

CHOW'S MOVING LEMMA AND THE HOMOTOPY CONIVEAU TOWER

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ABSTRACT. We show how the classical moving lemma of Chow extends to give functoriality for the spectra arising in the homotopy coniveau tower. This generalizes and extends the known functoriality of Bloch's cycle complexes to the setting of presheaves of spectra.

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INTRODUCTION

This paper is part of a program to understand and generalize the works of Bloch-Lichtenbaum [4] and Friedlander-Suslin [8], which present a construction of the spectral sequence from motivic cohomology to K -theory. We have started a study of this approach in our paper [17], where, relying on some of the results of Friedlander-Suslin, we gave a method for globalizing the Bloch-Lichtenbaum spectral sequence to schemes of finite type over a Dedekind domain. This method is a bit different from the one used by Friedlander-Suslin, and is somewhat better adapted to localization properties, while being rather less convenient regarding questions of functoriality.

It turns out that the approach of [17] can be applied to a wide variety of cohomology theories on algebraic varieties, including all the cohomology theories arising from objects in the Morel-Voevodsky \mathbb{A}^1 -stable homotopy category. Just as for K -theory, localization properties follow rather easily from the existing technology. The purpose of this paper is to handle the problem of functoriality in a fairly general setting.

The functoriality obtained here is used in our paper [18] to extend the main results of [4, 8] to a general setting and, at the same time, give a new proof of the main results of [4]. This yields, we think, a much more conceptual approach to the Bloch-Lichtenbaum-Friedlander-Suslin spectral sequence.

In somewhat more detail, one can consider a functor $E : \mathbf{Sm}/B^{\text{op}} \rightarrow \mathbf{Spt}$ from smooth B -schemes to spectra (for example $X \mapsto K(X)$, the K -theory spectrum of a B -scheme X). For such a functor, and for an X in \mathbf{Sm}/B , we construct the

homotopy coniveau tower

$$(0.0.1) \quad \dots \rightarrow E^{(p+1)}(X, -) \rightarrow E^{(p)}(X, -) \rightarrow \dots \rightarrow E^{(0)}(X, -)$$

where the $E^{(p)}(X, -)$ are simplicial spectra with n -simplices the limit of the spectra with support $E^W(X \times \Delta^n)$, where W is a closed subset of codimension $\geq p$ in “good position” with respect to the faces of Δ^n . This is just the evident extension of the tower used by Friedlander-Suslin in [8]. One can consider this tower as the algebraic analog of the one in topology formed by applying a cohomology theory to the skeletal filtration of a CW complex.

Our main purpose in this paper is construct a functorial model of the presheaf of spectra $U \mapsto E^{(p)}(U, -)$ on X_{Nis} , denoted $E^{(p)}(X_{\text{Nis}}, -)$. Our construction is based on first a “local functoriality”, which is essentially a generalization of the classical method used (e.g. in [24]) to prove Chow’s moving lemma for cycles modulo rational equivalence. This local functoriality yields a weak version of functoriality to the homotopy category of presheaves on X , for varying X , which we promote to give a full functoriality in the homotopy category of presheaves on \mathbf{Sm}/B (with respect to Nisnevich-local weak equivalences) by a formal rectification process. Naturally, we need to impose some additional (but quite natural) hypotheses on our functor E to achieve this.

One can view our results as a generalization and extension of the known functoriality for Bloch’s cycle complexes $z^q(X, *)$ to presheaves of spectra, just as in homotopy theory a spectrum generalizes an abelian group, viewed as an Eilenberg–MacLane spectrum.

The homotopy coniveau tower (0.0.1) leads directly to the spectral sequence

$$(0.0.2) \quad E_1^{p,q} := \pi_{-p-q}(E^{(p/p+1)}(X, -)) \implies \pi_{-p-q}(E^{(0)}(X, -)),$$

which, in the case of the K -theory spectrum was the main object of interest in the papers [4, 8]; here $E^{(p/p+1)}(X, -)$ is the homotopy cofiber of the map $E^{(p+1)}(X, -) \rightarrow E^{(p)}(X, -)$. One major goal of this paper is to make this spectral sequence functorial on \mathbf{Sm}/B . In fact, our results yield directly the functoriality of an associated spectral sequence, where one replaces the spectra $E^{(p)}(X, -)$ with fibrant models for $E^{(p)}(X_{\text{Nis}}, -)$. The localization properties of $E^{(p)}(X, -)$ mentioned above allow us to recover the functoriality of (0.0.2) from this. See §7.2 for further details.

This paper also has roots in our preprint [15]. In another paper, we will complete our detailed investigation of the spectral sequence from motivic cohomology to K -theory, relying on the results of this paper for the functoriality. To keep some contact with our original problem, we will occasionally illustrate our results by referring to the example $E = K$.

One topic that we do not address in this paper is that of multiplicative structure, including the multiplicative properties of the homotopy coniveau spectra sequence. The works of Hovey-Shipley-Smith [26] and Jardine [12, 13] make clear that the category of presheaves of *symmetric spectra* is the appropriate setting to discuss the multiplicative properties of the homotopy coniveau tower. As this paper is already too long, we will leave to the future the task of adapting the results here to presheaves of symmetric spectra.

I have had a great deal of help in developing the techniques that went in to this paper. Fabien Morel played a crucial role in numerous discussions on the \mathbb{A}^1 -stable homotopy category and related topics. Conversations with Bruno Kahn, Jens

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1. PRELIMINARIES

1.1. Simplicial sets. We fix some notation and recall some standard constructions.

We have the ordered sets $[p] := \{0 < \dots < p\}$. \mathbf{Ord} is the category with objects $[p]$, $p = 0, 1, \dots$, and with maps the order-preserving maps of sets. For a category \mathcal{C} , the category of functors $F : \mathbf{Ord} \rightarrow \mathcal{C}$ is the category of *cosimplicial* objects of \mathcal{C} , and that of functors $F : \mathbf{Ord}^{\mathrm{op}} \rightarrow \mathcal{C}$ the category of *simplicial* objects of \mathcal{C} . Let $\mathbf{Ord}_{\leq N}$ denote the full subcategory of the category \mathbf{Ord} , with objects $[p]$, $0 \leq p \leq N$. We have the category of N -truncated cosimplicial objects of \mathcal{C} , namely, the functors $F : \mathbf{Ord}_{\leq N} \rightarrow \mathcal{C}$, and similarly the category of N -truncated simplicial objects of \mathcal{C} .

The category \mathbf{Spc} of simplicial sets is called the category of *spaces*; if $F : \mathbf{Ord}^{\mathrm{op}} \rightarrow \mathbf{Spc}$ is a functor, we have the geometric realization $|F| \in \mathbf{Spc}$. This is defined as the quotient simplicial set

$$|F| := \coprod_n F(n) \times \Delta[n] / \sim$$

where $\Delta[n] := m \mapsto \mathrm{Hom}_{\mathbf{Ord}}([m], [n])$, and \sim is the standard relation

$$(x, g_*(y)) \sim (F(g)(x), y)$$

for $g : [n] \rightarrow [n']$ in \mathbf{Ord} , $x \in F([n'])$. We have as well the category of pointed spaces \mathbf{Spc}^* and geometric realization functors from pointed simplicial spaces to \mathbf{Spc}^* .

A partially ordered set (\mathcal{S}, \leq) may be considered as a small category with set of objects \mathcal{S} and with $\mathrm{Hom}(i, j)$ the singleton set if $i \leq j$ and empty otherwise. This defines a functor from the category \mathbf{pos} of partially ordered sets (with morphisms the order-preserving maps of sets) to small categories. In particular, sending $[n]$ to the associated category \underline{n} gives us the functor $\underline{*} : \mathbf{Ord} \rightarrow \mathbf{Cat}$.

If \mathcal{A} is a small category, we thus have the simplicial set $\mathcal{N}(\mathcal{A}) : \mathbf{Ord}^{\mathrm{op}} \rightarrow \mathbf{Sets}$, the *nerve* of \mathcal{A} , defined as the functor $\mathrm{Hom}_{\mathbf{Cat}}(-, \mathcal{A}) \circ \underline{*}$:

$$[n] \mapsto \mathrm{Hom}_{\mathbf{Cat}}(\underline{n}, \mathcal{A}).$$

1.2. Presheaves of simplicial sets. For a category \mathcal{C} , we have the category p.s.s. \mathcal{C} of functors $F : \mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Spc}$ (presheaves of simplicial sets), and p.s.s. $_*\mathcal{C}$ of functors $F : \mathcal{C}^{\mathrm{op}} \rightarrow \mathbf{Spc}^*$ (presheaves of pointed simplicial sets).

We give \mathbf{Spc} and \mathbf{Spc}^* the standard model structures: cofibrations are (pointed) monomorphisms, weak equivalences are weak equivalences on the geometric realization, and fibrations are determined by the RLP with respect to trivial cofibrations; the fibrations are then exactly the Kan fibrations. We let $|A|$ denote the geometric realization, and $[A, B]$ the homotopy classes of (pointed) maps $|A| \rightarrow |B|$.

We give $s\mathcal{C}$ and $s_*\mathcal{C}$ the model structure of functor categories described by Bousfield-Kan [6]. That is, the cofibrations and weak equivalences are the pointwise ones, and the fibrations are determined by the RLP with respect to trivial cofibrations. We let $\mathcal{H}s\mathcal{C}$ and $\mathcal{H}s_*\mathcal{C}$ denote the associated homotopy categories (see [10] for details).

1.3. Presheaves of spectra. Let \mathbf{Spt} denote the category of spectra. To fix ideas, a spectrum will be a sequence of pointed simplicial sets E_0, E_1, \dots together with maps of pointed simplicial sets $\epsilon_n : S^1 \wedge E_n \rightarrow E_{n+1}$. Maps of spectra are maps of the underlying simplicial sets which are compatible with the bonding maps ϵ_n . The stable homotopy groups $\pi_n^s(E)$ are defined by

$$\pi_n^s(E) := \lim_{m \rightarrow \infty} [S^{m+n}, E_m]$$

where $[-, -]$ is the set of homotopy classes of maps in \mathbf{Spc}^* .

The category \mathbf{Spt} has the following model structure (see for example [5]): Cofibrations are maps $f : E \rightarrow F$ such that $E_0 \rightarrow F_0$ is a cofibration, and for each $n \geq 0$, the map

$$E_{n+1} \coprod_{S^1 \wedge E_n} S^1 \wedge F_n \rightarrow F_{n+1}$$

is a cofibration. Weak equivalences are the stable weak equivalences, i.e., maps $f : E \rightarrow F$ which induce an isomorphism on π_n^s for all n . Fibrations are characterized by having the RLP with respect to trivial cofibrations.

Let \mathcal{C} be a category. We say that a natural transformation $f : E \rightarrow E'$ of functors $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Spt}$ is a weak equivalence if $f(X) : E(X) \rightarrow E'(X)$ is a stable weak equivalence for all X .

We will assume that the category \mathcal{C} has an initial object \emptyset and admits finite coproducts over \emptyset , denoted $X \amalg Y$. A functor $E : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Spt}$ is called *additive* if for each X, Y in \mathcal{C} , the canonical map

$$E(X \amalg Y) \rightarrow E(X) \oplus E(Y)$$

is a weak equivalence. An additive functor $E : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Spt}$ is called a *presheaf of spectra* on \mathcal{C} . This forms a full subcategory of the functor category.

We use the following model structure on the category of presheaves of spectra (see [11]): Cofibrations and weak equivalences are given pointwise, and fibrations are characterized by having the RLP with respect to trivial cofibrations. We denote this model category by $\mathbf{Spt}(\mathcal{C})$, and the associated homotopy category by $\mathcal{H}\mathbf{Spt}(\mathcal{C})$. We write \mathcal{SH} for the homotopy category of \mathbf{Spt} .

For the remainder of the paper, we let B denote a noetherian separated scheme of finite Krull dimension. We let \mathbf{Sm}/B denote the category of smooth B -schemes of finite type over B . We often write $\mathbf{Spt}(B)$ and $\mathcal{H}\mathbf{Spt}(B)$ for $\mathbf{Spt}(\mathbf{Sm}/B)$ and $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/B)$.

For $Y \in \mathbf{Sm}/B$, a subscheme $U \subset Y$ of the form $Y \setminus \cup_{\alpha} F_{\alpha}$, with $\{F_{\alpha}\}$ a possibly infinite set of closed subsets of Y , is called *essentially smooth over B* ; the category of essentially smooth B -schemes is denoted \mathbf{Sm}^{ess} .

1.4. Local model structure. If the category \mathcal{C} has a topology, there is often another model structure on $\mathbf{Spt}(\mathcal{C})$ which takes this into account. We consider the case of the small Nisnevich site X_{Nis} on a scheme X (assumed to be noetherian, separated and of finite Krull dimension), and on the big Nisnevich sites $\mathbf{Sm}/B_{\text{Nis}}$ or $\mathbf{Sch}/B_{\text{Nis}}$, as well as the Zariski versions X_{Zar} , etc. We describe the Nisnevich version below; the definitions and results for the Zariski topology are exactly parallel.

Definition 1.4.1. A map $f : E \rightarrow F$ of presheaves of spectra on X_{Nis} is a *local weak equivalence* if the induced map on the Nisnevich sheaf of stable homotopy groups $f_* : \pi_m^s(E)_{\text{Nis}} \rightarrow \pi_m^s(F)_{\text{Nis}}$ is an isomorphism of sheaves for all m . A map $f : E \rightarrow F$ of presheaves of spectra on $\mathbf{Sm}/B_{\text{Nis}}$ or $\mathbf{Sch}/B_{\text{Nis}}$ is a local weak equivalence if the restriction of f to X_{Nis} is a local weak equivalence for all $X \in \mathbf{Sm}/B$ or $X \in \mathbf{Sch}/B$. \square

The Nisnevich-local model structure on the category of presheaves of spectra on X_{Nis} has cofibrations given pointwise, weak equivalences the local weak equivalences and fibrations are characterized by having the RLP with respect to trivial cofibrations. We denote this model structure $\mathbf{Spt}_{\text{Nis}}(X)$ and the associated homotopy category $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(X)$. The same definitions yield the Nisnevich local model structures $\mathbf{Spt}_{\text{Nis}}(\mathbf{Sm}/B)$ and $\mathbf{Spt}_{\text{Nis}}(\mathbf{Sch}/B)$, with respective homotopy categories $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(\mathbf{Sm}/B)$, $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(\mathbf{Sch}/B)$. For details, we refer the reader to [11].

1.5. Simplicial spectra. Let $E : \mathbf{Ord}^{\text{op}} \rightarrow \mathbf{Spt}$ be a simplicial spectrum. Writing $E = (E_0, E_1, \dots)$ with bonding maps $\epsilon_n : S^1 \wedge E_n \rightarrow E_{n+1}$, we may take the geometric realizations $|E_n|$ and form the *total spectrum* of E as the spectrum $(|E_0|, |E_1|, \dots)$ with structure maps $|\epsilon_n| : S^1 \wedge |E_n| = |S^1 \wedge E_n| \rightarrow |E_{n+1}|$.

We use the geometric realization functor to define weak equivalences and cofibrations of simplicial spectra, giving us the model category of simplicial spectra, as well as the local model structure (for the Nisnevich or Zariski topology) on the category of simplicial spectra.

When the context makes the meaning clear, we will often omit the separate notation for the total spectrum, and freely pass between a simplicial spectrum and its associated total spectrum.

2. THE HOMOTOPY CONIVEAU TOWER

In this section, we give the construction of the homotopy coniveau tower in the general setting. Throughout this section, the base scheme B will be a noetherian separated scheme.

2.1. Spectra with support. In order to axiomatize the situation, we introduce some categories which describe “spectra with support in codimension $\geq q$ ”.

Definition 2.1.1. Let B be a scheme and fix an integer $q \geq 0$. Let $\mathbf{Sm}/B^{(q)}$ be the category with objects pairs (Y, W) , where Y is in \mathbf{Sm}/B and W is a closed subset of Y with $\text{codim}_Y W \geq q$. A morphism $f : (Y, W) \rightarrow (Y', W')$ is a B -morphism $f : Y \rightarrow Y'$ such that $W \supset f^{-1}(W')$. \square

We have the model category of presheaves $\mathbf{Spt}(\mathbf{Sm}/B^{(q)})$ and the homotopy category $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/B^{(q)})$, denoted $\mathbf{Spt}(B^{(q)})$ and $\mathcal{H}\mathbf{Spt}(B^{(q)})$. For $E \in \mathbf{Spt}(B^{(q)})$, we often write $E^W(Y)$ for $E(Y, W)$.

Take $E \in \mathbf{Spt}(B^{(q)})$, $(Y, W) \in \mathbf{Sm}/B^{(q)}$ and $W' \subset W$ a closed subset. Then (Y, W') is also in $\mathbf{Sm}/B^{(q)}$ and the identity on Y gives the morphism $\text{id}_Y : (Y, W) \rightarrow (Y, W')$. We thus have the map of spectra

$$i_{W, W'} := \text{id}_Y^* : E^{W'}(X) \rightarrow E^W(X)$$

with $i_{W, W'} \circ i_{W', W''} = i_{W, W''}$ for $W'' \subset W' \subset W$.

Take integers $q' \geq q$. Sending $(Y, W) \in \mathbf{Sm}/B^{(q')}$ to $(Y, W) \in \mathbf{Sm}/B^{(q)}$ identifies $\mathbf{Sm}/B^{(q')}$ with a full subcategory of $\mathbf{Sm}/B^{(q)}$; we denote the inclusion functor by

$$\rho_{q, q'} : \mathbf{Sm}/B^{(q')} \rightarrow \mathbf{Sm}/B^{(q)}.$$

2.2. Examples. Our primary source of spectra with supports is a presheaf of spectra on \mathbf{Sm}/B . Given $E \in \mathbf{Spt}(B)$, and $(Y, W) \in \mathbf{Sm}/B^{(q)}$, we set

$$E^W(Y) := \text{fib}(E(Y) \xrightarrow{j^*} E(Y \setminus W)),$$

where “fib” is the homotopy fiber and $j : Y \setminus W \rightarrow Y$ is the inclusion. This clearly defines functors $\mathbf{Spt}(B) \rightarrow \mathbf{Spt}(B^{(q)})$, compatible in q . As a particular example, we can take E to be the K -theory spectrum $X \mapsto K(X)$.

Another source of examples comes from algebraic cycles. Take the base-scheme B to be regular and of finite Krull dimension. For (Y, W) in $\mathbf{Sm}/B^{(q)}$, let $z_W^q(Y)$ denote the group of codimension q algebraic cycles on Y with support contained in W . Sending (Y, W) to $z_W^q(Y)$ defines a presheaf of abelian groups on $\mathbf{Sm}/B^{(q)}$; taking the associated Eilenberg-MacLane spectrum gives us the presheaf $z^q \in \mathbf{Spt}(B^{(q)})$.

Remark 2.2.1. It may well be that the “spectrum with supports” functor $\mathbf{Spt}(B) \rightarrow \mathbf{Spt}(B^{(q)})$ defined above is in fact essentially surjective, so the introduction of category $\mathbf{Spt}(B^{(q)})$ does not yield any examples beyond those coming from $\mathbf{Spt}(B)$, but we have not investigated this issue. \square

2.3. Support conditions. Let $X : \mathbf{Ord} \rightarrow \mathbf{Sm}/B$ be a smooth cosimplicial B -scheme. For $E : \mathbf{Sm}/B^{\text{op}} \rightarrow \mathbf{Spt}$ in $\mathbf{Spt}(B)$, we may evaluate E on X , forming the simplicial spectrum $E(X) : \mathbf{Ord}^{\text{op}} \rightarrow \mathbf{Spt}$. This is our basic method for constructing simplicial spectra; we will also need an extension of this construction by introducing *support conditions*.

Let X be a smooth cosimplicial B -scheme. We have the simplicial set $\mathcal{S}_X : \mathbf{Ord}^{\text{op}} \rightarrow \mathbf{Sets}$ with $\mathcal{S}_X(n)$ the set of closed subsets of $X(n)$.

Definition 2.3.1. (1) A *support condition* on X is a simplicial subset \mathcal{S} of \mathcal{S}_X such that, for each n ,

- (a) $\mathcal{S}(n)$ is closed under finite union
 - (b) If W is in $\mathcal{S}(n)$ and $W' \subset W$ is a closed subset, then W' is in $\mathcal{S}(n)$.
- (2) An assignment $n \mapsto W_n$, with $W_n \in \mathcal{S}_X(n)$ is a *simplicial closed subset* of X if for each $g : [m] \rightarrow [n]$ in \mathbf{Ord} , we have $X(g)^{-1}(W_n) \subset W_m$.
- (3) For a support condition \mathcal{S} on a smooth cosimplicial B -scheme X , and a simplicial closed subset W of X , we write $W \in \mathcal{S}$ if W_n is in $\mathcal{S}(n)$ for all n .
- (4) If $\text{codim}_{X(n)} W \geq q$ for all $W \in \mathcal{S}(n)$, we say that \mathcal{S} is *supported in codimension* $\geq q$. \square

Remarks 2.3.2. Fix a smooth cosimplicial B -scheme X .

- (1) A collection of closed subsets $W_n \subset X(n)$ forms a simplicial closed subset of X exactly when the complements $U(n) := X(n) \setminus W_n$ form an open cosimplicial

subscheme of X .

(2) Suppose we have closed subsets $T_1 \subset X(n_1), \dots, T_s \subset X(n_s)$. Then there is a unique minimal (in the sense of containment of closed subsets) simplicial closed subset W of X with $T_j \subset W_{n_j}$ for all $j = 1, \dots, s$; this is clear since the intersection of simplicial closed subsets is a simplicial closed subset. We call W the simplicial closed subset generated by the T_j . The explicit formula for W_m is

$$W_m = \cup_{i=1}^s \cup_{g: [m] \rightarrow [n_i]} X(g)^{-1}(T_i);$$

note that the union is finite, so W_m is indeed a closed subset of $X(m)$.

(3) Let $n \mapsto W_n$ be a simplicial closed subset of X , supported in codimension $\geq q$. Then $n \mapsto (X(n), W_n)$ defines a cosimplicial object of $\mathbf{Sm}/B^{(q)}$. \square

Lemma 2.3.3. *Let \mathcal{S} be a support condition on a smooth cosimplicial B -scheme X , and let T_i be in $\mathcal{S}(n_i)$, $i = 1, \dots, s$. Then the simplicial closed subset W generated by the T_i satisfies*

- (1) $T_i \subset W_{n_i}$ for $i = 1, \dots, s$.
- (2) $W_m \in \mathcal{S}(m)$ for all m .

Proof. Obvious from Remark 2.3.2(2). \square

Definition 2.3.4. Let W be a simplicial closed subset of X with $\text{codim}_{X(n)} W_n \geq q$ for all n , and take E in $\mathbf{Spt}(B^{(q)})$. Let $E^W(X)$ denote the simplicial spectrum

$$n \mapsto E^{W_n}(X(n)).$$

\square

Let \mathcal{S} be a support condition on X , supported in codimension $\geq q$. We make \mathcal{S} into a small category by ordering the $W \in \mathcal{S}$ by inclusion. For $E \in \mathbf{Spt}(B^{(q)})$, and $W \subset W'$ an inclusion of simplicial closed subsets of X , we have the map of simplicial spectra $i_{W',W} : E^W(X) \rightarrow E^{W'}(X)$, giving the functor $W \mapsto E^W(X)$ from \mathcal{S} to simplicial spectra. We set

$$E^{\mathcal{S}}(X) := \text{hocolim}_{W \in \mathcal{S}} E^W(X).$$

By Lemma 2.3.3, we have the canonical homotopy equivalence

$$E^{\mathcal{S}}(X)(n) \cong \text{hocolim}_{T \in \mathcal{S}(n)} E^T(X(n)).$$

2.4. The homotopy coniveau tower. We introduce our main object of study.

We have the cosimplicial scheme Δ^* , with

$$\Delta^r = \text{Spec}(\mathbb{Z}[t_0, \dots, t_r] / \sum_j t_j - 1).$$

The *vertices* of Δ^r are the closed subschemes v_i^r defined by $t_i = 1, t_j = 0$ for $j \neq i$. A *face* of Δ^r is a closed subscheme defined by equations of the form $t_{i_1} = \dots = t_{i_s} = 0$.

Definition 2.4.1. For X in \mathbf{Sm}/B , we let $\mathcal{S}_X^{(q)}(p)$ denote the set of closed subsets W of $X \times \Delta^p$ such that

$$\text{codim}_{X \times F} W \cap (X \times F) \geq q$$

for all faces F of Δ^p . \square

Clearly, sending p to $\mathcal{S}_X^{(q)}(p)$ defines a support condition $\mathcal{S}_X^{(q)}$ on $X \times \Delta^*$, supported in codimension $\geq q$. We let $X^{(q)}(p)$ be the set of codimension q points x of $X \times \Delta^p$ with closure $\bar{x} \in \mathcal{S}_X^{(q)}(p)$.

Definition 2.4.2. Take $E \in \mathbf{Spt}(B^{(q)})$. We set

$$\begin{aligned} E^{(q)}(X, p) &:= E^{\mathcal{S}_X^{(q)}(p)}(X \times \Delta^p) \\ E^{(q)}(X, -) &:= E^{\mathcal{S}_X^{(q)}}(X \times \Delta^*). \end{aligned}$$

Similarly, if W is a simplicial closed subset of $X \times \Delta^*$, supported in codimension $\geq q$, we write $E^W(X, -)$ for $E^W(X \times \Delta^*)$. \square

Take $q' \geq q$ and let E be in $\mathbf{Spt}(B^{(q)})$. We may form the restriction $E \circ \rho_{q, q'} \in \mathbf{Spt}(B^{(q')})$. Similarly, for $X \in \mathbf{Sm}/B$, we have the simplicial spectrum $(E \circ \rho_{q, q'})^{(q')}(X, -)$, which we write simply as $E^{(q')}(X, -)$.

The inclusions $\mathcal{S}_X^{(q')}(p) \subset \mathcal{S}_X^{(q)}(p)$ and the maps

$$\begin{aligned} i_{W, W'} &: E^{W'}(X \times \Delta^p) \rightarrow E^W(X \times \Delta^p); \\ W' &\in \mathcal{S}_X^{(q')}(p), W \in \mathcal{S}_X^{(q)}(p), W' \subset W \end{aligned}$$

define the natural map of simplicial spectra

$$\iota_{q, q'} : E^{(q')}(X, -) \rightarrow E^{(q)}(X, -)$$

with $\iota_{q, q'} \circ \iota_{q', q''} = \iota_{q, q''}$ for $q \leq q' \leq q''$. This gives us the *homotopy coniveau tower* for E :

$$(2.4.1) \quad \dots \rightarrow E^{(q+r+1)}(X, -) \rightarrow E^{(q+r)}(X, -) \rightarrow \dots \rightarrow E^{(q)}(X, -).$$

This is our main object of study.

2.5. First properties. We give a list of elementary properties of the spectra $E^{(q)}(X, -)$ and the tower (2.4.1)

- (1) Sending X to $E^{(q)}(X, -)$ is functorial for equi-dimensional (e.g. flat) maps $Y \rightarrow X$ in \mathbf{Sm}/B ; the tower (2.4.1) is natural with respect to equi-dimensional maps.
- (2) Sending E to $E^{(q)}(X, -)$ is functorial in E .
- (3) The functor $E \mapsto E^{(q)}(X, -)$ sends weak equivalences to weak equivalences, and sends homotopy (co)fiber sequences to homotopy (co)fiber sequences.

We let $\mathbf{Sm} // B$ denote the subcategory of \mathbf{Sm}/B with the same objects, but with morphisms $Y \rightarrow X$ being the *smooth* B -morphisms. We let $E_{\mathbf{sm}}^{(q)}$ be the presheaf on $\mathbf{Sm} // B$: $E_{\mathbf{sm}}^{(q)}(X) := E^{(q)}(X, -)$. We thus have the tower of presheaves on $\mathbf{Sm} // B$

$$(2.5.1) \quad \dots \rightarrow E_{\mathbf{sm}}^{(q+r+1)} \rightarrow E_{\mathbf{sm}}^{(q+r)} \rightarrow \dots \rightarrow E_{\mathbf{sm}}^{(q)}$$

which is natural in $E \in \mathbf{Spt}(\mathbf{Sm}/B^{(q)})$.

2.6. Good position. For the rest of §2, we will assume that the base-scheme B is regular, noetherian, separated and has Krull dimension at most one. For a B -scheme $f : X \rightarrow B$ and a point $b \in B$, we set $X_b := f^{-1}(b)$.

The generalized Chow's moving lemma (Theorem 2.6.2) asserts that the spectrum $E^{(q)}(X, -)$ defined by the support conditions $\mathcal{S}_X^{(q)}$ is weakly equivalent to another spectrum $E^{(q)}(X, -)_C$ defined with respect to a somewhat finer support condition $\mathcal{S}_{X, \mathcal{C}, e}^{(q)} \subset \mathcal{S}_X^{(q)}$, which we define in this section.

Definition 2.6.1. Let X be in \mathbf{Sm}/B , let \mathcal{C} be a finite set of locally closed subsets of X , and let $e : \mathcal{C} \rightarrow \mathbb{N}$ be a function. We assume that each $C \in \mathcal{C}$ has pure codimension in X . Let $\mathcal{S}_{X, \mathcal{C}, e}^{(q)}(p)$ be the subset of $\mathcal{S}_X^{(q)}(p)$ consisting of those W such that, for each $C \in \mathcal{C}$ and each face F of Δ^p , we have

$$\mathrm{codim}_{C \times F}(C \times F) \cap W \geq q - e(C).$$

□

It is more or less obvious that $p \mapsto \mathcal{S}_{X, \mathcal{C}, e}^{(q)}(p)$ defines a support condition on $X \times \Delta^*$. We write $\mathcal{S}_{X, \mathcal{C}}^{(q)}(p)$ for $\mathcal{S}_{X, \mathcal{C}, 0}^{(q)}(p)$.

Given $E \in \mathbf{Spt}(B^{(q)})$, we set

$$E^{(q)}(X, -)_{\mathcal{C}, e} := E^{\mathcal{S}_{X, \mathcal{C}, e}^{(q)}}(X \times \Delta^*).$$

The inclusion $\mathcal{S}_{X, \mathcal{C}, e}^{(q)} \rightarrow \mathcal{S}_X^{(q)}$ defines the natural map

$$E^{(q)}(X, -)_{\mathcal{C}, e} \rightarrow E^{(q)}(X, -);$$

similarly, if $\mathcal{C} \subset \mathcal{C}'$ and $e' \leq e$, we have the natural map

$$E^{(q)}(X, -)_{\mathcal{C}', e'} \rightarrow E^{(q)}(X, -)_{\mathcal{C}, e}.$$

We write $E^{(q)}(X, -)_C$ for $E^{(q)}(X, -)_{\mathcal{C}, 0}$.

Given a function $e : \mathcal{C} \rightarrow \mathbb{N}$, we let $e - 1 : \mathcal{C} \rightarrow \mathbb{N}$ be the function $(e - 1)(C) = \max(e(C) - 1, 0)$.

Let $f : Y \rightarrow X$ be a morphism in \mathbf{Sm}/B , \mathcal{C} a finite set of locally closed subsets of X such that $f^{-1}(C)$ has pure codimension on Y for each $C \in \mathcal{C}$. We let $f^*\mathcal{C}$ be the set of irreducible components of the locally closed subsets $f^{-1}(C)$, $C \in \mathcal{C}$ and for $D \in f^*\mathcal{C}$, define $f^*(e)(D)$ to be the minimum of the numbers $e(C)$, with D an irreducible component of $f^{-1}(C)$. If f happens to be the identity morphism, then, even though $f^*\mathcal{C}$ may not be equal to \mathcal{C} , the support conditions $\mathcal{S}_{X, \mathcal{C}, e}$ and $\mathcal{S}_{X, f^*\mathcal{C}, f^*e}$ are equal.

If f is smooth, the pull-backs by $f \times \mathrm{id}_{\Delta^p}$ defines the map of simplicial sets

$$f^{-1} : \mathcal{S}_{X, \mathcal{C}, e}^{(q)} \rightarrow \mathcal{S}_{Y, f^*\mathcal{C}, f^*e}^{(q)}$$

and the map of simplicial spectra

$$f^* : E^{(q)}(X, -)_{\mathcal{C}, e} \rightarrow E^{(q)}(Y, -)_{f^*\mathcal{C}, f^*e}.$$

In particular, we have the presheaf of simplicial spectra (for the Nisnevich topology) on X , $E^{(p)}(X_{\mathrm{Nis}}, -)_{\mathcal{C}, e}$.

We now state the main result of this paper; for a description of the axioms (4.1.1), (4.1.2) and (4.1.3) we refer the reader to §4:

Theorem 2.6.2 (Generalized Chow's moving lemma). *Let B be a regular noetherian separated scheme of Krull dimension at most one and $X \rightarrow B$ a smooth B -scheme of finite type. Let \mathcal{C} be a finite set of irreducible locally closed subsets of X , $e : \mathcal{C} \rightarrow \mathbb{N}$ a function and $q \geq 0$ an integer. Take $E \in \mathbf{Spt}(B^{(q)})$ satisfying axioms (4.1.1) (localization), (4.1.2) (Nisnevich excision) and (4.1.3) (only in the case of finite residue fields). Suppose that each $C \in \mathcal{C}$ dominates an irreducible component of B .*

(1) *Let x be a point of X . Then there is a basis of Nisnevich neighborhoods $p : U \rightarrow X$ of x , depending only on x and X , such that for each U in this basis, $E^{(q)}(U, -)_{p^*\mathcal{C}, p^*e} \rightarrow E^{(q)}(U, -)$ is a weak equivalence. In particular, the map of presheaves on X_{Nis}*

$$E^{(q)}(X_{\text{Nis}}, -)_{\mathcal{C}, e} \rightarrow E^{(q)}(X_{\text{Nis}}, -)$$

is a local weak equivalence.

(2) *Suppose that $B = \text{Spec } k$, k a field, and let $j : U \rightarrow X$ be an affine open subscheme. Then the map of spectra*

$$E^{(q)}(U, -)_{j^*\mathcal{C}, j^*e} \rightarrow E^{(q)}(U, -)$$

is a weak equivalence. In particular, the map of presheaves on X_{Zar}

$$E^{(q)}(X_{\text{Zar}}, -)_{\mathcal{C}, e} \rightarrow E^{(q)}(X_{\text{Zar}}, -)$$

is a local weak equivalence.

The proof of Theorem 2.6.2 is in two main steps, just as the classical case (cf. [24]): One needs a ‘‘moving by translation’’ result to handle the case of affine n -space, and then one uses the method of the projecting cone to handle the general case. The moving by translation result is discussed in the following section §3. The extension of the standard technology for the projecting cone to the case at hand is done in §5, with the proof being completed in §6.

3. EASY MOVING

In this section, we deal with the moving by translation portion of the argument. This result also allows us to prove the homotopy invariance of the spectra $E^{(p)}(X, -)$, following Bloch's argument for the cycle complexes [2]. We assume throughout this section that the base scheme B is of the form $B = \text{Spec } A$, with A a semi-local PID.

3.1. Some added generality. In fact, neither the moving-by-translation technique nor the proof of homotopy invariance require smoothness, so both are applicable to presheaves of spectra on other categories besides $\mathbf{Sm}/B^{(q)}$. To adapt to this more general setting, we first need to give a definition of the ‘‘dimension’’ of a finite-type B -scheme.

Definition 3.1.1. Let B be a regular irreducible scheme of Krull dimension ≤ 1 . Let $f : X \rightarrow B$ be an irreducible finite type B -scheme. If $B = \text{Spec } k$, k a field, set

$\dim X = \dim_B X$. If B has Krull dimension 1, let η be the generic point of B . Set

$$\dim X := \begin{cases} \dim_{k(\eta)} X_\eta + 1 & \text{if } X_\eta \neq \emptyset \\ \dim_{k(b)} X & \text{if } f(X) = b \in B^{(1)}. \end{cases}$$

We extend the definition to the case of reducible B in the obvious way.

If $W \subset X$ is a closed subset of some $X \in \mathbf{Sch}/B$, we write $\dim W \leq n$ if each irreducible component W' of W has $\dim W' \leq n$. \square

Note that, if $W' \subset W \subset X$ are closed subsets of X , then

$$\dim W \leq n \implies \dim W' \leq n.$$

Let X be in \mathbf{Sch}/B , $Y \in \mathbf{Sm}/B$ and W a closed subset of $X \times_B Y$. We write $\dim\text{-rel}_Y W \leq q$ if for each irreducible component Y_j of Y , we have

$$\dim W \cap X \times_B Y_j \leq q + \dim_B Y_j.$$

Definition 3.1.2. Let X be a finite type B -scheme. Fix an integer q . Let $X/B_{(q)}$ be the category of pairs (Y, W) , with Y in \mathbf{Sm}/B , quasi-projective over B , and W a closed subset of $X \times_B Y$ such that $\dim\text{-rel}_Y W \leq q$.

A morphism $f : (Y, W) \rightarrow (Y', W')$ is a morphism $f : Y \rightarrow Y'$ in \mathbf{Sm}/B such that $W \supset (\text{id}_X \times f)^{-1}(W')$. \square

Remark 3.1.3. If B has pure Krull dimension zero, then $B/B_{(q)}$ is the full subcategory of $\mathbf{Sm}/B^{(-q)}$ consisting of (Y, W) with Y quasi-projective. If B has pure Krull dimension one, then $B/B_{(q)}$ is the similarly defined full subcategory of $\mathbf{Sm}/B^{(1-q)}$. \square

We have the category $\mathbf{Spt}(X/B_{(q)})$ of presheaves of spectra on $X/B_{(q)}$ and the homotopy category $\mathcal{H}\mathbf{Spt}(X/B_{(q)})$. As before, we write $E^W(Y)$ for $E(Y, W)$ if E is in $\mathbf{Spt}(X/B_{(q)})$ and (Y, W) is in $X/B_{(q)}$.

Let C be a constructible subset of $X \in \mathbf{Sch}/B$ with closure \bar{C} . If \bar{C} has irreducible components $\bar{C}_1, \dots, \bar{C}_s$, we call the constructible sets $C \cap \bar{C}_1, \dots, C \cap \bar{C}_s$ the irreducible components of C and say that C is irreducible if $s = 1$, i.e. if \bar{C} is irreducible.

If C is irreducible define $\dim C := \dim \bar{C}$; similarly, we say that C has *pure codimension* d on X if for each irreducible component X_i of X containing C , we have $\text{codim}_{X_i} \bar{C} = d$. We write this as $\text{codim}_X C = d$. If C has irreducible components C_1, \dots, C_s , we say $\dim C \leq m$ if $\dim C_i \leq m$ for each i . In particular, for W a closed subset of X , the statement $\dim W \cap C \leq m$ makes sense. Finally, if C is an irreducible constructible subset of X , we say that C dominates a component of B if this is so for \bar{C} .

Definition 3.1.4. (1) Let X be in \mathbf{Sch}/B . Let $\mathcal{S}_{(q)}^X(p)$ be the set of closed subsets $W \subset X \times \Delta_B^p$ such that

$$\dim\text{-rel}_F W \cap (X \times F) \leq q$$

for each face F of Δ_B^p .

(2) Let X be in \mathbf{Sch}/B , let \mathcal{C} be a finite set of constructible subsets of X , and $e : \mathcal{C} \rightarrow \mathbb{N}$ a function. We assume that each $C \in \mathcal{C}$ has pure codimension on X . Let

$\mathcal{S}_{(q)}^{X,\mathcal{C},e}(p)$ be the subset of $\mathcal{S}_{(q)}^X(p)$ consisting of those W such that, for each $C \in \mathcal{C}$ and each face F of Δ_B^p , we have

$$\dim\text{-rel}_F(C \times F) \cap W \leq q + e(C) - \text{codim}_X C.$$

□

Definition 3.1.5. Take $X \in \mathbf{Sch}/B$, E in $\mathbf{Spt}(X/B_{(q)})$, $q \in \mathbb{Z}$. Set

$$E_{(q)}(X, -) := E^{\mathcal{S}_{(q)}^X}(\Delta^*).$$

If we have a finite set \mathcal{C} of constructible subsets of X (with pure codimension) and a function $e : \mathcal{C} \rightarrow \mathbb{N}$, set

$$E_{(q)}(X, -)^{\mathcal{C},e} := E^{\mathcal{S}_{(q)}^{X,\mathcal{C},e}}(\Delta^*).$$

□

Remarks 3.1.6. (1) Suppose that each irreducible component X_i of X has $\dim X_i = d$; we call such an X *equi-dimensional of dimension d* . Then we can index by codimension, as in Definitions 2.4.1 and 2.6.1, and we have

$$\mathcal{S}_{(q)}^{X,\mathcal{C},e}(p) = \mathcal{S}_{X,\mathcal{C},e}^{(d-q)}(p).$$

(2) Suppose that X is irreducible and that $X \rightarrow B$ is dominant, so we can index using codimension. Let \mathcal{C} be a finite set of irreducible locally closed subsets of X , $e : \mathcal{C} \rightarrow \mathbb{N}$ a function and W an irreducible closed subset of $X \times \Delta^n$. Let $i : \eta \rightarrow B$ be the generic point of B . Suppose that each $C \in \mathcal{C}$ dominates a component of B . Then W is in $\mathcal{S}_{X,\mathcal{C},e}^{(q)}(n)$ if and only if W_η is in $\mathcal{S}_{X_\eta, i^*\mathcal{C}, i^*(e)}^{(q)}(n)$ and W_b is in $\mathcal{S}_{X, i_b^*\mathcal{C}, i_b^*(e)}^{(q-1)}(n)$ for all codimension one points $i_b : b \rightarrow B$ of B .

(3) The extension from locally closed subsets to constructible subsets of X is only a notational convenience, at least if X is quasi-projective over B . Indeed, if C is a constructible subset of X of pure codimension, we can write C as a finite disjoint union of locally closed subsets $C_i \subset X$ of pure codimension $d_i \geq d$; if we set $e(C_i) := e(C) + d_i - d$, we can replace $\mathcal{C} = \{C\}, e$ with $\mathcal{D} := \{C_i \mid C \in \mathcal{C}\}, e$ and $\mathcal{S}_{X,\mathcal{C},e} = \mathcal{S}_{X,\mathcal{D},e}$. Similarly if X is irreducible and an irreducible constructible subset C dominates a component of B , then we can find a decomposition $C = \coprod_i C_i$ into locally closed subsets such that each C_i dominates a component of B , and we may then apply the comments of (2). The same holds if $X = Y \times_B Z$, with $Z \rightarrow B$ quasi-projective, $\mathcal{C} = \{Y \times C \mid C \in \mathcal{C}_0\}$, $e(Y \times C) = e_0(C)$ for some collection \mathcal{C}_0 of constructible subsets of Z and function $e_0 : \mathcal{C}_0 \rightarrow \mathbb{N}$. □

Notation 3.1.7. For X, E, \mathcal{C} and e as above, let Y be in \mathbf{Sm}/B of pure dimension d over B . Sending $W \subset (X \times_B Y) \times_B T$ to $W \subset X \times_B (Y \times_B T)$ defines the functor

$$p_Y : X \times_B Y/B_{(q+d)} \rightarrow X/B_{(q)}$$

and by pull-back the functor

$$p_Y^* : \mathbf{Spt}(X/B_{(q)}) \rightarrow \mathbf{Spt}(X \times_B Y/B_{(q+d)}).$$

We write $E_{(q)}(X_Y, -)^{\mathcal{C},e}$ for $(p_Y^* E)_{(q+d)}(X \times_B Y, -)^{p_2^*\mathcal{C}, p_2^*e}$. The projection $X \times_B Y \rightarrow X$ defines the “base extension” map

$$p_2^* : E_{(q)}(X, -)^{\mathcal{C},e} \rightarrow E_{(q)}(X_Y, -)^{\mathcal{C},e}.$$

We occasionally need to extend the assignment $Y \mapsto E_{(q)}(X_Y, -)^{\mathcal{C}, e}$ to essentially smooth subschemes $U \subset Y$, i.e., $Y \setminus U$ is a possibly infinite union of closed subsets. We define

$$E_{(q)}(X_U, -)^{\mathcal{C}, e} := \operatorname{hocolim}_F E_{(q)}(X_{Y \setminus F}, -)^{\mathcal{C}, e}$$

where F runs over closed subsets of Y contained in $Y \setminus U$.

In case $p : B' \rightarrow B$ is in $\mathbf{Sm}/B^{\text{ess}}$, and B' is also a regular scheme of Krull dimension one, we have

$$E_{(q)}(X_{B'}, -)^{\mathcal{C}, e} = E_{(q)}(X_{B'}, -)^{p_2^* \mathcal{C}, p_2^*(e)},$$

where on the right-hand side, we consider $X_{B'}$ as in \mathbf{Sch}/B' , and we extend E to a presheaf on $\mathbf{Sm}/B'_{(q)}$ by taking limits as above. Also, we have the base-extension map

$$p^* : E_{(q)}(X, -)^{\mathcal{C}, e} \rightarrow E_{(q)}(X_{B'}, -)^{\mathcal{C}, e}.$$

□

Remark 3.1.8. In this section E will be a functor $E : X/B_{(q)}^{\text{op}} \rightarrow \mathbf{Spt}$. In case the residue fields of B are infinite, the results of this section are valid for every such E . In the case of finite residue fields, one needs to impose an additional hypothesis on E which allows one to increase the size of the residue fields at will. As this is a technical point, we postpone the description of this additional hypothesis (axiom (4.1.3)) until §4. □

Examples 3.1.9. Our primary examples are:

(1) X is in \mathbf{Sm}/B of dimension d over B , and E is in $\mathbf{Spt}(B^{(d-a)})$. Then we have $(Y, W \subset X \times_B Y) \mapsto E^W(X \times_B Y)$, giving the functor

$$E_X : X/B_{(q)}^{\text{op}} \rightarrow \mathbf{Spt}.$$

As particular examples, we have the functor $E^{(d-a)} \in \mathbf{Spt}(B^{(d-a)})$ induced by a functor $E \in \mathbf{Spt}(B)$, e.g. E the K -theory spectrum $Y \mapsto K(Y)$.

(2) X is a quasi-projective B -scheme and E is the G -theory spectrum with supports. More precisely, let Y be a quasi-projective B -scheme and let $\mathcal{M}_X(Y)$ be the exact category of coherent sheaves on $X \times_B Y$ which are flat as \mathcal{O}_Y -modules. We let $G_X(Y)$ denote the K -theory spectrum of the exact category $\mathcal{M}_X(Y)$. Let $G(X \times_B Y)$ denote the K -theory spectrum of the category of coherent sheaves on $X \times_B Y$.

For Y quasi-projective over B , each $\mathcal{F} \in \mathcal{M}_X(Y)$ admits a surjection $\mathcal{E} \rightarrow \mathcal{F}$ with \mathcal{E} a locally free coherent sheaf on $X \times_B Y$; if Y is smooth over B and has Krull dimension $\leq d$, then each \mathcal{F} admits a resolution

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E}_d \rightarrow \dots \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0$$

with the \mathcal{E}_j locally free coherent sheaves and \mathcal{K} a flat \mathcal{O}_Y -module. By Quillen's resolution theorem [23, §4, Theorem 3], this implies that the canonical map $G_X(Y) \rightarrow G(X \times_B Y)$ is a weak equivalence.

By the flatness condition, $Y \mapsto \mathcal{M}_X(Y)$ is an exact pseudo-functor on \mathbf{Sm}/B . By the usual trick (see e.g. [23, §7, proof of Proposition 2.2]), we replace $\mathcal{M}_X(Y)$ with an equivalent exact category and change notation so that $Y \mapsto \mathcal{M}_X(Y)$ is an exact functor on \mathbf{Sm}/B . Thus, we have the functor $Y \mapsto G_X(Y)$ on \mathbf{Sm}/B , and the weak equivalence $G_X(Y) \rightarrow G(X \times_B Y)$, natural in Y for flat morphisms. In

particular, if $W \subset X \times_B Y$ is a closed subset, Quillen's localization theorem [23, Thm. 5, §5] yields a weak equivalence

$$G_X^W(Y) \rightarrow G^W(X \times_B Y) \sim G(W).$$

We have thus succeeded in extending (up to weak equivalence) the assignment

$$(Y, W \subset X \times_B Y) \mapsto G(W)$$

to a functor

$$G_{X,(q)} : X/B_{(q)} \rightarrow \mathbf{Spt}.$$

(3) Let X be a finite type B -scheme. For $(Y, W) \in X/B_{(q)}$, Y irreducible, let $z_q(Y, W)$ be the free abelian group on the irreducible components W_i of W with

$$\dim W_i = q + \dim_B Y.$$

We extend to arbitrary Y by taking the direct sum over the irreducible components of Y .

Let $f : Y' \rightarrow Y$ be a morphism in \mathbf{Sm}/B . Since f has finite Tor-dimension, the same is true for $\text{id} \times f : X \times_B Y' \rightarrow X \times_B Y$. For $(Y, W), (Y', W') \in X/B_{(q)}$ such that f extends to $f : (Y', W') \rightarrow (Y, W)$, and W_i an irreducible component of W as above, each component of $(\text{id} \times f)^{-1}(W_i)$ has the proper dimension, and so the cycle-theoretic pull-back

$$(\text{id} \times f)^* : z_q(Y, W) \rightarrow z_q(Y', W')$$

is well-defined. We thus have the functor

$$z_q : X/B_{(q)} \rightarrow \mathbf{Ab};$$

taking the associated Eilenberg-MacLane spectrum gives us the functor

$$z_q : X/B_{(q)} \rightarrow \mathbf{Spt}.$$

□

3.2. The homotopy. As a preliminary for our construction, we recall how one constructs a homotopy of simplicial sets.

We give the product $[p] \times [q]$ the product partial order

$$(a, b) \leq (a', b') \Leftrightarrow a \leq a' \text{ and } b \leq b'.$$

In what follows, we work in the the category \mathbf{pos} of partially ordered sets, i.e. maps are order-preserving maps of partially ordered sets.

Let $F : \mathbf{Ord}^{\text{op}} \rightarrow \mathbf{Spc}$, $G : \mathbf{Ord}^{\text{op}} \rightarrow \mathbf{Spc}$ be functors, with geometric realizations $|F|$ and $|G|$. Suppose we have, for each $h : [p] \rightarrow [q] \times [1]$, a morphism

$$H(h) : G([q]) \rightarrow F([p])$$

such that, for $g : [q] \rightarrow [r]$, $f : [s] \rightarrow [p]$, we have

$$H(h \circ f) = F(f) \circ H(h), \quad H((g \times \text{id}) \circ h) = H(h) \circ G(g).$$

Restricting the morphisms h to those of the form $i_\epsilon^p : [p] \rightarrow [p] \times [1]$,

$$i_\epsilon^p(j) = (j, \epsilon), \quad \epsilon = 0, 1,$$

the maps $H(i_\epsilon^p) : G([p]) \rightarrow F([p])$ define the maps of simplicial spaces

$$H_0, H_1 : G \rightarrow F.$$

Lemma 3.2.1. *The maps $H(h)$ define the map in \mathbf{Spc}*

$$H : |G| \times I \rightarrow |F|,$$

with

$$H_{|G| \times 0} = |H_0|, \quad H_{|G| \times 1} = |H_1|.$$

Proof. Indeed, $|G| \times I$ is the geometric realization of the simplicial space

$$(G \times I)([p]) = \coprod_{[q]} G([q]) \times \mathrm{Hom}_{\mathbf{pos}}([p], [q] \times [1]) / \sim,$$

where \sim is the equivalence relation $(G(g)(x), h) \sim (x, (g \times \mathrm{id}) \circ h)$, with the evident map on morphisms; to see this, one first considers the case $G = \Delta[n]$, from which the general case follows using the canonical isomorphism

$$G \cong \mathrm{colim}_{[n], \sigma: \Delta[n] \rightarrow G} \Delta[n].$$

Our assertion follows directly from this. \square

We can now proceed to the construction of the homotopy we will need.

Let v_i^p be the vertex $t_j = 0$, $j \neq i$ of Δ^p , and let $g = (g_1, g_2) : [n] \rightarrow [p] \times [q]$ be an order-preserving map. We have the affine linear map $T(g) : \Delta^n \rightarrow \Delta^p \times \Delta^q$ with $T(g)(v_i^n) = v_{g_1(i)}^p \times v_{g_2(i)}^q$. If g is injective, then $T(g)$ is a closed embedding. A face F of $\Delta^p \times \Delta^q$ is a subscheme of the form $T(g)(\Delta^n)$ for an injective g .

Definition 3.2.2. Let \mathcal{C} be a finite set of constructible subsets of X , with each $C \in \mathcal{C}$ having pure codimension on X , $e : \mathcal{C} \rightarrow \mathbb{N}$ a function. We let $\mathcal{S}_{(q),h}^{X,\mathcal{C},e}(n) \subset \mathcal{S}_{(q)}^{X \times \Delta^1}(n)$ be the set of $W \in \mathcal{S}_{(q)}^{X \times \Delta^1}(n)$ such that

- (1) For each face F of $\Delta^1 \times \Delta^n$, we have

$$\dim\text{-rel}_F W \cap X \times F \leq q$$

- (2) For each face F of $\Delta^1 \times \Delta^n$ and each $C \in \mathcal{C}$, we have

$$\dim\text{-rel}_F W \cap C \times F \leq q - \mathrm{codim}_X C + e(C).$$

\square

$n \mapsto \mathcal{S}_{(q),h}^{X,\mathcal{C},e}(n)$ defines a support condition on $X \times \Delta^1$; we set

$$E_{(q)}(X \times \Delta^1, -)_h^{\mathcal{C},e} := E_{(q),h}^{\mathcal{S}_{(q),h}^{X,\mathcal{C},e}}(X \times \Delta^1 \times \Delta^*).$$

Remarks 3.2.3. (1) Letting $0 = \delta_0(\Delta^0)$, $1 = \delta_1(\Delta^0) \in \Delta^1$, the subschemes $0 \times \Delta^n$ and $1 \times \Delta^n$ are faces of $\Delta^1 \times \Delta^n$. Thus the pull-back maps

$$\delta_0^*, \delta_1^* : E_{(q)}(X \times \Delta^1, -)_h^{\mathcal{C},e} \rightarrow E_{(q)}(X, -)^{\mathcal{C},e}$$

are defined.

- (2) The projection $p_1 : X \times \Delta^1 \rightarrow X$ has

$$p_1^{-1}(\mathcal{S}_{(q)}^{X,\mathcal{C},e}) \subset \mathcal{S}_{(q),h}^{X,\mathcal{C},e}.$$

Thus we have a well-defined pull-back map

$$p_1^* : E_{(q)}(X, -)^{\mathcal{C},e} \rightarrow E_{(q)}(X \times \Delta^1, -)_h^{\mathcal{C},e}.$$

\square

Lemma 3.2.4. *The maps $T(g)$ define a homotopy of the maps*

$$\delta_0^*, \delta_1^* : E_{(q)}(X \times \Delta^1, -)_h^{C,e} \rightarrow E_{(q)}(X, -)^{C,e},$$

natural in (C, e) and E , and natural with respect to base-change $X \mapsto X_{B'}$ for B -schemes $B' \rightarrow B$ in $\mathbf{Sm}/B^{\text{ess}}$.

Proof. This follows immediately from Lemma 3.2.1. \square

3.3. Action by a group-scheme. The “easy” moving lemma requires a transitive action by a connected linear algebraic group. We will only need the case of \mathbb{A}^n , with the action of translation on itself.

Let \mathcal{G} be a group scheme over B , and let $X \in \mathbf{Sch}/B$ be a B -scheme with a (right) \mathcal{G} -action $\rho : X \times_B \mathcal{G} \rightarrow X$. Let $B' \rightarrow B$ be in $\mathbf{Sm}/B^{\text{ess}}$, $\psi : \mathbb{A}_{B'}^1 \rightarrow \mathcal{G}$ a B -morphism with $\psi(0) = \text{id}_{\mathcal{G}}$. We let $\phi : X_{B'} \times \Delta^1 \rightarrow X$ be the composition

$$X_{B'} \times \Delta^1 \xrightarrow{\sim} X \times_B \mathbb{A}_{B'}^1 \xrightarrow{\text{id} \times \psi} X \times_B \mathcal{G} \xrightarrow{\rho} X,$$

where the isomorphism $X_{B'} \times \Delta^1 \xrightarrow{\sim} X \times_B \mathbb{A}_{B'}^1$ is induced by the isomorphism $(t_0, t_1) \mapsto t_0$ of Δ^1 with \mathbb{A}^1 . Let $\pi : X_{B'} \rightarrow X$ be the base-change map. We have the composition

$$X_{B'} \xrightarrow{\cdot \psi(1)} X_{B'} \xrightarrow{\pi} X$$

which we denote by $\phi(1)$. Note that

$$\begin{aligned} \phi(1) &= \phi \circ (\text{id} \times \delta_1) \\ \pi &= \phi \circ (\text{id} \times \delta_0). \end{aligned}$$

We will take \mathcal{G} to be the additive group \mathbb{A}^n , and ψ to be the linear map ψ_x , $\psi_x(t) = t \cdot x$, where $x : B' \rightarrow \mathbb{A}^n$ is a B -morphism. We take $X = Y \times \mathbb{A}^n$, $Y \in \mathbf{Sch}/B$, with \mathcal{G} acting on \mathbb{A}^n by translation and on Y trivially.

Notation 3.3.1. We write $A(x_1, \dots, x_n)$ for the localization of the polynomial ring $A[x_1, \dots, x_n]$ with respect to the multiplicatively closed set of $f = \sum_I a_I x^I$ such that the ideal in A generated by the coefficients a_I is the unit ideal.

The inclusion $A \rightarrow A(x_1, \dots, x_n)$ is a faithfully flat extension of semi-local PID's. We set

$$B(x) := \text{Spec } A(x_1, \dots, x_n),$$

and let $x : B(x) \rightarrow \mathbb{A}^n$ be the morphism with $x^* : A[x_1, \dots, x_n] \rightarrow A(x_1, \dots, x_n)$ the evident inclusion. Note that $B(x)$ is in $\mathbf{Sm}/B^{\text{ess}}$. \square

Lemma 3.3.2. *Let \mathcal{C}_0 be a finite set of irreducible constructible subsets of \mathbb{A}_B^n , $e : \mathcal{C}_0 \rightarrow \mathbb{N}$ a function and let Y be in \mathbf{Sch}/B . Let $X = Y \times_B \mathbb{A}^n$, $\mathcal{C} = p_2^* \mathcal{C}_0$ and $e = p_2^*(e_0)$. Let W be in $\mathcal{S}_X^{(q)}(p)$, and let $\psi := \psi_x : B(x) \rightarrow \mathbb{A}^n$. Suppose that each $C \in \mathcal{C}_0$ dominates an irreducible component of B . Then*

- (1) $\phi(1) \times \text{id}^{-1}(W)$ is in $\mathcal{S}_{(q)}^{X_{B(x)}, \mathcal{C}}(p)$.
- (2) Suppose that W is in $\mathcal{S}_{(q)}^{X, \mathcal{C}, e}(p)$. Then $\phi \times \text{id}^{-1}(W)$ is in $\mathcal{S}_{(q), h}^{X_{B(x)}, \mathcal{C}, e}(p)$.

Proof. By Remark 3.1.6(3), we may assume that \mathcal{C}_0 is a finite set of irreducible, locally closed subsets of \mathbb{A}_B^n , such that each $C \in \mathcal{C}_0$ dominates an irreducible component of B . Since the translation action preserves all the irreducible components of X , we may assume that X is irreducible. If $d = \dim X$, then $\mathcal{S}_{(q)}^X = \mathcal{S}_X^{(d-q)}$, etc., so we may index with codimension.

In case A is a field, or if $X = X_b$ for some $b \in B^{(1)}$, the result follows directly from [2, Lemma 2.2].

Suppose that A is not a field and that $X \rightarrow B$ is dominant. Let η be the generic point of B . Since the inclusion $i : \eta \rightarrow B$ is flat, W_η is in $\mathcal{S}_{X_\eta}^{(q)}$, and if W is in $\mathcal{S}_{X, \mathcal{C}, e}^{(q)}$, then W_η is in $\mathcal{S}_{X_\eta, i^* \mathcal{C}, i^*(e)}^{(q)}$. From the case of a field, it follows that (1) and (2) are valid with η replacing B , and $i^* \mathcal{C}, i^*(e)$ replacing \mathcal{C}, e .

For a codimension one point $i_b : b \rightarrow B$, we have the set of irreducible constructible subsets $i_b^* \mathcal{C}$ of X_b and the function $i_b^* e : i_b^* \mathcal{C} \rightarrow \mathbb{N}$.

To complete the proof it suffices by Remark 3.1.6(2) to show that, for each codimension one point $i_b : b \rightarrow B$ of B that

- (1) $\phi(1)_b^{-1}(W_b)$ is in $\mathcal{S}_{X_{b(x)}, i_b^* \mathcal{C}}^{(q-1)}$.
- (2) Suppose that W is in $\mathcal{S}_{X, \mathcal{C}, e}^{(q)}$. Then $\phi^{-1}(W_b)$ is in $\mathcal{S}_{X_{b(x)}, i_b^* \mathcal{C}, i_b^*(e)}^{(q-1), h}$.

By Remark 3.1.6(2)

$$W \in \mathcal{S}_{X, \mathcal{C}, e}^{(q)} \implies W_b \in \mathcal{S}_{X_b, i_b^* \mathcal{C}, i_b^* e}^{(q-1)}$$

and similarly without the (\mathcal{C}, e) . We note that $B(x) \rightarrow B$ is a bijection on points and that the $b(x)$ -valued point $x \times_B b$ of \mathbb{A}_b^n has \mathbb{A}_b^n as Zariski closure. Thus (1) and (2) follow from the case of fields. \square

We let $[E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X)$ be the homotopy cofiber of the map of simplicial spectra

$$E_{(q)}(X, -)^{\mathcal{C}, e} \rightarrow E_{(q)}(X, -).$$

We similarly let $[E_{(q)}/E_{(q)}^{\mathcal{C}, e}]_h(X \times \Delta^1)$ be the homotopy cofiber of the map of simplicial spectra

$$E_{(q)}(X \times \Delta^1, -)_h^{\mathcal{C}, e} \rightarrow E_{(q)}(X \times \Delta^1, -)_h.$$

Let $\pi : X_{B(x)} \rightarrow X$ denote the projection.

Lemma 3.3.3. *Let X, \mathcal{C} and e be as in Lemma 3.3.2. Then, for each $q \in \mathbb{Z}$, the base-change map*

$$\pi^* : [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X) \rightarrow [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X_{B(x)})$$

is homotopic to zero.

Proof. Consider the commutative diagram, well-defined by Lemma 3.3.2(2),

$$\begin{array}{ccc} [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X) & \xrightarrow{\phi^*} & [E_{(q)}/E_{(q)}^{\mathcal{C}, e}]_h(X \times \Delta_{B(x)}^1) \\ \pi^* \downarrow \Downarrow \phi(1)^* & & \delta_0^* \downarrow \Downarrow \delta_1^* \\ [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X_{B(x)}) & \xlongequal{\quad} & [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X_{B(x)}) \end{array}$$

By Lemma 3.2.4, $\phi(1)^*$ and π^* are homotopic, but by Lemma 3.3.2(1), $\phi(1)^*$ is homotopic to zero. \square

Proposition 3.3.4 (Moving by translation). *Let A be a semi-local PID and let $B = \text{Spec } A$. Let \mathcal{C}_0 be a finite set of irreducible constructible subsets of \mathbb{A}_B^n , $e : \mathcal{C}_0 \rightarrow \mathbb{N}$ a function and let Y be in \mathbf{Sch}/B . Let $X = Y \times_B \mathbb{A}^n$, $\mathcal{C} = p_2^* \mathcal{C}_0$ and $e = p_2^*(e_0)$. In case B has a point b with finite residue field $k(b)$, we assume that E*

satisfies axiom (4.1.3). Suppose further that each $C \in \mathcal{C}_0$ dominates a component of B . Then, for each $q \in \mathbb{Z}$, the map

$$E_{(q)}(X, -)^{\mathcal{C}, e} \rightarrow E_{(q)}(X, -)$$

is a weak equivalence.

Proof. It suffices to show that $[E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X)$ is weakly equivalent to a point. By Lemma 3.3.3, it suffices to show that the base-change map

$$(3.3.1) \quad \pi^* : [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X) \rightarrow [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X_{B(x)})$$

is injective on homotopy groups.

For this, we note that the scheme $B(x)$ is a filtered inverse limit of open subschemes U of \mathbb{A}_B^n , with U faithfully flat and of finite type over B . As

$$[E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X_{B(x)}) = \operatorname{hocolim}_{B(x) \subset U \subset \mathbb{A}_B^n} [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X_U)$$

by definition, it suffices to show that the map

$$p_2^* : [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X) \rightarrow [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X_U)$$

is injective on homotopy groups for each such U .

For this, it suffices to show that, for each U and each collection (W', W, V', V) of simplicial closed subsets with

- (1) $W' \subset W, V' \subset V, p_2^*W' \subset V'$ and $p_2^*W \subset V$
- (2) $W' \in \mathcal{S}_{(q)}^{X, \mathcal{C}, e}, W \in \mathcal{S}_{(q)}^X, V' \in \mathcal{S}_{(q)}^{X_U, \mathcal{C}, e}, V \in \mathcal{S}_{(q)}^{X_U}$

the map

$$(3.3.2) \quad p_2^* : [E^W(X \times \Delta^*)/E^{W'}(X \times \Delta^*)] \rightarrow [E^{V'}(X_U \times \Delta^*)/E^V(X_U \times \Delta^*)]$$

has the property that, if $p_2^*(x) = 0$ for $x \in \pi_m([E^W(X \times \Delta^*)/E^{W'}(X \times \Delta^*)])$, then x goes to zero in $\pi_m([E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X))$. We may also assume that each of W', W, V', V are generated by finitely many closed subsets W'_i, W_i, V'_i, V_i of $X \times \Delta^{n_i}, X_U \times \Delta^{n_i}$, $i = 1, \dots, s$.

Suppose first that each residue field $k(b)$, $b \in B$, is infinite. As B is semi-local, and U is an open subscheme of \mathbb{A}_B^n with $\mathbb{A}_{k(b)}^n \cap U \neq \emptyset$ for all b , the projection $U \rightarrow B$ admits a section $s : B \rightarrow U$ such that

$$(s \times \operatorname{id})^{-1}(V') \in \mathcal{S}_{(q)}^{\mathcal{C}, e, X}$$

$$(s \times \operatorname{id})^{-1}(V) \in \mathcal{S}_{(q)}^X$$

Thus we have the map

$$s^* : [E^{V'}(X_U \times \Delta^*)/E^V(X_U \times \Delta^*)] \rightarrow [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X_U)$$

with $s^* \circ p_2^*$ the canonical map $[E^W(X \times \Delta^*)/E^{W'}(X \times \Delta^*)] \rightarrow [E_{(q)}/E_{(q)}^{\mathcal{C}, e}](X)$, which suffices for our purpose.

If some $k(b)$ is finite, then E satisfies axiom (4.1.3). Take distinct primes p_1, p_2 , an integer $n \geq 1$ and let $k(b) \rightarrow k(b_1), k(b) \rightarrow k(b_2)$ be field extensions of respective degree p_1^n, p_2^n , for each b with $k(b)$ finite. There exist finite Galois étale B -schemes $f_1 : B_1 \rightarrow B, f_2 : B_2 \rightarrow B$, of relatively prime degrees n_1, n_2 over B , such that, for each b such that $k(b)$ is finite, each point $c \in B_j \times_B b$ has $k(c) \supset k(b_j)$ (see [17, Lemma 6.1]). By axiom (4.1.3), the maps $E \rightarrow f_{j*}^{G_j} f_j^* E$ are weak homotopy

equivalences after inverting n_j . One can then argue as above to show that p_2^* is injective on homotopy groups. \square

Theorem 3.3.5 (Homotopy invariance). *Let A be a semi-local PID and let $B = \text{Spec } A$. Let X be in \mathbf{Sch}/B and let $E : X/B_{(q)}^{\text{op}} \rightarrow \mathbf{Spt}$ be in $\mathbf{Spt}(X/B_{(q)})$. If $k(b)$ is finite for some $b \in B$, we assume in addition that E satisfies axiom (4.1.3). Then for $X \in \mathbf{Sch}/B$ the map*

$$p^* : E_{(q)}(X, -) \rightarrow E_{(q)}(X \times \mathbb{A}^1, -)$$

is a weak equivalence.

Proof. One repeats the argument used to prove [2, Theorem 2.1], replacing homology with homotopy, chain homotopy equivalence with homotopy equivalence and using Lemma 3.3.3 in place of [2, Lemma 2.3]. For the reader's convenience, we give a sketch.

The canonical map $E_{(q)}(X \times \mathbb{A}^1, -)_{\{X \times 0\}} \rightarrow E_{(q)}(X \times \mathbb{A}^1, -)$ is a weak equivalence by Proposition 3.3.4, and p^* factors through

$$p^* : E_{(q)}(X, -) \rightarrow E_{(q)}(X \times \mathbb{A}^1, -)_{\{X \times 0\}}.$$

As this map is split by i_0^* , p^* is injective on π_n , so we need only show surjectivity.

One first shows:

Lemma 3.3.6. *Let $i_0, i_1 : X \rightarrow X \times \mathbb{A}^1$ be the 0- and 1-sections, respectively. Then the maps*

$$i_0^*, i_1^* : E_{(q)}(X \times \mathbb{A}^1, -)_{\{X \times 0, X \times 1\}} \rightarrow E_{(q)}(X, -)$$

induce the same map on homotopy groups π_n , $n \in \mathbb{Z}$.

proof of lemma. Let \mathbb{A}^1 act on Δ^1 by translation: $t \cdot (t_0, t_1) = (t_0 + t, t_1 - t)$. Let $\psi : \mathbb{A}_{B(x)}^1 \rightarrow \mathbb{A}^1 := \text{Spec } A[t]$ be the map with $\psi^*(t) = xt$. As in the proof of Lemma 3.3.3, the maps

$$\phi(1)^*, \pi \times \text{id}^* : E_{(q)}(X \times \mathbb{A}^1, -)_{\{X \times 0, X \times 1\}} \rightarrow E_{(q)}(X \times \mathbb{A}_{B(x)}^1, -)_{\{X \times 0, X \times 1\}}$$

are homotopic (π^* the base-change morphism), hence $\pi^* \circ i_\epsilon^* = i_\epsilon^* \circ (\pi \times \text{id})^*$ and $i_\epsilon^* \circ \phi(1)^*$ are homotopic, $\epsilon = 0, 1$.

Identifying \mathbb{A}^1 with Δ^1 by $(t_0, t_1) \mapsto t_0$, one shows as in the proof of Lemma 3.3.2 that $\phi(1)^{-1}$ maps $\mathcal{S}_{(q)}^{X \times \Delta^1}$ into $\mathcal{S}_{(q), h}^{X_{B(x)}}$, giving us the map

$$\phi(1)^* : E_{(q)}(X \times \mathbb{A}^1, -)_{\{X \times 0, X \times 1\}} \rightarrow E_{(q)}(X \times \Delta_{B(x)}^1, -)_h$$

By Lemma 3.2.4, the maps $\delta_0^* \circ \phi(1)^*$ and $\delta_1^* \circ \phi(1)^*$ are homotopic. Thus $\pi^* \circ i_0^*$ and $\pi^* \circ i_1^*$ are homotopic. But π^* is injective on π_n (cf. proof of Proposition 3.3.4), whence the lemma. \square

The next step is to consider the multiplication map

$$\mu : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$$

$\mu(x, y) = xy$; note that μ is flat. We have the commutative diagram

$$\begin{array}{ccc} E_{(q)}(X \times \mathbb{A}^1, -)_{X \times 0} & \xrightarrow{\mu^*} & E_{(q)}(X \times \mathbb{A}^1 \times \mathbb{A}^1, -)_{X \times \mathbb{A}^1 \times \{0,1\}} \\ p^* i_0^* \downarrow \downarrow \iota & & i_0^* \downarrow \downarrow i_1^* \\ E_{(q)}(X \times \mathbb{A}^1, -) & \xlongequal{\quad} & E_{(q)}(X \times \mathbb{A}^1, -) \end{array}$$

where $\iota : E_{(q)}(X \times \mathbb{A}^1, -)_{X \times 0} \rightarrow E_{(q)}(X \times \mathbb{A}^1, -)$ is the canonical map. Using the lemma above and Proposition 3.3.4, it follows that $p^* i_0^*$ is an isomorphism on π_n , hence p^* is surjective. \square

4. THE AXIOMS

We have seen that the moving-by-translation results are valid for an arbitrary functor $E : X/B_{(q)}^{\text{op}} \rightarrow \mathbf{Spt}$ (except in the case of finite residue fields, which requires the axiom (4.1.3)). To proceed further, however, we will need to have some additional properties of E : localization and Nisnevich excision. In this section, we list the axioms we need; throughout this section, the base-scheme B will be a noetherian separated scheme.

4.1. Localization and excision. Let E be in $\mathbf{Spt}(B^{(q)})$. We consider the following conditions on E :

(4.1.1)

Axiom 1. (Localization)

- (a) Let \mathbf{Sm}/B^\emptyset be the full subcategory of $\mathbf{Sm}/B^{(q)}$ with objects (Y, \emptyset) . The restriction of E to \mathbf{Sm}/B^\emptyset is contractible. We fix a contraction h , i.e., h is a homotopy between the identity and the map to the base-point on the restriction of E to \mathbf{Sm}/B^\emptyset .
- (b) Let (Y, W) be in $\mathbf{Sm}/B^{(q)}$, let $W' \subset W$ be a closed subset and let $j : Y \setminus W' \rightarrow Y$ be the inclusion. The sequence

$$E^{W'}(Y) \xrightarrow{i_{W',W}} E^W(Y) \xrightarrow{j^*} E^{W \setminus W'}(Y \setminus W')$$

together with h gives a canonical morphism

$$\phi_{W',W,Y} : E^{W \setminus W'}(Y \setminus W') \rightarrow \text{cofib}(i_{W',W})$$

Then $\phi_{W',W,Y}$ is a weak equivalence.

To explain: We have the commutative diagram

$$\begin{array}{ccc} E^{W'}(Y) & \xrightarrow{i_{W',W}} & E^W(Y) \\ j^* \downarrow & & \downarrow j^* \\ E^\emptyset(Y) & \xrightarrow{i_{\emptyset, W \setminus W'}} & E^{W \setminus W'}(Y \setminus W') \end{array}$$

The contraction h gives a contraction of $j^* \circ i_{W',W}$, which in turn induces the map $\phi_{W',W,Y}$. In particular, $\phi_{W',W,Y}$ is natural with respect to the category of morphisms $(Y, W) \rightarrow (Y, W')$ in $\mathbf{Sm}/B^{(q)}$.

(4.1.2)

Axiom 2. (Nisnevich excision) Let $f : Y \rightarrow X$ be an étale map in \mathbf{Sm}/B , $f : (Y, W') \rightarrow (X, W)$ an extension of f to a morphism in $\mathbf{Sm}/B^{(q)}$ with $W' = f^{-1}(W)$. Suppose that $f : W' \rightarrow W$ is an isomorphism (of schemes with the reduced structure). Then

$$f^* : E^W(X) \rightarrow E^{W'}(Y)$$

is a weak equivalence.

Lemma 4.1.1. *Let E be in $\mathbf{Spt}(B^{(q)})$, let (Y, W) be in $\mathbf{Sm}/B^{(q)}$ such that W is a disjoint union of closed subsets, $W = W' \amalg W''$. Suppose that E satisfies axiom 2. Then the natural map*

$$i_{W',W} + i_{W'',W} : E^{W'}(Y) \oplus E^{W''}(Y) \rightarrow E^W(Y)$$

is a weak equivalence.

Proof. We have the étale morphism $p : (Y \setminus W') \amalg (Y \setminus W'') \rightarrow Y$, induced by the inclusions. Clearly $p^{-1}(W) \rightarrow W$ is an isomorphism, so

$$p^* : E^W(Y) \rightarrow E^{p^{-1}(W)}((Y \setminus W') \amalg (Y \setminus W''))$$

is a weak equivalence by axiom 2. As

$$((Y \setminus W') \amalg (Y \setminus W''), p^{-1}(W)) = (Y \setminus W', W'') \amalg (Y \setminus W'', W')$$

and E is a presheaf, the map

$$E^{W''}(Y \setminus W') \oplus E^{W'}(Y \setminus W'') \rightarrow E^{p^{-1}(W)}((Y \setminus W') \amalg (Y \setminus W''))$$

is a weak equivalence. Using axiom 2 again, the restriction map

$$E^{W''}(Y) \oplus E^{W'}(Y) \rightarrow E^{W''}(Y \setminus W') \oplus E^{W'}(Y \setminus W'')$$

is a weak equivalence. The commutativity of the diagram

$$\begin{array}{ccc} E^W(Y) & \longrightarrow & E^{p^{-1}(W)}((Y \setminus W') \amalg (Y \setminus W'')) \\ \uparrow & & \uparrow \\ E^{W''}(Y) \oplus E^{W'}(Y) & \longrightarrow & E^{W''}(Y \setminus W') \oplus E^{W'}(Y \setminus W'') \end{array}$$

completes the proof. \square

For E , (Y, W) and W' as in Lemma 4.1.1, we let $p_{W'} : E^W(Y) \rightarrow E^{W'}(Y)$ be the composition in \mathcal{SH}

$$E^W(Y) \xrightarrow{(i_{W',W} + i_{W'',W})^{-1}} E^{W'}(Y) \oplus E^{W''}(Y) \xrightarrow{p_1} E^{W'}(Y).$$

We introduce one final axiom, to handle the case of finite residue fields. Suppose we have a finite étale Galois cover $\pi : B' \rightarrow B$ with group G . Given $E \in \mathbf{Spt}(B^{(q)})$, define $\pi_*^G \pi^* E$ by

$$\pi_*^G \pi^* E^W(X) := (E^{W_{B'}}(X_{B'}))^G,$$

where $(-)^G$ denotes a functorial model for the G homotopy fixed point spectrum. We have as well the natural transformation

$$\pi^* : E \rightarrow \pi_*^G \pi^* E.$$

(4.1.3)

Axiom 3. (Finite fields) Suppose that B has some closed point b with finite residue field $k(b)$. Let $\pi : B' \rightarrow B$ be a finite Galois étale cover of B with group G . Then after inverting $|G|$, the natural transformation $\pi^* : E \rightarrow \pi_*^G \pi^* E$ is a weak equivalence.

5. MOVING BY THE PROJECTING CONE

The method of moving by translation takes care of the case $X = \mathbb{A}^n$; for a general smooth affine B -scheme, we need to apply the classical method of the projecting cone. Since most of the known results (*cf.*[16, Part I, Chap. II, §3], [24]) are restricted to the case of varieties over a field, we will need to prove some technical results which enable the extension to base-schemes $B = \text{Spec } A$, A a semi-local PID.

5.1. The projecting cone. Although we will eventually need to assume that $B = \text{Spec } A$, with A a semi-local PID, we will work in a somewhat more general setting for a while, assuming only that A is semi-local and noetherian.

Let $i : X \rightarrow \mathbb{A}_B^n$ be a closed subscheme of \mathbb{A}_B^n with $X \in \mathbf{Sm}/B$. We assume that X is equi-dimensional over B ; let $m = \dim_B X$. Let $\bar{X} \subset \mathbb{P}_B^n$ be the closure of X in \mathbb{P}_B^n and let $\mathbb{P}_B^{n-1}(\infty) = \mathbb{P}_B^n \setminus \mathbb{A}_B^n$. Using homogeneous coordinates X_0, \dots, X_n for \mathbb{P}^n , $\mathbb{P}^{n-1}(\infty)$ is defined by $X_0 = 0$.

Let $M_{n \times m}$ be the affine space $\text{Spec } B[\{X_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}]$; we view $M_{n \times m} \times \mathbb{A}^m$ as the parameter space for affine-linear maps $\pi : \mathbb{A}^n \rightarrow \mathbb{A}^m$ by defining

$$\pi_{(x_{ij}), (v_j)}(y_1, \dots, y_n) = (y_1, \dots, y_n) \cdot (x_{ij}) + (v_1, \dots, v_m).$$

Let $\mathcal{V} \subset M_{n \times m}$ be the matrices of rank m . Each B' -valued point $x := (x_{ij})$ of \mathcal{V} yields the codimension m linear subspace $L(x) \subset \mathbb{P}_{B'}^{n-1}(\infty)$ defined by the equations

$$\sum_i x_{i1} X_i = \dots = \sum_i x_{im} X_i = 0.$$

We let $\mathcal{V}_X \subset \mathcal{V}$ be the open subscheme with $L(x) \cap \bar{X} = \emptyset$ for $x \in \mathcal{V}_X$ and set

$$\mathcal{U}_X := \mathcal{V}_X \times \mathbb{A}^m.$$

For $\pi \in \mathcal{U}_X$, let $\pi_X : X \rightarrow \mathbb{A}^m$ be the restriction of π to X .

For $x \in X$, we let $\mathcal{U}_X(x)$ be the open subscheme of \mathcal{U}_X of those π for which π_X is étale over a neighborhood of $\pi(x)$.

Let $B' \rightarrow B$ be a B -scheme and $y := (x, v) : B' \rightarrow \mathcal{U}_X$ a B -morphism. We write $L(y)$ for $L(x)$. The affine-linear projection $\pi_y : \mathbb{A}_{B'}^n \rightarrow \mathbb{A}_{B'}^m$ extends to the linear projection $\bar{\pi}_y : \mathbb{P}_{B'}^n \setminus L(y) \rightarrow \mathbb{P}_{B'}^m$. Since $L(y) \cap \bar{X}_{B'} = \emptyset$, the map $\bar{\pi}_y$ restricts to a finite surjective morphism $\bar{X}_{B'} \rightarrow \mathbb{P}_{B'}^m$. Thus the restriction of π_y to

$$\pi_{y,X} : X_{B'} \rightarrow \mathbb{A}_{B'}^m$$

is finite and surjective and the maps

$$\pi_{y,X} \times \text{id} : X_{B'} \times \Delta^p \rightarrow \mathbb{A}_{B'}^m \times \Delta^p$$

are also finite and surjective for each p .

We make the following basic assumptions:

(5.1.1)

- (1) The structure morphism $\mathcal{U}_X \rightarrow B$ is surjective.
- (2) For $x \in X$, $\mathcal{U}_X(x)$ is non-empty.

Remarks 5.1.1. (1) In case $\dim B \leq 1$, (5.1.1)(1) is equivalent to the assumption that $\mathbb{P}_B^{n-1}(\infty) \cap \bar{X}$ is equi-dimensional (of relative dimension $\dim_B X - 1$) over B , equivalently, for each $b \in B^{(1)}$,

$$(\bar{X})_b = \overline{X_b}$$

where $\overline{X_b}$ is the closure of X_b in \mathbb{P}_b^n . In particular, (5.1.1)(1) holds if A is a field.

(2) It is easy to construct $X \in \mathbf{Sm}/B$ which is affine and equi-dimensional over B , of dimension say d over B , but does not admit *any* finite surjective morphism $X_B \rightarrow \mathbb{A}_B^d$, even if all residue fields of B are infinite. However, we will show (Theorem 10.0.1) that this property holds locally on (X, B) in the Nisnevich topology.

(3) Suppose $i_0 : X \rightarrow \mathbb{A}_B^n$ satisfies (5.1.1)(1). Let

$$i_d : \mathbb{A}^n \rightarrow \mathbb{A}^N = \text{Spec } \mathbb{Z}[Y_1, \dots, Y_N]$$

be the embedding with $\{i_d^*(Y_j)\}$ a \mathbb{Z} -basis of the monomials of degree d for some $d > 1$. Then $i := (i_d \times_{\text{id}} B) \circ i_0$ satisfies (5.1.1)(2). Also, if all residue fields of B have characteristic zero, then (5.1.1)(2) is true for all embeddings i which satisfy (5.1.1)(1). See for example [1, XI 2.1] for a proof.

(4) For $y : B' \in \mathcal{U}_X$, $\pi_{y,X} : X_{B'} \rightarrow \mathbb{A}_{B'}^m$ is flat. Indeed, since X and \mathbb{A}^m are smooth over B , we may assume that $B = B' = \text{Spec } F$ for a field F . Note that X and \mathbb{A}^m are both Cohen-Macaulay schemes. Let z be a point of \mathbb{A}^m . Since $\pi_{y,X}$ is étale over a dense open subset of \mathbb{A}^m , we can find a system of parameters x_1, \dots, x_r at w such that $\pi_{y,X}$ is étale over each generic point of the subscheme Z_i defined by (x_1, \dots, x_i) , $1 \leq i \leq r-1$. Since X is Cohen-Macaulay, this implies that $\pi_{y,X}^{-1}(Z_i)$ is reduced for all $i = 1, \dots, r-1$; as $\pi_{y,X}$ is finite this implies that $\pi_{y,X}^*(x_1), \dots, \pi_{y,X}^*(x_r)$ is a regular sequence at $\pi_{y,X}^{-1}(w)$. \square

5.2. Notation and elementary properties. We assume in this section that (5.1.1) holds.

In particular, \mathcal{U}_X is an open subscheme of some $\mathbb{A}_B^r := \text{Spec } B[y_1, \dots, y_r]$ over B , of finite type and faithfully flat over B . Generalizing the construction of §3.3, we let $A(y)$ denote the localization of $A[y_1, \dots, y_r]$ with respect to the set of $f = \sum_I a_I y^I$ such that the ideal in A generated by the a_I is the unit ideal. This gives us the semi-local noetherian ring $A(y)$, and the B -scheme $B(y) := \text{Spec } A(y)$, together with a B -morphism $y : B(y) \rightarrow \mathcal{U}_X$ having Zariski dense image; in fact, for each point b of B , the restriction $y_b : b(y) \rightarrow \mathcal{U}_X \times_B b$ has Zariski dense image.

Notation 5.2.1. For each p , let $f_p : X_{B(y)} \times \Delta^p \rightarrow \mathbb{A}_{B(y)}^m \times \Delta^p$ denote the morphism $\pi_{y,X} \times \text{id}$. We let $R_y \subset X_{B(y)}$ denote the ramification locus of $\pi_{y,X}$.

If $W \subset X \times \Delta^p$ is a pure codimension q closed subset with irreducible components W_1, \dots, W_s , let $\text{cyc}(W) := \sum_{i=1}^s W_i$ and set

$$W^c := \text{Supp}(f_p^*(f_{p*}(\text{cyc}(W))) - \text{cyc}(W)).$$

Let $W^+ \subset X_{B(y)} \times \Delta^p$ denote the closure of $f_p^{-1}(f_p(W_{B(y)})) \setminus W_{B(y)}$. \square

Remark 5.2.2. For $W \subset X \times \Delta^p$ a pure codimension q closed subset, W^+ is also a pure codimension q closed subset of $X \times \Delta^p$, and $W^+ \subset W^c$. In addition $W^+ \cup W = W^c \cup W$, and $W^+ = W^c$ if and only if W^c and W have no irreducible components in common. \square

Lemma 5.2.3. *Suppose that A is a field. Let X be in \mathbf{Sm}/B with a closed embedding $i : X \rightarrow \mathbb{A}_B^n$, \mathcal{C} a finite set of irreducible locally closed subsets of X , and $e : \mathcal{C} \rightarrow \mathbb{N}$ a function. Let $q > 0$ be an integer.*

- (1) *Let Z be an effective codimension q cycle on X with support $|Z| \in \mathcal{S}_{X,\mathcal{C},e}^{(q)}(0)$. Then $\pi_{y,X}^*(\pi_{y,X*}(Z)) = Z + Z'$, with Z' effective. In addition, Z' has no component in common with Z , and $|Z'|$ is in $\mathcal{S}_{X,\mathcal{C},e-1}^{(q)}(0)$.*
- (2) *Let $W \subset X \times \Delta^p$ be a closed subset in $\mathcal{S}_{X,\mathcal{C},e}^{(q)}(p)$. Then*
 - (a) *$W_{B(y)} \cap (R_y \times \Delta^p)$ is in $\mathcal{S}_{X_{B(y)},\mathcal{C},e}^{(q+1)}(p)$.*
 - (b) *W^c is in $\mathcal{S}_{X_{B(y)},\mathcal{C},e-1}^{(q)}(p)$.*

Proof. We note that $B(y)$ is Zariski dense in \mathcal{U}_X . The result is a direct consequence (after a translation in the notation) of [16, Part I, Chap. II, Lemma 3.5.4 and Lemma 3.5.6]. \square

5.3. Birationality. We now proceed to analyze the maps $f_p : X \times \Delta^p \rightarrow \mathbb{A}_B^m \times \Delta^p$. For most $W \in \mathcal{S}_{X,\mathcal{C},e}^{(q)}(p)$, the map $f_p : W \rightarrow f_p(W)$ is birational, but there are some special cases that need to be excluded. For the remainder of §5, we will assume that $B = \text{Spec } A$, with A a semi-local PID.

Definition 5.3.1. For a B -scheme $B' \rightarrow B$ and a codimension q closed subset Z of $X_{B'} \times \Delta^p$, we call Z *induced* if $Z \subset X \times Z'$ for some codimension q closed subset Z' of Δ_B^p . If $W \subset X_{B'} \times \Delta^p$ is a pure codimension q closed subset, let $\text{Ind}(W)$ denote the union of the induced irreducible components of W . \square

Remarks 5.3.2. (1) If $Z \subset X \times \Delta^p$ is induced, then $(\pi_{y,X} \times \text{id})(Z)$ is an induced closed subset of $\mathbb{A}_{B(y)}^m \times \Delta^p$.

(2) Suppose $Z \subset X \times \Delta^p$ is induced and in $\mathcal{S}_X^{(q)}(p)$. Then for each finite set of irreducible locally closed subsets \mathcal{C} , Z is in $\mathcal{S}_{X,\mathcal{C}}^{(q)}(p)$. Indeed, each irreducible component of Z is an irreducible component of a subset of the form $p_2^{-1}(Z_0)$ for some $Z_0 \in \mathcal{S}_B^{(q)}(p)$, from which our assertion follows.

(3) Let $W = \{W_p \subset X \times \Delta^p\}$ be a simplicial closed subset, with each $W_p \in \mathcal{S}_X^{(q)}$. Then $p \mapsto \text{Ind}(W_p)$ forms a simplicial closed subset $\text{Ind}(W)$ in $\mathcal{S}_X^{(q)}$ and $p \mapsto (W_p)^+$ and $p \mapsto (W_p)^c$ form simplicial closed subsets W^+ and W^c in $\mathcal{S}_{X_{B(y)}}^{(q)}$. This follows directly from the finiteness of $\pi_{y,X}$. \square

Lemma 5.3.3. *Let W be a proper closed subscheme of $X \times \Delta^p$ of pure codimension $q > 0$. The map*

$$f_p : W_{B(y)} \setminus f_p^{-1}(f_p(\text{Ind}(W)_{B(y)})) \rightarrow f_p(W_{B(y)}) \setminus f_p(\text{Ind}(W)_{B(y)})$$

is birational.

Proof. We delete the subscripts $_{B(y)}$ indicating base-change unless they are needed for clarity. We write f for f_p .

If $Z' \subset \mathbb{A}^m \times \Delta^p$ is induced, then clearly $f^{-1}(Z')$ is induced, hence, if W' is an irreducible component of W , then $f(W')$ is induced if and only if W' is induced. In particular, if W' is an irreducible component of W which is not induced, then $f(W')$ is not contained in $f(\text{Ind}(W))$. Thus, we may assume that $\text{Ind}(W) = \emptyset$.

Write W as

$$W = W_{\text{gen}} \cup \coprod_{b \in B^{(1)}} W(b),$$

where W_{gen} is the union of the irreducible components of W which dominate some component of B , and $W(b)$ is the union of irreducible components of W contained in $X_b \times \Delta^p$. As f_p is a B -morphism, it suffices to prove the result for $W = W_{\text{gen}}$ and $W = W(b)$. In the first case, it suffices to replace W with W_η , η a generic point of B and in the second, we may replace X with X_b . Thus, it suffices to prove the result for A a field F .

Let $W_0 \subset X$ be the closure of the image of W under the projection $p_X : X \times \Delta^p \rightarrow X$. We proceed by induction on the maximum d of the dimension of an irreducible component of W_0 , starting with $d = -1$ (i.e. $W_0 = \emptyset$). Suppose we know our result for $d - 1$. Let $W' \subset W$ be the union of the irreducible components of W whose projections to X have dimension at most $d - 1$, and let W'' be the union of the remaining components.

Let W_1 be an irreducible component of W'' and W_2 an irreducible component of W' . Suppose $f(W_1) \subset f(W_2)$ or $f(W_2) \subset f(W_1)$. Since the projection $\pi_{y,X}$ is finite, and each component of W has the same dimension, it follows that $f(W_1) = f(W_2)$.

Let $p_{\mathbb{A}^m} : \mathbb{A}^m \times \Delta^p \rightarrow \mathbb{A}^m$ be the projection. Then

$$p_{\mathbb{A}^m}(f(W_1)) = p_{\mathbb{A}^m}(f(W_2)),$$

which is impossible, since

$$\begin{aligned} \dim_B p_{\mathbb{A}^m}(f(W_1)) &= \dim_B \pi_{y,X}(p_X(W_1)) = d, \\ \dim_B p_{\mathbb{A}^m}(f(W_2)) &= \dim_B \pi_{y,X}(p_X(W_2)) \leq d - 1. \end{aligned}$$

In particular, it follows from our induction hypothesis that

$$f : W' \setminus f^{-1}(f(W'')) \rightarrow f(W') \setminus f(W'')$$

is birational. Similarly, $f : W \rightarrow f(W)$ is birational if and only if $f : W'' \rightarrow f(W'')$ is birational. Thus, we may assume that $W = W''$.

We first consider the case $p = 0$. Let W be a pure codimension q proper closed subset of X , let W_1, \dots, W_r be the irreducible components of W and let $Z = \sum_i W_i$. By Lemma 5.2.3(1), the cycle $f^*(f_*(Z)) - Z$ is effective, and has no W_i as component.

In particular, each component of $f_*(Z)$ has multiplicity one, hence the irreducible cycles $f_*(W_i)$ each have multiplicity one, and are pair-wise distinct. Since the multiplicity of $f_*(W_i)$ is the degree of the field extension $[F(y)(W_i) : F(y)(f(W_i))]$, this implies that the map $W_{B(y)} \rightarrow f(W_{B(y)})$ is birational, as desired.

The case $p = 0$ easily implies the general case, in case the image subset W_0 is a proper subset of X . Thus we have only to consider the case in which each component of W dominates X .

Fix an \bar{F} -point z of X . Suppose that, for all \bar{F} -points x in an open subset of X , the fiber $W_x \subset x \times \Delta^p = \Delta_{\bar{F}}^p$ of W over x has an irreducible component Z in common with the fiber W_z . Then clearly $W_{\bar{F}}$ contains $X \times Z$, from which it follows

that W contains an induced component, namely, the union of the conjugates of $X \times Z$ over F . Since we have assumed that this is not the case, it follows that, for all pairs (z, x) in an open subset U of $X \times X$, the fibers W_z and W_x have no components in common.

Let η be a geometric generic point of \mathbb{A}^m over F , such that η remains a generic point over $F(y)$. If z and x are distinct points in $\pi_{y,X}^{-1}(\eta)$, then a dimension count implies that (z, x) is a \bar{F} -generic point of $X \times_F X$ and hence (z, x) is a geometric point of U . Thus, W_z and W_x have no common components. Since

$$f(W)_\eta = \cup_{x \in \pi_{y,X}^{-1}(\eta)} W_x \subset \Delta_{F(y,\eta)}^p$$

as closed subsets of $\Delta_{F(y,\eta)}^p$, it follows that the map $W_{\pi_{y,X}^{-1}(\eta)} \rightarrow f(W)_\eta$ is is generically one to one.

As $f : W_{F(y)} \rightarrow f(W_{F(y)})$ is a map over $\mathbb{A}_{F(y)}^m$, this map is one to one over a dense open subset of $f(W_{F(y)})$. Since, by Lemma 5.2.3(2), $W_{F(y)}$ is generically étale over $f(W_{F(y)})$, $W_{F(y)} \rightarrow f(W_{F(y)})$ is thus birational, completing the proof. \square

5.4. Behavior in codimension one. We proceed to extend our analysis to behavior at the codimension one points of B . For $b \in B$, we let $b(y) = B(y) \times_B b$, and write $f_{p,b}$ for the fiber of f_p over $b(y)$.

Lemma 5.4.1. *Let X be in \mathbf{Sm}/B with a closed embedding $i : X \rightarrow \mathbb{A}_B^n$ satisfying (5.1.1). Let $W \subset X \times \Delta^p$ be a closed subset of pure codimension $q > 0$. Then*

- (1) R_y is a pure codimension one subscheme of $X_{B(y)}$ and is flat over $B(y)$.
- (2) For all $b \in B$, $R_y \times \Delta^p \cap W_{b(y)}$ has pure codimension 1 on $W_{b(y)}$.

Proof. Since $X_{B(y)}$ and $\mathbb{A}_{B(y)}^m$ are smooth over $B(y)$, R_y is the subscheme defined by the vanishing of

$$\det(d\pi_{y,X}) : \det \Omega_{X_{B(y)}/B(y)} \rightarrow \pi_{y,X}^*(\det \Omega_{\mathbb{A}_{B(y)}^m}).$$

In particular, R_y is a Cartier divisor on X . From the case of fields, we know ([24], [16, Part I, Chap. II, Lemma 3.5.4]) that $\pi_{y_b, X_b} : X_{b(y)} \rightarrow \mathbb{A}_{b(y)}^m$ is generically étale for each point $b \in B$; since $R_{y_b} = R_y \times_{B(y)} b(y)$, it follows that R_y is flat over B . This proves (1).

For (2), it suffices to show that $R_{y_b} \times \Delta^p \cap W_{b(y)}$ has pure codimension 1 on $W_{b(y)}$ for all $b \in B$, which reduces us to the case of fields. This follows from [16, *loc. cit.*]. \square

Remark 5.4.2. Let $W \subset X \times \Delta^p$ be an irreducible closed subset. If W is not induced, it follows from Lemma 5.3.3 and Lemma 5.4.1 that $f_{p*}(W) = 1 \cdot f_p(W)$, thus

$$f_p^*(f_{p*}(W)) = 1 \cdot W + Z^+,$$

where Z^+ is an effective cycle with support W^+ . Similarly, if W is induced, then $f_{p*}(W) = d_W \cdot f_p(W)$ for some integer $d_W \geq 1$, and

$$f_p^*(f_{p*}(W)) = d_W \cdot W + Z^+.$$

where Z^+ as above is an effective cycle with support W^+ , and with $Z^+ \geq d_W \cdot \text{cyc}(W^+)$. Thus if W is an arbitrary closed subset of $X \times \Delta^p$ of pure codimension, then

$$\text{cyc}(W^c) - \text{cyc}(W^+) = \sum_Z 1 \cdot Z,$$

where the sum is over the induced irreducible components Z of W with $d_Z > 1$. \square

Lemma 5.4.3. *Let $W \subset X \times \Delta^p$ be a pure codimension q closed subset, and let Z be an irreducible codimension q component of $W \cap W^c$. Then Z is induced.*

Proof. This follows directly from Remark 5.4.2. \square

Lemma 5.4.4. *Let $W \subset X \times \Delta^p$ be a pure codimension q closed subset. Suppose that each irreducible component of W dominates B . Then for each $b \in B$ we have the identity of closed subsets*

$$(W^c)_{b(y)} = (W_b)^c.$$

Proof. Let $W_b^* \subset X_b \times \Delta^p$ be an effective cycle with support equal to W_b . It follows from Remark 5.4.2 (applied with b replacing B) that $(W_b)^c$ is the support of $f_{p,c}^*(f_{p,c^*}(W_b^*)) - W_b^*$. For instance, we may take $W_b^* = i^*(\text{cyc } W)$, where $i : X_{b(y)} \times \Delta^p \rightarrow X_{B(y)} \times \Delta^p$ is the inclusion.

We have

$$i^*(f_p^*(f_{p^*}(\text{cyc } W)) - \text{cyc } W) = f_{p,b}^*(f_{p,b^*}(i^*(\text{cyc } W)) - i^*(\text{cyc } W));$$

by our remarks above, the right-hand side has support $(W_b)^c$; the left-hand side has support $(W^c)_{b(y)}$ by Remark 5.4.2. \square

We can now verify the fundamental properties of the projecting cone construction for a one-dimensional base B .

Proposition 5.4.5. *Let A be a semi-local PID, $B = \text{Spec } A$. Let X be in \mathbf{Sm}/B with a closed embedding $i : X \rightarrow \mathbb{A}_B^n$ satisfying (5.1.1), \mathcal{C} a finite set of irreducible locally closed subsets of X such that each $C \in \mathcal{C}$ dominates B , and $e : \mathcal{C} \rightarrow \mathbb{N}$ a function. Take $q > 0$ and $W \in \mathcal{S}_{X,\mathcal{C},e}^{(q)}(p)$. Then W^c is in $\mathcal{S}_{X_{B(y)},\mathcal{C},e-1}^{(q)}(p)$.*

Proof. We may assume that W is irreducible. If $W \subset X_b$ for some closed point $b \in B$, the result follows from the case of a fields (cf. Lemma 5.2.3), so we may suppose that W dominates B . By Remark 3.1.6(2), it suffices to show

- (1) $(W^c)_{b(y)} \in \mathcal{S}_{X_{b(y)},i_b^*\mathcal{C},i_b^*e-1}^{(q)}(p)$ for b a generic point of B .
- (2) $(W^c)_{b(y)} \in \mathcal{S}_{X_{b(y)},i_b^*\mathcal{C},i_b^*e-1}^{(q-1)}(p)$ for $b \in B^{(1)}$.

Take $b \in B$. If b is the generic point of B , then W_b is in $\mathcal{S}_{X_b,i_b^*\mathcal{C},i_b^*e}^{(q)}(p)$, so by Lemma 5.2.3, $(W_b)^c = (W^c)_b$ is in $\mathcal{S}_{X_{b(y)},i_b^*\mathcal{C},i_b^*e-1}^{(q)}(p)$. If b is a codimension one point of B , then by Remark 3.1.6(2), W_b is in $\mathcal{S}_{X_b,i_b^*\mathcal{C},i_b^*e}^{(q-1)}(p)$, so as above, $(W_b)^c$ is in $\mathcal{S}_{X_{b(y)},i_b^*\mathcal{C},i_b^*e-1}^{(q-1)}(p)$. By Lemma 5.4.4, $(W^c)_{b(y)} = (W_b)^c$ (as closed subsets), which completes the proof. \square

Proposition 5.4.6. *Let A be a semi-local PID, $B = \text{Spec } A$. Let X be in \mathbf{Sm}/B with a closed embedding $i : X \rightarrow \mathbb{A}_B^n$ satisfying (5.1.1). Take $q > 0$ and $W \in \mathcal{S}_X^{(q)}(p)$. Then*

- (1) f_p is étale in a neighborhood of $W_{B(y)} \setminus (\text{Ind}(W)_{B(y)} \cup W^+)$,
- (2) the map

$$W_{B(y)} \setminus (\text{Ind}(W)_{B(y)} \cup W^+) \xrightarrow{f_p} f(W_{B(y)}) \setminus f_p((\text{Ind}(W)_{B(y)} \cup W^+) \cap W_{B(y)})$$

is an isomorphism.

Proof. Set

$$W^0 := W_{B(y)} \setminus (\text{Ind}(W)_{B(y)} \cup W^+)$$

and

$$W^1 := f(W_{B(y)}) \setminus f_p((\text{Ind}(W)_{B(y)} \cup W^+) \cap W_{B(y)}).$$

Let $R_y \subset X_{B(y)}$ be the ramification locus of $\pi_{y,X}$.

We note that

$$f_p^{-1}(f_p(\text{Ind}(W))) \subset \text{Ind}(W) \cup W^+.$$

Thus, as in the proof of Lemma 5.3.3, we may assume that $\text{Ind}(W) = \emptyset$.

Take a point $x \in W$ and let $x' = f_p(x)$. The assertions are local for the étale topology on $\mathbb{A}^m \times \Delta^p$, so, after changing notation, we may replace $\mathbb{A}_{B(y)}^m \times \Delta^p$ with its Henselization V at x' . We set

$$U := X_{B(y)} \times \Delta^p \times_{\mathbb{A}_{B(y)}^m \times \Delta^p} V.$$

Let $x_1 = x$ and let x_2, \dots, x_s be the other points of $f_p^{-1}(x')$. We order the x_i so that x_2, \dots, x_r are not in $R_y \times \Delta^p$ and x_{r+1}, \dots, x_s are. Since V is Hensel, we have a decomposition of U as a disjoint union

$$U = \coprod_{i=1}^s U_i$$

with $x_i \in U_i$. Let $W_i = W_{B(y)} \times_{X \times \Delta^p} U_i$, $W_i^+ = W^+ \times_{X \times \Delta^p} U_i$.

We claim that, if $x_i \in W_{B(y)}$ for some $i > 1$, or if x is in $R_y \times \Delta^p$, then x is in W^+ . We temporarily write f_p for the map $f_p \times \text{id}_V$. Suppose first that x is not in $R_y \times \Delta^p$. By Lemma 5.3.3, $W_{B(y)}$ maps birationally to its image $f_p(W_{B(y)})$, so the same is true after pull-back by V . Thus, W_i maps birationally to its image $f_p(W_i)$, and $f_p(W_i)$ has no component in common with $f_p(W_1)$. Therefore $f_p^{-1}(W_i) \cap U_1$ is a union of components of W_1^+ , so $x = x_1$ is in W^+ .

If x is in $R_y \times \Delta^p$, let d be the degree of $f_p : U_1 \rightarrow V$. Since x is a point of ramification for $f_p|_{U_1}$, d is at least 2. Since R_y is a divisor on X , $f_p|_{U_1}$ is generically étale; since $f_p : W_1 \rightarrow f_p(W_1)$ is birational, $f_p^{-1}(f_p(W_1))$ must strictly contain W_1 . Since $f_p^{-1}(f_p(W_1)) = W_1 \cup W_1^+$, it follows that W_1^+ is non-empty; since U_1 is local with closed point x , we must have $x \in W_1^+$.

From this, it follows that

- (a) $f_p(W_{B(y)} \setminus W^+) \subset f_p(W_{B(y)}) \setminus f_p(W^+ \cap W_{B(y)})$,
- (b) $W_{B(y)} \cap R_y \times \Delta^p \subset W_{B(y)} \cap W^+$.
- (c) $f_p : W_{B(y)} \setminus W^+ \rightarrow f_p(W_{B(y)}) \setminus f_p(W^+ \cap W_{B(y)})$ is set-theoretically one-to-one.

Since f_p is flat (Remark 5.1.1(4)), (b) implies that f_p is étale in a neighborhood of $W_{B(y)} \setminus W^+$, verifying (1). The statements (a), (c) and (1) together imply (2). \square

We need one last technical result.

Notation 5.4.7. Let $\text{Br}_y = \pi_{y,X}(R_y)$. Since $\text{codim}_{X_{B(y)}} R_y = 1$, Br_y is a divisor on $\mathbb{A}_{B(y)}^m$. \square

Lemma 5.4.8. *Let W be in $\mathcal{S}_X^{(q)}(p)$. Then $f_p(W)$ is in $\mathcal{S}_{\mathbb{A}_{B(y)}^m, \{\text{Br}_y\}}^{(q)}(p)$.*

Proof. Noting that each face F of Δ^p is a Δ^s for some s , it suffices to prove that

$$\text{codim}_{f_p(W)}(f_p(W) \cap \text{Br}_y \times \Delta^p) \geq 1.$$

for each pure codimension q closed subset W of $X \times \Delta^p$. We may assume that W is irreducible.

Let $W^0 \subset X$ be the closure of the image of W . It suffices to show that $\pi_{y,X}(W_{B(y)}^0)$ is not contained in Br_y . Let w be the generic point of W^0 . Since the image of $y \times \text{id} : B(y) \times_B k(w) \rightarrow \mathcal{U}_X \times_B k(w)$ is Zariski dense, it follows from (5.1.1)(2) that $\pi_{y,X}$ is étale at all the points of $\pi_{y,X}^{-1}(\pi_{y,X}(w))$. Thus $\pi_{y,X}(w)$ is not in Br_y , as desired. \square

6. CHOW'S MOVING LEMMA

We can now present the proof of the generalized Chow's moving lemma, Theorem 2.6.2. Part of the proof relies on Theorem 10.0.1; as this latter is a technical result which is completely disjoint from the main line of argument, we postpone its statement and proof to §10.

In this section, the base-scheme is $B = \text{Spec } A$, with A a semi-local PID. We fix a smooth irreducible B -scheme X with a closed embedding $i : X \rightarrow \mathbb{A}_B^n$. We let $m = \dim_B X$, and we assume that i satisfies (5.1.1). It follows from Theorem 10.0.1 that this case suffices to prove Theorem 2.6.2.

We fix a presheaf $E \in \mathbf{Spt}(B^{(q)})$. We extend the definition of the simplicial spectra $E^{(q)}(X, -)$ to $X \in \mathbf{Sm}^{\text{ess}}/B$ by taking a homotopy colimit, as in §3. We assume throughout this section that E satisfies the axioms (4.1.1) and (4.1.2).

6.1. Excision and its consequences.

Lemma 6.1.1. *Let B' be a B -scheme and $x : B' \rightarrow \mathcal{U}_X$ a B -morphism. Let $W \supset W'$ be simplicial closed subsets of $X \times \Delta^*$. We denote $\pi_{x,X} \times \text{id}_\Delta$ by f . Suppose that for each $p \geq 0$*

- (1) f_p is étale along $W_p \setminus W'_p$
- (2) $f_p : W_p \setminus W'_p \rightarrow f_p(W_p) \setminus f_p(W'_p)$ is an isomorphism (of reduced schemes).

Let $V \subset X_{B'} \times \Delta^$ be the simplicial closed subset $f^{-1}(f(W))$ and let $W'' = f^{-1}(f(W'))$. Then the composition (which we denote by f^*)*

$$\begin{aligned} E^{f(W) \setminus f(W')}(\mathbb{A}_{B'}^m \times \Delta^* \setminus f(W')) \\ \xrightarrow{f^*} E^{V \setminus W''}(X_{B'} \times \Delta^* \setminus W'') \\ = E^{W \setminus W'}(X_{B'} \times \Delta^* \setminus W') \oplus E^{V \setminus (W \cup W'')}(X_{B'} \times \Delta^* \setminus W'') \\ \xrightarrow{\pi} E^{W \setminus W'}(X_{B'} \times \Delta^* \setminus W') \end{aligned}$$

is an isomorphism in \mathcal{SH} . Here π is the projection, and the direct sum decomposition of $E^{V \setminus W''}(X_{B'} \times \Delta^ \setminus W'')$ arises from the decomposition of $V \setminus W''$ into disjoint closed simplicial subsets*

$$V \setminus W'' = W \setminus W' \amalg V \setminus (W \cup W'')$$

and Lemma 4.1.1.

Proof. This follows from axiom (4.1.2). \square

Remark 6.1.2. For $B' = B(y)$, $x : B' \rightarrow \mathcal{U}_X$ the canonical morphism $y : B(y) \rightarrow \mathcal{U}_X$, $W \in \mathcal{S}_X^{(g)}$ and W' the simplicial closed subset defined by

$$W'_p := W_p \cap (W_p^+ \cup \text{Ind}(W)_p),$$

the pair (W, W') satisfies the conditions of Lemma 6.1.1. This follows from Proposition 5.4.6 \square

Definition 6.1.3. Let $E, X, x : B' \rightarrow \mathcal{U}_X, W$ and W', W'', V and p be as in Lemma 6.1.1. We define the push-forward morphism

$$f_* : E^{W \setminus W'}(X_{B'} \times \Delta^* \setminus W') \rightarrow E^{f(W) \setminus f(W')}(A_{B'}^m \times \Delta^* \setminus f(W'))$$

to be the inverse (in \mathcal{SH}) of the pull-back

$$f^* : E^{f(W) \setminus f(W')}(A_{B'}^m \times \Delta^* \setminus f(W')) \rightarrow E^{W \setminus W'}(X_{B'} \times \Delta^* \setminus W').$$

\square

Lemma 6.1.4. Let $E, x : B' \rightarrow \mathcal{U}_X, W$ and W' be as in Lemma 6.1.1. Then

$$\text{id} - f^* \circ f_* : E^{W \setminus W'}(X_{B'} \times \Delta^* \setminus W') \rightarrow E^{W \setminus W'}(X_{B'} \times \Delta^* \setminus W')$$

is the zero map (in \mathcal{SH}).

Proof. This follows directly from Lemma 6.1.1 and the definition of f_* . \square

We will require the following notation:

Definition 6.1.5. Let $\pi : X \rightarrow Y$ be a finite morphism in \mathbf{Sm}/B , \mathcal{C} a finite set of irreducible locally closed subsets of X and $e : \mathcal{C} \rightarrow \mathbb{N}$ a function. We let $\mathcal{S}_{Y, \mathcal{C}, e}^{(g)}$ be the set of $W \in \mathcal{S}_Y^{(g)}(p)$ with $(\pi \times \text{id}_{\Delta^p})^{-1}(W) \in \mathcal{S}_{X, \mathcal{C}, e}^{(g)}(p)$. \square

The $\mathcal{S}_{Y, \mathcal{C}, e}^{(g)}(p)$ form a support condition on Y ; we set

$$E^{(g)}(Y, -)_{\mathcal{C}, e} := E^{\mathcal{S}_{Y, \mathcal{C}, e}^{(g)}}(Y \times \Delta^*).$$

Remark 6.1.6. For each $C_i \in \mathcal{C}$, $\pi(C_i)$ is an irreducible constructible subset of Y . Setting $\mathcal{D} := \{\pi(C), C \in \mathcal{C}\}$, we define $d(\pi(C_i))$ to be the minimum of the $e(C)$ with $\pi(C) = D_i$. Then $\mathcal{S}_{Y, \mathcal{C}, e}^{(g)} = \mathcal{S}_{Y, \mathcal{D}, d}^{(g)}$. \square

6.2. The proof of Theorem 2.6.2. Let $X \rightarrow \text{Spec } B$ and $i : X \rightarrow \mathbb{A}_B^n$ be as above. Let \mathcal{C} be a finite set of irreducible locally closed subsets of $X \in \mathbf{Sm}/B$ such that each $C \in \mathcal{C}$ dominates B and $e : \mathcal{C} \rightarrow \mathbb{N}$ a function. As mentioned at the beginning of §6, we assume that $B = \text{Spec } A$, with A a semi-local PID. We will show that

$$(6.2.1) \quad E^{(g)}(X, -)_{\mathcal{C}, e} \rightarrow E^{(g)}(X, -)$$

is a weak equivalence. This and Theorem 10.0.1 implies Theorem 2.6.2.

As above, we have the B -morphism $y : B(y) \rightarrow \mathcal{U}_X$, and the morphism of cosimplicial $B(y)$ -schemes $f : X_{B(y)} \times \Delta^* \rightarrow \mathbb{A}_{B(y)}^m \times \Delta^*$.

If $e(C) \geq \dim_B X$ for all C , then $\mathcal{S}_{X, \mathcal{C}, e}^{(g)} = \mathcal{S}_X^{(g)}$, so by descending induction on e it suffices to show

$$(6.2.2) \quad E^{(g)}(X, -)_{\mathcal{C}, e-1} \rightarrow E^{(g)}(X, -)_{\mathcal{C}, e}$$

is a weak equivalence. Letting $E^{(q)}(X, -)_{\mathcal{C}, e/e-1}$ denote the homotopy cofiber of the map $E^{(q)}(X, -)_{\mathcal{C}, e-1} \rightarrow E^{(q)}(X, -)_{\mathcal{C}, e}$, it thus suffices to show that

$$(6.2.3) \quad E^{(q)}(X, -)_{\mathcal{C}, e/e-1} \text{ is weakly contractible.}$$

Lemma 6.2.1. *The base-change*

$$\pi^* : E^{(q)}(X, -)_{\mathcal{C}, e/e-1} \rightarrow E^{(q)}(X_{B(y)}, -)_{\pi^*\mathcal{C}, e/e-1}$$

induces the zero-map on homotopy groups.

Proof. By axiom (4.1.1), we have the weak equivalence of $E^{(q)}(X, -)_{\mathcal{C}, e/e-1}$ with the homotopy colimit

$$(6.2.4) \quad E^{(q)}(X, -)_{\mathcal{C}, e/e-1} \sim \operatorname{hocolim}_{W_1, W_2} E^{W_1 \setminus W_1 \cap W_2}(X \times \Delta^* \setminus W_1 \cap W_2)$$

where the limit is over simplicial closed subsets $W_1 \in \mathcal{S}_{X, \mathcal{C}, e}^{(q)}$, $W_2 \in \mathcal{S}_{X, \mathcal{C}, e-1}^{(q)}$. We call a such pair (W_1, W_2) *allowable* for X, \mathcal{C}, e . Similarly, we call a pair (W_1, W_2) of closed simplicial subsets of $\mathbb{A}^m \times \Delta^*$ allowable for $\mathbb{A}^m, \mathcal{C}, e$ if W_1 is in $\mathcal{S}_{\mathbb{A}^m, \mathcal{C}, e}^{(q)}$ and W_2 is in $\mathcal{S}_{\mathbb{A}^m, \mathcal{C}, e-1}^{(q)}$ (see Definition 6.1.5 for the notation).

Let (W_1, W_2) be an allowable pair for X, \mathcal{C}, e . Set

$$E^{(W_1, W_2)}(X, -) := E^{W_1 \setminus W_1 \cap W_2}(X \times \Delta^* \setminus W_1 \cap W_2).$$

We set

$$\pi^*(W_1, W_2) := (W_{1B(y)}, W_{2B(y)} \cup W_1^+ \cup \operatorname{Ind}(W_1)_{B(y)}),$$

and

$$\begin{aligned} f(\pi^*(W_1, W_2)) \\ := (f(W_{1B(y)}), f(W_{2B(y)} \cup [(W_1^+ \cup \operatorname{Ind}(W_1)_{B(y)}) \cap W_{1B(y)}])). \end{aligned}$$

By Proposition 5.4.5, Remark 5.4.2 and Remark 5.3.2, $\pi^*(W_1, W_2)$ is allowable for $X_{B(y)}, \pi^*\mathcal{C}, e$ and $f(\pi^*(W_1, W_2))$ is allowable for $\mathbb{A}_{B(y)}^m, \mathcal{C}, e$.

To complete the proof, it suffices to show that for each allowable pair (W_1, W_2) , the composition

$$E^{(W_1, W_2)}(X, -) \xrightarrow{\iota_{W_1, W_2}} E^{(q)}(X, -)_{\mathcal{C}, e/e-1} \xrightarrow{\pi^*} E^{(q)}(X_{B(y)}, -)_{\pi^*\mathcal{C}, e/e-1}$$

induces zero on the homotopy groups, where ι_{W_1, W_2} is the canonical map given by the description of $E^{(q)}(X, -)_{\mathcal{C}, e/e-1}$ in (6.2.4).

By Lemma 6.1.1, Remark 6.1.2 and Lemma 6.1.4, we may enlarge W_2 so that $f(W_1 \setminus W_2) = f(W_1) \setminus f(W_2)$, f is étale on a neighborhood of $W_1 \setminus W_2$ and $f : W_1 \setminus W_2 \rightarrow f(W_1) \setminus f(W_2)$ is an isomorphism of reduced schemes.

Thus, using Lemma 6.1.1, the pull-back

$$f^* : E^{f(\pi^*(W_1, W_2))}(\mathbb{A}_{B(y)}^m, -) \rightarrow E^{\pi^*(W_1, W_2)}(X_{B(y)}, -)$$

admits the inverse

$$f_* : E^{\pi^*(W_1, W_2)}(X_{B(y)}, -) \rightarrow E^{f(\pi^*(W_1, W_2))}(\mathbb{A}_{B(y)}^m, -)$$

in \mathcal{SH} .

We consider the composition

$$\begin{aligned} E^{(W_1, W_2)}(X, -) &\xrightarrow{\pi^*} E^{(W_{1B(y)}, W_{2B(y)})}(X_{B(y)}, -) \\ &\rightarrow E^{\pi^*(W_1, W_2)}(X_{B(y)}, -) \xrightarrow{f_*} E^{f(\pi^*(W_1, W_2))}(\mathbb{A}_{B(y)}^m, -) \\ &\xrightarrow{f^*} E^{\pi^*(W_1, W_2)}(X_{B(y)}, -) \xrightarrow{\iota} E^{(q)}(X_{B(y)}, -)_{\pi^* \mathcal{C}, e/e-1}; \end{aligned}$$

by Lemma 6.1.4, this composition induces the same map on homotopy groups as $\pi^* \circ \iota_{W_1, W_2}$. Thus it suffices to show that $\iota \circ f^*$ induces zero on homotopy groups.

We can factor $\iota \circ f^*$ as

$$\begin{aligned} E^{f(\pi^*(W_1, W_2))}(\mathbb{A}_{B(y)}^m, -) &\xrightarrow{\iota'} E^{(q)}(\mathbb{A}_{B(y)}^m, -)_{\pi^* \mathcal{C}, e/e-1} \\ &\xrightarrow{f^*} E^{(q)}(X_{B(y)}, -)_{\pi^* \mathcal{C}, e/e-1}. \end{aligned}$$

By Proposition 3.3.4 and Remark 6.1.6 the middle term is weakly contractible, which completes the proof. \square

The argument we used in the proof of Proposition 3.3.4 shows that the base-change π^* is injective on homotopy groups. This verifies (6.2.3), completing the proof of Theorem 2.6.2.

7. FUNTORIALITY

We complete the program by showing how Theorem 2.6.2 yields functoriality for the presheaves $E^{(q)}(X_{\text{Nis}}, -)$ and $E^{(q)}(X_{\text{Zar}}, -)$. We actually prove two different levels of functoriality, a weak form, being the extension to a functor

$$E^{(q)} : \mathbf{Sm}/B^{\text{op}} \rightarrow \mathcal{SH},$$

and a strong form, which gives a functor

$$E^{(q)} : \mathbf{Sm}/B^{\text{op}} \rightarrow \mathbf{Spt}.$$

For the strong form, we require that $B = \text{Spec } F$, F a field.

Let E be in $\mathbf{Spt}(B^{(q)})$. Recall from §2.5 the subcategory $\mathbf{Sm} // B$ of \mathbf{Sm}/B with the same objects, but with morphisms being the smooth morphisms in \mathbf{Sm}/B , the presheaf

$$E_{\text{sm}}^{(q)} : \mathbf{Sm} // B^{\text{op}} \rightarrow \mathbf{Spt}$$

and the homotopy coniveau tower (2.5.1). This is the starting point for full functoriality.

Throughout this section, we assume that B is a regular noetherian separated scheme of Krull dimension at most one.

7.1. Functoriality in \mathcal{SH} . In order to give a natural setting for the form of functoriality one achieves over a general base B , we introduce some notation.

Let $f : Y \rightarrow X$ be a morphism in \mathbf{Sm}/B . We have the functor $f_* : \mathbf{Spt}_{\text{Nis}}(Y) \rightarrow \mathbf{Spt}_{\text{Nis}}(X)$ with left adjoint f^* . Checking stalkwise, one sees that f^* preserves cofibrations and weak equivalences, hence f_* preserves fibrations.

Since $\mathcal{HSpt}_{\text{Nis}}(Y)$ is equivalent to the homotopy category of fibrant objects in $\mathbf{Spt}_{\text{Nis}}(Y)$, we have the functor

$$f_* : \mathcal{HSpt}_{\text{Nis}}(Y) \rightarrow \mathcal{HSpt}_{\text{Nis}}(X)$$

sending a fibrant object $F \in \mathbf{Spt}_{\text{Nis}}(Y)$ to $f_* F \in \mathcal{HSpt}_{\text{Nis}}(X)$.

For morphisms $f : Y \rightarrow X$, $g : Z \rightarrow Y$, there is a canonical isomorphism $\theta_{f,g} : f_*g_* \rightarrow (fg)_*$, satisfying the standard cocycle condition. Indeed, if F is fibrant in $\mathbf{Spt}_{\mathrm{Nis}}(Z)$, then f_*F is fibrant in $\mathbf{Spt}_{\mathrm{Nis}}(Y)$, so the canonical isomorphism $g_*f_*F \rightarrow (gf)_*F$ in $\mathbf{Spt}_{\mathrm{Nis}}(X)$ defines the natural isomorphism $\theta_{f,g}$. The cocycle condition is checked in a similar fashion.

Form the category $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}/B$ with objects (X, \mathcal{E}) , with $X \in \mathbf{Sm}/B$ and with $\mathcal{E} \in \mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(X)$. A morphism $\alpha : (X, \mathcal{E}) \rightarrow (Y, \mathcal{F})$ is a morphism $f : Y \rightarrow X$ in \mathbf{Sm}/B together with a morphism $\phi : \mathcal{E} \rightarrow f_*\mathcal{F}$ in $\mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}(X)$. Composition is given by

$$(g, \psi) \circ (f, \phi) := (fg, \theta_{g,f} \circ g_*(\psi) \circ \phi).$$

Sending (X, \mathcal{E}) to X defines the functor $\pi : \mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}/B \rightarrow \mathbf{Sm}/B^{\mathrm{op}}$.

The functor $E_{\mathrm{sm}}^{(q)}$ defines a section to π over the subcategory $\mathbf{Sm} // B^{\mathrm{op}}$:

$$\tilde{E}_{\mathrm{sm}}^{(q)} : \mathbf{Sm} // B^{\mathrm{op}} \rightarrow \mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}/B,$$

with $\tilde{E}_{\mathrm{sm}}^{(q)}(X)$ the presheaf $E^{(q)}(X_{\mathrm{Nis}}, -)$.

With these notations, we can state our first main result on functoriality:

Theorem 7.1.1. *Let E be in $\mathbf{Spt}(B^{(q)})$, satisfying axioms (4.1.1), (4.1.2) and (4.1.3). Then the section $\tilde{E}_{\mathrm{sm}}^{(q)}$ extends (up to isomorphism) to a section*

$$\tilde{E}^{(q)} : \mathbf{Sm}/B^{\mathrm{op}} \rightarrow \mathcal{H}\mathbf{Spt}_{\mathrm{Nis}}/B.$$

The remainder of this section is devoted to the proof of this result. As a matter of notation, if we have composable morphisms $f : Y \rightarrow X$, $g : Z \rightarrow Y$, objects $E_X \in \mathbf{Spt}_{\mathrm{Nis}}(X)$, $E_Y \in \mathbf{Spt}_{\mathrm{Nis}}(Y)$ and $E_Z \in \mathbf{Spt}_{\mathrm{Nis}}(Z)$ and morphisms $\phi : E_X \rightarrow f_*E_Y$, $\psi : E_Y \rightarrow g_*E_Z$, we let $\psi \circ \phi : E_X \rightarrow (fg)_*E_Z$ denote the composition $\theta_{f,g} \circ g_*(\psi) \circ \phi$.

We consider the following situation: Suppose B is irreducible. Let $f : Y \rightarrow X$ be a morphism in \mathbf{Sm}/B with X and Y equi-dimensional over B . Let $X^f \subset X$ be the subset of X of points over which f is equi-dimensional with fiber dimension $\dim_B Y - \dim_B X$. Let \mathcal{C} be a finite set of irreducible locally closed subsets of X such that there is a subset $\mathcal{C}_0 \subset \mathcal{C}$ satisfying

- (1) $X \setminus X^f \subset \cup_{C \in \mathcal{C}_0} C$
- (2) For each $C \in \mathcal{C}_0$, the restriction of f to $f : f^{-1}(C) \rightarrow C$ is equi-dimensional.
- (3) Each $C \in \mathcal{C}_0$ dominates a component of B .

We call such a \mathcal{C} *good for f* . We extend this notion to the general case by taking disjoint unions.

Suppose that \mathcal{C} is good for f . Then

$$W \in \mathcal{S}_{X, \mathcal{C}}^{(q)}(p) \implies (f \times \mathrm{id})^{-1}(W) \in \mathcal{S}_Y^{(q)}(p),$$

so we have the well-defined map of simplicial spectra

$$f^* : E^{(q)}(X, -)_{\mathcal{C}} \rightarrow E^{(q)}(Y, -).$$

This construction is compatible with pull-back by a smooth morphism $Z \rightarrow X$, so we have the map of presheaves on X_{Nis} :

$$f^* : E^{(q)}(X_{\mathrm{Nis}}, -)_{\mathcal{C}} \rightarrow f_*E^{(q)}(Y_{\mathrm{Nis}}, -).$$

If we vary q , we have the well-defined map of towers of presheaves

$$f^* : E^{(-)}(X_{\mathrm{Nis}}, -)_{\mathcal{C}} \rightarrow f_*E^{(-)}(Y_{\mathrm{Nis}}, -).$$

By Theorem 2.6.2(1), the map $E^{(q)}(X_{\text{Nis}}, -)_{\mathcal{C}} \rightarrow E^{(q)}(X_{\text{Nis}}, -)$ is a local weak equivalence in $\mathbf{Spt}_{\text{Nis}}(X)$, so the diagram

$$E^{(q)}(X_{\text{Nis}}, -) \leftarrow E^{(q)}(X_{\text{Nis}}, -)_{\mathcal{C}} \xrightarrow{f^*} f_* E^{(-)}(Y_{\text{Nis}}, -)$$

defines the map in the homotopy category $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(X)$

$$(7.1.1) \quad \tilde{f}^* : E^{(q)}(X_{\text{Nis}}, -) \rightarrow f_* E^{(q)}(Y_{\text{Nis}}, -).$$

Lemma 7.1.2. *The map (7.1.1) is independent of the choice of good \mathcal{C} .*

Proof. Let \mathcal{C}' be another collection which is good for f . Then $\mathcal{C} \cup \mathcal{C}'$ is clearly good for f , so we may assume that $\mathcal{C}' \supset \mathcal{C}$. This yields the commutative diagram of presheaves

$$\begin{array}{ccc} E^{(q)}(X_{\text{Nis}}, -) & \longleftarrow & E^{(q)}(X_{\text{Nis}}, -)_{\mathcal{C}} \\ \uparrow & \nearrow & \downarrow f^* \\ E^{(q)}(X_{\text{Nis}}, -)_{\mathcal{C}'} & \xrightarrow{f^*} & f_* E^{(-)}(Y_{\text{Nis}}, -), \end{array}$$

whence the result. \square

For a general f , there may be no \mathcal{C} which is good for f . However, if f is smooth, then every \mathcal{C} is good for f . If $i : Y \rightarrow X$ is a closed embedding, and \mathcal{C} is any collection containing $i(Y)$, then \mathcal{C} is good for i . Thus f^* is defined for smooth morphisms and for closed embeddings. This justifies the following

Definition 7.1.3. Let $f : Y \rightarrow X$ be a morphism in \mathbf{Sm}/B . Factor f as $f = p \circ i$, where $p : Y \times_B X \rightarrow X$ is the projection, and $i : Y \rightarrow Y \times_B X$ is the graph, $i = (\text{id}_Y, f)$. Define the map f^* in $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(X)$ by

$$f^* := i^* \circ p^* : E^{(q)}(X_{\text{Nis}}, -) \rightarrow f_* E^{(q)}(Y_{\text{Nis}}, -).$$

\square

We proceed to show the functoriality $(fg)^* = g^* f^*$.

Lemma 7.1.4. *Let*

$$\begin{array}{ccc} T & \xrightarrow{i_2} & Y \\ f_2 \downarrow & & \downarrow f_1 \\ Z & \xrightarrow{i_1} & X \end{array}$$

be a commutative diagram in \mathbf{Sm}/B , with f_1, f_2 smooth morphisms, and i_1, i_2 closed embeddings. Then $\tilde{i}_2^ \circ \tilde{f}_1^* = \tilde{f}_2^* \circ \tilde{i}_1^*$ in $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(X)$.*

Proof. Let $g = f_1 i_2 = f_2 i_1$. Let

$$\mathcal{C}_X := \{i_1(Z)\}, \quad \mathcal{C}_Y := \{i_2(T)\}.$$

One checks that

$$W \in \mathcal{S}_{X, \mathcal{C}_X}^{(q)}(p) \implies (f_1 \times \text{id})^{-1}(W) \in \mathcal{S}_{Y, \mathcal{C}_Y}^{(q)}(p).$$

Thus, we have the pull-back

$$f_1^* : E^{(q)}(X_{\text{Nis}}, -)_{\mathcal{C}_X} \rightarrow f_{1*} E^{(q)}(Y_{\text{Nis}}, -)_{\mathcal{C}_Y},$$

giving the commutative diagram

$$\begin{array}{ccc}
 & & f_{1*}E^{(q)}(Y_{\text{Nis}}, -) \\
 & \nearrow f_1^* & \uparrow \\
 E^{(q)}(X_{\text{Nis}}, -)_{\mathcal{C}_X} & \xrightarrow{f_1^*} & f_{1*}E^{(q)}(Y_{\text{Nis}}, -)_{\mathcal{C}_Y} \\
 \downarrow i_1^* & & \downarrow f_{1*}(i_2^*) \\
 i_{1*}E^{(q)}(Z_{\text{Nis}}, -) & \xrightarrow{i_{1*}(f_2^*)} & g_*E^{(q)}(T_{\text{Nis}}, -)
 \end{array}$$

The result follows easily from this. \square

Lemma 7.1.5. *Let $f : Y \rightarrow X$, $g : Z \rightarrow Y$ be morphisms in \mathbf{Sm}/B . If f and g are both smooth morphisms, or both closed embeddings, then*

$$\widetilde{fg}^* = \tilde{g}^* \tilde{f}^* : E^{(q)}(X_{\text{Nis}}, -) \rightarrow (fg)_*E^{(q)}(Z_{\text{Nis}}, -).$$

in $\mathcal{HSpt}_{\text{Nis}}(X)$.

Proof. For smooth morphisms f and g , fg is also smooth, so we can represent \tilde{f}^* , \tilde{g}^* and \widetilde{fg}^* by the maps f^* , g^* and $fg^* = g^* \circ f^*$ in $\mathbf{Spt}_{\text{Nis}}(X)$, verifying the result in this case.

If f and g are closed embeddings, take $\mathcal{C}_X = \{f(Y), f(g(Z))\}$ and $\mathcal{C}_Y = \{g(Z)\}$. Then

$$W \in \mathcal{S}_{X, \mathcal{C}_X}^{(q)}(p) \implies (f \times \text{id})^{-1}(W) \in \mathcal{S}_{Y, \mathcal{C}_Y}(p),$$

giving us the pull-back map

$$f^* : E^{(q)}(X_{\text{Nis}}, -)_{\mathcal{C}_X} \rightarrow f_*E^{(q)}(Y_{\text{Nis}}, -)_{\mathcal{C}_Y}.$$

In addition, \mathcal{C}_X is good for f and for fg and we have the commutative diagram of spectra

$$\begin{array}{ccc}
 & & f_*E^{(q)}(Y_{\text{Nis}}, -) \\
 & \nearrow f^* & \uparrow \\
 E^{(q)}(X_{\text{Nis}}, -)_{\mathcal{C}_X} & \xrightarrow{f^*} & f_*E^{(q)}(Y_{\text{Nis}}, -)_{\mathcal{C}_Y} \\
 & \searrow (fg)^* & \downarrow f_*(g^*) \\
 & & (fg)_*E^{(q)}(Z_{\text{Nis}}, -).
 \end{array}$$

\square

Proposition 7.1.6. *Let $f : Y \rightarrow X$, $g : Z \rightarrow Y$ be morphisms in \mathbf{Sm}/B . Then*

$$(fg)^* = g^* f^* : E^{(q)}(X_{\text{Nis}}, -) \rightarrow (fg)_*E^{(q)}(Z_{\text{Nis}}, -).$$

in $\mathcal{HSpt}_{\text{Nis}}(X)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
 & & i & & \\
 & \curvearrowright & & \curvearrowleft & \\
 Z & \xrightarrow{(\text{id}_Z, g)} & Y \times_B Z & \xrightarrow{(\text{id}_Y, f) \times \text{id}_Z} & X \times_B Y \times_B Z \\
 & & \downarrow p_1 & & \downarrow p_{12} \\
 & & Y & \xrightarrow{(\text{id}_Y, f)} & X \times_B Y \\
 & & & & \downarrow p_1 \\
 & & & & X
 \end{array}$$

with $i := [(\text{id}_Y, f) \times \text{id}_Z] \circ (\text{id}_Z, g)$ and $p := p_1 \circ p_{12}$. Applying Lemma 7.1.4 and Lemma 7.1.5 to this diagram shows that

$$g^* \circ f^* = \tilde{i}^* \circ \tilde{p}^*.$$

Applying Lemma 7.1.4 and Lemma 7.1.5 to the commutative diagram

$$\begin{array}{ccccc}
 Z & \xrightarrow{i} & X \times_B Y \times_B Z & & \\
 \parallel & & \downarrow p_{13} & \searrow p & \\
 Z & \xrightarrow{(\text{id}_Z, g)} & X \times_B Z & \xrightarrow{p_1} & X
 \end{array}$$

yields

$$\tilde{i}^* \circ \tilde{p}^* = (fg)^*.$$

□

To construct the section $\tilde{E}^{(q)}$, send X to $E^{(q)}(X_{\text{Nis}}, -) \in \mathcal{H}\mathbf{Spt}_{\text{Nis}}(X)$ and $f : Y \rightarrow X$ to $(f, f^*) : (X, E^{(q)}(X_{\text{Nis}}, -)) \rightarrow (Y, E^{(q)}(Y_{\text{Nis}}, -))$. By Proposition 7.1.6, this gives a well-defined section to π , extending $\tilde{E}_{\text{sm}}^{(q)}$. Since the construction of f^* is natural with respect to change in q , we have the tower of functors asserted in Theorem 7.1.1. This completes the proof of Theorem 7.1.1.

Sending $E \in \mathcal{H}\mathbf{Spt}_{\text{Nis}}(X)$ to the global sections of a fibrant model defines the functor

$$R\Gamma(X, -) : \mathcal{H}\mathbf{Spt}_{\text{Nis}}(X) \rightarrow \mathcal{SH};$$

varying X , we have the functor

$$R\Gamma : \mathcal{H}\mathbf{Spt}_{\text{Nis}}/B \rightarrow \mathcal{SH}.$$

We have for example the functor

$$R\Gamma E_{\text{sm}}^{(q)} : \mathbf{Sm} // B^{\text{op}} \rightarrow \mathcal{SH},$$

defined by $R\Gamma E_{\text{sm}}^{(q)} := R\Gamma \circ \tilde{E}_{\text{sm}}^{(q)}$.

Corollary 7.1.7. *Let E be in $\mathbf{Spt}(B)$, satisfying (4.1.1), (4.1.2) and (4.1.3). The functor $R\Gamma E_{\text{sm}}^{(q)}$ extends to the functor*

$$R\Gamma E^{(q)} : \mathbf{Sm}/B^{\text{op}} \rightarrow \mathcal{SH}$$

by setting $R\Gamma E^{(q)} := R\Gamma \circ \tilde{E}^{(q)}$, and we have the tower of functors

$$\dots \rightarrow R\Gamma E^{(q+1)} \rightarrow R\Gamma E^{(q)} \rightarrow \dots \rightarrow R\Gamma E^{(0)}.$$

This result suffices to make the sheafified version of the coniveau spectral sequence functorial; see the next section §7.2 for details.

7.2. The homotopy coniveau spectral sequence.

Fix $E \in \mathbf{Spt}(B)^*$. Corollary 7.1.7 gives us the tower of functors on \mathbf{Sm}/B

$$(7.2.1) \quad \dots \rightarrow R\Gamma E^{(p+1)} \rightarrow R\Gamma E^{(p)} \rightarrow \dots \rightarrow R\Gamma E^{(0)}.$$

from $\mathbf{Sm}/B^{\text{op}}$ to \mathcal{SH} . We let $R\Gamma E^{(p/p+1)}$ denote the functor

$$X \mapsto \text{cofib}[R\Gamma E^{(p+1)}(X) \rightarrow R\Gamma E^{(p)}(X)]$$

and set

$$\widehat{R\Gamma E^{(0)}}(X) := \text{holim}_p \text{cofib}[R\Gamma E^{(p)}(X) \rightarrow R\Gamma E^{(0)}(X)].$$

Let

$$\begin{aligned} \mathbb{H}^q(X, E^{(p)}) &:= \pi_{-q} R\Gamma E^{(p)}(X) \\ \mathbb{H}^q(X, E^{(p/p+1)}) &:= \pi_{-q} R\Gamma E^{(p/p+1)}(X) \\ \mathbb{H}^q(X, \widehat{E}^{(0)}) &:= \pi_{-q} \widehat{R\Gamma E^{(0)}}(X). \end{aligned}$$

Evaluating the tower (7.2.1) at X and taking the spectral sequence of a tower gives us the weakly convergent spectral sequence

$$(7.2.2) \quad E_1^{p,q} := \mathbb{H}^{p+q}(X, E^{(p/p+1)}) \implies \mathbb{H}^{p+q}(X, \widehat{E}^{(0)}).$$

which is functorial on \mathbf{Sm}/B .

Similarly, set

$$E^{(p/p+1)}(X, -) := \text{cofib}[E^{(p+1)}(X, -) \rightarrow E^{(p)}(X, -)],$$

etc. Applying the same spectral sequence construction to the homotopy coniveau tower (2.5.1) yields the weakly convergent spectral sequence

$$(7.2.3) \quad E_1^{p,q} := \pi_{-p-q}(E^{(p/p+1)}(X, -)) \implies \pi_{-p-q}(\widehat{E}^{(0)}(X, -)).$$

functorial on \mathbf{Sm}/B .

The canonical maps $E^{(p)}(X, -) \rightarrow R\Gamma E^{(p)}(X)$ give a map of spectral sequences (7.2.2) \rightarrow (7.2.3), natural in X over \mathbf{Sm}/B .

Remark 7.2.1. The main results of [17] can be applied to $E \in \mathbf{Spt}(B)^*$. One result that can be proved is that, for $X \in \mathbf{Sm}/B$, the presheaf $E^{(p)}(X_{\text{Nis}}, -)$ satisfies a Mayer-Vietoris property, namely, for $U \subset X$ an open subscheme $V \rightarrow X$ étale such that $\{U \rightarrow X, V \rightarrow X\}$ is a Nisnevich cover, the square of restriction maps

$$\begin{array}{ccc} E^{(p)}(X, -) & \longrightarrow & E^{(p)}(U, -) \\ \downarrow & & \downarrow \\ E^{(p)}(V, -) & \longrightarrow & E^{(p)}(U \times_X V, -) \end{array}$$

is homotopy cartesian. By a basic result of Thomason [11, 27], this property implies that the canonical map $E^{(p)}(X, -) \rightarrow R\Gamma E^{(p)}(X)$ is a weak equivalence.

In particular, the map of spectral sequences (7.2.2) \rightarrow (7.2.3) is an isomorphism of spectral sequences. We have therefore extended the functoriality of the spectral sequence (7.2.3) from \mathbf{Sm}/B to \mathbf{Sm}/B .

If E satisfies the additional property of homotopy invariance:

$$p_1^* : E(X) \rightarrow E(X \times \mathbb{A}^1)$$

is a weak equivalence for all $X \in \mathbf{Sm}/B$, then the canonical map

$$E(X) \rightarrow E^{(0)}(X, -)$$

is well-known to be a weak equivalence.

Finally, suppose in addition that E is N -connected for some N , i.e., for each $Y \in \mathbf{Sm}/B$, $\pi_n(E(Y)) = 0$ for $n \leq N$. Then the canonical map $E^{(0)}(X, -) \rightarrow \hat{E}^{(0)}(X, -)$ is a weak equivalence and hence the spectral sequence (7.2.3) becomes

$$(7.2.4) \quad E_1^{p,q} := \pi_{-p-q}(E^{(p/p+1)}(X, -)) \implies \pi_{-p-q}(E(X)).$$

The proof of the localization theorem for $E \in \mathbf{Spt}(B)^*$ and the convergence properties asserted above are given in our preprint [18]. \square

7.3. Straightening a homotopy limit. In order to pass from functoriality in the homotopy category to a strict functoriality, we first take a certain homotopy limit. This yields a functor “up to homotopy and all higher homotopies”; we then apply a construction of Dwyer-Kan to rectify this to a strict functor. In this section, we present these constructions in the categorical setting.

As the first ingredient in our construction, we recall the notations and results of Dwyer-Kan [7] on which the construction is based.

Recall that a *simplicial category* is a category enriched in simplicial sets. We make the category of simplicial sets \mathbf{Spc} a simplicial category as usual by setting $\mathcal{H}om_{\mathbf{Spc}}(A, B)_n := \text{Hom}_{\mathbf{Spc}}(A \times \Delta[n], B)$. For simplicial categories \mathcal{A}, \mathcal{B} , (with \mathcal{A} small) we have the category of simplicial functors $\mathcal{A} \rightarrow \mathcal{B}$, denoted $\mathcal{B}^{\mathcal{A}}$. Given $X, Y \in \mathbf{Spc}^{\mathcal{A}}$, a morphism $\theta : X \rightarrow Y$ is called a weak equivalence if $\theta(a) : X(a) \rightarrow Y(a)$ is a weak equivalence of simplicial sets for each object $a \in \mathcal{A}$.

For a simplicial category \mathcal{A} , we have the underlying category \mathcal{A}_0 , i.e.,

$$\text{Hom}_{\mathcal{A}_0}(a, b) = \text{Hom}_{\mathcal{A}}(a, b)_0.$$

A simplicial functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is a *weak equivalence* if

- (1) For each pair of objects of \mathcal{A} , the map of simplicial sets $f : \text{Hom}_{\mathcal{A}}(x, y) \rightarrow \text{Hom}_{\mathcal{B}}(f(x), f(y))$ is a weak equivalence.
- (2) $\pi_0(f) : \pi_0\mathcal{N}(\mathcal{A}_0) \rightarrow \pi_0\mathcal{N}(\mathcal{B}_0)$ is surjective, \mathcal{N} the nerve.

Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a simplicial functor. We have the pull-back functor $f^* : \mathbf{Spc}^{\mathcal{B}} \rightarrow \mathbf{Spc}^{\mathcal{A}}$, i.e., $f^*X(a) := X(f(a))$. Dwyer-Kan define the *homotopy push-forward* hocolim $f_* : \mathbf{Spc}^{\mathcal{A}} \rightarrow \mathbf{Spc}^{\mathcal{B}}$ as follows: Take $X \in \mathbf{Spc}^{\mathcal{A}}$. For $b \in \mathcal{B}$, hocolim $f_*X(b)$ is the diagonal simplicial set associated to the bisimplicial set $(\text{hocolim } f_*X)_{**}(b)$ with

$$(7.3.1) \quad (\text{hocolim } f_*X)_{n,*}(b) := \coprod X(a_0) \times \text{Hom}_{\mathcal{A}}(a_0, a_1) \times \dots \times \text{Hom}_{\mathcal{A}}(a_{n-1}, a_n) \times \text{Hom}_{\mathcal{B}}(f(a_n), b),$$

with the evident face and degeneracy maps, and the evident functoriality in b . For $a \in \mathcal{A}$, we have the evident map

$$i_a : X(a) \rightarrow \text{hocolim } f_*X(f(a)).$$

This map is natural in X , but not in a .

The functor $\text{hocolim } f_*$ admits a right adjoint

$$\text{hocolim } f^* : \mathcal{B}^{\mathbf{Spc}} \rightarrow \mathcal{A}^{\mathbf{Spc}}$$

and there is a natural transformation $f^* \rightarrow \text{hocolim } f^*$. For $Y \in \mathcal{B}^{\mathbf{Spc}}$, we have the co-unit $\text{hocolim } f_*(\text{hocolim } f^*Y) \rightarrow Y$; we let

$$r_Y : \text{hocolim } f_*(f^*Y) \rightarrow Y$$

denote the composition

$$\text{hocolim } f_*(f^*Y) \rightarrow \text{hocolim } f_*(\text{hocolim } f^*Y) \rightarrow Y.$$

The main result of [7] is

Theorem 7.3.1 ([7, Theorem 2.1]). *Suppose that $f : \mathcal{A} \rightarrow \mathcal{B}$ is a weak equivalence of small simplicial categories. Then for each $X \in \mathbf{Spc}^{\mathcal{A}}$ and $a \in \mathcal{A}$, the map*

$$i_a : X(a) \rightarrow \text{hocolim } f_*X(f(a))$$

is a weak equivalence. Also, for each $Y \in \mathbf{Spc}^{\mathcal{B}}$, the map

$$r_Y : \text{hocolim } f_*f^*Y \rightarrow Y$$

is a weak equivalence, natural in Y ,

We will apply the Dwyer-Kan construction to the 2-category of factorizations in a small category \mathcal{B} .

Let I be a 2-category. We form the simplicial category $\mathcal{N}I$ by setting

$$\text{Hom}_{\mathcal{N}I}(a, b) := \mathcal{N}\text{Hom}_I(a, b)$$

for each pair of objects $a, b \in I$. A simplicial functor $X : \mathcal{N}I \rightarrow \mathbf{Spc}$ is called a *continuous* functor $X : I \rightarrow \mathbf{Spc}$; given a continuous functor $X : I \rightarrow \mathbf{Spc}$ and a 2-functor $\pi : I \rightarrow J$, we thus have the continuous functor $\text{hocolim } \pi_*X : J \rightarrow \mathbf{Spc}$.

Let B be a small category. Form the 2-category $\text{Fac}B$ as follows: $\text{Fac}B$ has the same objects as B . A morphism $b \rightarrow b'$ in $\text{Fac}B$ consists of a sequence of composable morphisms in B ,

$$b = b_0 \xrightarrow{f_1} b_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} b_n = b',$$

where if some $f_i = \text{id}$, we identify the sequence with the one obtained by deleting f_i (unless $n = 1$). Composition is just by concatenation. We write such a sequence as $f_n * \dots * f_1$. Given such a sequence, we can form a new (shorter) sequence by changing some of the $*$'s to compositions in the category B ; this operation defines a 2-morphism. We have the functor $\pi : \text{Fac}B \rightarrow B$ which is the identity on objects, changes the concatenation $*$ to composition in B , and sends all 2-morphisms to the identity. This yields the functor of simplicial categories $\pi : \mathcal{N}\text{Fac}B \rightarrow B$ (with B discrete).

The category $\text{Hom}_{\text{Fac}B}(b, b')$ is a partially ordered set with maximal elements $\text{Hom}_B(b, b')$, and each $\tilde{f} \in \text{Hom}_{\text{Fac}B}(b, b')$ is dominated by a unique maximal element, namely $\pi(\tilde{f})$. Thus, we have

Lemma 7.3.2. $\pi : \mathcal{N}\text{Fac}B \rightarrow B$ *is a weak equivalence of simplicial categories.*

We may therefore apply the Dwyer-Kan theorem: For a continuous functor $X : \text{Fac}B \rightarrow \mathbf{Spc}$, we denote $\text{hocolim } \pi_*X : B \rightarrow \mathbf{Spc}$ by C_BX , or just CX if the context makes the meaning clear. We have for each $b \in B$ the weak equivalence

$$i_b : X(b) \rightarrow C_BX(b).$$

Thus, the Dwyer-Kan hocolim construction transforms a "functor up to homotopy and all higher homotopies" on B , i.e., a continuous functor on $\text{Fac}B$, to an honest functor on B .

The second ingredient in our construction involves some aspects of the holim construction, which we now recall (for details, see [6]).

Let \mathcal{A} be a small category. For $a \in \mathcal{A}$, we have the category of objects over a , \mathcal{A}/a ; sending a to the nerve $\mathcal{N}(\mathcal{A}/a)$ defines the functor

$$\mathcal{N}(\mathcal{A}/-) : \mathcal{A} \rightarrow \mathbf{Spc}.$$

Given a functor $F : \mathcal{A} \rightarrow \mathbf{Spc}$, we have the homotopy limit $\text{holim}_{\mathcal{A}} F$, defined by

$$\text{holim}_{\mathcal{A}} F := \mathcal{H}om(\mathcal{N}(\mathcal{A}/-), F),$$

i.e., $\text{holim}_{\mathcal{A}} F$ is the simplicial set with n -simplicies the set of natural transformations $\sigma : \mathcal{N}(\mathcal{A}/-) \times \Delta[n] \rightarrow F$.

From the definition, it is clear that $\text{holim}_{\mathcal{A}} F$ is natural in F , i.e., if $\theta : F \rightarrow G$ is a natural transformation of functors, we have the map of spaces $\theta_* : \text{holim}_{\mathcal{A}} F \rightarrow \text{holim}_{\mathcal{A}} G$. Similarly, if $i : \mathcal{B} \rightarrow \mathcal{A}$ is a functor of small categories, we have the map of spaces $i^* : \text{holim}_{\mathcal{A}} F \rightarrow \text{holim}_{\mathcal{B}}(F \circ i)$.

Remark 7.3.3. Suppose $F : \mathcal{A} \rightarrow \mathbf{Spc}$ has the properties

- (1) $F(a)$ is fibrant for all $a \in \mathcal{A}$.
- (2) $F(a) \rightarrow F(a')$ is a weak equivalence for all morphisms $a \rightarrow a'$ in \mathcal{A} .
- (3) \mathcal{A} has a final object a_0 .

Then the canonical map $\text{holim}_{\mathcal{A}} F \rightarrow F(a_0)$ is a weak equivalence. Indeed, by [6, XI(5.6)] the natural transformation of F to the constant functor $c_{F(a_0)}$ with value $F(a_0)$ induces a weak equivalence

$$\text{holim}_{\mathcal{A}} F \rightarrow \text{holim}_{\mathcal{A}} c_{F(a_0)}.$$

By [6, IX(4.2)]

$$\text{holim}_{\mathcal{A}} c_{F(a_0)} = \mathcal{H}om(\mathcal{N}(\mathcal{A}), F(a_0)).$$

But since a_0 is the final object of \mathcal{A} , the inclusion $a_0 \rightarrow \mathcal{A}$ induces a homotopy equivalence $pt = \mathcal{N}(a_0) \rightarrow \mathcal{N}(\mathcal{A})$, hence a weak equivalence

$$\mathcal{H}om(\mathcal{N}(\mathcal{A}), F(a_0)) \rightarrow F(a_0).$$

□

For a category \mathcal{A} , we have the 2-category \mathbf{Cat}/\mathcal{A} , with objects functors $\alpha : \mathfrak{a} \rightarrow \mathcal{A}$, with \mathfrak{a} small. A morphism $[\alpha : \mathfrak{a} \rightarrow \mathcal{A}] \rightarrow [\alpha' : \mathfrak{a}' \rightarrow \mathcal{A}]$ is a pair (f, θ) with $f : \mathfrak{a} \rightarrow \mathfrak{a}'$ a functor, and $\theta : \alpha' \circ f \rightarrow \alpha$ a natural transformation. The composition is

$$(f', \theta') \circ (f, \theta) := (f'f, \theta' \circ f^*(\theta)).$$

A 2-morphism $\vartheta : (f, \theta) \rightarrow (f', \theta')$ is a natural transformation $\vartheta : f \rightarrow f'$ such that $\theta' \circ \alpha'_*(\vartheta) = \theta$. Composition of 2-morphisms is by composition of natural transformations.

Let $F : \mathcal{A} \rightarrow \mathbf{Spc}$ be a functor. For $\alpha : \mathfrak{a} \rightarrow \mathcal{A}$, we have the space $\text{holim}_{\mathfrak{a}} F \circ \alpha$; for a morphism

$$(f, \theta) : [\alpha : \mathfrak{a} \rightarrow \mathcal{A}] \rightarrow [\alpha' : \mathfrak{a}' \rightarrow \mathcal{A}]$$

we have the map

$$\mathrm{holim} F(f, \theta) := \theta_* \circ f^* : \mathrm{holim}_{\mathbf{a}'} F \circ \alpha' \rightarrow \mathrm{holim}_{\mathbf{a}} F \circ \alpha.$$

This defines a functor (on the underlying category)

$$\mathrm{holim} F : (\mathbf{Cat}/\mathcal{A})_0^{\mathrm{op}} \rightarrow \mathbf{Spc}.$$

Let $\mathbf{Cat}/\mathcal{A}^{\mathrm{op}}$ be the 2-category with underlying category $(\mathbf{Cat}/\mathcal{A})_0^{\mathrm{op}}$, and with

$$\mathrm{Hom}_{\mathbf{Cat}/\mathcal{A}^{\mathrm{op}}}(\alpha, \alpha') := \mathrm{Hom}_{\mathbf{Cat}/\mathcal{A}}(\alpha', \alpha).$$

The functor $\mathrm{holim} F$ extends to a continuous functor

$$\mathrm{holim} F : \mathbf{Cat}/\mathcal{A}^{\mathrm{op}} \rightarrow \mathbf{Spc}.$$

To see this, let \underline{n} be the category associated to the ordered set $\{0 < 1 < \dots < n\}$. If $G : \mathcal{B} \rightarrow \mathbf{Spc}$ is a functor, we have the functor $G \circ p_2 : \underline{n} \times \mathcal{B} \rightarrow \mathbf{Spc}$ and the natural isomorphism of simplicial sets

$$\mathcal{H}om_{\mathbf{Spc}}(\Delta[n], \mathrm{holim}_{\mathcal{B}} G) \cong \mathrm{holim}_{\underline{n} \times \mathcal{B}} G \circ p_2,$$

where $\mathcal{H}om$ denotes the simplicial set of morphisms in \mathbf{Spc} . Indeed, it follows directly from the definition of holim that

$$\begin{aligned} \mathrm{holim}_{\underline{n} \times \mathcal{B}} G \circ p_2 &= \lim_{\substack{\leftarrow \\ i \in \underline{n}}} \mathcal{H}om(\mathcal{N}(\underline{n}/i), \mathrm{holim}_{\mathcal{B}} G) \\ &\cong \mathcal{H}om(\lim_{\substack{\rightarrow \\ i \in \underline{n}}} \mathcal{N}(\underline{n}/i), \mathrm{holim}_{\mathcal{B}} G) \\ &\cong \mathcal{H}om(\Delta[n], \mathrm{holim}_{\mathcal{B}} G). \end{aligned}$$

Now suppose we have functors $\alpha : \mathbf{a} \rightarrow \mathcal{A}$, $\beta : \mathbf{b} \rightarrow \mathcal{A}$ and an n -simplex

$$\sigma := [(f_0, \theta_0) \xrightarrow{\vartheta_1} (f_1, \theta_1) \xrightarrow{\vartheta_2} \dots \xrightarrow{\vartheta_n} (f_n, \theta_n)].$$

in $\mathrm{Hom}_{\mathbf{Cat}/\mathcal{A}}(\alpha, \beta)$. Define the morphism in $(\mathbf{Cat}/\mathcal{A})_0$

$$(f_{\bullet}, \theta_{\bullet}) : [\alpha \circ p_2 : \underline{n} \times \mathbf{a} \rightarrow \mathcal{A}] \rightarrow [\beta : \mathbf{b} \rightarrow \mathcal{A}]$$

by

$$\begin{aligned} (f_{\bullet})|_{i \times \mathbf{a}} &:= f_i \\ f_{\bullet}((i-1, x) \rightarrow (i, x)) &:= \vartheta_i(x) : f_{i-1}(x) \rightarrow f_i(x) \\ \theta_{\bullet}(i, x) &:= \theta_i(x) \end{aligned}$$

This yields the map

$$\mathrm{holim} F(f_{\bullet}, \theta_{\bullet}) : \mathrm{holim} F(\beta) \rightarrow \mathcal{H}om(\Delta[n], \mathrm{holim} F(\alpha));$$

taking the adjoint gives

$$F(\sigma) : \Delta[n] \times \mathrm{holim} F(\beta) \rightarrow \mathrm{holim} F(\alpha).$$

One checks this respects the simplicial structure in $\mathcal{N}\mathrm{Hom}_{\mathbf{Cat}/\mathcal{A}}(\alpha, \beta)$ and the composition in $\mathcal{N}\mathbf{Cat}/\mathcal{A}$ and \mathbf{Spc} .

For the last ingredient, we recall the notion of a lax fibering of categories.

Definition 7.3.4. Let $\pi : \mathcal{A} \rightarrow B$ be a functor of small categories. For each $b \in B$, we have the subcategory $\pi^{-1}(b)$ of \mathcal{A} consisting of objects over b and morphisms over id_b ; let $\alpha_b : \pi^{-1}(b) \rightarrow \mathcal{A}$ be the inclusion. A *lax fibering* of π consists of the assignment:

- (1) for each morphism $h : b \rightarrow b'$ in B , a functor $h^* : \pi^{-1}(b') \rightarrow \pi^{-1}(b)$, and a natural transformation $\theta_h : \alpha_b \circ h^* \rightarrow \alpha_{b'}$
- (2) for each pair of composable morphisms $b \xrightarrow{h} b' \xrightarrow{g} b''$ in B , a natural transformation $\vartheta_{g,h} : h^* g^* \rightarrow (gh)^*$

which satisfies

- a) $\pi(\theta_h(x)) = h$ for $h : b \rightarrow b'$ in B , and $x \in \pi^{-1}(b')$.
- b) for composable morphisms $h : b \rightarrow b'$, $g : b' \rightarrow b''$, and for $x \in \pi^{-1}(b'')$,

$$\theta_g(x) \circ \theta_h(g^*x) = \theta_{gh}(x) \circ \vartheta_{g,h}(x).$$

- c) for composable morphisms $b \xrightarrow{h} b' \xrightarrow{g} b'' \xrightarrow{f} b'''$

$$\vartheta_{fgh} \circ (h^*)_*(\vartheta_{f,g}) = \vartheta_{f,gh} \circ (f^*)_*(\vartheta_{g,h}).$$

If the natural transformations $\vartheta_{g,h}$ are all identity maps, we call $(\pi : \mathcal{A} \rightarrow B, \theta)$ a *strict fibering* of π . \square

We apply these constructions to the following situation: Given a lax fibering $(\pi : \mathcal{A} \rightarrow B, h \mapsto h^*, \theta, \vartheta)$, we have the 2-functor s from $\text{Fac}B$ to $\mathbf{Cat}/\mathcal{A}^{\text{op}}$ defined by sending b to $\alpha_b : \pi^{-1}(b) \rightarrow \mathcal{A}$, $h : b \rightarrow b'$ in B to (h^*, θ_h) , and the 2-morphism $h * g \rightarrow hg$ to $\vartheta_{h,g}$. Thus, if we have a functor $F : \mathcal{A} \rightarrow \mathbf{Spc}$, we can pull back $\text{holim} F$ by s , giving the continuous functor

$$\text{holim}_{\pi} F : \text{Fac}B \rightarrow \mathbf{Spc}.$$

Applying the Dwyer-Kan machinery thus gives us the functor

$$C\text{holim}_{\pi} F : B \rightarrow \mathbf{Spc},$$

and for each $b \in B$ the weak equivalence

$$(7.3.2) \quad i_b : \text{holim}_{\pi} F(b) \rightarrow C\text{holim}_{\pi} F(b).$$

Noting that

$$\text{holim}_{\pi} F(b) = \text{holim}_{\pi^{-1}(b)} F \circ \alpha_b,$$

we have made the assignment $b \mapsto \text{holim}_{\pi^{-1}(b)} F \circ \alpha_b$ into a functor on B , after adjusting by a weak equivalence.

Remark 7.3.5. Suppose we have a subcategory $i_0 : B_0 \rightarrow B$ of B such that the restriction of ϑ to pairs of maps in B_0 is the identity transformation, i.e., the restriction of π to $\pi_{B_0} : \pi^{-1}(B_0) \rightarrow B_0$ is a strict fibering. Then $b \mapsto (\text{holim}_{\pi} F)(b)$ extends to a functor $\overline{\text{holim}}_{\pi} F|_{B_0} : B_0 \rightarrow \mathbf{Spc}$. Applying Theorem 7.3.1 to $Y := \overline{\text{holim}}_{\pi} F|_{B_0}$, and noting that $\text{holim}_{\pi} F|_{B_0} = \pi_{B_0}^* Y$, we have the weak equivalence

$$r_{B_0} : C_{B_0} \text{holim}_{\pi} F|_{B_0} \rightarrow \overline{\text{holim}}_{\pi} F|_{B_0}.$$

The evident inclusion of $n, *$ -simplices (see (7.3.1)) gives the weak equivalence

$$\iota_{B_0} : C_{B_0} \text{holim}_{\pi} F|_{B_0} \rightarrow (C_B \text{holim}_{\pi} F)|_{B_0},$$

which finally yields the isomorphism in the homotopy category $\mathcal{H}\mathbf{Spc}^{B_0}$

$$\psi : \overline{\mathrm{holim}}_{\pi} F|_{B_0} \rightarrow (C_B \mathrm{holim}_{\pi} F)|_{B_0}$$

by setting $\psi := \iota_{B_0} \circ r_{B_0}^{-1}$.

In words: if $b \mapsto \mathrm{holim}_{\pi^{-1}(b)} F \circ \alpha_b$ is already a functor on the subcategory B_0 of B , then the functor $C_B \mathrm{holim}_{\pi} F$ extends this functor, after a modification by a pointwise weak equivalence in the category of functors $B_0 \rightarrow \mathbf{Spc}$. \square

7.4. Strict functoriality. We apply the categorical constructions of §7.3 to the problem of functoriality of the homotopy coniveau tower.

We consider the case $B = \mathrm{Spec} k$, k a field, and show how to extend the object $E_{\mathrm{sm}}^{(q)}$ of $\mathcal{H}\mathbf{Spt}_{\mathrm{Zar}}(\mathbf{Sm} // B)$ to an object $E^{(q)}$ of $\mathcal{H}\mathbf{Spt}_{\mathrm{Zar}}(B)$. The rough idea is to replace $E^{(q)}(X, -)$ with the homotopy limit of the spectra $E^{(q)}(X, -)_{\mathcal{C}, e}$ over a good index category and use Theorem 2.6.2 to conclude that the resulting homotopy limit is locally weakly equivalent to $E^{(q)}(X, -)$. The index category of the (\mathcal{C}, e) is chosen so that there are well-defined pull-back maps between the respective homotopy limits for each morphism $f : Y \rightarrow X$, however, due to a lack of strict functoriality in the indexing categories, one does not have a strict functoriality on the resulting pull-back maps. We then use the results of §7.3 to complete the construction.

The main result of this section is

Theorem 7.4.1. *Let k be a field, and take $E \in \mathbf{Spt}(k)$ satisfying axioms (4.1.1), (4.1.2) and (4.1.3). Let $\rho : \mathbf{Sm} // k \rightarrow \mathbf{Sm} / k$ be the inclusion. Then there is a canonical object $E_{\mathrm{Zar}}^{(q)} \in \mathbf{Spt}_{\mathrm{Zar}}(k)$ and a canonical isomorphism $\psi_{E, q} : E_{\mathrm{sm}}^{(q)} \rightarrow E_{\mathrm{Zar}}^{(q)} \circ \rho$ in $\mathcal{H}\mathbf{Spt}_{\mathrm{Zar}}(\mathbf{Sm} // k)$.*

In addition, the assignment $E \mapsto E_{\mathrm{Zar}}^{(q)}$ is functorial in E and natural with respect to change in q , and $(E, q) \mapsto \psi_{E, q}$ is natural in E and q .

We begin the construction by introducing the category $\mathcal{L}(\mathbf{Sm} / k)$ (see [16] for a slightly different version). Objects are morphisms $f : X' \rightarrow X$ in \mathbf{Sm} / k such that X' can be written as a disjoint union $X' = X'_0 \amalg X'_1$ with $f : X'_0 \rightarrow X$ an isomorphism. The choice of this decomposition is not part of the data. We identify $f : X' \rightarrow X$ and $f \amalg p : X' \amalg X'' \rightarrow X$ if $p : X'' \rightarrow X$ is smooth, and we identify $f_1 : X_1 \rightarrow X$ and $f_2 : X_2 \rightarrow X$ if there is an X -isomorphism $X_1 \cong X_2$. We note that a choice of component X'_0 of X' isomorphic to X via f determines a smooth section $s : X \rightarrow X'$ to f .

A morphism $g : (f_X : X' \rightarrow X) \rightarrow (f_Y : Y' \rightarrow Y)$ is a morphism $g : X \rightarrow Y$ in \mathbf{Sm} / k such that there exists a smooth morphism $g' : X' \rightarrow Y'$ with $f_Y \circ g' = g \circ f_X$. Using the smooth section $s_X : X \rightarrow X'$, one easily sees that this condition also respects the identifications $f_X \sim f_X \amalg p$ for $p : X'' \rightarrow X$ smooth: we extend $g' : X' \rightarrow Y'$ to $g'' : X' \amalg X'' \rightarrow Y'$ by the map $g' s_X p$ on X'' . Similarly, this condition is unaltered by adding a smooth morphism to $Y' \rightarrow Y$ and also respects the identification of isomorphisms over X or over Y . The choice of g' is not part of the data, so

$$\mathrm{Hom}_{\mathcal{L}(\mathbf{Sm} / k)}((f_X : X' \rightarrow X), (f_Y : Y' \rightarrow Y)) \subset \mathrm{Hom}_{\mathbf{Sm} / k}(X, Y).$$

Composition is induced by the composition in \mathbf{Sm} / k ; this is well-defined since smooth morphisms are closed under composition.

Sending $f_X : X' \rightarrow X$ to X defines the faithful functor

$$\pi : \mathcal{L}(\mathbf{Sm} / k)^{\mathrm{op}} \rightarrow \mathbf{Sm} / k^{\mathrm{op}}.$$

We make $\mathcal{L}(\mathbf{Sm}/k)^{\text{op}}$ a lax fibered category over $\mathbf{Sm}/k^{\text{op}}$ as follows: For each morphism $f : X \rightarrow Y$ in $\mathbf{Sm}/k^{\text{op}}$ (i.e. $f : Y \rightarrow X$ in \mathbf{Sm}/k), let $f^* : \pi^{-1}(Y) \rightarrow \pi^{-1}(X)$ be the functor

$$f^*(g : Y' \rightarrow Y) := (fg \amalg \text{id}_X : Y' \amalg X \rightarrow X).$$

The map $\theta_f(g) : f^*(g : Y' \rightarrow Y) \rightarrow (g : Y' \rightarrow Y)$ in $\mathcal{L}(\mathbf{Sm}/k)^{\text{op}}$ is the opposite of the map $f : (g : Y' \rightarrow Y) \rightarrow (fg \amalg \text{id}_X : Y' \amalg X \rightarrow X)$; we see that this latter is a map in $\mathcal{L}(\mathbf{Sm}/k)$ by using the inclusion $Y' \rightarrow Y' \amalg X$ as the required smooth morphism. Given maps $f : X \rightarrow Y$ and $f' : Y \rightarrow Z$ in $\mathbf{Sm}/k^{\text{op}}$, and $h : Z' \rightarrow Z$ in $\mathcal{L}(\mathbf{Sm}/k)$, set $\vartheta_{f',f}$ to be the map

$$f^*(f'^*(h : Z' \rightarrow Z)) \rightarrow (f'f)^*(h : Z' \rightarrow Z)$$

being the opposite of the map

$$(ff'g \amalg \text{id}_X : Z' \amalg X \rightarrow X) \rightarrow (ff'g \amalg f \amalg \text{id}_X : Z' \amalg Y \amalg X \rightarrow X)$$

induced by the identity on X . One easily checks that this data does indeed define a lax fibering.

Remark 7.4.2. Let $\rho : \mathbf{Sm} // k \rightarrow \mathbf{Sm}/k$ be the inclusion. It is easy to see that the pull-back $\rho^{-1}\mathcal{L}(\mathbf{Sm}/k)^{\text{op}} \rightarrow (\mathbf{Sm} // k)^{\text{op}}$ is a strict fibering. \square

Let $f : Y \rightarrow X$ be a morphism in \mathbf{Sm}/k . We let

$$\mathcal{S}_f^{(q)}(p) = \{W \in \mathcal{S}_X^{(q)}(p) \mid (f \times \text{id}_{\Delta^p})^{-1}(W) \text{ is in } \mathcal{S}_Y^{(q)}\}.$$

For $E \in \mathbf{Spt}(k)$, set

$$E^{(q)}(X, -)_f := E^{\mathcal{S}_f^{(q)}}(X \times \Delta^*).$$

Lemma 7.4.3. *Let $g : (f_X : X' \rightarrow X) \rightarrow (f_Y : Y' \rightarrow Y)$ be a morphism in $\mathcal{L}(\mathbf{Sm}/k)$. Then, for $W \in \mathcal{S}_{f_Y}^{(p)}(n)$, $(g \times \text{id}_{\Delta^n})^{-1}(W)$ is in $\mathcal{S}_{f_X}^{(p)}(n)$.*

Proof. Choose a component X'_0 of X' for which $f_X : X'_0 \rightarrow X$ is an isomorphism, and let $s : X \rightarrow X'$ be the composition

$$X \xrightarrow{f_X^{-1}} X'_0 \subset X'.$$

Then s is a smooth section to $f_X : X' \rightarrow X$, and we can factor g as $g = f_Y \circ g' \circ s$, with $g' : X' \rightarrow Y'$ smooth. Since $(f_Y \times \text{id})^{-1}$ maps $\mathcal{S}_{f_Y}^{(p)}(n)$ to $\mathcal{S}_{Y'}^{(p)}(n)$, by definition, and $g' \circ s$ is smooth, it follows that $W' := (g \times \text{id}_{\Delta^n})^{-1}(W)$ is in $\mathcal{S}_X^{(p)}(n)$. Similarly, since $g \circ f_X = f_Y \circ g'$ and

$$\begin{aligned} (f_X \times \text{id})^{-1}(W') &= ((g \circ f_X) \times \text{id})^{-1}(W) \\ &= ((f_Y \circ g') \times \text{id})^{-1}(W) = (g' \times \text{id})^{-1}((f_Y \times \text{id})^{-1}(W)), \end{aligned}$$

we see that W' is in $\mathcal{S}_{f_X}^{(p)}(n)$. \square

The use of Theorem 2.6.2 for the problem of functoriality is based on the following elementary result:

Lemma 7.4.4. *Let $f : Y \rightarrow X$ be a morphism in \mathbf{Sm}/k . Then there is a finite set \mathcal{C} of irreducible locally closed subsets of X and a function $e : \mathcal{C} \rightarrow \mathbb{N}$ such that $\mathcal{S}_f^{(q)} = \mathcal{S}_{\mathcal{C},e}^{(q)}$.*

Proof. We may assume that X and Y are irreducible. For a subset S of Y , we write

$$\text{codim}_Y S \leq d$$

if there is a point $s \in S$ whose closure $\bar{s} \subset Y$ has $\text{codim}_Y \bar{y} \leq d$.

For each $n \in \mathbb{N}$, let

$$D_n := \{x \in X \mid \text{codim}_Y f^{-1}(x) \leq \text{codim}_X x - n\}.$$

By Chevalley's theorem [14], each D_n is a constructible subset of X . Write $D_n \setminus D_{n+1}$ as a disjoint union of irreducible locally closed subsets

$$D_n \setminus D_{n+1} = \coprod_j \mathcal{C}_j^{(n)},$$

and let $e(\mathcal{C}_j^{(n)}) = \text{codim}_X \mathcal{C}_j^{(n)} - n$. If x is the generic point of $\mathcal{C}_j^{(n)}$, then $e(\mathcal{C}_j^{(n)})$ is the codimension of some irreducible component of $f^{-1}(x)$, so $e(\mathcal{C}_j^{(n)}) \geq 0$. Letting $\mathcal{C} := \{\mathcal{C}_j^{(n)}\}$, it is an easy exercise to show that $\mathcal{S}_f^{(q)} = \mathcal{S}_{\mathcal{C},e}^{(q)}$. \square

We define the functor

$$E^{(q)}(? , -)_? : \mathcal{L}(\mathbf{Sm}/k)^{\text{op}} \rightarrow \mathbf{Spt}$$

by sending $(f : X' \rightarrow X)$ to the functorial fibrant model (in \mathbf{Spt}) $\widetilde{E^{(q)}(X, -)}_f$ of $E^{(q)}(X, -)_f$. It follows from Lemma 7.4.3 that this really does define a functor. Let $hE^{(q)}(X)$ be the homotopy limit

$$hE^{(q)}(X) := \text{holim}_{(X,f) \in \pi^{-1}(X)} E^{(q)}(X, -)_f.$$

By the construction of §7.3, we have the continuous functor

$$hE^{(q)} : \text{FacSm}/k^{\text{op}} \rightarrow \mathbf{Spt},$$

the functor

$$ChE^{(q)} : \mathbf{Sm}/k^{\text{op}} \rightarrow \mathbf{Spt},$$

$$X \mapsto ChE^{(q)}(X)$$

and for each $X \in \mathbf{Sm}/k$ the weak equivalence

$$i_X : hE^{(q)}(X) \rightarrow ChE^{(q)}(X).$$

Thus, for fixed $X \in \mathbf{Sm}/k$, we have the diagram

$$(7.4.1) \quad \begin{array}{ccc} hE^{(q)}(X) & \xrightarrow{i_X} & ChE^{(q)}(X) \\ & & \downarrow w_X \\ E^{(q)}(X, -) & \xrightarrow{j_X} & \widetilde{E^{(q)}(X, -)}_{\text{id}} \end{array}$$

Using Remarks 7.4.2 and 7.3.5 we have the diagram of functors on $\mathbf{Sm} // k$:

$$(7.4.2) \quad \begin{array}{ccc} \bar{h}E_{\text{sm}}^{(q)} & \xleftarrow{r} & C(hE_{\text{sm}}^{(q)}) \xrightarrow{\iota} ChE^{(q)} \circ \rho \\ & & \downarrow w \\ E_{\text{sm}}^{(q)} & \xrightarrow{j} & \widetilde{E_{\text{sm}}^{(q)}} \end{array}$$

r , ι and j are pointwise weak equivalences.

Note that the fiber category $\pi^{-1}(X) \subset \mathcal{L}(\mathbf{Sm}/k)^{\text{op}}$ has the final object id_X . By Theorem 2.6.2, Remark 7.3.3 and Lemma 7.4.4, w_X is a weak equivalence for each affine X , hence w is a weak equivalence in $\mathbf{Spt}_{\text{Zar}}(\mathbf{Sm}/k)$. Thus, we have isomorphism in $\mathcal{HSpt}_{\text{Zar}}(\mathbf{Sm}/k)$

$$\psi_{E,q} : E_{\text{sm}}^{(q)} \rightarrow \text{Ch}E^{(q)} \circ \rho.$$

One easily checks that $\text{Ch}E^{(q)}$ is functorial in E and the remainder of the construction is natural in E and q . Setting $E^{(q)} := \text{Ch}E^{(q)}$ completes the proof of Theorem 7.4.1.

7.5. Naturality. We conclude our discussion of functoriality with various naturality issues.

The following result is obvious from the construction of $E^{(q)}(X, -)$.

Lemma 7.5.1. *The assignment $E \mapsto E_{\text{sm}}^{(q)}$ defines a functor $?_{\text{sm}}^{(q)} : \mathbf{Spt}(\mathbf{Sm}/B^{(q)}) \rightarrow \mathbf{Spt}(\mathbf{Sm}/B)$. $?_{\text{sm}}^{(q)}$ preserves weak equivalences and homotopy cofiber sequences.*

Similarly, we have

Lemma 7.5.2. *Fix $X \in \mathbf{Sm}/B$, a finite set of irreducible locally closed subsets \mathcal{C} of X and a function $e : \mathcal{C} \rightarrow \mathbb{N}$. The assignment $E \mapsto E^{(q)}(X, -)_{\mathcal{C},e}$ defines a functor $?^{(q)}(X, -)_{\mathcal{C},e} : \mathbf{Spt}(\mathbf{Sm}/B^{(q)}) \rightarrow \mathbf{Spt}$, preserving weak equivalences and homotopy cofiber sequences.*

and

Lemma 7.5.3. *Let k be a field. The assignment*

$$E \mapsto [(X, f : X' \rightarrow X) \mapsto E^{(q)}(X, -)_f]$$

defines a functor $?^{(q)} : \mathbf{Spt}(\mathbf{Sm}/k^{(q)}) \rightarrow \mathbf{Spt}(\mathcal{L}(\mathbf{Sm}/k))$, preserving weak equivalences and homotopy cofiber sequences.

Let $\mathbf{Sec}_{\pi}(\mathbf{Sm}/B^{\text{op}}, \mathcal{HSpt}_{\text{Nis}}/B)$ be the category of sections to $\pi : \mathcal{HSpt}_{\text{Nis}}/B \rightarrow \mathbf{Sm}/B^{\text{op}}$. This category inherits a structure of homotopy fiber sequences from $\mathcal{HSpt}_{\text{Nis}}/B$. Let $\mathbf{Spt}(B^{(q)})^*$ be the full subcategory of $\mathbf{Spt}(B^{(q)})$ of objects satisfying the axioms (4.1.1), (4.1.2) and (4.1.3). A weak equivalence (resp. homotopy fiber sequence) in $\mathbf{Spt}(B^{(q)})^*$ is a weak equivalence (resp. homotopy fiber sequence) in $\mathbf{Spt}(B^{(q)})$ involving objects in $\mathbf{Spt}(B^{(q)})^*$. Note that we do not require compatibility with the data in the various axioms, for instance, the contraction in axiom (4.1.1).

From the three lemmas above follows easily

Theorem 7.5.4. *Let B be a regular noetherian separated scheme of Krull dimension at most one. The assignment $E \mapsto \tilde{E}^{(q)}$ defines a functor*

$$\tilde{?}^{(q)} : \mathbf{Spt}(B^{(q)})^* \rightarrow \mathbf{Sec}_{\pi}(\mathbf{Sm}/B^{\text{op}}, \mathcal{HSpt}_{\text{Nis}}/B^*).$$

$\tilde{?}^{(q)}$ sends weak equivalences to isomorphisms and homotopy fiber sequences to point-wise distinguished triangles. In case $B = \text{Spec } k$, k a field, the assignment $E \mapsto E^{(q)}$ defines a functor

$$?^{(q)} : \mathbf{Spt}(B^{(q)})^* \rightarrow \mathbf{Spt}(B)^*.$$

$?^{(q)}$ sends weak equivalences to isomorphisms and homotopy fiber sequences to weak homotopy fiber sequences.

Remark 7.5.5. In case of an infinite field, we have similar functor $?^{(q)}$ on the category $\mathbf{Spt}(B^{(q)})_2^*$ of presheaves satisfying axioms (4.1.1) and (4.1.2). \square

We also have naturality with respect to change in q . Take $q' > q$, giving us the functor $\rho_{q,q'} : \mathbf{Sm}/B^{(q')} \rightarrow \mathbf{Sm}/B^{(q)}$, $(Y, W) \mapsto (Y, W)$. The inclusion of support conditions $\mathcal{S}_X^{(q')} \subset \mathcal{S}_X^{(q)}$ gives the natural transformation

$$\phi_{q,q'}(E) : (E \circ \rho_{q,q'})_{\text{sm}}^{(q')} \rightarrow E_{\text{sm}}^{(q)},$$

natural in E , i.e., the natural transformation $\phi_{q,q'} : ?_{\text{sm}}^{(q')} \circ \rho_{q,q'}^* \rightarrow ?_{\text{sm}}^{(q)}$. We have the identity

$$\phi_{q,q'} \circ (\phi_{q',q''} \circ \rho_{q,q'}^*) = \phi_{q,q''}.$$

Lemma 7.5.6. *For $q' > q$ there is a natural transformation $\tilde{\Phi}_{q,q'} : \tilde{?}^{(q')} \circ \rho_{q,q'}^* \rightarrow \tilde{?}^{(q)}$ extending $\phi_{q,q'}$. For $q'' > q' > q$, we have $\tilde{\Phi}_{q,q'} \circ (\tilde{\Phi}_{q',q''} \circ \rho_{q,q'}^*) = \tilde{\Phi}_{q,q''}$. In case $B = \text{Spec } k$, k a field, we have a natural transformation $\Phi_{q,q'} : ?^{(q')} \circ \rho_{q,q'}^* \rightarrow ?^{(q)}$ extending $\phi_{q,q'}$, with the same compatibility.*

Proof. For each X, \mathcal{C}, e , we have the natural inclusion of support conditions $\mathcal{S}_{X,\mathcal{C},e}^{(q')} \subset \mathcal{S}_{X,\mathcal{C},e}^{(q)}$, giving the natural map

$$\rho_{q,q'}^{X,\mathcal{C},e} : E^{(q')}(X, -)_{\mathcal{C},e} \rightarrow E^{(q)}(X, -)_{\mathcal{C},e}$$

with $\rho_{q,q'}^{X,\mathcal{C},e} \circ \rho_{q',q''}^{X,\mathcal{C},e} = \rho_{q,q''}^{X,\mathcal{C},e}$. These maps feed through the construction of $\tilde{E}^{(q)}$ and $E^{(q)}$, respectively, to give the result. \square

Corollary 7.5.7. *Let B be a regular noetherian separated scheme of Krull dimension at most one. The homotopy coniveau tower (2.5.1) extends to the tower*

$$\dots \rightarrow \tilde{?}^{(q+r+1)} \rightarrow \tilde{?}^{(q+r)} \rightarrow \dots \rightarrow \tilde{?}^{(q)}$$

of functors $\mathbf{Spt}(B^{(q)})^ \rightarrow \mathbf{Sec}_\pi(\mathbf{Sm}/B^{\text{op}}, \mathcal{HSpt}_{\text{Nis}}/B)$ In case $B = \text{Spec } k$, k a field, there is a tower*

$$\dots \rightarrow ?^{(q+r+1)} \rightarrow ?^{(q+r)} \rightarrow \dots \rightarrow ?^{(q)}$$

of functors $\mathbf{Spt}(B^{(q)})^ \rightarrow \mathbf{Spt}_{\text{Nis}}(B)$ whose composition with the canonical functor $\mathbf{Spt}_{\text{Nis}}(B) \rightarrow \mathcal{HSpt}_{\text{Nis}}(B)$ extends the tower (2.5.1), considered as a tower of functors $\mathbf{Spt}(B^{(q)})^* \rightarrow \mathcal{HSpt}_{\text{Nis}}(\mathbf{Sm}/B)$.*

8. EQUI-DIMENSIONAL SUPPORTS

Over a base-scheme B of positive Krull dimension, especially in the case of mixed characteristic, it is often useful to consider an *equi-dimensional* version $\mathcal{S}_{X/B}^{(q)}$ of the support conditions $\mathcal{S}_X^{(q)}$. These have two main advantages:

- (1) The support conditions $\mathcal{S}_{X/B}^{(q)}$ admit external products.
- (2) The support conditions $\mathcal{S}_{X/B}^{(q)}$ admit an extended functoriality.

Fortunately, one needs to make only minor modifications to the arguments used for the support conditions $\mathcal{S}_X^{(q)}$ to cover the equi-dimensional case. In this section, we describe the equi-dimensional theory, and explain these minor changes. We will not go into detail explaining the application to products in this paper, but we will discuss the extended functoriality.

Of course, it would be best if the two theories agreed. Unfortunately, this is at present not known in the most important case, that of mixed characteristic. For this reason, we need to present both theories.

We fix a noetherian separated base-scheme B .

8.1. Basic definitions.

Definition 8.1.1. Let X be in \mathbf{Sm}/B .

(1) Let $\mathcal{S}_{X/B}^{(q)}(p) \subset \mathcal{S}_X(p)$ be the set of closed subsets $W \subset X \times \Delta^p$ such that, for all $b \in B$, $W_b \subset X_b \times \Delta^p$ is in $\mathcal{S}_{X_b}^{(q)}(p)$.

(2) Let \mathcal{C} be a finite set of irreducible locally closed subsets of X , $e : \mathcal{C} \rightarrow \mathbb{N}$ a function. Let $\mathcal{S}_{X/B, \mathcal{C}, e}^{(q)}(p) \subset \mathcal{S}_{X/B}^{(q)}(p)$ be the set of closed subsets $W \subset X \times \Delta^p$ such that, for all $b \in B$, $W_b \subset X_b \times \Delta^p$ is in $\mathcal{S}_{X_b, i_b^* \mathcal{C}, i_b^* e}^{(q)}(p)$. \square

It is clear that $p \mapsto \mathcal{S}_{X/B, \mathcal{C}, e}^{(q)}(p)$ defines a support condition on $X \times \Delta^*$ and that $\mathcal{S}_{X/B, \mathcal{C}, e}^{(q)}(p) = \mathcal{S}_{X/B, \emptyset}^{(q)}(p) = \mathcal{S}_{X/B}^{(q)}(p)$ if $e(C) \geq \dim_B X$ for all $C \in \mathcal{C}$.

Definition 8.1.2. Let E be a presheaf of spectra on $\mathbf{Sm}/B^{(q)}$, $X \in \mathbf{Sm}/B$. We set

$$E^{(q)}(X/B, -) := E^{\mathcal{S}_{X/B}^{(q)}}(X \times \Delta^*)$$

$$E^{(q)}(X/B, -)_{\mathcal{C}, e} := E^{\mathcal{S}_{X/B, \mathcal{C}, e}^{(q)}}(X \times \Delta^*)$$

\square

We have as well the equi-dimensional versions of the definitions presented in §3.1:

Definition 8.1.3. Take $X \in \mathbf{Sch}/B$ and $q \in \mathbb{Z}$.

(1) The category $X//B_{(q)}$ is defined to have objects (Y, W) , with $Y \in \mathbf{Sm}/B$ and $W \subset X \times_B Y$ a closed subset such that (Y_b, W_b) is in $X_b/b_{(q)}$ for all $b \in B$. A morphism $(Y, W) \rightarrow (Y', W')$ is a B -morphism $f : Y \rightarrow Y'$ with $W \supset f^{-1}(W')$.

(2) Let $\mathcal{S}_{(q)}^{X/B}(p) \subset \mathcal{S}_X(p)$ be the set of closed subsets $W \subset X \times \Delta^p$ such that, for all $b \in B$, $W_b \subset X_b \times \Delta^p$ is in $\mathcal{S}_{(q)}^{X_b}(p)$.

(3) Let \mathcal{C} be a finite set of locally closed subsets of X , $e : \mathcal{C} \rightarrow \mathbb{N}$ a function. Suppose that, for each $b \in B$, $C \in \mathcal{C}$, the locally closed subset C_b of X_b has pure codimension (which may depend on b). Let $\mathcal{S}_{(q)}^{X/B, \mathcal{C}, e}(p) \subset \mathcal{S}_{(q)}^{X/B}(p)$ be the set of closed subsets $W \subset X \times \Delta^p$ such that, for all $b \in B$, $W_b \subset X_b \times \Delta^p$ is in $\mathcal{S}_{(q)}^{X_b, i_b^* \mathcal{C}, i_b^* e}(p)$. \square

Definition 8.1.4. Take $X \in \mathbf{Sch}/B$, $E \in \mathbf{Spt}(X//B^{(q)})$. We set

$$E_{(q)}(X/B, -) := E^{\mathcal{S}_{(q)}^{X/B}}(\Delta^*)$$

$$E_{(q)}(X/B, -)^{\mathcal{C}, e} := E^{\mathcal{S}_{(q)}^{X/B, \mathcal{C}, e}}(\Delta^*)$$

\square

8.2. The basic principle. The transfer of our main results to the setting of equi-dimensional support conditions is based on the obvious fact that the support conditions $\mathcal{S}_{X/B, \mathcal{C}, e}^{(q)}$ and $\mathcal{S}_{(q)}^{X/B, \mathcal{C}, e}$ are defined point-wise over B , and these support conditions agree with the support conditions $\mathcal{S}_{X, \mathcal{C}, e}^{(q)}$ and $\mathcal{S}_{(q)}^{X, \mathcal{C}, e}$, respectively, in case $B = \text{Spec } F$, F a field.

Next, our two basic moving operations: translation of \mathbb{A}^n by \mathbb{A}^n and taking the cone with respect to a projection, are defined by generic translations or generic projections. In order to make the necessary generic extension $B(x)$ of B , we assume that $B = \text{Spec } A$, A a noetherian semi-local ring. For polynomial variables x_1, \dots, x_n , we let $A(x)$ be the localization of $A[x_1, \dots, x_n]$ with respect to the multiplicatively closed subset of polynomials $f = \sum_I a_I x^I$ such that the ideal of A generated by the a_I is the unit ideal, and we set $B(x) := \text{Spec } A(x)$, as in §5.2.

With this notation, we have as in §3.3 the generic translation $\psi_x : \mathbb{A}_{B(x)}^1 \rightarrow \mathbb{A}_{B(x)}^n$, $\psi_x(t) = t \cdot x$, which restricts to the generic translation $\psi_{x,b} : \mathbb{A}_{b(x)}^1 \rightarrow \mathbb{A}_{b(x)}^n$ for all $b \in B$. Similarly, if $X \subset \mathbb{A}_B^n$ is a closed subscheme, smooth over B and satisfying the conditions (5.1.1), then we have the generic linear projection $\pi_X : X_{B(x)} \rightarrow \mathbb{A}_{B(x)}^m$, such that the restriction $\pi_{X_b} : X_{b(x)} \rightarrow \mathbb{A}_{b(x)}^m$ remains generic for all $b \in B$.

8.3. The main results. We conclude this section with a statement of the main results for equi-dimensional support conditions, along with a sketch of their proofs.

Proposition 8.3.1 (Moving by translation). *Let B be a semi-local noetherian ring with infinite residue fields. Let \mathcal{C}_0 be a finite set of irreducible locally closed subsets of \mathbb{A}_B^n , $e : \mathcal{C}_0 \rightarrow \mathbb{N}$ a function and let Y be in \mathbf{Sch}/B . Let $X = Y \times_B \mathbb{A}^n$, $\mathcal{C} = p_2^* \mathcal{C}_0$ and $e = p_2^*(e_0)$. Take $E \in \mathbf{Spt}(X//B_{(q)})$. Then the map*

$$E_{(q)}(X/B, -)^{\mathcal{C}, e} \rightarrow E_{(q)}(X/B, -)$$

is a weak equivalence.

Theorem 8.3.2 (Homotopy invariance). *Let B be a semi-local noetherian ring with infinite residue fields, take $X \in \mathbf{Sch}/B$ and $E \in \mathbf{Spt}(X//B_{(q)})$. Then the map*

$$p^* : E_{(q)}(X/B, -) \rightarrow E_{(q+1)}(X \times \mathbb{A}^1/B, -)$$

is a weak equivalence.

To prove these two results, one just repeats the arguments of §3, using the equi-dimensional support conditions instead of the ones used in §3.1. Since one only needs to verify the various properties of these support conditions point-wise, the case $B = \text{Spec } F$, F a field, applied to the support conditions used in §3.1, suffices to prove all the results we need for the equi-dimensional support conditions. Note that one does not need to allow constructible subsets instead of locally closed subsets as elements of \mathcal{C} , as one simply decomposes a constructible subset into a disjoint union of locally closed subsets and modifies a given function e as in Remark 3.1.6(3).

Finally, we have the main result in the equi-dimensional case:

Theorem 8.3.3. *Let B be a noetherian separated scheme, let X be in \mathbf{Sm}/B , let \mathcal{C} be a finite set of irreducible locally closed subsets of X , $e : \mathcal{C} \rightarrow \mathbb{N}$ a function and $q \geq 0$ an integer. Take E in $\mathbf{Spt}(B^{(q)})$ satisfying axioms (4.1.1), (4.1.2) and (4.1.3).*

Let x be a point of X . Then there is a basis of Nisnevich neighborhoods $p : U \rightarrow X$ of x , depending only on x and X , such that $E^{(q)}(U/B, -)_{p^*c, p^*e} \rightarrow E^{(q)}(U/B, -)$ is a weak equivalence for each U . In particular, the map of presheaves on X_{Nis}

$$E^{(q)}(X_{\text{Nis}}/B, -)_{c,e} \rightarrow E^{(q)}(X_{\text{Nis}}/B, -)$$

is a local weak equivalence.

The proof of this result is accomplished by using the argument of §5, in the case of $B = \text{Spec } F$, F a field, and our basic principle, to show that, with $f : X \rightarrow \mathbb{A}_{B(x)}^m$ the generic finite projection, we have

$$W \in \mathcal{S}_{X/B, c, e}^{(q)}(p) \implies W^+ \cup \text{Ind}W \text{ is in } \mathcal{S}_{X/B, c, e-1}^{(q)}(p).$$

In addition, the map

$$X \times \Delta^p \setminus W^+ \cup \text{Ind}W \rightarrow \mathbb{A}_{B(x)}^m \times \Delta^p \setminus f(W^+ \cup \text{Ind}W)$$

induced by $f \times \text{id}$ is étale on a neighborhood of $|W| \setminus (W^+ \cup \text{Ind}W)$, and the restriction

$$|W| \setminus (W^+ \cup \text{Ind}W) \rightarrow f(|W| \setminus (W^+ \cup \text{Ind}W))$$

is an isomorphism. Once we have this, the argument used to prove Theorem 2.6.2 proves the equi-dimensional analog above.

8.4. Functoriality. Theorem 8.3.3 yields the same functoriality for the spectra $E^{(q)}(X/B, -)$ (over an arbitrary base B , or even over a category of such B) that we have for the spectra $E^{(q)}(X, -)$ in case $B = \text{Spec } k$, k a field (Theorem 7.5.4), as well as the naturality of the homotopy coniveau tower.

Indeed, let $\mathcal{V} \subset \mathbf{Sch}$ be a small category of separated noetherian schemes. Form the category \mathbf{Sm}/\mathcal{V} with objects being $Y \rightarrow B$ in \mathbf{Sm}/B with $B \in \mathcal{V}$, and morphisms $(f, f_0) : (Y \rightarrow B) \rightarrow (Y' \rightarrow B')$ being a pair of morphisms $f_0 : B \rightarrow B'$ in \mathcal{V} and $f : Y \rightarrow Y'$ a morphisms of schemes over f_0 .

For an integer $q \geq 0$, we have the category $\mathbf{Sm}/\mathcal{V}^{(q)}$, consisting of pairs $(Y \rightarrow B, W)$ with $(Y \rightarrow B, W) \in \mathbf{Sm}/B^{(q)}$, where a morphism $(Y \rightarrow B, W) \rightarrow (Y' \rightarrow B', W')$ is a morphism $(f, f_0) : (Y \rightarrow B) \rightarrow (Y' \rightarrow B')$ in \mathbf{Sm}/\mathcal{V} , with $f^{-1}(W') \subset W$. Both $\mathbf{Sm}/\mathcal{V}^{(q)}$ and \mathbf{Sm}/\mathcal{V} are fibered over \mathcal{V} by the evident forgetful functor, with fiber over $B \in \mathcal{V}$ being $\mathbf{Sm}/B^{(q)}$ and \mathbf{Sm}/B , respectively.

Let \mathcal{V}_{sm} be the subcategory of \mathcal{V} consisting of the smooth morphisms. We have as well the subcategory $\mathbf{Sm}//\mathcal{V}$ of \mathbf{Sm}/\mathcal{V} with the same objects, and with morphisms consisting of (f, f_0) with f and f_0 smooth morphisms. $\mathbf{Sm}//\mathcal{V}$ is fibered over \mathcal{V}_{sm} by the forgetful functor as above. A similar construction defines the subcategory $\mathbf{Sm}//\mathcal{V}^{(q)}$ of $\mathbf{Sm}/\mathcal{V}^{(q)}$ with forgetful functor to \mathcal{V}_{sm} as above. We let $\rho : \mathbf{Sm}//\mathcal{V} \rightarrow \mathbf{Sm}/\mathcal{V}$ denote the inclusion.

Definition 8.4.1. The category $\mathbf{Spt}(\mathbf{Sm}/\mathcal{V})$ of presheaves of spectra on \mathbf{Sm}/\mathcal{V} is the category of functors $E : \mathbf{Sm}/\mathcal{V}^{\text{op}} \rightarrow \mathbf{Spt}$ such that the restriction of E to each fiber $\mathbf{Sm}/B^{\text{op}}$ is a presheaf (i.e. additive). We have as well the categories of presheaves of spectra on $\mathbf{Sm}//\mathcal{V}^{(q)}$ and on $\mathbf{Sm}//\mathcal{V}$, denoted $\mathbf{Spt}(\mathbf{Sm}//\mathcal{V}^{(q)})$ and $\mathbf{Spt}(\mathbf{Sm}//\mathcal{V})$, respectively.

We let $\mathbf{Spt}(\mathbf{Sm}/\mathcal{V})^*$, respectively $\mathbf{Spt}(\mathbf{Sm}/\mathcal{V}^{(q)})^*$, be the full subcategory of $\mathbf{Spt}(\mathbf{Sm}/\mathcal{V})$, respectively $\mathbf{Spt}(\mathbf{Sm}/\mathcal{V}^{(q)})$, consisting of E satisfying axioms (4.1.1), (4.1.2) and (4.1.3). \square

Via the Nisnevich topology on Y , for $Y \rightarrow B$ in \mathcal{V} , we may define the Nisnevich-local model structure on $\mathbf{Spt}(\mathbf{Sm}/\mathcal{V})$, $\mathbf{Spt}(\mathbf{Sm}/\mathcal{V})$, etc., denoted $\mathbf{Spt}_{\text{Nis}}(\mathbf{Sm}/\mathcal{V})$, $\mathbf{Spt}_{\text{Nis}}(\mathbf{Sm}/\mathcal{V})$, and so on. We have the homotopy categories $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(\mathbf{Sm}/\mathcal{V})$, $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(\mathbf{Sm}/\mathcal{V})$, etc.

Take $E \in \mathbf{Spt}(\mathbf{Sm}/\mathcal{V}^{(q)})$. Sending X to $E^{(q)}(X/B, -)$ defines the presheaf $E_{\text{sm}}^{(q)}/B \in \mathbf{Spt}(\mathbf{Sm}/\mathcal{V})$, and the homotopy coniveau tower

$$\dots \rightarrow E_{\text{sm}}^{(q+r+1)}/B \rightarrow E_{\text{sm}}^{(q+r)}/B \rightarrow \dots \rightarrow E_{\text{sm}}^{(q)}/B$$

Sending E to $E_{\text{sm}}^{(q)}/B$ is functorial in E , and preserves weak equivalences and homotopy fiber sequences. Similarly, the homotopy coniveau tower is natural in E . We denote the functor $E \mapsto E_{\text{sm}}^{(q)}/B$ by $?_{\text{sm}}^{(q)}/B$ and the functor sending E to the homotopy coniveau tower by $?_{\text{sm}}^{(*\leq q)}/B$.

Using equi-dimensional supports throughout, we have the support conditions $\mathcal{S}_f^{(q)}/f_0 \subset \mathcal{S}_X/B^{(q)}$:

$$\mathcal{S}_f^{(q)}/f_0(p) = \{W \in \mathcal{S}_X/B^{(q)}(p) \mid (f \times \text{id})^{-1}(W) \text{ is in } \mathcal{S}_Y/B'^{(q)}\}$$

We set $E^{(q)}(X/B, -)_f := E_{\mathcal{S}_f^{(q)}/f_0}^{(q)}(X \times \Delta^*)$, and we have the well-defined pull-back

$$f^* : E^{(q)}(X/B, -)_f \rightarrow E^{(q)}(Y/B', -).$$

Using the evident analog of the definition of the collection \mathcal{C} and the function e from Lemma 7.4.4, this lemma extends directly to the equi-dimensional setting, for an arbitrary morphism f over a morphism f_0 . The line of argument in Theorem 7.5.4 yields

Theorem 8.4.2.

- (1) For $E \in \mathbf{Spt}(\mathbf{Sm}/\mathcal{V}^{(q)})^*$, there is a canonically defined object $\tilde{E}^{(q)}/B \in \mathbf{Spt}(\mathbf{Sm}/\mathcal{V})$ and a natural isomorphism

$$\phi_{E,q} :: E_{\text{sm}}^{(q)}/B \rightarrow \tilde{E}^{(q)}/B$$

in $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(\mathbf{Sm}/\mathcal{V})$.

- (2) The assignment $E \mapsto \tilde{E}^{(q)}/B$ defines a functor

$$?^{(q)}/B : \mathbf{Spt}(\mathbf{Sm}/\mathcal{V}^{(q)})^* \rightarrow \mathbf{Spt}(\mathbf{Sm}/\mathcal{V})^*.$$

$?^{(q)}$ preserves weak equivalences and homotopy fiber sequences.

- (3) The isomorphism ϕ define a natural isomorphism of functors $?_{\text{sm}}^{(q)}/B \rightarrow ?^{(q)}/B \circ \rho$

- (4) We have as well naturality with respect to change in q , yielding the homotopy coniveau tower functor $?^{(*\leq q)}/B$, and the isomorphism of towers $?_{\text{sm}}^{(*\leq q)}/B \rightarrow ?^{(*\leq q)}/B \circ \rho$ in $\mathcal{H}\mathbf{Spt}_{\text{Nis}}(\mathbf{Sm}/\mathcal{V})$.

9. THE EXAMPLES

In this section, we give conditions on a presheaf of spectra $E \in \mathbf{Spt}(B)$ which imply the axioms (4.1.1), (4.1.2) and (4.1.3) in §4.

Let E be a presheaf of spectra on \mathbf{Sm}/B . We have the associated presheaf with supports in $\mathbf{Spt}(\mathbf{Sm}/B^{(q)})$, defined by $E(Y, W) := E^W(Y) := \text{fib}(E(Y) \rightarrow E(Y \setminus W))$. Note that this construction is compatible with the restriction functors $\rho_{q,q'} : \mathbf{Sm}/B^{(q')} \rightarrow \mathbf{Sm}/B^{(q)}$ for $q' \geq q$. Thus, it suffices to consider the extension

of E to a presheaf on $\mathbf{Sm}/B^{(0)}$, and so we need not refer to the codimension of W in Y .

9.1. Localization. For $W = \emptyset$, we have $E^\emptyset(Y) = \text{cofib}(\text{id} : E(Y) \rightarrow E(Y))$, hence we have a canonical contraction of $E^\emptyset(Y)$. Let $W' \subset W \subset Y$ be closed subsets of $Y \in \mathbf{Sm}/B$. The Quetzalcoatl lemma gives us the canonical isomorphism

$$\text{cofib}(i_{W',W} : E^{W'}(Y) \rightarrow E^W(Y)) \cong E^{W \setminus W'}(Y \setminus W')$$

which verifies the localization axiom (4.1.1).

9.2. Nisnevich excision. We need to impose this as an axiom:

(9.2.1)

Axiom II (Nisnevich excision). Let $f : Y \rightarrow X$ be an étale morphism in \mathbf{Sm}/B and $W \subset X$ a closed subset. Let $W' = f^{-1}(W)$ and suppose that $f : W' \rightarrow W$ is an isomorphism of reduced schemes. Then $f^* : E^W(X) \rightarrow E^{W'}(Y)$ is a weak equivalence.

Remarks 9.2.1. (1) The condition (9.2.1) on some $E \in \mathbf{Spt}(B)$ is equivalent to requiring that, for $\phi : E \rightarrow \tilde{E}$ a fibrant model for the Nisnevich-local model structure, the map ϕ is a pointwise weak equivalence. In particular, if E is already fibrant in $\mathbf{Spt}_{\text{Nis}}(B)$, then (9.2.1) is satisfied.

(2) Quillen's localization theorem for G -theory together with the weak equivalence $K(X) \rightarrow G(X)$ for X regular [23, Thm. 5, §5] implies that the K -theory functor $X \mapsto K(X)$ satisfies (9.2.1), at least if the base-scheme B is regular; indeed, the localization theorem yields the weak equivalence $G(W) \rightarrow K^W(X)$, natural with respect to pull-back by smooth morphisms. \square

9.3. Finite fields and \mathbb{P}^1 -spectra. In case the base-scheme B is $\text{Spec } F$ for a finite field F , axiom (4.1.3) is valid in case E is the 0-spectrum of a \mathbb{P}^1 -spectrum; we proceed to explain.

Let $\mathcal{SH}(F)$ denote the stable homotopy category of \mathbb{P}^1 -spectra over F . We will give a quick overview of this construction, see [19, 20, 21] for details.

For $E \in \mathbf{Spt}(F)$, let $E^{\mathbb{A}^1}$ be the presheaf $X \mapsto E(X \times \mathbb{A}^1)$. The projections $X \times \mathbb{A}^1 \rightarrow X$ give the natural map $p_1^* : E \rightarrow E^{\mathbb{A}^1}$. The \mathbb{A}^1 -local model structure on $\mathbf{Spt}(F)$ is formed by taking the Bousfield localization of the Nisnevich model structure $\mathbf{Spt}_{\text{Nis}}(F)$ with respect to the maps $p_1^* : E \rightarrow E^{\mathbb{A}^1}$. That is, define $E \in \mathbf{Spt}(F)$ to be \mathbb{A}^1 -local if $p_1^* : E \rightarrow E^{\mathbb{A}^1}$ is a weak equivalence in $\mathbf{Spt}_{\text{Nis}}(F)$, and define $f : A \rightarrow B$ in $\mathbf{Spt}(F)$ to be an \mathbb{A}^1 -weak equivalence if

$$f^* : \text{Hom}_{\mathcal{H}\mathbf{Spt}_{\text{Nis}}(F)}(B, E) \rightarrow \text{Hom}_{\mathcal{H}\mathbf{Spt}_{\text{Nis}}(F)}(A, E)$$

is an isomorphism for all \mathbb{A}^1 -local E . Letting the cofibrations be the same as those in $\mathbf{Spt}_{\text{Nis}}(F)$, the weak equivalences be the \mathbb{A}^1 -weak equivalences and the fibrations be determined by the RLP with respect to trivial cofibrations defines the model category $\mathbf{Spt}_{S^1}(F)$ with underlying category $\mathbf{Spt}(F)$. We denote the homotopy category of $\mathbf{Spt}_{S^1}(F)$ by $\mathcal{H}_{S^1}(F)$.

One can form the \mathbb{A}^1 -local model structure on $\mathbf{Spc}(F)$ similarly, by a Bousfield localization of the Nisnevich-local model structure $\mathbf{Spc}_{\text{Nis}}(F)$. The resulting homotopy category is denoted $\mathcal{H}_{\mathbb{A}^1}(F)$.

For E a presheaf of spectra on \mathbf{Sm}/F , we have the presheaf $\Omega_T E$ defined by

$$\Omega_T(E)(X) := E^{X \times 0}(X \times \mathbb{A}^1).$$

We define the category $\mathbf{Spt}_T(F)$: An object \mathcal{E} of $\mathbf{Spt}_T(F)$ is a sequence of objects $E_n \in \mathbf{Spt}(F)$, together with maps $\epsilon_n : E_n \rightarrow \Omega_T E_{n+1}$. Maps $\mathcal{E} \rightarrow \mathcal{E}'$ in $\mathbf{Spt}_T(F)$ are collections of maps $E_n \rightarrow E'_n$ in $\mathbf{Spt}(F)$ which intertwine the bonding maps ϵ_* , ϵ'_* .

One can form the *stable \mathbb{A}^1 -homotopy sheaves* $\pi_{a,b}(\mathcal{E})$ by taking the Nisnevich sheaf associated to the presheaf

$$U \mapsto \lim_{n \rightarrow \infty} \pi_{a-2b}((\Omega_T^{n+b} \tilde{E}_n)(U)),$$

where \tilde{E}_n is the functorial fibrant model of E_n in $\mathbf{Spt}_{S^1}(F)$. A weak equivalence $f : \mathcal{E} \rightarrow \mathcal{E}'$ is a map in $\mathbf{Spt}_T(F)$ which induces an isomorphism on the sheaves $\pi_{a,b}$ for all a, b .

For $\mathcal{E} \in \mathbf{Spt}_T(F)$, define the *0-spectrum* \mathcal{E}_0 to be the homotopy colimit of the $\Omega_T^n \tilde{E}_n$ where \tilde{E}_n is a fibrant model for E_n in the \mathbb{A}^1 -local model structure $\mathbf{Spt}_{S^1}(F)$. If the E_n are all fibrant objects in the category $\mathbf{Spt}(F)$ (for the pointwise model structure), all satisfy the Nisnevich excision axiom II, all satisfy homotopy invariance:

$$p^* : E(X) \rightarrow E(X \times \mathbb{A}^1)$$

is a weak equivalence for all X , and all the maps ϵ_n are point-wise weak equivalences, then \mathcal{E}_0 itself is the 0-spectrum of \mathcal{E} (up to pointwise weak equivalence).

In addition, the 0-spectrum \mathcal{E}_0 of a \mathbb{P}^1 -spectrum \mathcal{E} admits weak push-forward maps in the following situation: Take $X \in \mathbf{Sm}/F$ and let $Y \subset X \times_F \mathbb{A}^n$ be a closed subscheme, étale over X . Suppose that the normal bundle N_Y of Y in $X \times_F \mathbb{A}^n$ is trivial; fix an isomorphism $\psi : \mathcal{O}_Y^n \rightarrow N_Y$. Then there is a push-forward morphism

$$f_*^\psi : \mathcal{E}_0(Y) \rightarrow \mathcal{E}_0(X),$$

natural (in \mathcal{SH}) in the triple (X, Y, ψ) .

Remark 9.3.1. Some authors use the \mathbb{G}_m -loops functor $\Omega_{\mathbb{G}_m}$ instead of Ω_T in their definition of the objects in $\mathcal{SH}(F)$, or use a \mathbb{P}^1 -suspension to form bonding maps in a sequence of spaces over F . All these constructions lead to equivalence homotopy categories, because $\mathbb{P}^1 \cong \mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \cong S^1 \wedge \mathbb{G}_m$ in $\mathcal{H}_{\mathbb{A}^1}(F)$. \square

9.4. Norms in finite extensions. For a field F , we have the Grothendieck-Witt ring $\mathrm{GW}(F)$ with augmentation $\dim : \mathrm{GW}(F) \rightarrow \mathbb{Z}$ and augmentation ideal $I(F) := \ker \dim$.

Let $F \rightarrow L$ be a separable field extension of degree n ; let $f : \mathrm{Spec} L \rightarrow \mathrm{Spec} F$ be the map of schemes. We fix an embedding $i : \mathrm{Spec} L \rightarrow \mathbb{A}^1$ and a defining equation g for $i(\mathrm{Spec} L)$. For $X \in \mathbf{Sm}_F$, we thus have the embedding $i_X : X_L \rightarrow X \times \mathbb{A}_F^1$ with defining equation g . The 1-form dg defines a trivialization of the normal bundle of X_L in $X \times \mathbb{A}_F^1$, giving the push-forward map $f_*^g : E(X_L) \rightarrow E(X)$ for E the 0-spectrum of a \mathbb{P}^1 -spectrum \mathcal{E} .

Lemma 9.4.1. *Let F be a field of characteristic $\neq 2$ such that each element $x \in I(F)$ is nilpotent. Suppose that E is the 0-spectrum of a \mathbb{P}^1 -spectrum $\mathcal{E} \in \mathcal{SH}(F)$. Then, for $X \in \mathbf{Sm}_F$, the map $f_*^g f^* : E(X)[1/n] \rightarrow E(X)[1/n]$ is an isomorphism in \mathcal{SH} .*

Proof. For an element $u \in L^\times$, we have the quadratic form $t(u)$ on the F -vector space L defined by

$$(x, y) \mapsto \mathrm{Tr}_{L/F}(uxy).$$

This gives the class $[t(u)]$ in $\mathrm{GW}(F)$. In particular, the defining equation $g(x)$ for $i(\mathrm{Spec} L)$ gives the element $dg/dx \in L^\times$, and the class $[t(dg/dx)] \in \mathrm{GW}(F)$.

\mathcal{E} is canonically a module for the sphere spectrum \mathbb{S} , so it suffices to prove the result for $\mathcal{E} = \mathbb{S}$. Let S_F denote the motivic sphere spectrum over F . By results of Morel [20, 19] there is a ring homomorphism $\rho : \mathrm{GW}(F) \rightarrow \mathrm{Hom}_{\mathcal{SH}(F)}(S_F, S_F)$, and $f_*f^* : S_F \rightarrow S_F$ is given by $\rho([t(dg/dx)])$. Since $t(dg/dx)$ has dimension n , this element is invertible in $\mathrm{GW}(F)[1/n]$ by our assumption on F , hence f_*f^* is invertible in $\mathrm{Hom}_{\mathcal{SH}(F)}(\mathbb{S}, \mathbb{S})[1/n]$. \square

Remark 9.4.2. As pointed out by the referee, the hypothesis on F in Lemma 9.4.1 is just that -1 is a sum of squares in F . See for example [25, Chap. 2, Para. 7, Thm. 7.6]. \square

Corollary 9.4.3. *Let F be a finite field of characteristic $\neq 2$, E the 0-spectrum of some $\mathcal{E} \in \mathcal{SH}(F)$. Then E satisfies axiom (4.1.3).*

Proof. For a finite field F of characteristic $\neq 2$, one has $I(F)^2 = 0$. Let $F \rightarrow L$ be a finite (hence Galois) extension of degree say n . By Lemma 9.4.1, the map $f^* : E(X) \rightarrow E(X_L)$ is injective on homotopy groups after inverting n . After inverting n , the homotopy groups of $E(X_L)^G$ are just the G -invariants in $\pi_*(E(X_L))$, which is the same as the image of f_*f^* (after inverting n). Thus $E \rightarrow \pi_*^G \pi^* E$ is a weak equivalence after inverting n . \square

Remark 9.4.4. As the 0-spectrum \mathcal{E}_0 of a \mathbb{P}^1 -spectrum \mathcal{E} is a filtered homotopy colimit of fibrant objects in $\mathbf{Spt}_{\mathrm{Nis}}(B)$, \mathcal{E}_0 satisfies axiom (9.2.1). Thus, all our functoriality results are available for \mathcal{E}_0 , at least if the finite residue fields of B all have characteristic different from 2. \square

Remark 9.4.5. The codimension q cycles presheaf $z^q \in \mathbf{Spt}(B^{(q)})$ satisfies the axioms (4.1.1), (4.1.2) and (4.1.3). For the localization axiom, let $W' \subset W \subset Y$ be closed subsets of $Y \in \mathbf{Sm}/B$ with $\mathrm{codim}_Y W \geq q$. The sequence (of abelian groups)

$$0 \rightarrow z_{W'}^q(Y) \xrightarrow{i_*} z_W^q(Y) \xrightarrow{j^*} z_{W \setminus W'}^q(Y \setminus W') \rightarrow 0$$

is well-known to be exact, which verifies the localization property for the associated Eilenberg-MacLane spectra.

The Nisnevich excision property follows from the isomorphism

$$f^* : z_W^q(X) \rightarrow z_{W'}^q(Y)$$

for $f : Y \rightarrow X$ étale with $W' = f^{-1}(W)$ and $f : W' \rightarrow W$ an isomorphism.

The usual push-forward of cycles for proper morphisms gives a well-defined push-forward map $f_* : z_W^q(Y) \rightarrow z_{f(W)}^q(X)$ for $f : Y \rightarrow X$ a finite morphism in \mathbf{Sm}/B . If $f : Y \rightarrow X$ has degree n , then f_*f^* is multiplication by n , from which one easily verifies axiom (4.1.3).

The simplicial spectrum $z^q(X, -)$ is the simplicial Eilenberg-MacLane spectrum associated to the simplicial abelian group $z_{ab}^q(X, -)$, whose associated chain complex $z^q(X, *)$ is just Bloch's cycle complex (defined in [2], at least for X quasi-projective over a field). We have as well the equi-dimensional version $z^q(X/B, *)$,

being the chain complex associated to the simplicial abelian group $z_{ab}^q(X/B, -)$, with associated Eilenberg-MacLane spectrum $z^q(X/B, -)$.

Thus, our results are applicable to Bloch's cycle complexes $z^q(X, *)$, even in the case of $X \in \mathbf{Sm}/B$ with B a regular scheme of Krull dimension one, as well as for the equi-dimensional version $z^q(X/B, *)$ over an arbitrary noetherian base B . Since the presheaf $z^q(X_{\text{Nis}}, *)$ satisfies localization (see [3]), this yields, in particular, functorial models for $z^q(X, *)$ on \mathbf{Sm}/k for k a field. Of course, the results of Voevodsky-Suslin-Friedlander [29], extended to positive characteristic by Voevodsky [28], give a functorial model for $z^q(X, *)$ as the Suslin complex of the presheaf with transfers $z_{\text{q.fin.}}(\mathbb{A}^q \times X/X)$, this latter being the presheaf of abelian groups on \mathbf{Sm}/k with $z_{\text{q.fin.}}(\mathbb{A}^q \times X/X)(Y)$ the group of cycles $Z = \sum_i n_i Z_i$ on $\mathbb{A}^q \times X \times Y$ such that each Z_i is quasi-finite and dominant over some component of $X \times Y$. \square

Remark 9.4.6. As we have already remarked, assuming B regular, the K -theory presheaf $X \mapsto K(X)$ on \mathbf{Sm}/B satisfies localization, hence the canonical map $K(X) \rightarrow R\Gamma K(X)$ is a weak equivalence for all $X \in \mathbf{Sm}/B$. Also, for $p : Y \rightarrow X$ a finite étale map of degree n , we have $p_* : K(Y) \rightarrow K(X)$ and $p_* p^*$ induces $\times n$ on the K -groups $K_n(X)$. Thus, our full package of functoriality is available for the simplicial spectra $K^{(q)}(X, -)$ and the equi-dimensional version $K^{(q)}(X/B, -)$. \square

Remark 9.4.7. The referee pointed out that, for $\pi^* : F \rightarrow L$ a Galois extension with group G , the map $E \rightarrow \pi_*^G \pi^* E$ is a weak equivalence after inverting $|G|$ (for a given base-field F , not necessarily finite) if E is the 0-spectrum of some $\mathcal{E} \in \mathcal{SH}(F)$, assuming that the Hopf map $\eta : \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$ is zero after smashing with \mathcal{E} . The referee remarks that, by [22, Theorem 14] and discussion following this result, $[S_F, \text{cone}(\eta)] = \mathbb{Z}$, where S_F is the motivic sphere spectrum, and that it is easy to see that the action of $\pi_* \pi^*$ on $[S_F, \text{cone}(\eta)]$ is by multiplication by $|G|$. In particular, if \mathcal{E} is an MGL-module (in the naive sense, just as an object in $\mathcal{SH}(F)$), then \mathcal{E} has this property. For such \mathcal{E} , we may therefore apply our main result to the 0-spectrum \mathcal{E}_0 , even if some residue field is finite of characteristic 2. \square

10. GOOD COMPACTIFICATIONS

This section is devoted to proving the following result:

Theorem 10.0.1. *Let B be a noetherian separated scheme with infinite residue fields, and let $\pi : X \rightarrow B$ be in \mathbf{Sm}/B , with X equi-dimensional over B . For each point $x \in X$, there is a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (Y, x) & \xrightarrow{f} & (X, x) \\ q \downarrow & & \downarrow \pi \\ (B', q(x)) & \xrightarrow{g} & (B, p(x)) \end{array}$$

and a closed embedding $i : Y \rightarrow \mathbb{A}_{B'}^n$, such that

- (1) $f : (Y, x) \rightarrow (X, x)$ is a Nisnevich neighborhood of x in X , i.e., f is étale.
- (2) g is étale
- (3) $Y \rightarrow B'$ in \mathbf{Sm}/B'
- (4) i satisfies (5.1.1)

In this section, all schemes are assumed to be separated. We fix a noetherian scheme B as base-scheme, and work throughout in the category of B -schemes.

10.1. Families of curves. Let $d = \dim_B X$. The first step is to compactify a neighborhood of (X, x) to form a family of curves over \mathbb{P}_B^{d-1} . We then try to find a good line bundle on the compactification which will contract undesirable components. We begin by reducing to the case in which the fiber $X_{p(x)}$ is geometrically irreducible.

Lemma 10.1.1. *Let $\pi : X \rightarrow B$ be in \mathbf{Sm}/B , $x \in X$. Then there is a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (Y, x) & \xrightarrow{f} & (X, x) \\ q \downarrow & & \downarrow \pi \\ (B', b') & \xrightarrow{g} & (B, b) \end{array}$$

such that

- (1) f and g are étale
- (2) $Y \rightarrow B'$ in \mathbf{Sm}/B'
- (3) The fiber $Y_{b'}$ is geometrically irreducible.

Proof. In general, since we are allowed to pass to a Zariski open neighborhood of x , we may assume that B is local, $B = \text{Spec } \mathcal{O}$, and that $p(x)$ is the closed point b of B .

Let L be the algebraic closure of $k(b)$ in $k(X_b)$. Then $X_b \rightarrow b$ factors through $X_b \rightarrow \text{Spec } L$, and X_b is geometrically irreducible over L . Now let $B' \rightarrow B$ be a local étale cover with closed point b' having $k(b') = L$. Then $X \times_B B' \rightarrow B'$ has fiber $X_b \times_{k(b)} L$ over b' , hence contains X_b as a connected component and thus contains x as a closed point. Therefore, the projection $X \times_B B' \rightarrow X$ defines a Nisnevich neighborhood of (X, x) . Removing the complement of X_b in $X_b \times_{k(b)} L$ from $X \times_B B'$ and letting $Y \rightarrow B'$ be the resulting open subscheme completes the construction. \square

We therefore assume that $B = \text{Spec } \mathcal{O}$, \mathcal{O} noetherian and local, with closed point $b = \pi(x)$, and that the fiber X_b is geometrically irreducible. We may also assume that X is connected.

For a commutative ring R and an ideal $I \subset R$, we let $V(I)$ denote the closed subscheme $\text{Spec } R/I$ of $\text{Spec } R$. Similarly, if S is a graded R algebra and $J \subset S$ a homogeneous ideal, we let $V_h(J)$ denote the closed subscheme $\text{Proj}(S/J)$ of $\text{Proj}(S)$. For a coherent sheaf \mathcal{F} on a scheme S , we have the sheaf of symmetric algebras $\text{Sym}_{\mathcal{O}_S}^* \mathcal{F}$ and the projective S -scheme $\mathbb{P}_S(\mathcal{F}) := \text{Proj}_S(\text{Sym}_{\mathcal{O}_S}^* \mathcal{F})$.

Definition 10.1.2. Let T be a B -scheme of finite type, $Y \rightarrow T$ a T -scheme. We call Y a *hypersurface* over T if Y is flat over T and there is a locally free coherent sheaf \mathcal{E} on T , a closed embedding $Y \subset \mathbb{P}_T(\mathcal{E})$ and integer d such that each $t \in T$ has an open neighborhood $j : U \rightarrow T$ such that $j^*Y \subset \mathbb{P}_U(j^*\mathcal{E})$ is the subscheme defined by a non-zero section of $\mathcal{O}_{\mathcal{E}}(d)$ over $\mathbb{P}_U(j^*\mathcal{E})$. \square

Remark 10.1.3. A closed subscheme Y of $P := \mathbb{P}_U(j^*\mathcal{E})$ defined by a section $F \neq 0$ of $\mathcal{O}_{\mathcal{E}}(d)$ over P is flat over U if and only if for each point $u \in U$, the restriction i_u^*F to P_u is non-zero. Indeed, $\text{Tor}_i^{\mathcal{O}_U}(\mathcal{O}_Y, k(u))$ is the i th homology sheaf of the complex

$$\mathcal{O}_{P_u}(-d) \xrightarrow{\times i_u^*F} \mathcal{O}_{P_u}.$$

□

Lemma 10.1.4. *There is a Zariski open neighborhood (U, x) of (X, x) , a projective B -scheme \bar{U} , an open immersion $j : U \rightarrow \bar{U}$ and projective morphisms $p : \bar{U} \rightarrow \mathbb{P}_B^{d-1}$, $q : \bar{U} \rightarrow \mathbb{P}_B^{d+1}$ such that*

- (1) *p has fibers of dimension one.*
- (2) *\bar{U} is a hypersurface over \mathbb{P}_B^{d-1} .*
- (3) *For each $y \in \mathbb{P}_B^{d-1}$, the map $q_y : p^{-1}(y) \rightarrow \mathbb{P}_{k(y)}^{d+1}$ is a closed embedding.*
- (4) *$p|_U : U \rightarrow \mathbb{P}_B^{d-1}$ is smooth.*
- (5) *Let $y = p(x)$. Then U_y is geometrically irreducible.*

Proof. The construction is more or less standard: We first show that there is an open neighborhood of x in X which is isomorphic to an open subscheme of a hypersurface $V(F)$ in \mathbb{A}_B^{d+1} , flat over B , i.e. F is in $\mathcal{O}[X_1, \dots, X_{d+1}]$ and F is not zero modulo the maximal ideal \mathfrak{m} of \mathcal{O} .

For this, it suffices to replace x with any closed point x_0 of X in the closure of x . Thus, we may temporarily assume that x is a closed point of X . Let $\mathfrak{m}_{X,x}$ denote the maximal ideal in $\mathcal{O}_{X,x}$. As the fiber X_b is smooth of dimension d over $k(b)$, we can find d elements $\bar{f}_1, \dots, \bar{f}_d \in \mathfrak{m}_{X_b,x}$ which generate $\mathfrak{m}_{X_b,x}$ and form a regular sequence in $\mathfrak{m}_{X_b,x}$. Lifting the \bar{f}_j to elements $f_j \in \mathfrak{m}_{X,x}$, we have the regular sequence f_1, \dots, f_d . As the morphism

$$\bar{f} := (\bar{f}_1, \dots, \bar{f}_d) : \text{Spec } \mathcal{O}_{X_b,x} \rightarrow \mathbb{A}_{k(b)}^d$$

is étale, the morphism

$$f := (f_1, \dots, f_d) : \text{Spec } \mathcal{O}_{X,x} \rightarrow \mathbb{A}_B^d$$

is also étale. Note that $f(x)$ is the origin $0_b \in \mathbb{A}_b^d$.

By the presentation theorem for étale morphisms, there are elements $F, G \in \mathcal{O}_{\mathbb{A}^d, 0_b}[X_{d+1}]$ such that the localization $(\mathcal{O}_{\mathbb{A}^d, 0_b}[X_{d+1}]/F)_G$ is étale over $\mathcal{O}_{\mathbb{A}^d, 0_b}$, and a further localization of $(\mathcal{O}_{\mathbb{A}^d, 0_b}[X_{d+1}]/F)_G$ at some closed point is isomorphic to $\mathcal{O}_{X,x}$ (as an algebra over $\mathcal{O}_{\mathbb{A}^d, 0_b}$). We may assume that F and G are actually in $\mathcal{O}[X_1, \dots, X_{d+1}]$ and that the open subscheme $G \neq 0$ of the hypersurface $V(F)$ in \mathbb{A}_B^{d+1} is isomorphic to an open neighborhood of x in X . Since X_b has dimension d over $k(b)$, it follows that F is not in $\mathfrak{m}[X_1, \dots, X_{d+1}]$, so the hypersurface $F = 0$ is flat over B . Changing notation, we may assume that $X = V(F) \setminus V(G)$.

Let $\bar{F} \in \mathcal{O}[X_0, X_1, \dots, X_{d+1}]$ be the homogenization of F and $\bar{X} \subset \mathbb{P}_B^{d+1}$ the subscheme defined by \bar{F} . Clearly \bar{X} is a hypersurface over B which contains X as an open subscheme. We extend $\pi : X \rightarrow B$ to $\pi : \bar{X} \rightarrow B$. Let $\mathbb{P}_\infty^d \subset \mathbb{P}_B^{d+1}$ be the hyperplane in defined by $X_0 = 0$.

Let $P \subset \mathbb{P}_B^{d+1}$ be a general \mathbb{P}^2 containing x . P will intersect \bar{X}_b transversely at x ; by Bertini's theorem [14], the intersection $P \cap X_b$ is geometrically irreducible for all sufficiently general P . For such a P , let $\ell_b := P \cap \mathbb{P}_\infty^d$. We may choose P so that $\ell_b \cap \bar{X}_b$ has dimension 0; let $\ell \subset \mathbb{P}_\infty^d$ be any \mathbb{P}_B^1 with fiber ℓ_b over b . Then $\ell \cap \bar{X} \rightarrow B$ is finite. For $z \in \bar{X} \setminus \ell$, let P_z be the \mathbb{P}^2 spanned by z and $\ell_{\pi(z)}$.

Let $\mu : \tilde{\mathbb{P}}_B^{d+1} \rightarrow \mathbb{P}_B^{d+1}$ be the blow-up of \mathbb{P}_B^{d+1} along ℓ . The projection $\mathbb{P}_B^{d+1} \setminus \ell \rightarrow \mathbb{P}_B^{d-1}$ with center ℓ defines a morphism $\Pi : \tilde{\mathbb{P}}_B^{d+1} \rightarrow \mathbb{P}_B^{d-1}$ making $\tilde{\mathbb{P}}_B^{d+1}$ into a Zariski locally trivial \mathbb{P}^2 -bundle over \mathbb{P}_B^{d-1} , so $\tilde{\mathbb{P}}_B^{d+1} = \text{Proj}_{\mathbb{P}_B^{d-1}}(\mathcal{E})$ for some locally free coherent sheaf \mathcal{E} of rank three on \mathbb{P}_B^{d-1} .

Let $\tilde{X} := \mu^{-1}(\bar{X})$, and let $p : \tilde{X} \rightarrow \mathbb{P}_B^{d-1}$ be the restriction of Π . Since $\ell \cap \bar{X} \rightarrow B$ is finite, p is equi-dimensional with fibers of dimension one. Also, $\bar{X} \subset \mathbb{P}_B^{d+1}$ is a hypersurface over B , so $\tilde{X} \subset \tilde{\mathbb{P}}_B^{d+1}$ is a Cartier divisor on $\tilde{\mathbb{P}}_B^{d+1}$. As p has fiber dimension one, this implies that $\tilde{X} \subset \tilde{\mathbb{P}}_B^{d+1}$ is a hypersurface over \mathbb{P}_B^{d-1} .

By construction, $p^{-1}(p(z)) = P_z \cap \tilde{X}$ for all $z \in \tilde{X} \setminus \ell$. Thus, the fiber $p^{-1}(p(x))$ admits an open neighborhood of x which is smooth over $k(p(x))$ and is geometrically irreducible. Since p is flat, there is a neighborhood U of $x \in \tilde{X}$ which is smooth over \mathbb{P}_B^{d-1} , and with fiber $U_{p(x)}$ geometrically irreducible.

Also, the identity $p^{-1}(p(z)) = P_z \cap \tilde{X}$ implies that the projection $\mu : p^{-1}(y) \rightarrow \mathbb{P}_B^{d+1}$ is a closed embedding for all $y \in \mathbb{P}_B^{d-1}$. We may therefore take $\bar{U} := \tilde{X}$ and q the restriction of μ , which completes the proof. \square

The next step is to construct a “numerically contracting” invertible sheaf on \bar{U} . For a B -morphism $T \rightarrow \mathbb{P}_B^{d-1}$, we denote the fiber product $\bar{U} \times_{\mathbb{P}_B^{d-1}} T$ by \bar{U}_T , and similarly use U_T to denote $U \times_{\mathbb{P}_B^{d-1}} T$.

Lemma 10.1.5. *Let T be a noetherian scheme. Let $p : \bar{U} \rightarrow T$ be a hypersurface over T with fibers of dimension one, 0 a point of T , $x \in \bar{U}_0$. Suppose*

- (a) *There is an open neighborhood U of x such that $p : U \rightarrow T$ is smooth.*
- (b) *There is a Cartier divisor $\mathcal{D} \subset U$ such that $p : \mathcal{D} \rightarrow T$ is finite.*
- (c) *The fiber U_0 is geometrically irreducible.*

Then there is a neighborhood $V \subset U$ of x and an open neighborhood T_0 of $0 \in T$ such that

- (1) $p(V) = T_0$,
- (2) $\mathcal{D} \cap p^{-1}(T_0) \subset V$,
- (3) *the fibers of $p : V \rightarrow T$ are geometrically irreducible.*

Moreover, there is a closed subset $\bar{U}^\emptyset \subset \bar{U} \setminus V$ so that the fiber \bar{U}_t^\emptyset is a union of irreducible components of \bar{U}_t and $\bar{U}_t = \bar{U}_t^\emptyset \cup \bar{V}_t$ for all $t \in T_0$, where \bar{V}_t is the closure of V_t in \bar{U} .

Proof. We may replace T with any open neighborhood of 0 . Since \bar{U} is a hypersurface over T of relative dimension one, we may assume that \bar{U} is a closed subscheme of \mathbb{P}_T^2 , defined by an $F \in H^0(\mathbb{P}_T^2, \mathcal{O}(d))$ with the property that $i_t^* F \neq 0$ for all points $i_t : t \rightarrow T$ of T .

Let d_1, \dots, d_r be positive integers with $\sum_j d_j = d$, and let T_{d_1, \dots, d_r} be the subset of T consisting of those points $t \in T$ such that $i_t^* F$ factors in $\overline{k(t)}[X_0, X_1, X_2]$ as

$$i_t^* F = \prod_j F_{t,j},$$

with $\deg F_{t,j} = d_j$. Then T_{d_1, \dots, d_r} is a closed subset of T , as follows from the fact that the set of $F \in \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d)))$ which admit such a factorization is the closed subset given as the image of the multiplication map

$$\prod_j \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d_j))) \rightarrow \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d))).$$

Similarly, the set of t for which $i_t^* F$ admits such a factorization with the $F_{t,j}$ irreducible in $\overline{k(t)}[X_0, X_1, X_2]$ is an open subset $T_{d_1, \dots, d_r}^{\text{irr}}$ of T_{d_1, \dots, d_r} . Clearly the $T_{d_1, \dots, d_r}^{\text{irr}}$ form a stratification of T by locally closed subsets.

Let $V \subset U$ be an open subscheme. Then the set $\text{Red}(V)$ of $t \in T$ such that the fiber V_t is not geometrically irreducible is a constructible subset of T . This follows from the remark above, noticing that t is not in $\text{Red}(V) \cap T_{d_1, \dots, d_r}^{\text{irr}}$ if and only if $V(F_{t,j}) \subset \bar{U} \setminus V$ for all but at most one index j , so $\text{Red}(V) \cap T_{d_1, \dots, d_r}^{\text{irr}}$ is open in $T_{d_1, \dots, d_r}^{\text{irr}}$.

Now let W be an irreducible closed subset of T containing 0. Let w be the generic point of W . Denote the geometric fiber of $\bar{U} \rightarrow T$ over w by \bar{U}_w . We claim that there is a unique irreducible component $\bar{U}_w^{\mathcal{D}}$ which has non-empty intersection with \mathcal{D} .

Indeed, note that the hypotheses on $\bar{U} \rightarrow T$, $U \subset \bar{U}$ and \mathcal{D} are preserved under arbitrary base-change $(T', 0') \rightarrow (T, 0)$. Thus, we may assume that each irreducible component of \bar{U}_w is geometrically irreducible and similarly for \bar{U}_0 . By assumption, there is a unique irreducible component $\bar{U}_0^{\mathcal{D}}$ of \bar{U}_0 with non-empty intersection with \mathcal{D} , namely, the closure of U_0 . Suppose are two components, \bar{U}_w^1 and \bar{U}_w^2 , of \bar{U}_w which intersect \mathcal{D} . Extend the specialization $w \rightarrow 0$ to a specialization of cycles $\bar{U}_w^i \rightarrow Z^i \subset \bar{U}_0$, $i = 1, 2$. Then both Z^1 and Z^2 intersect \mathcal{D} , so both Z^1 and Z^2 contain $\bar{U}_0^{\mathcal{D}}$. Thus the cycle \bar{U}_w specializes over $w \rightarrow 0$ to a cycle Z supported in \bar{U}_0 , containing $\bar{U}_0^{\mathcal{D}}$ with multiplicity at least two. This contradicts the smoothness of $U \rightarrow T$.

Since the irreducible component $\bar{U}_w^{\mathcal{D}}$ is unique, this component must be the base-extension to $k(\bar{w})$ of a unique irreducible component $\bar{U}_w^{\mathcal{D}}$ of \bar{U}_w . Thus there is a unique irreducible component $\bar{U}_w^{\mathcal{D}}$ of \bar{U}_w intersecting \mathcal{D} , and $\bar{U}_w^{\mathcal{D}}$ is geometrically irreducible. Let \bar{U}_w^{\emptyset} be the union of the other irreducible components of \bar{U}_w , and let \bar{U}_W^{\emptyset} be the closure of \bar{U}_w^{\emptyset} in \bar{U} .

Let \bar{U}^{\emptyset} be the union of the \bar{U}_W^{\emptyset} , where W runs over irreducible closed subsets of T containing 0. We claim that \bar{U}^{\emptyset} is a closed subset of \bar{U} . To see this, take such a W and let w be the generic point. There is a unique sequence $d_1 \geq \dots \geq d_r > 0$ with $\sum_j d_j = d$ and $w \in T_{d_1, \dots, d_r}^{\text{irr}}$. There is thus an irreducible closed subset W' of T containing W such that W' is the closure of an irreducible component of $T_{d_1, \dots, d_r}^{\text{irr}}$; it suffices to show that $\bar{U}_W^{\emptyset} \subset \bar{U}_{W'}^{\emptyset}$. This is clear, since our condition implies that \bar{U}_w and $\bar{U}_{w'}$ have r irreducible components (counted with multiplicity), hence \bar{U}_w^{\emptyset} is in the closure of $\bar{U}_{w'}^{\emptyset}$.

Let $V_1 = U \setminus \bar{U}^{\emptyset}$. By our construction, the closure $\overline{\text{Red}(V_1)}$ does not contain 0. Thus, taking $V = V_1 \setminus p^{-1}(\overline{\text{Red}(V_1)})$, $T_0 = T \setminus \overline{\text{Red}(V_1)}$ proves the result. \square

We suppose we have $U, p : \bar{U} \rightarrow \mathbb{P}_B^{d-1}$ and $q : \bar{U} \rightarrow \mathbb{P}_B^{d-1}$ as in Lemma 10.1.4. For a B -morphism $T \rightarrow \mathbb{P}_B^{d-1}$, and a point $t \in T$ lying over $p(x)$, we let C_t denote the closure of the geometrically irreducible curve V_t in \bar{U}_t .

Lemma 10.1.6. *Let $j : U \rightarrow \bar{U}$, $p : \bar{U} \rightarrow \mathbb{P}_B^{d-1}$, $x \in U$ and $q : \bar{U} \rightarrow \mathbb{P}_B^{d+1}$ be as in Lemma 10.1.4. There is a Nisnevich neighborhood $(T, 0) \rightarrow (\mathbb{P}_B^{d-1}, p(x))$ and an effective Cartier divisor $\mathcal{D} \subset \bar{U}_T$ such that*

- (1) T is affine and $\mathcal{D} \rightarrow T$ is finite and étale
- (2) \mathcal{D} is supported in $U_T \subset \bar{U}_T$.
- (3) Let $\mathcal{L} := \mathcal{O}_{\bar{U}}(\mathcal{D})$. Then $\mathcal{L} \otimes \mathcal{O}_{C_0} \cong \mathcal{O}_{C_0}(1)$.

Proof. Let H be a general hyperplane in \mathbb{P}_B^{d+1} and let $D := q^{-1}(H)$. Choosing H sufficiently general, we may assume that $D \cap C_{p(x)}$ is contained in U and that the

intersection is transverse. We may also assume that D intersects the entire fiber $\bar{U}_{p(x)}$ properly.

Thus there is a neighborhood U_0 of $p(x)$ in \mathbb{P}_B^{d-1} such that $D \cap \bar{U}_y$ is finite for all $y \in U_0$. Let $D_0 := D \cap p^{-1}(U_0)$, giving us the finite U_0 -scheme $p : D_0 \rightarrow U_0$.

Let \mathcal{O} be the strict Henselization $\mathcal{O}_{U_0, p(x)}^h$. Since $D \cap C_{p(x)}$ is contained in U and the intersection is transverse, it follows that $D_0 \times_{U_0} \text{Spec}(\mathcal{O})$ breaks up as a disjoint union

$$D_0 \times_{U_0} \text{Spec}(\mathcal{O}) = \coprod_{w \in D \cap C_{p(x)}} \text{Spec}(\mathcal{O}) \coprod \mathcal{D}''$$

where $\mathcal{D}'' \cap C_{p(x)} = \emptyset$ and the $\text{Spec}(\mathcal{O})$ indexed by $w \in D \cap C_{p(x)}$ is the unique component Y of $D_0 \times_{U_0} \text{Spec}(\mathcal{O})$ with $Y \cap C_{p(x)} = w$. We label this component as $\text{Spec}(\mathcal{O})_w$.

Let $e = \deg_{\mathbb{P}^{d+1}}(q(C_{p(x)}))$. We thus have the section

$$s_1 : \text{Spec}(\mathcal{O}) \rightarrow S^e(U/\mathbb{P}_B^{d-1})$$

with $s_1(y) = \sum_w p^{-1}(y) \cap \text{Spec}(\mathcal{O})_w$. Here $S^e(U/\mathbb{P}_B^{d-1})$ is the relative symmetric product:

$$S^e(U/\mathbb{P}_B^{d-1}) := U \times_{\mathbb{P}_B^{d-1}} \dots \times_{\mathbb{P}_B^{d-1}} U / \Sigma_e.$$

The zero-cycle $D \cap C_{p(x)}$ is defined over $k(p(x))$ and supported in the smooth locus U , so the section s descends to a section on the Henselization $\mathcal{O}_{U_0, p(x)}^h$:

$$s_0 : \text{Spec}(\mathcal{O}_{U_0, p(x)}^h) \rightarrow S^e(U/\mathbb{P}_B^{d-1}).$$

Since U is of finite type, there is a Nisnevich neighborhood $(T, 0) \rightarrow (U_0, p(x))$ such that s_0 descends to a section

$$s : T \rightarrow S^e(U/\mathbb{P}_B^{d-1}).$$

Thus the Cartier divisor $D_0 \times_{U_0} T \subset U \times_{U_0} T$ has an open and closed subscheme \mathcal{D} such that we have the identities of cycles

$$\mathcal{D} \cap U_t = s(t)$$

for all $t \in T$. Since D_0 is finite over U_0 , $\mathcal{D} \rightarrow T$ is finite.

Since

$$\mathcal{D} \times_T \text{Spec}(\mathcal{O}) = \coprod_{w \in D \cap C_{p(x)}} \text{Spec}(\mathcal{O})_w,$$

$\mathcal{D} \rightarrow T$ is étale over a neighborhood of 0; shrinking T , we may assume that T is affine and $\mathcal{D} \rightarrow T$ is étale, completing the proof. \square

Lemma 10.1.7. *We retain the notation from Lemma 10.1.6. There is an integer m_0 such that $\mathcal{L}^{\otimes m}$ is generated by global sections for all $m \geq m_0$.*

Proof. The base-scheme T is affine, so write $T = \text{Spec} A$.

Let $s \in H^0(\bar{U}_T, \mathcal{L})$ be the canonical section of $\mathcal{L} = \mathcal{O}_{\bar{U}_T}(\mathcal{D})$ with divisor \mathcal{D} , and let $m > 0$ be an integer. We have the standard sheaf sequence

$$0 \rightarrow \mathcal{L}^{\otimes m-1} \xrightarrow{\times s} \mathcal{L}^{\otimes m} \xrightarrow{i_{\mathcal{D}}^*} \mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{D}} \rightarrow 0$$

which yields the long exact cohomology sequence

$$(10.1.1) \quad 0 \rightarrow H^0(\bar{U}_T, \mathcal{L}^{\otimes m-1}) \rightarrow H^0(\bar{U}_T, \mathcal{L}^{\otimes m}) \rightarrow \\ H^0(\mathcal{D}, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{D}}) \rightarrow H^1(\bar{U}_T, \mathcal{L}^{\otimes m-1}) \rightarrow \\ H^1(\bar{U}_T, \mathcal{L}^{\otimes m}) \rightarrow H^1(\mathcal{D}, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{D}}) \rightarrow \dots$$

Since $\mathcal{D} \rightarrow T$ is finite and T is affine, \mathcal{D} is affine as well. Thus $H^1(\mathcal{D}, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{D}}) = 0$ for all $m \geq 0$, and we therefore have the tower of surjections

$$\dots \rightarrow H^1(\bar{U}_T, \mathcal{L}^{\otimes m-1}) \rightarrow H^1(\bar{U}_T, \mathcal{L}^{\otimes m}) \rightarrow \dots$$

Since $\bar{U}_T \rightarrow T$ is projective, the cohomology groups $H^1(\bar{U}_T, \mathcal{L}^{\otimes m})$ are finitely generated A -modules for all m , and hence this tower is eventually constant, say

$$H^1(\bar{U}_T, \mathcal{L}^{\otimes m-1}) \xrightarrow{\sim} H^1(\bar{U}_T, \mathcal{L}^{\otimes m})$$

for all $m \geq m_0$.

Looking back at our sequence (10.1.1), we thus have the surjection

$$H^0(\bar{U}_T, \mathcal{L}^{\otimes m}) \rightarrow H^0(\mathcal{D}, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{D}})$$

for $m \geq m_0$. Since \mathcal{D} is affine, $\mathcal{L}^{\otimes m} \otimes \mathcal{O}_{\mathcal{D}}$ is globally generated for all $m \geq 0$, hence $\mathcal{L}_x^{\otimes m}$ is generated by global sections of $\mathcal{L}^{\otimes m}$ for all $x \in \mathcal{D}$ and $m \geq m_0$. Since $s^{\otimes m}$ generates $\mathcal{L}_x^{\otimes m}$ for all $x \notin \mathcal{D}$, it follows that $\mathcal{L}^{\otimes m}$ is generated by global sections for all $m \geq m_0$, as desired. \square

10.2. Finishing the proof.

Proposition 10.2.1. *Let $q : (X, x) \rightarrow (B, b)$ be smooth. Then there is a Nisnevich neighborhood $(X_1, x) \rightarrow (X, x)$, a B -scheme \bar{X}_1 , an open immersion $j : X_1 \rightarrow \bar{X}_1$, a smooth B -scheme $T \rightarrow B$ and a projective B -morphism $p : (\bar{X}_1, x) \rightarrow (T, 0)$ such that*

- (1) p has geometrically irreducible fibers of dimension one.
- (2) $p \circ j$ is smooth.

Furthermore, \bar{X}_1 admits a finite T -morphism $\bar{X}_1 \rightarrow T \times_B \mathbb{P}^1$.

Proof. By Lemma 10.1.1, we may assume that the fiber X_b is geometrically irreducible. Take U , $q : \bar{U} \rightarrow \mathbb{P}_B^{d+1}$, $(T, 0)$, $\pi : (\bar{U}_T, x) \rightarrow (T, 0)$, V and \mathcal{L} as given by Lemma 10.1.6. By Lemma 10.1.7, the sheaf $\mathcal{L}^{\otimes m}$ is generated by global sections for some $m > 0$. As $\bar{U}_T \rightarrow T$ has fibers of dimension one, we may find two global sections s_0, s_1 of $\mathcal{L}^{\otimes m}$ which generate $\mathcal{L}^{\otimes m}$, after shrinking T . This yields the T -morphism

$$f := (s_0 : s_1) : \bar{U} \rightarrow T \times_B \mathbb{P}^1.$$

Since $\mathcal{L}^{\otimes m} = \mathcal{O}_{\bar{U}}(m\mathcal{D})$, we may assume that s_1 is the canonical section with divisor $m\mathcal{D}$.

Let

$$\begin{array}{ccc} \bar{U} & \xrightarrow{g} & \bar{X}_1 \\ & \searrow f & \downarrow h \\ & & T \times_B \mathbb{P}^1 \end{array}$$

be the Stein factorization of f (cf. [9, III, Corollary 11.5]).

By Lemma 10.1.5, we may shrink T and find an open neighborhood V of x in U_T containing \mathcal{D} such the fibers of $V \rightarrow T$ are geometrically irreducible. For $t \in T$, we

let C_t denote the closure in \bar{U}_t of the geometrically irreducible curve V_t . We let C_t^\emptyset denote the union of the remaining components of \bar{U}_t . By Lemma 10.1.5, $\cup_{t \in T} C_t^\emptyset$ forms a closed subset \bar{U}_T^\emptyset of \bar{U}_T with \mathcal{D} is supported in $\bar{U}_T \setminus \bar{U}_T^\emptyset$.

Take $t \in T$. Since $\mathcal{L}^{\otimes m} = \mathcal{O}_{\bar{U}}(m\mathcal{D})$ and \mathcal{D} is supported in $\bar{U}_T \setminus \bar{U}_T^\emptyset$, $f(C_t^\emptyset)$ is a finite subset of $t \times \mathbb{P}^1$, and so $f(\bar{U}_T^\emptyset)$ is a closed subset W of $T \times_B \mathbb{P}^1$, finite over T . On the other hand, \mathcal{L} restricted to C_t has positive degree, so the restriction of f to $f_t : C_t \rightarrow t \times \mathbb{P}^1$ is finite. Furthermore $f(\mathcal{D}) \subset T \times (1 : 0)$. Since $C_t^\emptyset \cap \mathcal{D} = \emptyset$, s_1 is nowhere zero on C_t^\emptyset . Thus $f(C_t^\emptyset)$ does not contain $t \times (1 : 0)$, hence $f(\mathcal{D}) \cap W = \emptyset$. Shrinking V if necessary, we may assume that $V \subset \bar{U} \setminus f^{-1}(W)$.

It follows that the restriction of f to

$$\tilde{f} : \bar{U} \setminus f^{-1}(W) \rightarrow T \times_B \mathbb{P}^1 \setminus W$$

is finite. By the universal property of the Stein factorization, the restriction of g to $\bar{U} \setminus f^{-1}(W)$ is an open immersion, and thus the restriction of g to V is an open immersion. We have the morphism $p_1 \circ h : \bar{X}_1 \rightarrow T$.

Since $p : \bar{U} \rightarrow T$ is a hypersurface, the fibers of p are all connected. Since g is surjective, it follows that $C_t \rightarrow \bar{X}_{1t}$ is surjective for all $t \in T$. Since C_t is geometrically irreducible for all t , it follows that the fibers of $p_1 \circ h$ are geometrically irreducible. We take $X_1 := g(V) \subset \bar{X}_1$; since $g : V \rightarrow \bar{X}_1$ is an open immersion and $V \rightarrow T$ is smooth, $X_1 \rightarrow T$ is smooth. Since $(V, x) \rightarrow (X, x)$ is a Nisnevich neighborhood of (X, x) , the proof is complete. \square

We now proceed by induction.

Theorem 10.2.2. *Let B be a noetherian scheme with infinite residue fields, let $\pi : X \rightarrow B$ be in \mathbf{Sm}/B of relative dimension d , x a point of X . Then there is a Nisnevich neighborhood $(Y, x) \rightarrow (X, x)$ of x in X , a local, étale B -scheme $(B', 0) \rightarrow (B, \pi(x))$, a projective B' -scheme $q : \bar{Y} \rightarrow B'$ and an open immersion $j : Y \rightarrow \bar{Y}$ such that*

- (1) \bar{Y} has geometrically irreducible fibers of dimension d .
- (2) $j(Y)$ is dense in \bar{Y} .
- (3) $q(j(x)) = 0$.

Proof. We may assume from the start that B is local and that x lies over the closed point b of B . If $d = 0$, then $X \rightarrow B$ is étale, so we just take $B' = Y = \bar{Y} = X$. In general, assume the result for $X' \rightarrow B$ smooth of dimension $< d$. By Proposition 10.2.1, we have a Nisnevich neighborhood $(X_1, x) \rightarrow (X, x)$, a dense open immersion $j_1 : X_1 \rightarrow \bar{X}_1$, a smooth B -scheme $V \rightarrow B$ and a finite morphism $f : \bar{X}_1 \rightarrow V \times \mathbb{P}^1$ such that $p_1 \circ f$ has geometrically irreducible fibers of dimension one and $p_1 \circ f \circ j_1$ is smooth. Let $0 = p_1 \circ f \circ j_1(x)$.

We apply our induction hypothesis to $(V, 0) \rightarrow (B, b)$, giving the Nisnevich neighborhood $(V', 0) \rightarrow (V, 0)$, the dense open immersion $j' : V' \rightarrow \bar{V}'$ and the projective morphism $\pi' : \bar{V}' \rightarrow B'$ with geometrically irreducible fibers of dimension $d-1$. Taking a base-change with respect to $V' \rightarrow V$, we may simply replace V with V' and change notation, so we may assume that $V = V'$. In particular $V \rightarrow B'$ has geometrically irreducible fibers of dimension $d-1$ and so $\bar{X}_1 \rightarrow B'$ has geometrically irreducible fibers of dimension d .

Since f is finite, f is projective, so there is a factorization of f as a closed embedding $i : \bar{X}_1 \rightarrow V \times \mathbb{P}^1 \times \mathbb{P}^N$, followed by the projection $V \times \mathbb{P}^1 \times \mathbb{P}^N \rightarrow V \times \mathbb{P}^1$.

Let \bar{X}_1^c be the closure of $i(\bar{X}_1)$ in $\bar{V} \times \mathbb{P}^1 \times \mathbb{P}^N$, giving us the open immersion $j_1^c : \bar{X}_1 \rightarrow \bar{X}_1^c$ and the projective morphism $p_{12} : \bar{X}_1^c \rightarrow \bar{V} \times \mathbb{P}^1$.

Form the Stein factorization of p_{12} :

$$\begin{array}{ccc} \bar{X}_1^c & \xrightarrow{q} & \bar{Y} \\ & \searrow p_{12} & \downarrow h \\ & & \bar{V} \times \mathbb{P}^1 \end{array}$$

Since $\bar{X}_1 \rightarrow \bar{V} \times \mathbb{P}^1$ is finite, and $\bar{X}_1 = \bar{X}_1^c \times_{\bar{V}} V$, it follows that the composition

$$\bar{X}_1 \rightarrow \bar{X}_1^c \rightarrow \bar{Y}$$

is an open immersion with dense image; we identify \bar{X}_1 with its image in \bar{Y} . Let $\bar{Y} \rightarrow B'$ be the composition

$$\bar{Y} \rightarrow \bar{V} \times \mathbb{P}^1 \rightarrow \bar{V} \rightarrow B'.$$

We claim that $\bar{Y} \rightarrow B'$ has geometrically irreducible fibers.

Indeed, it is easy to see that each irreducible component of \bar{Y} dominates some irreducible component of B' , and that the fiber of $\bar{Y} \rightarrow B'$ over a generic point $\eta \in B'$ has pure dimension d over $k(\eta)$. Take $b' \in B'$. By the upper-semi-continuity of the fiber dimension, each irreducible component of $\bar{Y}_{b'}$ has dimension $\geq d$ over $k(b')$. Since $\bar{Y}_{b'} \rightarrow \bar{V}_{b'} \times \mathbb{P}^1$ is finite, each geometrically irreducible component P of $\bar{Y}_{b'}$ must have

$$\dim_{\overline{k(b')}} P = d.$$

Set $\bar{b}' := \text{Spec } \overline{k(b')}$.

$\bar{V}_{\bar{b}'}$ is irreducible of dimension $d - 1$. Thus $P \rightarrow \bar{V}_{\bar{b}'} \times \mathbb{P}^1$ is surjective and the fibers of $P \rightarrow \bar{V}_{\bar{b}'}$ are pure dimension one. The fibers of $\bar{Y} \rightarrow \bar{V}$ agree with the fibers of $\bar{X}_1 \rightarrow \bar{V}$ over V . Thus, if η is the generic point of $\bar{V}_{\bar{b}'}$, we have $P_\eta = (\bar{X}_1)_\eta$, and hence $(\bar{X}_1)_{\bar{b}'}$ is dense in P . Since P was arbitrary, $\bar{Y}_{\bar{b}'}$ is geometrically irreducible. Taking $Y := X_1$ completes the proof. \square

Proof of Theorem 10.0.1. Take $(Y, x) \rightarrow (X, x)$, $j : Y \rightarrow \bar{Y}$ and $\bar{Y} \rightarrow B' \rightarrow B$ as given by Theorem 10.2.2.

$\bar{Y} \rightarrow B'$ is projective, so there is a closed embedding $i_1 : \bar{Y} \rightarrow \mathbb{P}_{B'}^N$. Let $W = \bar{Y} \setminus j(Y)$. Since the fibers of Y are dense in the fibers of \bar{Y} , there is a hypersurface $H \subset \mathbb{P}_{B'}^N$ over B' of sufficiently high degree m containing W and avoiding the point x . We may replace Y with $\bar{Y} \setminus H$; taking the m -fold Veronese embedding, we may assume that $H = \mathbb{P}_\infty^{N-1}$. The resulting closed embedding

$$i : Y \rightarrow \mathbb{A}_{B'}^N = \mathbb{P}_{B'}^N \setminus \mathbb{P}_\infty^{N-1}$$

satisfies the condition (5.1.1) since the fibers of \bar{Y} are irreducible. \square

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