X \times_k Y, *, *)$. By [5, Lemma 2.5], the inclusion

$$\mathrm{Tot}\, z^{q+q'}(X \times_k Y, *, *)_{\mathcal{T}} \xrightarrow{\iota} \mathrm{Tot}\, z^{q+q'}(X \times_k Y, *, *) \tag{2.2}$$

is a quasi-isomorphism. The map

$$(\mathrm{id} \times T_{n,m})^* : z^{q+q'}(X \times_k Y, (n,m))_{\mathcal{T}} \to z^{q+q'}(X \times_k Y, n+m)$$

is well-defined, and the sum of the $(\mathrm{id} \times T_{n,m})^*$ defines the map of complexes

$$(\mathrm{id} \times T)^* : \mathrm{Tot}\, z^{q+q'}(X \times_k Y, *, *)_{\mathcal{T}} \to z^{q+q'}(X \times_k Y, *). \tag{2.3}$$

The external product $\boxtimes$ is defined as the composition

$$(\mathrm{id} \times T)^* \circ \iota^{-1} \circ \boxtimes_1,$$

after reindexing.

2.3. **Cycle classes.** We have the étale sheaf $\mu_n^{\otimes q}$, giving the étale cohomology groups $H^p_{\text{ét}}(X, \mu_n^{\otimes q})$. A construction of natural cycle classes

$$\mathrm{cl}^{p;q}_{X,n} \colon H^p(X, \mathbb{Z}/n(q)) \to H^q_{\text{ét}}(X, \mu_n^{\otimes q})$$

is given in [5, §3]. A quick description of the construction of $\mathrm{cl}^{p;q}_{X,n}$ is as follows: Let $G^*_{\text{ét}}(Y, \mu_n^{\otimes q})$ denote the global sections of the Godement resolution of the étale sheaf $\mu_n^{\otimes q}$ on Y. We extend this definition to cosimplicial schemes Y^* by taking the extended total complex (§7.1) of the double complex $(a,b) \mapsto G^a_{\text{ét}}(Y^{-b}, \mu_n^{\otimes q})$.

For a closed codimension $\geq q$ subscheme W of a smooth k-scheme Y, purity for étale cohomology gives rise to a canonical isomorphism of the étale cohomology with supports, $H^{2q}_W(Y, \mu_n^{\otimes q})$, with the free $\mathbb{Z}/n$-module on the reduced irreducible codimension q subschemes of Y contained in W. This gives us the class $\mathrm{cyc}^q_W(Z) \in H^{2q}_W(Y, \mu_n^{\otimes q})$ for a cycle Z on Y with support in W. In addition, we have $H^p_W(Y, \mu_n^{\otimes q}) = 0$ for $p < 2q$. Applying this to the cosimplicial scheme $X \times \Delta^*$ and forgetting the support gives the natural map in the derived category $\mathbf{D}(\mathbf{Ab})$

$$\mathrm{cl}^q_{X \times \Delta^*, n} \colon \mathcal{Z}^q(X,*)/n \to G^*_{\text{ét}}(X \times \Delta^*, \mu_n^{\otimes q}).$$

By the homotopy property for étale cohomology, the map $G^*_{\text{ét}}(X, \mu_n^{\otimes q}) \xrightarrow{\epsilon_X^*} G^*_{\text{ét}}(X \times \Delta^*, \mu_n^{\otimes q})$ induced by the augmentation $\epsilon_X : X \times \Delta^* \to X$ is a quasi-isomorphism, giving us the natural map in $\mathbf{D}(\mathbf{Ab})$

$$\mathrm{cl}^q_{X,n} := (\epsilon^*)^{-1} \circ \mathrm{cl}^q_{X \times \Delta^*, n} : \mathcal{Z}^q(X,*)/n \to G^*_{\text{ét}}(X, \mu_n^{\otimes q}). \tag{2.4}$$

Taking cohomology gives the desired cycle class map

$$\mathrm{cl}^{p,q}_{X,n}\colon H^p(X,\mathbb{Z}/n(q)) \to H^p_{\text{ét}}(X,\mu_n^{\otimes q}).$$

By [5, Proposition 4.2], the maps (2.4) are natural in X. By [5, Proposition 4.7], the maps (2.4) are compatible with the products on the complexes $\mathcal{Z}^q(X,*)/n$ and $G^*_{\text{ét}}(X,\mu_n^{\otimes q})$, i.e., the diagram

$$\begin{array}{ccc} [\mathcal{Z}^q(X,*)/n]\otimes^L[\mathcal{Z}^{q'}(Y,*)/n] & \xrightarrow{\cup_{X,Y}} & \mathcal{Z}^q(X\times_k Y,*)/n \\ {\scriptstyle \mathrm{cl}^q_{X,n}\otimes \mathrm{cl}^{q'}_{Y,n}}\downarrow & & \downarrow{\scriptstyle \mathrm{cl}^{q+q'}_{X\times_k Y,n}} \\ G^*_{\text{ét}}(X,\mu_n^{\otimes q})\otimes^L G^*_{\text{ét}}(Y,\mu_n^{\otimes q'}) & \xrightarrow[\cup_{X,Y}]{} & G^*_{\text{ét}}(X\times_k Y,\mu_n^{\otimes q+q'}) \end{array}$$

commutes in $\mathbf{D}(\mathbf{Ab})$. In particular, the maps $\mathrm{cl}^{*,*}$ define a homomorphism of bi-graded rings $z^q(X\times_k Y,(n,m))_{\mathcal{T}}$

$$\mathrm{cl}^{*,*}_{X,n}\colon H^*(X,\mathbb{Z}/n(*)) \to H^*_{\text{ét}}(X,\mu_n^{\otimes *}).$$

3. Transfers and inductors

3.1. We have the additive category $\mathbf{C}(\mathbf{Ab})$ of complexes of abelian groups; let $\mathbf{C}^b(\mathbf{Ab})$ denote the full sub-category of complexes with bounded cohomology. Similarly, we have the full sub-categories $\mathbf{C}^+(\mathbf{Ab})$ and $\mathbf{C}^-(\mathbf{Ab})$ of complexes with cohomology bounded below (resp. bounded above).

3.2. Let G be a profinite group, and M a complex of continuous G-modules. Define the complex $\mathbb{H}^*(G;M)$ as the extended total complex of the double complex

$$C^0(G;M)\to C^1(G;M)\to \dots,$$

where $C^*(G;-)$ is the standard complex of continuous $\mathrm{Gal}(L'/L)$ cochains, i.e., in degree n, $\mathbb{H}^*(G;M)$ is $\prod_{a+b=n} C^a(G,M^b)$. Using the complex $C_*(G;-)$ of cocontinuous $\mathrm{Gal}(L'/L)$ chains, we define the complex $\mathbb{H}_*(G;M)$ to be the usual total complex of the double complex

$$\dots\to C_1(G;M)\to C_0(G;M).$$

The G-hypercohomology and hyperhomology of M are then given by

$$\mathbb{H}^p(G;M) = H^p(\mathbb{H}^*(G;M)),$$
$$\mathbb{H}_p(G;M) = H^{-p}(\mathbb{H}_*(G;M)).$$

We note that the complexes $\mathbb{H}^*(G;M)$ and $\mathbb{H}_*(G;M)$ take quasi-isomorphisms in M to quasi-isomorphisms of complexes of abelian groups, even for unbounded M; this follows from Lemma 7.3 with $N_n = M_n = 0$.

We may also form the complexes $H_0(G;M)$ and $H^0(G;M)$ by applying $H_0(G;-)$ and $H^0(G;-)$ termwise. We have the canonical projections and inclusions

$$M \xrightarrow{\iota_*} \mathbb{H}_*(G;M) \xrightarrow{\pi_*} H_0(G;M)$$
$$H^0(G;M) \xrightarrow{\iota^*} \mathbb{H}^*(G;M) \xrightarrow{\pi^*} M.$$

More generally, let $U \subset G$ be a closed normal subgroup, and suppose that U acts trivially on M. We have the natural inclusion and projection

$$\mathbb{H}_*(G;M) \xrightarrow{\pi_*} \mathbb{H}_*(G/U;M)$$
$$\mathbb{H}^*(G/U;M) \xrightarrow{\iota^*} \mathbb{H}^*(G;M).$$

3.3. For a Galois field extension L^*/L, let $\mathbf{F}_{L^*/L}$ be the category of subextensions L'/L, with maps homomorphisms of fields over the identity on L. We denote $\mathbf{F}_{L_{\mathrm{sep}}/L}$ by $\mathbf{F}_L$

Let $F\colon \mathbf{F}_{L^*/L} \to \mathbf{C}(\mathbf{Ab})$ be a functor which takes direct limits in $\mathbf{F}_{L^*/L}$ to direct limits in $\mathbf{C}(\mathbf{Ab})$, and let L'/L be in $\mathbf{F}_{L^*/L}$, with L' Galois over L. The functoriality of F gives $F(L')$ a $\mathrm{Gal}(L'/L)$-action, respecting the differential. The compatibility of F with direct limits implies that $F(L')$ (with the discrete topology) is a complex of continuous $\mathrm{Gal}(L'/L)$-modules. Define

$$\mathbb{H}^*(L'/L;F(L')) := \mathbb{H}^*(\mathrm{Gal}(L'/L);F(L')),$$
$$\mathbb{H}_*(L'/L;F(L')) := \mathbb{H}_*(\mathrm{Gal}(L'/L);F(L')).$$

We similarly write $\mathbb{H}^p(L'/L;F(L'))$ for $\mathbb{H}^p(\mathrm{Gal}(L'/L);F(L'))$ and $\mathbb{H}_p(L'/L;F(L'))$ for $\mathbb{H}_p(\mathrm{Gal}(L'/L);F(L'))$. In case $L' = L_{\mathrm{sep}}$ we set

$$\mathbb{H}^*(L;F(L_{\mathrm{sep}})) := \mathbb{H}^*(L_{\mathrm{sep}}/L;F(L_{\mathrm{sep}})),$$

etc.

Let $L \subset L' \subset L''$ be a tower of Galois extensions of L. If $F(L'')$ is in $\mathbf{C}^+(\mathbf{Ab})$, we have the canonical isomorphism in the derived category

$$\mathbb{H}^*(L'/L, \mathbb{H}^*(L''/L';F(L''))) \cong \mathbb{H}^*(L''/L, F(L'')). \tag{3.1}$$

Similarly, if $F(L'')$ is in $\mathbf{C}^-(\mathbf{Ab})$, we have the canonical isomorphism in the derived category

$$\mathbb{H}_*(L'/L, \mathbb{H}_*(L''/L';F(L''))) \cong \mathbb{H}_*(L''/L, F(L'')). \tag{3.2}$$

3.4. For a category $\mathcal{C}$, let $\mathrm{Iso}\mathcal{C}$ be the subcategory with the same objects, and with morphisms the isomorphisms in $\mathcal{C}$. We let $\mathrm{Func}(\mathcal{C},\mathbf{Ab})$ denote the abelian category of functors from $\mathcal{C}$ to $\mathbf{Ab}$, giving rise to the category of complexes $\mathbf{C}^*(\mathrm{Func}(\mathcal{C},\mathbf{Ab}))$ ($* = \emptyset, b, +, -$ a boundedness condition), and the derived category $\mathbf{D}^*(\mathrm{Func}(\mathcal{C},\mathbf{Ab}))$. We note that $\mathbf{C}(\mathrm{Func}(\mathcal{C},\mathbf{Ab}))$ is the same as the category of functors from $\mathcal{C}$ to $\mathbf{C}(\mathbf{Ab})$.

Sending an isomorphism $f : A \to B$ to the inverse $f^{-1} : B \to A$ defines the isomorphism of categories $\iota : \mathrm{Iso}\mathcal{C} \to \mathrm{Iso}\mathcal{C}^{\mathrm{op}}$. Thus, we have the isomorphism of functor categories

$$\iota : \mathbf{C}^*(\mathrm{Func}(\mathrm{Iso}\mathcal{C},\mathbf{Ab})) \to \mathbf{C}^*(\mathrm{Func}(\mathrm{Iso}\mathcal{C}^{\mathrm{op}},\mathbf{Ab}));$$
$$\iota : \mathbf{D}^*(\mathrm{Func}(\mathrm{Iso}\mathcal{C},\mathbf{Ab})) \to \mathbf{D}^*(\mathrm{Func}(\mathrm{Iso}\mathcal{C}^{\mathrm{op}},\mathbf{Ab})).$$

In particular, if we have a complex of functors $F \in \mathbf{C}^*(\mathrm{Func}(\mathcal{C},\mathbf{Ab}))$, we let $F^{\hat{}}$ denote the element $\iota(F_{|\mathrm{Iso}\mathcal{C}})$ of $\mathbf{C}^*(\mathrm{Func}(\mathrm{Iso}\mathcal{C}^{\mathrm{op}},\mathbf{Ab}))$.

We let $\mathbf{F}^{\mathrm{fin}}_{L^*/L}$ denote the full subcategory of $\mathbf{F}_{L^*/L}$ consisting of the finite extensions of L. We let F_{fin} denote the restriction of a functor $F : \mathbf{F}_{L^*/L} \to \mathbf{C}(\mathbf{Ab})$ to $\mathbf{F}^{\mathrm{fin}}_{L^*/L}$. The category $\mathrm{Iso}\mathbf{F}^{\mathrm{fin}}_{L^*/L}$ is equivalent to the disjoint union of the one-point categories $\mathrm{Iso}(L'/L)$, with $L \subset L' \subset L^*$, L' finite over L, where $\mathrm{Iso}(L'/L)$ is the group of isomorphisms of L' over L. Thus $\mathbf{D}(\mathrm{Func}(\mathrm{Iso}(\mathbf{F}^{\mathrm{fin}}_{L^*/L}), \mathbf{Ab}))$ is equivalent to the product of the derived categories $\mathbf{D}(\mathbf{Mod}_{\mathrm{Iso}(L'/L)})$, L' as above.

Definition 3.5. Let L^*/L be a Galois extension of fields, and let $F : \mathbf{F}_{L^*/L} \to \mathbf{C}(\mathbf{Ab})$ be a functor. A *transfer* for F is an extension of the image of $F_{\hat{\mathrm{fin}}}$ in $\mathbf{D}(\mathrm{Func}(\mathrm{Iso}(\mathbf{F}^{\mathrm{fin}}_{L^*/L})^{\mathrm{op}}, \mathbf{Ab}))$ to an element F^{op} of $\mathbf{D}(\mathrm{Func}((\mathbf{F}^{\mathrm{fin}}_{L^*/L})^{\mathrm{op}}, \mathbf{Ab}))$.

A *strict* transfer for F is an extension of $F_{\hat{\mathrm{fin}}} \in \mathbf{C}(\mathrm{Func}(\mathrm{Iso}(\mathbf{F}^{\mathrm{fin}}_{L^*/L})^{\mathrm{op}}, \mathbf{Ab}))$ to an element F^{op} of $\mathbf{C}(\mathrm{Func}((\mathbf{F}^{\mathrm{fin}}_{L^*/L})^{\mathrm{op}}, \mathbf{Ab}))$.

Suppose that $F_{\hat{\mathrm{fin}}}$ is in the subcategory $\mathbf{C}^*(\mathrm{Func}(\mathrm{Iso}(\mathbf{F}^{\mathrm{fin}}_{L^*/L})^{\mathrm{op}}, \mathbf{Ab}))$. A transfer with boundedness condition $* = b, +, -$ is is an extension of the image of $F_{\hat{\mathrm{fin}}}$ in $\mathbf{D}^*(\mathrm{Func}(\mathrm{Iso}(\mathbf{F}^{\mathrm{fin}}_{L^*/L})^{\mathrm{op}}, \mathbf{Ab}))$ to an element F^{op} of $\mathbf{D}^*(\mathrm{Func}((\mathbf{F}^{\mathrm{fin}}_{L^*/L})^{\mathrm{op}}, \mathbf{Ab}))$. A strict transfer with boundedness condition $*$ is defined similarly.

Concretely, a strict transfer consists in giving maps $\mathrm{Tr}_{L''/L'} : F(L'') \to F(L')$ of complexes with the functorialities $\mathrm{Tr}_{L''/L'} \circ \mathrm{Tr}_{L'''/L''} = \mathrm{Tr}_{L'''/L'}$ for each tower $L'''/L''/L'/L$, and with $\mathrm{Tr}_{M''/M'} \circ F(\sigma) = F(\tau) \circ \mathrm{Tr}_{L''/L'}$ for each commutative diagram

$$\begin{array}{ccc} L'' & \xrightarrow{\sigma} & M'' \\ \cup & & \cup \\ L' & \xrightarrow[\tau]{} & M' \end{array}$$

with σ and τ isomorphisms. The bounded versions are described similarly.

Example 3.6. Let X be a variety over a field k. Sending a field extension L/k to the complex $\mathcal{Z}^q(X \times_k L, *)$ defines the functor

$$\mathcal{Z}^q(X) : \mathbf{F}_k \to \mathbf{C}^-(\mathbf{Ab}).$$

If L'/L is finite, the projection $\pi_{L'/L} : X \times_k L' \to X \times_k L$ induces the push-forward map

$$\pi_{L'/L*} : \mathcal{Z}^q(X \times_k L', *) \to \mathcal{Z}^q(X \times_k L, *),$$

which defines a strict bounded above transfer for the functor $\mathcal{Z}^q(X)$.

3.7. Suppose that a functor $F : \mathbf{F}_{L^*/L} \to \mathbf{C}(\mathbf{Ab})$ has a strict bounded above transfer Tr, and let L''/L' be a finite Galois extension in $\mathbf{F}_{L^*/L}$, with L'' finite over L. We have the extension of $\mathrm{Tr}_{L''/L'} : F(L'') \to F(L')$ to the map

$$\mathrm{Tr}_{L''/L'} : \mathbb{H}_*(L''/L'; F(L'')) \to F(L'),$$

defined by the composition

$$\mathbb{H}_*(L''/L'; F(L'')) \xrightarrow{\mathbb{H}_*(L''/L'; \mathrm{Tr}_{L''/L'})} \mathbb{H}_*(L''/L'; F(L')) \xrightarrow{\pi^{L'}_*} H_0(L''/L'; F(L')) = F(L').$$

If L'' and L' are Galois over L, we have the map

$$\mathbb{H}_*(\mathrm{Tr}_{L''/L'}) : \mathbb{H}_*(L''/L; F(L'')) \to \mathbb{H}_*(L'/L; F(L'))$$

defined as the composition

$$\mathbb{H}_*(L''/L; F(L'')) \xrightarrow{\mathbb{H}_*(L''/L'; \mathrm{Tr}_{L''/L'})} \mathbb{H}_*(L''/L; F(L')) \xrightarrow{\pi_*} \mathbb{H}_*(L'/L; F(L')).$$

One sees that these extended transfers are well-defined, functorial, and for $L' = L$, we have $\mathbb{H}_*(\mathrm{Tr}_{L''/L}) = \mathrm{Tr}_{L''/L}$.

Letting $\mathbf{F}^{\mathrm{Gal}}_{L^*/L}$ denote the category of finite Galois subextensions L'/L of L^*/L, the maps $\mathbb{H}_*(\mathrm{Tr}_{L''/L'})$ define the functor

$$\mathbb{H}_*(-/L, F(-)) : (\mathbf{F}^{\mathrm{Gal}}_{L^*/L})^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab}).$$

Taking the homotopy limit over $(\mathbf{F}^{\mathrm{Gal}}_{L^*/L})^{\mathrm{op}}$ gives us the complex $\mathrm{holim}\, \mathbb{H}_*(-/L, F(-))$. The map (7.3) (with i the trivial extension L/L) defines the natural map

$$\mathrm{Tr} : \mathrm{holim}\, \mathbb{H}_*(-/L, F(-)) \to F(L). \tag{3.3}$$

Definition 3.8. Let L be a field, let $F : \mathbf{F}_{L^*/L} \to \mathbf{C}(\mathbf{Ab})$ be a functor with strict bounded above transfer Tr. Let x be an element of $H^{-n}(F(L))$. A *family of inductors* for x is an element $\tilde{x}$ of $H^{-n}(\mathrm{holim}\, \mathbb{H}_*(-/L, F(-)))$ with $\mathrm{Tr}(\tilde{x}) = x$.

3.9. **The Tate isomorphism.** We recall some of the material of [10, Annexe]. Let G be a profinite group, let U be a normal subgroup of finite index, and let M be a continuous G-module of cohomological dimension $\leq n$. Then M has U-cohomological dimension $\leq n$, and we have the canonical isomorphism in $\mathbf{D}^b(\mathbf{Ab})$

$$\mathbb{H}_*(G/U; \mathbb{H}^*(U; M)) \cong \mathbb{H}^*(G; M). \tag{3.4}$$

This is gotten by first truncating the standard G-free resolution of M

$$0 \to M \to X^0 \to \ldots \to X^i \to \ldots$$

at degree n, yielding the resolution

$$0 \to M \to X^0 \to \ldots \to X^{n-1} \to Z^n \to 0,$$

with Z^n having G-cohomological dimension 0. One then takes U-invariants, giving the complex

$$(X^0)^U := Y_n \to \ldots \to (X^{n-1})^U := Y_1 \to (Z^n)^U := Y_0$$

canonically isomorphic to $\mathbb{H}^*(U; M)$ in $\mathbf{D}^b(\mathbf{Mod}_{G/U})$. Since the X^i and Z^n have cohomological dimension 0, it follows from [10, Annexe, Lemme 1] that the Y_j have G/U-cohomological dimension 0, and that the trace induces isomorphisms

$$H_0(G/U; Y_j) \to H^0(G/U; Y_j). \tag{3.5}$$

This readily implies that the complex

$$H_0(G/U; Y_n) \to \ldots \to H_0(G/U; Y_0)$$

is canonically isomorphic to $\mathbb{H}_*(G/U; \mathbb{H}^*(U; M))$ in $\mathbf{D}^b(\mathbf{Ab})$, and that the complex

$$H^0(G/U; Y_n) \to \ldots \to H^0(G/U; Y_0)$$

is canonically isomorphic to $\mathbb{H}^*(G/U; \mathbb{H}^*(U; M))$ in $\mathbf{D}^b(\mathbf{Ab})$. As we have the canonical isomorphism (3.1)

$$\mathbb{H}^*(G/U; \mathbb{H}^*(U; M)) \cong \mathbb{H}^*(G; M),$$

the isomorphisms (3.5) induce the isomorphism (3.4).

3.10. The Tate isomorphism gives us a method of constructing families of inductors. Let M be an abelian group, which we consider as a trivial discrete G-module for all groups G. We suppose that M has finite $\mathrm{Gal}(L^*/L)$-cohomological dimension d.

Let $G = \mathrm{Gal}(L^*/L)$, and let $\mathcal{C}^*(G, M)$ be the complex of homogeneous continuous G-cochains, i.e.,

$$\mathcal{C}^n(G, M) := \varinjlim_{U} \mathrm{Hom}((G/U)^{n+1}, M),$$

where the limit is over subgroups $U \subset G$ of finite index. By definition, $\mathbb{H}^*(L^*/L; M) = H^0(G, \mathcal{C}^*(G, M))$, where G acts diagonally. More generally, if $G(L') \subset G$ is the subgroup of finite index corresponding to a finite extension L'/L, the natural map

$$H^0(G(L'), \mathcal{C}^*(G, M)) \xrightarrow{\alpha_{L'}} H^0(G(L'), \mathcal{C}^*(G(L'), M)) = \mathbb{H}^*(L^*/L'; M)$$

is a quasi-isomorphism. The isomorphism (3.1) for a Galois extension L''/L' is induced by the quasi-isomorphisms $\alpha_{L'}$ and $\alpha_{L''}$, together with the identity of complexes

$$H^0(G(L'), \mathcal{C}^*(G, M)) = H^0(G(L')/G(L''), H^0(G(L''), \mathcal{C}^*(G, M))).$$

Let F be the functor

$$F(L') := H^0(G(L'), \mathcal{C}^*(G, M))$$

By [10, Annexe, Lemme 1], the natural map $\tau_{\le d} F(L') \to F(L')$ is a quasi-isomorphism for all L' finite over L.

Let $L_\beta/L_\alpha/L$ be a finite Galois subextension of L^*/L, and take an element f in $H^0(G(L_\beta), \mathcal{C}^n(G, M))$. Define $\mathrm{Tr}_{L_\beta/L_\alpha}(f) \in H^0(G(L_\alpha), \mathcal{C}^n(G, M))$ by

$$\mathrm{Tr}_{L_\beta/L_\alpha}(f)(\sigma_0, \ldots, \sigma_n) = \sum_{\sigma \in G(L_\alpha)/G(L_\beta)} f(\sigma_0\sigma, \ldots, \sigma_n\sigma).$$

This defines the map

$$\mathrm{Tr}_{L_\beta/L_\alpha} : F(L_\beta) \to F(L_\alpha)$$

with the functoriality required to define a strict (bounded) transfer.

Via the description of the isomorphism (3.1), we see that the Tate isomorphism (3.4)

$$\mathbb{H}_*(L'/L, \mathbb{H}^*(L', M)) \to \mathbb{H}^*(L, M)$$

is given by the map

$$\mathbb{H}_*(\mathrm{Tr}_{L'/L}) : \mathbb{H}_*(L'/L; H^0(G(L'), \mathcal{C}^*(G, M)) \to H^0(G, \mathcal{C}^*(G, M)).$$

combined with the quasi-isomorphism $\alpha_{L'}$. Thus, we see that the maps

$$\mathbb{H}_*(L_\beta/L; H^0(G(L_\beta), \mathcal{C}^*(G, M))) \xrightarrow{\mathbb{H}_*(\mathrm{Tr}_{L_\beta/L_\alpha})} \mathbb{H}_*(L_\alpha/L; H^0(G(L_\beta), \mathcal{C}^*(G, M))).$$

are quasi-isomorphisms.

From the definition of holim as the extended complex of a double complex, Lemma 7.3 shows that the natural map (3.3)

$$\mathrm{Tr} : \mathrm{holim}\, H_*(-/L; F(-)) \to F(L)$$

is a quasi-isomorphism. If we have an element x of $H^{-a}(L^*/L;M) \cong H^{-a}(F(L))$, the isomorphism

$$H^{-a}(\mathrm{Tr}) : H^{-a}(\mathrm{holim}\,\mathbb{H}_*(-/L;F(-)) \to H^{-a}(F(L))$$

uniquely defines the element

$$\tilde{x} \in H^{-a}(\mathrm{holim}\,\mathbb{H}_*(-/L;F(-))$$

with $\mathrm{Tr}(\tilde{x}) = x$, giving the desired family of inductors for x.

4. Descent for motivic cohomology

4.1. **The motivic Bott element.** We fix a prime l, and an integer $\nu \geq 1$.

For a field F, we have [1, Theorem 6.1]

$$H^p(\mathrm{Spec}\,F, \mathbb{Z}(1)) = \begin{cases} 0 & \text{for } p \neq 1, \\ F^\times & \text{for } p = 1. \end{cases}$$

By [5, Proposition 4.9 and Theorem 7.5], the cycle class induces an isomorphism in the derived category

$$\mathrm{cl}^1_{F,n} : \mathcal{Z}^1(\mathrm{Spec}\,F, *)/n \to \tau_{\leq 1}\mathbb{H}^*(F;\mu_n)$$

for all n prime to $\mathrm{char}\,F$. Thus, the cycle class maps gives the isomorphisms

$$H^p(\mathrm{Spec}\,F, \mathbb{Z}/n(1)) \cong \begin{cases} 0 & \text{for } p \neq 0,1, \\ F^\times/n & \text{for } p = 1, \\ \mu_n(F) & \text{for } p = 0, \end{cases} \tag{4.1}$$

where $\mu_n(F)$ is the group of nth roots of 1 in F. In particular, if F contains a primitive l^νth root of unity, then the choice of one such defines an element β_{l^ν} of $H^0(\mathrm{Spec}\,F, \mathbb{Z}/l^\nu(1))$, which we call the *motivic Bott element*.

Lemma 4.2. *Let L be a field of characteristic prime to l containing μ_{l^ν}. Let L^*/L be a Galois extension of l-cohomological dimension at most one. Then $\beta_{l^\nu} \in H^0(\mathcal{Z}^1(\mathrm{Spec}\,L, *)/l^\nu)$ admits a family of inductors*

$$\tilde{\beta}_{l^\nu} \in H^0(\mathrm{holim}\,\mathbb{H}_*(-/L, \mathcal{Z}^1(-,*)/l^\nu))$$

for the functor

$$L' \mapsto \mathcal{Z}^1(\mathrm{Spec}\,L', *)/l^\nu; \quad L' \in \mathbf{F}_{L^*/L}.$$

Proof. We let $F : \mathbf{F}_{L^*/L} \to \mathbf{C}(\mathbf{Ab})$ be the functor

$$L' \mapsto H^0(G(L'), \mathcal{C}^*(G, H^0(\mathcal{Z}^1(L_{\mathrm{sep}}, *)/l^\nu))).$$

Here $G = \mathrm{Gal}(L_{\mathrm{sep}}/L)$, and $G(L') \subset G$ is the subgroup $\mathrm{Gal}(L'/L)$. Similarly, letting $G_{L^*} = \mathrm{Gal}(L^*/L)$ and $G_{L^*}(L') = \mathrm{Gal}(L^*/L')$, we have the functor $F_{L^*} : \mathbf{F}_{L^*/L} \to \mathbf{C}(\mathbf{Ab})$,

$$L' \mapsto H^0(G_{L^*}(L'), \mathcal{C}^*(G_{L^*}, H^0(\mathcal{Z}^1(L^*, *)/l^\nu))).$$

By the construction of §3.10, the element β_{l^ν} of $H^0(\mathcal{Z}^1(L,*)/l^\nu) = H^0(F_{L^*}(L))$ admits a family of inductors $\tilde{x}$ in $H^0(\mathrm{holim}\,\mathbb{H}_*(-/L, F_{L^*}(-))$.

Since L^*/L has l-cohomological dimension at most one, so does L^*/L_α for all finite extensions L_α/L in $\mathbf{F}_{L^*/L}$. Thus, the canonical map

$$\tau_{\leq 1}F_{L^*}(L') \to F_{L^*}(L')$$

is a quasi-isomorphism for all such L_α, hence we have the family of inductors $\tilde{x}_1$ for β_{l^ν} for the functor $\tau_{\leq 1}F_{L^*}$.

The natural maps

$$i_{L'}: \tau_{\leq 1}F_{L^*}(L') \to \tau_{\leq 1}F(L')$$

define the natural transformation $i: \tau_{\leq 1}F_{L^*} \to \tau_{\leq 1}F$. Applying i to the family $\tilde{x}_1$ gives the family of inductors

$$\tilde{y} \in H^0(\operatorname{holim} \mathbb{H}_*(-/L, \tau_{\leq 1}F(-))$$

for $\beta_{l^\nu} \in H^0(\mathcal{Z}^1(L_{\rm sep}, *)/l^\nu)$.

By our computation of $H^p(\mathcal{Z}^1(L_{\rm sep}, *)/l^\nu)$ in (4.1), we see that the natural maps

$$H^0(\mathcal{Z}^1(L_{\rm sep}, *)/l^\nu) \cong \tau_{\leq 0}\tau_{\geq 0}\mathcal{Z}^1(L_{\rm sep}, *)/l^\nu \to \tau_{\geq 0}\mathcal{Z}^1(L_{\rm sep}, *)/l^\nu \leftarrow \mathcal{Z}^1(L_{\rm sep}, *)/l^\nu$$

are all quasi-isomorphisms. Thus, we have the family of inductors

$$\tilde{z} \in H^0(\operatorname{holim} \mathbb{H}_*(-/L, \tau_{\leq 1}H^0(G(-), \mathcal{C}^*(G, \mathcal{Z}^1(L_{\rm sep}, *)/l^\nu)))$$

for $\beta_{l^\nu} \in H^0(\mathcal{Z}^1(L_{\rm sep}, *)/l^\nu)$.

It follows directly from (4.1), together with the isomorphisms given by the Kummer sequence

$$H^0_{\text{ét}}(L', \mu_{l^\nu}) \cong \mu_{l^\nu}(L'),\ H^1_{\text{ét}}(L', \mu_{l^\nu}) \cong L'^*/(L'^*)^{l^\nu}$$

that the natural map

$$\mathcal{Z}^1(L', *)/l^\nu \to \tau_{\leq 1}H^0(G(L'), \mathcal{C}^*(G, \mathcal{Z}^1(L_{\rm sep}, *)/l^\nu))$$

is a quasi-isomorphism. Thus the family of inductors $\tilde{z}$ gives us the family of inductors

$$\tilde{\beta}_{l^\nu} \in \operatorname{holim} \mathbb{H}_*(-/L, \mathcal{Z}^1(-, *)/l^\nu).$$

for $\beta_{l^\nu} \in H^0(\mathcal{Z}^1(L, *)/l^\nu)$. □

Lemma 4.3. *Let L be a field of characteristic prime to l containing μ_{l^ν}, let $p: X \to \operatorname{Spec} L$ be an essentially smooth L-scheme, and let L^*/L be a Galois extension of l-cohomological dimension ≤ 1. Then there is a commutative diagram in $\mathbf{D}^-(\mathbf{Ab})$,*

(4.2)
$$\begin{array}{ccc}
\mathcal{Z}^{q+1}(X, *)/l^\nu & \xrightarrow{\eta'} & \mathbb{H}^*(L^*/L; \mathcal{Z}^{q+1}(X \times_L L^*, *)/l^\nu) \\
\uparrow{\scriptstyle \cup p^*\beta_{l^\nu}} & \nwarrow{\scriptstyle \phi(L^*/L;X)} & \uparrow{\scriptstyle \cup p^*\beta_{l^\nu}} \\
\mathcal{Z}^q(X, *)/l^\nu & \xrightarrow[\eta]{} & \mathbb{H}^*(L^*/L; \mathcal{Z}^q(X \times_L L^*, *)/l^\nu),
\end{array}$$

natural in X and in L^*/L, where η and η' are the canonical maps.

Proof. We use the notation from §2.2. We let $\mathcal{Z}^{s,t}(-, *, *)$ denote the reindexed cohomological version of $z^{s+t}(-, *, *)$, i.e.,

$$\mathcal{Z}^{s,t}(-, a, b) = z^{s+t}(-, 2s - a, 2t - b),$$

and define the cohomological version $\mathcal{Z}^{s,t}(-, *, *)_\mathcal{T}$ of $z^{s+t}(-, *, *)_\mathcal{T}$ similarly. The total complexes $\operatorname{Tot} \mathcal{Z}^{s,t}(-, *, *)$ and $\operatorname{Tot} \mathcal{Z}^{s,t}(-, *, *)_\mathcal{T}$ depend only on $s+t$, and are just reindexed cohomological versions of $\operatorname{Tot} z^{s+t}(X, *, *)$ and $\operatorname{Tot} z^{s+t}(X, *, *)_\mathcal{T}$, respectively.

Let L_α/L be a finite Galois extension. We have the map

$$\gamma: \mathcal{Z}^q(X \times_L L_\alpha, *) \otimes \mathcal{Z}^r(\operatorname{Spec} L_\alpha, *) \to \operatorname{Tot} \mathcal{Z}^{q,r}(X, *, *)$$

defined as the composition

$$\mathcal{Z}^q(X\times_L L_\alpha,p)\otimes\mathcal{Z}^r(\operatorname{Spec}L_\alpha,*)\xrightarrow{\boxtimes_1}\operatorname{Tot}\mathcal{Z}^{q,r}(X\times_L L_\alpha,*,*)\xrightarrow{p_{1*}}\operatorname{Tot}\mathcal{Z}^{q,r}(X,*,*).$$

Truncating and using the cap product of cohomology on homology, the map γ canonically extends to

$$\mathbb{H}^*(L_\alpha/L;\tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha,*))\otimes\mathbb{H}_*(L_\alpha/L;\mathcal{Z}^r(\operatorname{Spec}L_\alpha,*))\xrightarrow{\mathbb{H}\gamma}\mathbb{H}_*(L_\alpha/L;\operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,r}(X,*,*))$$

(see §7.2 for the definition of the truncation $\tau^1_{\geq -N}$).

Composed with the canonical map

$$\mathbb{H}_*(L_\alpha/L;\operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,r}(X,*,*))\to\operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,r}(X,*,*),$$

the map $\mathbb{H}\gamma$ induces the map of complexes

$$\mathbb{H}_*(L_\alpha/L;\mathcal{Z}^r(\operatorname{Spec}L_\alpha,*)/l^\nu)\xrightarrow{\psi_\alpha}\operatorname{Hom}(\mathbb{H}^*(L_\alpha/L;\tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha,*)/l^\nu),\operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,r}(X,*,*)/l^\nu)).$$

We consider $\mathbb{H}_*(L_\alpha/L;\mathcal{Z}^r(\operatorname{Spec}L_\alpha,*)/l^\nu)$ as a functor from $(\mathbf{F}^{\mathrm{Gal}}_{L^*/L})^{\mathrm{op}}$ to $\mathbf{C}(\mathbf{Ab})$ via the maps $\mathbb{H}_*(\mathrm{Tr}_{L_\beta/L_\alpha})$. Similarly, the natural maps

$$\mathbb{H}^*(L_\alpha/L;\tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha,*)/l^\nu)\to\mathbb{H}^*(L_\beta/L;\tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\beta,*)/l^\nu)$$

for an extension L_β/L_α makes

$$\operatorname{Hom}(\mathbb{H}^*(L_\alpha/L;\tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha,*)/l^\nu),\operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,r}(X,*,*)/l^\nu))$$

a functor from $(\mathbf{F}^{\mathrm{Gal}}_{L^*/L})^{\mathrm{op}}$ to $\mathbf{C}(\mathbf{Ab})$. The projection formula

$$p_*(p^*x\boxtimes_1 y)=x\boxtimes_1 p_*(y)$$

shows that the maps ψ_α define a natural transformation of functors from $(\mathbf{F}^{\mathrm{Gal}}_{L^*/L})^{\mathrm{op}}$ to $\mathbf{C}(\mathbf{Ab})$. Via the canonical isomorphism of complexes

$$\begin{aligned}&\operatorname*{holim}_\alpha\operatorname{Hom}(\mathbb{H}^*(L_\alpha/L;\tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha,*)/l^\nu),\operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,r}(X,*,*)/l^\nu))\\ \cong&\operatorname{Hom}(\operatorname*{hocolim}_\alpha\mathbb{H}^*(L_\alpha/L;\tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha,*)/l^\nu),\operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,r}(X,*,*)/l^\nu)),\end{aligned}$$

we arrive at the maps

$$\begin{aligned}&\operatorname*{holim}_\alpha\mathbb{H}_*(L_\alpha/L;\mathcal{Z}^r(\operatorname{Spec}L_\alpha,*)/l^\nu)\xrightarrow{\psi_N}\\ &\operatorname{Hom}(\operatorname*{hocolim}_\alpha\mathbb{H}^*(L_\alpha/L;\tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha,*)/l^\nu),\operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,r}(X,*,*)/l^\nu)),\end{aligned}$$

compatible in N.

Take $r=1$. Let $\tilde\beta_{l^\nu}$ be the family of inductors for β_{l^ν} constructed in Lemma 4.2; we consider $\tilde\beta_{l^\nu}$ as a cycle of degree zero in the complex

$$\operatorname*{holim}_\alpha\mathbb{H}_*(L_\alpha/L;\mathcal{Z}^r(\operatorname{Spec}L_\alpha,*)/l^\nu).$$

We thus have the map

$$\text{(4.3)}\quad \operatorname*{holim}_N \operatorname*{hocolim}_\alpha \mathbb{H}^*(L_\alpha/L; \tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha, *)/l^\nu) \xrightarrow{\operatorname{holim}_N \psi_N(\tilde\beta_{l^\nu})} \operatorname*{holim}_N \operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,1}(X,*,*)/l^\nu.$$

For fixed p, we have

$$\mathbb{H}^p(L^*/L; \tau_{\geq -N}\mathcal{Z}^q(X\times_L L^*, *)/l^\nu) = \mathbb{H}^p(L^*/L; \mathcal{Z}^q(X\times_L L^*, *)/l^\nu)$$

for all N sufficiently large, since L^*/L has finite l-cohomological dimension. Since

$$\operatorname{Tot}\mathcal{Z}^q(X\times_L L^*, *, *)/l^\nu \cong \varprojlim_N \operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^q(X\times_L L^*, *, *)/l^\nu$$

and

$$\mathbb{H}^*(L^*/L; \mathcal{Z}^q(X\times_L L^*, *)/l^\nu) \cong \varprojlim_N \mathbb{H}^*(L^*/L; \tau_{\geq -N}\mathcal{Z}^q(X\times_L L^*, *)/l^\nu),$$

it follows from Lemma 7.7 that the natural maps

$$\mathbb{H}^*(L^*/L; \mathcal{Z}^q(X\times_L L^*, *)/l^\nu) \to \operatorname*{holim}_N \mathbb{H}^*(L^*/L; \tau_{\geq -N}\mathcal{Z}^q(X\times_L L^*, *)/l^\nu),$$

and

$$\operatorname{Tot}\mathcal{Z}^{q,1}(X,*,*)/l^\nu \to \operatorname*{holim}_N \operatorname{Tot}\tau^1_{\geq -N}\mathcal{Z}^{q,1}(X,*,*)/l^\nu$$

are quasi-isomorphisms. Since

$$\mathbb{H}^*(L^*/L; \tau_{\geq -N}\mathcal{Z}^q(X\times_L L^*, *)/l^\nu) \cong \varinjlim_\alpha \mathbb{H}^*(L_\alpha/L; \tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha, *)/l^\nu),$$

Lemma 7.5 and Lemma 7.7 imply that the natural map

$$\operatorname*{holim}_N \operatorname*{hocolim}_\alpha \mathbb{H}^*(L_\alpha/L; \tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha, *)/l^\nu) \to \operatorname*{holim}_N \mathbb{H}^*(L^*/L; \tau_{\geq -N}\mathcal{Z}^q(X\times_L L^*, *)/l^\nu),$$

is a quasi-isomorphism. Thus, the map (4.3) gives us the natural map in $\mathbf{D}^-(\mathbf{Ab})$

$$\phi(\widetilde{L^*/L}; X) : \mathbb{H}^*(L^*/L; \mathcal{Z}^q(X\times_L L^*, *)/l^\nu) \to \operatorname{Tot}\mathcal{Z}^{q,1}(X,*,*)/l^\nu.$$

Composing $\phi(\widetilde{L^*/L}; X)$ with

$$\operatorname{Tot}\mathcal{Z}^{q,1}(X,*,*)/l^\nu \xrightarrow{\iota^{-1}} \operatorname{Tot}\mathcal{Z}^{q,1}(X,*,*)_{\mathcal{T}}/l^\nu \xrightarrow{(\mathrm{id}\times T)^*} \mathcal{Z}^{q+1}(X,*)/l^\nu$$

(see (2.2) and (2.3) for the notation) gives the natural map $\phi(L^*/L; X)$.

We have the natural map (7.3) (corresponding to the trivial extension L/L)

$$\rho_N : \mathcal{Z}^q(X,p)/l^\nu \to \operatorname*{hocolim}_\alpha \mathbb{H}^*(L_\alpha/L; \tau_{\geq -N}\mathcal{Z}^q(X\times_L L_\alpha, *)/l^\nu).$$

For $y\in \mathcal{Z}^q(X,p)/l^\nu$, we have $\psi_N(\tilde\beta_{l^\nu})(\rho_N(y)) = y\boxtimes_1 \beta_{l^\nu}$. The commutativity of the left-hand triangle in (4.2) follows directly from this.

For the commutativity of the right-hand triangle in (4.2), consider a simplicial scheme X_* whose boundary maps $d^n_j : X_{n+1}\to X_n$ are flat. We have the double complex $(a,b)\mapsto \mathcal{Z}^q(X_a, b)$, where the first differential is $\sum_j (-1)^j d^{a*}_j$, and the second is the differential in $\mathcal{Z}^q(X_a,*)$. We let $\mathcal{Z}^q(X_*,*)$ be the extended total complex (see §7.1) of this double complex.

We note that the contruction of $\phi(L^*/L; X)$ extends in the obvious way to simplicial schemes X_* with flat boundary maps. For each finite subextension L_α/L of L^*/L, let X_α be the Čech simplicial scheme

$$X \times_L L_\alpha \leftrightarrows X \times_L (L_\alpha \otimes_L L_\alpha) \substack{\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow} \dots\ .$$

We have the quasi-isomorphisms

$$\begin{array}{c} \varinjlim_\alpha \tau_{\geq -N}\mathcal{Z}^q(X_\alpha, *)/l^\nu \\ \downarrow \\ \varinjlim_\alpha \mathbb{H}^*(L^*/L; \tau_{\geq -N}\mathcal{Z}^q(X_\alpha \times_L L^*, *)/l^\nu) \\ \uparrow \\ \mathbb{H}^*(L^*/L; \tau_{\geq -N}\mathcal{Z}^q(X \times_L L^*, *)/l^\nu) \end{array}$$

and the commutative diagrams (in $\mathbf{D}^-(\mathbf{Ab})$)

$$\begin{array}{ccc} \mathcal{Z}^{q+1}(X_\alpha, *)/l^\nu & \longrightarrow & \mathbb{H}^*(L^*/L; \mathcal{Z}^{q+1}(X_\alpha \times_L L^*, *)/l^\nu) \\ {\scriptstyle \cup p^*\beta_{l^\nu}}\uparrow & & \uparrow{\scriptstyle \cup p^*\beta_{l^\nu}} \\ \mathcal{Z}^q(X_\alpha, *)/l^\nu & \longrightarrow & \mathbb{H}^*(L^*/L; \mathcal{Z}^q(X_\alpha \times_L L^*, *)/l^\nu) \end{array}$$

and

$$\begin{array}{ccc} \mathbb{H}^*(L^*/L; \mathcal{Z}^{q+1}(X \times_L L^*, *)/l^\nu) & \longrightarrow & \mathbb{H}^*(L^*/L; \mathcal{Z}^{q+1}(X_\alpha \times_L L^*, *)/l^\nu) \\ {\scriptstyle \cup p^*\beta_{l^\nu}}\uparrow & & \uparrow{\scriptstyle \cup p^*\beta_{l^\nu}} \\ \mathbb{H}^*(L^*/L; \mathcal{Z}^q(X \times_L L^*, *)/l^\nu) & \longrightarrow & \mathbb{H}^*(L^*/L; \mathcal{Z}^q(X_\alpha \times_L L^*, *)/l^\nu). \end{array}$$

Since the left-hand triangle of (4.2) commutes with X replaced by X_α, this gives the commutativity of the right-hand triangle in (4.2). □

Proposition 4.4. *Let L be a field of characteristic prime to l containing μ_{l^ν}, let $p: X \to \operatorname{Spec} L$ be an essentially smooth L-scheme, and let L^*/L be a Galois extension of l-cohomological dimension ≤ 1. Then the natural map*

$$\mathcal{Z}^*(X, *)/l^\nu[\beta_{l^\nu}^{-1}] \to \mathbb{H}^*(L^*/L, \mathcal{Z}^*(X \times_L L^*, *)/l^\nu[\beta_{l^\nu}^{-1}])$$

is a quasi-isomorphism.

Proof. This follows directly from the commutativity of the diagram in Lemma 4.3. □

Theorem 4.5. *Let k be a field containing a primitive l^νth root of unity, where l is a prime with $(l, \operatorname{char} k) = 1$. If $l = 2$, and $\operatorname{char} k = 0$, we assume that k contains a square root of -1. Let X be an essentially smooth k-scheme. Then the natural map*

$$\mathcal{Z}^*(X, *)/l^\nu[\beta_{l^\nu}^{-1}] \to \mathbb{H}^*(k, \mathcal{Z}^*(X \times_k k_{\mathrm{sep}}, *)/l^\nu[\beta_{l^\nu}^{-1}])$$

is a quasi-isomorphism.

Proof. By taking limits, we may assume that X is of finite type over k. In this case, X is defined over a subfield k' of k which is finitely generated over the prime field. If we have proved the theorem for all k finitely generated over the prime field, then the general result follows by taking limits; we may therefore assume that k is finitely generated over the prime field.

Let $L_0 = k$. If we construct a finite tower of fields

$$L_0 \subset L_1 \subset \ldots \subset L_N = k_{\mathrm{sep}}$$

such that each layer L_i/L_{i-1} has l-cohomological dimension ≤ 1, the result follows by induction on N, Proposition 4.4 and the isomorphism (3.1)

$$\mathbb{H}^*(k, \mathcal{Z}^*(X \times_k k_{\mathrm{sep}}, *)/l^\nu[\beta_{l^\nu}^{-1}]) \cong \mathbb{H}^*(L_1/k, \mathbb{H}^*(L_1, \mathcal{Z}^*(X \times_k k_{\mathrm{sep}}, *)/l^\nu[\beta_{l^\nu}^{-1}]).$$

To construct such a tower, let k_0 denote the algebraic closure of the prime field in L_0; k_0 is thus a finite extension of the prime field.

Suppose $\operatorname{char} k_0 = 0$. Let $k_1 = k_0(\mu_{l^\infty})$, and let k_2 be the the algebraic closure of k_1. Then $\mathrm{Gal}(k_1/k_0)$ is isomorphic to $\mathbb{Z}_l$, hence k_1/k_0 has l-cohomological dimension 1. By [10, II, Proposition 9] k_2/k_1 has l-cohomological dimension 1 as well. If $\operatorname{char} k_0 > 0$, let $k_1 = k_2$ be the algebraic closure of k_0. By [10, II, Example 3.3(a)], k_1/k_0 has cohomological dimension 1. Thus, in either case, we may then take

$$L_1 = L_0 k_1, \quad L_2 = L_1 k_2.$$

L_2 is the function field of a smooth affine variety Y of finite type over $\bar{k}_0$. Shrinking Y to a suitable open subset, we may assume that Y admits a smooth k_2-morphism $p : Y \to C$, where C is an open subscheme of $\mathbb{A}^1_{k_2}$, such that the fibers of p are geometrically irreducible. Letting k_3 be the separable closure of $k_2(C)$, the extension k_3/k_2 has l-cohomological dimension 1, hence we may take $L_3 := L_2 k_3$. Since

$$\mathrm{tr.dim}_{k_3} L_3 = \mathrm{tr.dim}_{k_2} L_2 - 1,$$

we may continue in this fashion, giving the tower

$$L_0 \subset L_1 \subset \ldots \subset L_{N-1},$$

with L_{N-1} separably generated and of transcendence dimension 1 over a separably closed subfield k_{N-1}. We may then take L_N to be the separable closure of L_{N-1}, completing the construction of the desired tower. □

5. The proof of Theorem 1.1

Let X be an essentially smooth scheme over a field k, let X_{Zar} and $X_{\text{ét}}$ be the small Zariski and étale sites on X, respectively, and let $\epsilon = \epsilon_X : X_{\text{ét}} \to X_{\mathrm{Zar}}$ be the change of topology morphism.

Let $\mathcal{Z}^q(*)_X$ denote the complex of sheaves $U \mapsto \mathcal{Z}^q(U, *)$ on X_{Zar}, and let $\mathcal{Z}^q_{\text{ét}}(*)_X$ denote the étale sheafification of $\mathcal{Z}^q(*)_X$. We have the natural map in $\mathbf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}))$

$$\eta : \mathcal{Z}^q(*)_X/n \to R\epsilon_* \mathcal{Z}^q_{\text{ét}}(*)_X/n. \tag{5.1}$$

Sheafifying the cycle class map, as in [5, §3.10], we have the natural map in $\mathbf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}))$

$$\mathrm{cl}^q : \mathcal{Z}^q(*)_X/n \to R\epsilon_* \mu_n^{\otimes q} \tag{5.2}$$

and the natural map in in $\mathbf{D}(\mathrm{Sh}_{X_{\text{ét}}})$

$$\mathrm{cl}^q_{\text{ét}} : \mathcal{Z}^q_{\text{ét}}(*)_X/n \to \mu_n^{\otimes q}. \tag{5.3}$$

As we shall soon see, the cycle class map (2.4) is gotten from (5.2) by applying the functor $R\Gamma(X, -)$.

Lemma 5.1. *Let $n \geq 2$ be an integer with $(n, \mathrm{char} k) = 1$, and let X be an essentially smooth k-scheme. Then the map* (5.3)

$$\mathrm{cl}^q_{\text{ét}} : \mathcal{Z}^q_{\text{ét}}(*)_X/n \to \mu_n^{\otimes q}$$

is an isomorphism in $\mathbf{D}(\mathrm{Sh}(X_{\text{ét}}))$. If k contains a primitive nth root of unity, then the maps $\mathrm{cl}^q_{\text{ét}}$ induce a map in $\mathbf{D}(\mathrm{Sh}(X_{\text{ét}}))$

$$\mathrm{cl}^*_{\text{ét}} : \mathcal{Z}^*_{\text{ét}}(*)_X/n[\beta_n^{-1}] \to \mu_n^{\otimes *} := \oplus_{q=-\infty}^{\infty} \mu_n^{\otimes q},$$

which is an isomorphism in $\mathbf{D}(\mathrm{Sh}(X_{\text{ét}}))$ as well.

Proof. Suppose that k contains μ_n, and let ζ be the primitive nth root of unity used to construct the Bott element β_n. Since $\times\zeta : \mu_{l^\nu}^{\otimes q} \to \mu_{l^\nu}^{\otimes q+1}$ is an isomorphism, and since $\zeta = \mathrm{cl}^{1,0}(\beta_n)$, the natural map

$$\mu_n^{\otimes *} \to \mu_n^{\otimes *}[\mathrm{cl}^{1,0}(\beta_n)^{-1}]$$

is an isomorphism. Thus, the maps $\mathrm{cl}^q_{\text{ét}}$ do induce a map

$$\mathrm{cl}^*_{\text{ét}} : \mathcal{Z}^*_{\text{ét}}(*)_X/n[\beta_n^{-1}] \to \mu_n^{\otimes *},$$

and it suffices to show that the map (5.3) is an isomorphism. This is proven in [5, Theorem 4.13], as a consequence of the Suslin-Voevodsky theorem [12]. □

Theorem 5.2. *Let k be a field, X an essentially smooth k-scheme, and $n \geq 2$ an integer with $(n, \mathrm{char} k) = 1$. Suppose that k contains a primitive nth root of unity. If $\mathrm{char}\, k = 0$, we suppose in addition that either n is odd or that k contains a square root of -1. Then the sheafified cycle class map* (5.2) *induces an isomorphism in* $\mathbf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}))$

$$\mathrm{cl}^* : \mathcal{Z}^*(*)_X/n[\beta_n^{-1}] \to R\epsilon_*(\mu_n^{\otimes *}). \tag{5.4}$$

Proof. Let ζ be the primitive nth root of unity used to construct β_n. Since $\mathrm{cl}^1(\beta_n) = \zeta$, we see that the maps cl^q do indeed induce a map of the desired form.

As in the proof of Theorem 4.5, we may assume that k is finitely generated over the prime field. Letting k_0 be the algebraic closure of the prime field in k, X is an essentially smooth k_0-scheme, hence we may replace k with k_0. In particular, we may assume that k is perfect.

Let $X^{(a)}$ denote the set of codimension a points of X; for $x \in X^{(a)}$, we let $i_x : x \to X$ denote the inclusion. We have the motivic cohomology (Zariski) sheaf $\mathcal{H}^p(\mathbb{Z}/n(q))$ and the étale cohomology (Zariski) sheaf $\mathcal{H}^p_{\text{ét}}(\mu_n^{\otimes q})$, associated to the presheaves $U \mapsto H^p(U, \mathbb{Z}/n(q))$ and $U \mapsto H^p_{\text{ét}}(U, \mu_n^{\otimes q})$, respectively. We have the Gersten resolution of $\mathcal{H}^p(\mathbb{Z}/n(q))$

$$0 \to \mathcal{H}^p(\mathbb{Z}/n(q)) \to \ldots \to \coprod_{x \in X^{(a)}} i_{x*}H^{p-a}(k(x), \mathbb{Z}/n(q-a)) \to \ldots \tag{5.5}$$

and the Gersten resolution of $\mathcal{H}^p_{\text{ét}}(\mu_n^{\otimes q})$

$$0 \to \mathcal{H}^p_{\text{ét}}(\mu_n^{\otimes q}) \to \ldots \to \coprod_{x \in X^{(a)}} i_{x*}H^{p-a}_{\text{ét}}(k(x), \mu_n^{\otimes q-a}) \to \ldots \tag{5.6}$$

(see [3, Theorem 4.2]; for the resolution (5.5), one needs the localization property of motivic cohomology, proved in [2]). Inverting β_n in (5.5) and (5.6) gives the Gersten resolutions

(5.7)
$$0 \to \mathcal{H}^p(\mathbb{Z}/n(*))[\beta_n^{-1}] \to \dots \to \coprod_{x \in X^{(a)}} i_{x*} H^{p-a}(k(x), \mathbb{Z}/n(*-a))[\beta_n^{-1}] \to \dots$$

and

$$0 \to \mathcal{H}^p_{\text{ét}}(\mu_n^{\otimes *}) \to \dots \to \coprod_{x \in X^{(a)}} i_{x*} H^{p-a}_{\text{ét}}(k(x), \mu_n^{\otimes * - a}) \to \dots \tag{5.8}$$

By [5, Propositions 4.5 and 4.7], the maps cl^* induce a map from (5.7) to (5.8). This reduces us to the case $X = \operatorname{Spec} F$, with F a field finitely generated over k.

We may factor (5.4) as the composition

$$\mathcal{Z}^*(*)_X/n[\beta_n^{-1}] \xrightarrow{\iota} R\epsilon_* \mathcal{Z}^*_{\text{ét}}(*)_X/n[\beta_n^{-1}] \xrightarrow{R\epsilon_* \mathrm{cl}^*_{\text{ét}}} R\epsilon_*(\mu_n^{\otimes *}),$$

where ι is the natural map. Since $R\epsilon_* \mathrm{cl}^*_{\text{ét}}$ is an isomorphism by Lemma 5.1, it suffices to show that

$$\iota : \mathcal{Z}^*(*)_X/n[\beta_n^{-1}] \to R\epsilon_* \mathcal{Z}^*_{\text{ét}}(*)_X/n[\beta_n^{-1}] \tag{5.9}$$

is an isomorphism.

For $X = \operatorname{Spec} F$, the map (5.9) is isomorphic in $\mathbf{D}(\mathbf{Ab})$ to the natural map

$$\mathcal{Z}^*(\operatorname{Spec} F, *)/n[\beta_n^{-1}] \to \mathbb{H}^*(F, \mathcal{Z}^*(\operatorname{Spec} F_{\text{sep}}, *)/n[\beta_n^{-1}]),$$

which is an isomorphism by Theorem 4.5. □

We now complete the proof of Theorem 1.1. We have the natural maps

$$\phi : \mathcal{Z}^q(X, *)/n \to R\Gamma(X, \mathcal{Z}^q(*)_X/n),$$
$$\phi_{\text{ét}} : G^*_{\text{ét}}(X, \mu_n^{\otimes q}) \to R\Gamma(X, R\epsilon_* \mu_n^{\otimes q}).$$

It follows from the Mayer-Vietoris property of motivic cohomology [2], [8] that ϕ is an isomorphism in $\mathbf{D}(\mathbf{Ab})$. Since the étale Godement resolution is a flasque resolution for the étale topology, $\phi_{\text{ét}}$ is an isomorphism in $\mathbf{D}(\mathbf{Ab})$.

In [5, §4.1], it is shown that the diagram

$$\begin{array}{ccc} \mathcal{Z}^q(X, *)/n & \xrightarrow{\phi} & R\Gamma(X, \mathcal{Z}^q(*)_X/n) \\ \downarrow{\scriptstyle \mathrm{cl}^q} & & \downarrow{\scriptstyle R\Gamma(X, \mathrm{cl}^q)} \\ G^*_{\text{ét}}(X, \mu_n^{\otimes q}) & \xrightarrow[\phi_{\text{ét}}]{} & R\Gamma(X, R\epsilon_* \mu_n^{\otimes q}) \end{array}$$

is commutative in $\mathbf{D}(\mathbf{Ab})$. Thus, Theorem 1.1 follows directly from Theorem 5.2.

6. An extension of Theorem 1.1

We conclude by proving an extension of Theorem 1.1, in which we remove the hypothesis that k contain the nth roots of unity, although we still need to assume $\sqrt{-1} \in k$ in case n is even and $\operatorname{char} k = 0$ (see however Remark 6.3).

We take $n = l^\nu$, with l prime. We have the standard Bockstein sequences

$$\to H^p(X, \mathbb{Z}/l^{\nu-r}(q)) \to H^p(X, \mathbb{Z}/l^\nu(q)) \to H^p(X, \mathbb{Z}/l^r(q)) \xrightarrow{\partial_r} H^{p+1}(X, \mathbb{Z}/l^{\nu-r}(q)) \to$$

Lemma 6.1. *Let X be a smooth, quasi-projective k-scheme. Suppose $\nu - r \geq r$. For $a \in H^p(X, \mathbb{Z}/l^r(q))$, $b \in H^{p'}(X, \mathbb{Z}/l^{\nu-r}(q'))$, we have*

$$\partial_r(a \cup b) = \partial_r(a) \cup b + (-1)^p a \cup \partial_{\nu-r}(b)$$

in $H^{p+p'}(X, \mathbb{Z}/l^r(q+q'))$.

Proof. It suffices to prove the similar formula for the external product

$$\boxtimes : H^*(X, \mathbb{Z}/l^r(q)) \otimes H^*(X, \mathbb{Z}/l^{\nu-r}(q')) \to H^*(X \times_k X, \mathbb{Z}/l^r(q+q')).$$

To simplify the notation, we replace p with $2q - p$, and p' with $2q' - p'$. Then a and b are represented by algebraic cycles A on $X \times \Delta^p$ and B on $X \times \Delta^{p'}$, intersecting all faces properly. We use the notation of §2.2.

Using Jouanolou's trick, we may suppose that X is affine; replacing X with $X \times \mathbb{A}^{\max(q,q')}$, and using the homotopy property of motivic cohomology, we may assume that $\dim_k X \geq \max(q, q')$. By Suslin's result [11], we may take A and B with A equidimensional over Δ^p and B equidimensional over $\Delta^{p'}$. Thus, the product $A \times B$ is in $z^{q+q'}(X \times X, (p, p'))_{\mathcal{T}}$. The desired Leibniz rule is now a consequence of the formal identity satisfied by the triangulation T which makes $(\mathrm{id} \times T)^*$ a map of complexes. □

Now suppose that k contains μ_l, and fix an integer $\nu > 1$. By Lemma 6.1, $\beta_l^{l^{\nu-1}} \in H^0(k, \mathbb{Z}/l(l^{\nu-1}))$ lifts to some x in $H^0(k, \mathbb{Z}/l^\nu(l^{\nu-1}))$. We define

$$\mathcal{Z}^*(X, *)/l^\nu[\beta^{-1}] := \mathcal{Z}^*(X, *)/l^\nu[x^{-1}];$$

it is easily seen that $\mathcal{Z}^*(X, *)/l^\nu[\beta^{-1}]$ is independent (in the derived category) of the choice of x lifting $\beta_l^{l^{\nu-1}}$. We write $\mathrm{cl}(\beta)$ for $\mathrm{cl}^{l^{\nu-1}}(x)$.

Cup product with $\mathrm{cl}(\beta)$ on étale cohomology is an isomorphism, and the cycle class map extends to a map

$$\mathrm{cl}^*_{X,l^\nu} : \mathcal{Z}^*(X, *)/l^\nu[\beta^{-1}] \to G^*_{\text{ét}}(X, \mu_{l^\nu}^{\otimes *})[\mathrm{cl}(\beta)^{-1}] \tag{6.1}$$

If k does not contain μ_l, we have the extension $k \subset k' := k(\mu_l)$ of degree prime to l. We may then define the complex $\mathcal{Z}^*(X_{k_l}, *)/l^\nu[\beta^{-1}]$ as above, and take the $\mathrm{Gal}(k'/k)$ invariants to define $\mathcal{Z}^*(X, *)/l^\nu[\beta^{-1}]$ and the map (6.1).

Theorem 6.2. *Let k be a field, $l \neq \mathrm{char}\, k$ a prime. In case $l = 2$ and $\mathrm{char}\, k = 0$, we assume that k contains $\sqrt{-1}$. Then the map* (6.1) *is an isomorphism in* $\mathbf{D}(\mathbf{Ab})$.

Proof. We may assume that k contains μ_l. The filtration of $\mathcal{Z}^*(X, *)/l^\nu[\beta^{-1}]$ by powers of l gives a sequence of distinguished triangles

$$\mathcal{Z}^*(X, *)/l^{i-1}[\beta^{-1}] \to \mathcal{Z}^*(X, *)/l^i[\beta^{-1}] \to \mathcal{Z}^*(X, *)/l[\beta_l^{-1}].$$

We have the similarly defined sequence for localized étale cohomology; the maps (6.1) give maps of distinguished triangles. From Theorem 1.1, the cycle class map for $i = 1$ is an isomorphism, so the five lemma implies that the cycle class map for $i = 1, \ldots, \nu$ are also isomorphisms. □

Remark 6.3. Theorem 1.1, Theorem 5.2 and Theorem 6.2 all remain true without the hypothosis that k contain $\sqrt{-1}$. This follows from the Milnor conjecture (proved by Voevodsky in [15]), which, together with [5], implies that the cycle class map

$$\mathrm{cl}_X^q : \mathcal{Z}^q(X,*)/2^\nu \to G^*_{\text{ét}}(X,\mu_{2^\nu}^{\otimes q})$$

induces an isomorphism in cohomology in degrees $\leq q$.

7. Some homological algebra

For the reader's convenience, we collect a few basic facts about homotopy limits and colimits.

7.1. **Total complexes.** Let

$$(C^{a,b}, d_1^{a,b} : C^{a,b} \to C^{a+1,b}, d_2^{a,b} : C^{a,b} \to C^{a,b+1})$$

be a double complex (of abelian groups), i.e., $d_1^2 = 0 = d_2^2$ and $d_1d_2 + d_2d_1 = 0$. The associated *total complex* $(\mathrm{Tot}\, C, d)$ is given by

$$(\mathrm{Tot}\, C)^n := \bigoplus_{a+b=n} C^{a,b},\ d^n := \sum_{a+b=n} d_1^{a,b} + d_2^{a,b}.$$

The *extended* total complex $(\widetilde{\mathrm{Tot}}C, d)$ is defined by

$$(\widetilde{\mathrm{Tot}}C)^n := \prod_{a+b=n} C^{a,b},\ d^n := \prod_{a+b=n} d_1^{a,b} + d_2^{a,b}.$$

Tot and $\widetilde{\mathrm{Tot}}$ obviously define functors from the category of double complexes to the category of complexes.

Given a double complex C^{**}, we may take cohomology with respect to d_1, forming for each a the complex $(H_1^a(C), d_2)$. Similarly, we may take cohomology with respect to d_2, forming for each b the complex $(H_2^b(C), d_1)$.

7.2. **Truncations.** For a complex of abelian groups (A^*, d), we have the truncations $\tau_{\leq N}A^*$ and $\tau_{\geq N}A^*$, with

$$(\tau_{\leq N}A^*)^n = \begin{cases} 0 & \text{for } n > N \\ \ker(d^N : A^N \to A^{N+1}) & \text{for } n = N \\ A^n & \text{for } n < N, \end{cases}$$

$$(\tau_{\geq N}A^*)^n = \begin{cases} 0 & \text{for } n < N \\ A^N/d^{N-1}(A^{N-1}) & \text{for } n = N \\ A^n & \text{for } n > N. \end{cases}$$

We have the natural inclusion $\iota_N : \tau_{\leq N}A^* \to A^*$ and projection $\pi_N : A^* \to \tau_{\geq N}A^*$; ι_N induces an isomorphism on H^n for $n \leq N$ and π_N induces an isomorphism on H^n for $n \geq N$.

We may apply the truncations $\tau_{\leq N}$ and $\tau_{\geq N}$ to a double complex A^{**} by truncating with respect to either the first or the second index, giving the sub-double complexes $\iota^i_N : \tau^i_{\le N}A^{**} \to A^{**}$, and the quotient double complexes $\pi^i_N : A^{**} \to \tau^i_{\ge N}A^{**}$, $i = 1, 2$. It is elementary to verify that the natural maps

$$\lim_{\overrightarrow{N}} H^n(\mathrm{Tot}\,\tau^i_{\le N}A^{**}) \to H^n(\mathrm{Tot}\,A^{**}) \tag{7.1}$$

$$H^n(\widetilde{\mathrm{Tot}}A^{**}) \to \lim_{\overleftarrow{N}} H^n(\widetilde{\mathrm{Tot}}\tau^i_{\ge N}A^{**}) \tag{7.2}$$

are isomorphisms for $i = 1, 2$.

The basic property of the functors Tot and $\widetilde{\mathrm{Tot}}$ we will need is the following:

Lemma 7.3. *Let $f : C^{**} \to D^{**}$ be a map of double complexes. such that either*

1. *For each a, the map of complexes $f^{a,*} : C^{a,*} \to D^{a,*}$ is a quasi-isomorphism, or*
2. *For each b, the map of complexes $H^b_2(f) : H^b_2(C) \to H^b_2(D)$ is a quasi-isomorphism.*

(i) Suppose that, for each n, there is an integer N_n such that $C^{a,b} = D^{a.b} = 0$ for $a + b = n$ and $a > N_n$. Then the map

$$\mathrm{Tot}\,(f) : \mathrm{Tot}\,(C) \to \mathrm{Tot}\,(D)$$

is a quasi-isomorphism.
(ii) Suppose that, for each n, there is an integer M_n such that $C^{a,b} = D^{a,b} = 0$ for $a + b = n$ and $a < M_n$. Then the map

$$\widetilde{\mathrm{Tot}}(f) : \widetilde{\mathrm{Tot}}(C) \to \widetilde{\mathrm{Tot}}(D)$$

is a quasi-isomorphism.

Proof. We note that the hypothesis (1) implies (2), so it suffices to prove the lemma under the hypothesis (2).

We have the spectral sequence $E(A^{**})$,

$$E^{p,q}_2(A^{**}) = H^p(H^q_2(A^{**}), d_1) \Longrightarrow H^{p+q}(\mathrm{Tot}\,(A^{**}))$$

and the spectral sequence $\tilde{E}(A^{**})$,

$$\tilde{E}^{p,q}_2(A^{**}) = H^p(H^q_2(A^{**}), d_1) \Longrightarrow H^{p+q}(\widetilde{\mathrm{Tot}}(A^{**})).$$

Also, the E_2-terms for the truncations of A^{**} are

$$E^{p,q}_2(\tau^2_{\le N}A^{**}) = \begin{cases} E^{p,q}_2(A^{**}) & \text{for } q \le N \\ 0 & \text{for } q > N, \end{cases}$$

$$\tilde{E}^{p,q}_2(\tau^2_{\ge N}A^{**}) = \begin{cases} E^{p,q}_2(A^{**}) & \text{for } q \ge N \\ 0 & \text{for } q < N. \end{cases}$$

Under the assumption (i), the spectral sequences $E(\tau^2_{\le N}C^{**})$ and $E(\tau^2_{\le N}D^{**})$ are strongly convergent for each N; the hypotheses (2) implies that the map of spectral sequences $E(\tau^2_{\le N}C^{**}) \to E(\tau^2_{\le N}D^{**})$ induced by f is an isomorphism on the E_2 terms. Thus

$$\mathrm{Tot}\,(\tau^2_{\le N}f) : \mathrm{Tot}\,(\tau^2_{\le N}C^{**}) \to \mathrm{Tot}\,(\tau^2_{\le N}D^{**})$$

is a quasi-isomorphism; taking the direct limit over N, and using (7.1), we see that $\mathrm{Tot}\,(f) : \mathrm{Tot}\,(C^{**}) \to \mathrm{Tot}\,(D^{**})$ is a quasi-isomorphism.

Similarly, under the assumption (ii), the spectral sequences $\tilde{E}(\tau^2_{\geq N}C^{**})$ and $\tilde{E}(\tau^2_{\geq N}D^{**})$ are strongly convergent for each N. Replacing the direct limit with an inverse limit and using (7.2), it follows as above that $\widetilde{\mathrm{Tot}}(f) : \widetilde{\mathrm{Tot}}(C^{**}) \to \widetilde{\mathrm{Tot}}(D^{**})$ is a quasi-isomorphism. □

7.4. **Homotopy limits and colimits of complexes.** Let I be a small category, We have the *nerve* of I, $\mathcal{N}(I)$, which is the simplicial set with n-simplices the set of composable sequences of maps

$$i_0 \xrightarrow{s_1} i_1 \to \ldots \to i_{n-1} \xrightarrow{s_n} i_n.$$

The face map $\delta^n_j : \mathcal{N}(I)_n \to \mathcal{N}(I)_{n-1}$ is given by composing s_{j+1} and s_j (for $0 < j < n$), or by deleting i_j (for $j = 0, n$). The degeneracy map $\sigma^n_j : \mathcal{N}(I)_n \to \mathcal{N}(I)_{n+1}$ is given by inserting an identity map on i_j.

Let $F : I \to \mathbf{C}(\mathbf{Ab})$ be a functor. Form the double complex $IF^{**}_{\rightarrow}$ with

$$IF^{-a,b}_{\rightarrow} := \bigoplus_{i_0\to\ldots\to i_a\in\mathcal{N}(I)_a} F(i_0)^b.$$

The differential d_2 is given by

$$d^{a,b}_2 := (-1)^a \bigoplus_{i_0\to\ldots\to i_a\in\mathcal{N}(I)_a} d^b_{F(i_0)}.$$

To define the differential d_1, fix $i_* := i_0 \to \ldots \to i_a \in \mathcal{N}(I)_a$. For $j = 1, \ldots , a$, let $\partial^a_j : IF^{-a,b}_{\rightarrow} \to IF^{-a+1,b}_{\rightarrow}$ be the map sending $F(i_0)^b$ in the summand i_* to $F(i_0)^b$ in the summand $\delta^a_j(i_*)$ by the identity. Let $\partial^a_0 : IF^{-a,b}_{\rightarrow} \to IF^{-a+1,b}_{\rightarrow}$ be the map sending $F(i_0)^b$ in the summand i_* to $F(i_1)^b$ in the summand $\delta^a_0(i_*)$ by $F(s_1)$. We then set

$$d^{-a,b}_1 := \sum_{j=0}^{a}(-1)^j\partial^a_j.$$

Finally, we define the complex $\mathrm{hocolim}_I F$ by

$$\operatorname*{hocolim}_I F := \mathrm{Tot}\,(IF_{\rightarrow}).$$

This defines a functor $\mathrm{hocolim}_I$ from the category of functors $F : I \to \mathbf{C}(\mathbf{Ab})$ to $\mathbf{C}(\mathbf{Ab})$.

The functor holim_I is defined dually. Let $IF^{**}_{\leftarrow}$ be the double complex with

$$IF^{a,b}_{\leftarrow} := \prod_{i_0\to\ldots\to i_a\in\mathcal{N}(I)_a} F(i_a)^b.$$

The differential d_2 is defined as for $IF^{**}_{\rightarrow}$. The map $d^a_j : IF^{a,b}_{\leftarrow} \to IF^{a+1,b}_{\leftarrow}$ is defined as follows: Fix $i_* := i_0 \to \ldots \to i_{a+1} \in \mathcal{N}(I)_{a+1}$. For $0 \leq j < a + 1$, the map d^a_j composed with the projection on the factor i_* is just the projection of $IF^{a,b}_{\leftarrow}$ on the factor $\delta^{a+1}_j(i_*)$. The map d^a_{a+1} composed with the projection on the factor i_* is the projection of $IF^{a,b}_{\leftarrow}$ on the factor $\delta^{a+1}_{a+1}(i_*)$, followed by the map $F(s_{a+1})$. We set

$$d^{a,b}_1 := \sum_{j=0}^{a+1}(-1)^j d^a_j.$$

We define $\mathrm{holim}_I F$ by

$$\operatorname*{holim}_I F := \widetilde{\mathrm{Tot}} IF_{\leftarrow}.$$

We map $IF^{0,b}$ to $\lim_{\substack{\to\\ I}} F^b$ by the sum of the canonical maps

$$F(i)^b \to \lim_{\substack{\to\\ I}} F^b.$$

This extends to the map of complexes

$$\pi_F : \operatorname{hocolim}_I F \to \lim_{\substack{\to\\ I}} F$$

by sending $IF_{\to}^{a,b}$ to zero for $a \neq 0$. Similarly, the canonical inclusion

$$\lim_{\substack{\leftarrow\\ I}} F^b \to \prod_{i\in I} F(i)^b = IF_{\leftarrow}^{0,b}$$

defines the map of complexes

$$\iota^F : \lim_{\substack{\leftarrow\\ I}} F \to \operatorname{holim}_I F.$$

Fix an object $i \in I$. The inclusion functor $i \to I$ gives us the natural maps

$$\iota_i : F(i) \to \operatorname{hocolim}_I F,\ \pi^i : \operatorname{holim}_I F \to F(i). \tag{7.3}$$

The compositions

$$F(i) \xrightarrow{\iota_i} \operatorname{hocolim}_I F \xrightarrow{\pi_F} \lim_{\substack{\to\\ I}} F$$

$$\lim_{\substack{\leftarrow\\ I}} F \xrightarrow{\iota^F} \operatorname{holim}_I F \xrightarrow{\pi^i} F(i)$$

are the respective canonical maps.

We call a natural transformation $\omega : F \to G$ of functors $F, G : I \to \mathbf{C}(\mathbf{Ab})$ a *quasi-isomorphism* if $\omega(i) : F(i) \to G(i)$ is a quasi-isomorphism for all $i \in I$.

Lemma 7.5. *Let $F, G : I \to \mathbf{C}(\mathbf{Ab})$ be functors, and let $\omega : F \to G$ be a quasi-isomorphism. Then*

$$\operatorname{hocolim}_I(\omega) : \operatorname{hocolim}_I(F) \to \operatorname{hocolim}_I(G)$$

and

$$\operatorname{holim}_I(\omega) : \operatorname{holim}_I(F) \to \operatorname{holim}_I(G)$$

are quasi-isomorphisms.

Proof. This follows directly from Lemma 7.3, the definitions of holim and hocolim, and the fact that both direct sums and products take families of quasi-isomorphisms to quasi-isomorphisms. □

We have the set $\mathbb{N}$ of natural numbers $n \geq 1$, with the usual order. This gives us the category $\mathbb{N}$ with the set of objects $\mathbb{N}$ and with unique morphism $i \to j$ if and only if $i \leq j$.

Definition 7.6. Let $G : \mathbb{N}^{\mathrm{op}} \to \mathbf{Ab}$ be a functor. We say that G satisfies the Mittag-Leffler conditions if for each $n \geq 1$, there is an $N \geq 0$ such that

$$\operatorname{Im}(G(n+N) \to G(n)) = \cap_{k=0}^{\infty} \operatorname{Im}(G(n+k) \to G(n)).$$

Lemma 7.7. *Let $F : I \to \mathbf{C}(\mathbf{Ab})$ be a functor.*
(i) Suppose that I is right-filtering. Then the map

$$\pi_F : \operatorname{hocolim}_I F \to \varinjlim_I F$$

is a quasi-isomorphism.
(ii) Suppose $I = \mathbb{N}^{\mathrm{op}}$ and suppose that, for each b the functors F^b and $H^b(F)$ satisfy the Mittag-Leffler conditions of Definition 7.6. Then the map

$$\iota^F : \varprojlim_I F \to \operatorname{holim}_I F$$

is a quasi-isomorphism.

Proof. We first prove (ii). From [4, Chapter IX, Proposition 6.2], the complex $\operatorname{holim}_I H^b(F)$ has ith cohomology equal to the ith right-derived functor of the functor $\varprojlim_I H^b(F)$. Since $H^b(F)$ satisfies the Mittag-Leffler conditions, we have

$$R^i \varprojlim_I H^b(F) = 0$$

for $i > 0$ (see [4, Chapter IX, Remark 6.4]). As forming products commutes with taking cohomology, we have the isomorphism of complexes

$$H_2^b(IF_{\leftarrow}) \cong \operatorname{holim}_I H^b(F).$$

Since the functors F^b satisfy Mittag-Leffler, we have the isomorphism

$$H^b(\varprojlim_I F) \cong \varprojlim_I H^b(F).$$

These two identities imply that ι induces a quasi-isomorphism of the complex $H_2^b(IF_{\leftarrow})$ with the complex (supported in degree zero) $H^b(\varprojlim_I F)$. By Lemma 7.3, this implies that ι is a quasi-isomorphism.

The proof of (i) is similar, but easier, since taking a filtered direct limit is an exact functor. □

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