Algebraic Cobordism

A. Algebraic cobordism of schemes

B. Cobordism motives

Motives and Periods
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Outline: Part A

- Oriented Borel-Moore homology
- Universality and Riemann-Roch
- Fundamental classes
Oriented Borel-Moore homology
Regular embeddings and l.c.i. morphisms  Recall:

A regular embedding of codimension $d$ is a closed immersion $i : Z \rightarrow X$ such that $I_Z$ is locally generated by a regular sequence of length $d$.

Example  A regular embedding of codimension 1 is a Cartier divisor

Definition  A morphism $f : Y \rightarrow X$ in $\text{Sch}_k$ is an l.c.i. morphism if $f$ can be factored as $p \circ i$, with $i : Y \rightarrow P$ a regular embedding and $p : P \rightarrow X$ smooth and quasi-projective.

$X \in \text{Sch}_k$ is an l.c.i. scheme if $X \rightarrow \text{Spec} \ k$ is an l.c.i. morphism.

$Lci_k \subset \text{Sch}_k$ is the full subcategory of l.c.i. schemes.

$\text{Sch}'_k :=$ the subcategory of projective morphisms in $\text{Sch}_k$. 
Oriented homology

An oriented Borel-Moore homology theory $A_*$ on $\text{Sch}_k$ consists of the following data:

(D1) An additive functor $A_*: \text{Sch}'_k \to \text{Ab}^*, \ X \mapsto A_*(X)$.

(D2) For $f: Y \to X$ an l.c.i. morphism in $\text{Sch}_k$, a homomorphism of graded groups $f^*: A_*(X) \to A_{*-d}(Y)$,

$d :=$ the codimension of $f$.

(D3) For each pair $(X,Y)$ in $\text{Sch}_k$, a (commutative, associative) bilinear graded pairing $A_*(X) \otimes A_*(Y) \to A_*(X \times_k Y)$

\[ u \otimes v \mapsto u \times v, \]

and a unit element $1 \in A_0(\text{Spec}(k))$. 
These satisfy six conditions:

(BM1) \(\text{id}_X^* = \text{id}_{A^*(X)}\). For composable l.c.i. morphism \(f\) and \(g\),\n\[ (f \circ g)^* = g^* \circ f^*. \]

(BM2) Given a Tor-independent cartesian square in \(\text{Sch}_k\):
\[
\begin{array}{ccc}
W & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y & \xrightarrow{g} & Z,
\end{array}
\]

with \(f\) projective, \(g\) l.c.i. . Then \(g^*f_* = f'_*g'^*\).

(BM3) For \(f\) and \(g\) morphisms in \(\text{Sch}_k\): If \(f\) and \(g\) are projective, then \((f \times g)_*(u \times v) = f_*(u) \times g_*(v)\).
If \(f\) and \(g\) are l.c.i. , then \((f \times g)^*(u \times v) = f^*(u) \times g^*(u')\).
(PB) For a line bundle $L$ on $Y \in \text{Sch}_k$ with zero section $s : Y \to L$ define $\tilde{c}_1(L) : A_*(Y) \to A_{*-1}(Y)$ by $\tilde{c}_1(L)(\eta) = s^*(s_*(\eta))$.

Let $E \to X$ be a rank $n+1$ vector bundle, with associated projective space bundle $q : \mathbb{P}(E) \to X$. Then

$$\bigoplus_{i=0}^n A_{*+i-n}(X) \xrightarrow{\Sigma_{i=0}^{n-1} \tilde{c}_1(O(1)_E)^i \circ q^*} A_*(\mathbb{P}(E))$$

is an isomorphism.

(EH) Let $p : V \to X$ be an affine space bundle. Then

$$p^* : A_*(X) \to A_{*+r}(V)$$

is an isomorphism.

(CD) ***.
Examples (1) The Chow group functor

\[ X \mapsto \text{CH}_*(X) \]

with projective push-forward and l.c.i. pull-back given by Fulton.

(2) The Grothendieck group of coherent sheaves

\[ X \mapsto G_0(X)[\beta, \beta^{-1}] \]

(\deg \beta = 1). L.c.i. pull-back exists because an l.c.i. morphism has finite Tor-dimension.

(3) Algebraic cobordism (char \( k \) = 0) \( X \mapsto \Omega_*(X) \).
L.c.i. pull-backs are similar to Fulton’s, but require a bit more work.

Note. There are “refined intersections” for \( \Omega_* \), similar to Fulton’s refined intersection theory for \( \text{CH}_* \).
Homology and cohomology

Every morphism in $\text{Sm} / k$ is l.c.i., hence:

**Proposition** Let $A_*$ be an O.B.M.H.T. on $\text{Sch}_k$. Then the restriction of $A$ to $\text{Sm} / k$, with

$$A^n(X) := A_{\dim X - n}(X),$$

defines an O.C.T. $A^*$ on $\text{Sm} / k$:

- The product $\cup$ on $A^*(X)$ is $x \cup y = \delta^*_X(x \times y)$.
- $1_X = p^*_X(1)$ for $p_X : X \to \text{Spec} k$ in $\text{Sm} / k$.
- $c_1(L) = \tilde{c}_1(L)(1_X)$ for $L \to X$ a line bundle.
Consequence:

Let $A_*$ be an O.B.M.H.T. on $\text{Sch}_k$. There is a unique formal group law $F_A \in A_*(k)[[u,v]]$ with

$$F_A(\tilde{c}_1(L), \tilde{c}_1(M))(f_*(1_Y)) = \tilde{c}_1(L \otimes M)(f_*(1_Y))$$

for all $X \in \text{Sch}_k$, all $(f: Y \to X) \in \mathcal{M}(X)$.

**Examples**

(1) $\text{CH}_*$ has the additive formal group law: $F_{\text{CH}}(u,v) = u + v$

(2) $G_0[\beta, \beta^{-1}]$ has the multiplicative formal group law:

$$F_{G_0}(u,v) = u + v - \beta uv.$$

(3) $\Omega_*$ has the universal formal group law: $(F_\Omega, \Omega_*(k)) = (F_{\mathbb{L}}, \mathbb{L}_*)$
Universality and Riemann-Roch
Universality

**Theorem** Algebraic cobordism $\Omega_*$ is the universal O.B.M.H.T. on $\text{Sch}_k$.

Also:

**Theorem** The canonical morphism $\vartheta_{\text{CH}} : \Omega_* \otimes_{\mathbb{L}} \mathbb{Z} \to \text{CH}_*$ is an isomorphism, so $\text{CH}_*$ is the universal additive theory on $\text{Sch}_k$.

A new result (due to S. Dai) is

**Theorem** The canonical morphism

$$\vartheta_{G_0} : \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \to G_0[\beta, \beta^{-1}]$$

is an isomorphism, so $G_0[\beta, \beta^{-1}]$ is the universal multiplicative theory on $\text{Sch}_k$. 
Twisting

The $\tau$-twisting construction is modified: one leaves $f_*$ alone and twists $f^*$ by $\overbrace{\text{Td}^{-1}_\tau(N_f)}$:  
\[ f^*_\tau = \overbrace{\text{Td}^{-1}_\tau(N_f)} \circ f^*. \]

Here $f : Y \to X$ is an l.c.i. morphism and $N_f \in K_0(Y)$ is the virtual normal bundle: If we factor $f$ as $p \circ i$, $p$ smooth, $i$ a regular embedding, then:

$p$ has a relative tangent bundle $T_p$
$i$ has a normal bundle $N_i$ and

\[ N_f := [N_i] - [i^*T_p]. \]

$\overbrace{\text{Td}^{-1}_\tau(N_f)}$ is the inverse Todd class operator, defined as we did $\text{Td}^{-1}_\tau$, using the operators $\tilde{c}_1$ instead of the classes $c_1$. 
Riemann-Roch for singular varieties

Twisting $\text{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}]$ to give it the multiplicative group law and using Dai’s theorem, we recover the Fulton-MacPherson Riemann-Roch transformation $\tau: G_0 \rightarrow \text{CH}_* \otimes \mathbb{Q}$:

Using the universal property of $G_0[\beta, \beta^{-1}]$ gives

$$\tau_\beta: G_0[\beta, \beta^{-1}] \rightarrow \text{CH}_* \otimes \mathbb{Q}[\beta, \beta^{-1}](\times);$$

$\tau$ is the restriction of $\tau_\beta$ to degree 0.
Adams operations

J. Malagon-Lopez has used the twisting construction to define Adams operations

\[ \psi_k : \Omega_* \to \Omega_*[1/k] \]

satisfying an “Adams-Riemann-Roch” formula. These recover the classical Adams operations on \( K_0 \) and \( G_0 \) (after inverting \( k \) for \( \psi_k \)).
Fundamental classes
Fundamental classes for l.c.i. schemes

**Definition** Let \( p_X : X \to \text{Spec} \, k \) be an l.c.i. scheme. For an O.B.M.H.T. \( A \) on \( \text{Sch}_k \), set

\[
1^A_X := p_X^*(1)
\]

If \( X \) has pure dimension \( d \) over \( k \), then \( 1^A_X \) is in \( A_d(X) \). \( 1_X \) is the *fundamental class* of \( X \).

Properties:

- For \( X = X_1 \amalg X_2 \in \text{Lci}_k \),
  \[
  1_X = i_1^*(1_{X_1}) + i_2^*(1_{X_2}).
  \]

- For \( f : Y \to X \) an l.c.i. morphism in \( \text{Lci}_k \), \( f^*(1_X) = 1_Y \).
Fundamental classes of non-l.c.i. schemes

Some theories have more extensive pull-back morphisms, and thus admit fundamental classes for more schemes.

**Example** Both $\text{CH}_*$ and $G_0[\beta, \beta^{-1}]$ admit pull-back for arbitrary flat maps, still satisfying all the axioms. Thus, functorial fundamental classes in $\text{CH}_*$ and $G_0[\beta, \beta^{-1}]$ exist for all $X \in \text{Sch}_k$.

This is NOT the case for all theories.

We give an example for $\Omega_*$. 
Let $S_1 \subset \mathbb{P}^5$ be $\mathbb{P}^2$ embedded by $\mathcal{O}(2)$.

Let $S_2 \subset \mathbb{P}^5$ be $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}(2,1)$.

Let $R_1 \subset S_1, R_2 \subset S_2$ be smooth hyperplane sections. Note:

1. $\deg c_1(\mathcal{O}(2))^2 = \deg c_1(\mathcal{O}(2,1))^2 = 4$

2. $R_1$ and $R_2$ are both $\mathbb{P}^1$'s

So: $R_1$ and $R_2$ are both rational normal curves of degree 4 in $\mathbb{P}^4$.

We may assume $R_1 = R_2 = R$. 
Let $C(S_1)$, $C(S_2)$ and $C(R)$ be the projective cones in $\mathbb{P}^6$.

**Proposition**  Let $A$ be a O.B.M.H.T. on $\text{Sch}_k$. If we can extend fundamental classes in $A$ for $\text{Sm}/k$ to $C(S_1)$, $C(S_2)$ and $C(R)$, functorial for l.c.i. morphisms, then

$$[\mathbb{P}^2] = [\mathbb{P}^1 \times \mathbb{P}^1] \text{ in } A_2(k)$$

This is NOT the case for $A = \Omega$, since

$$c_2(\mathbb{P}^2) = 3, \quad c_2(\mathbb{P}^1 \times \mathbb{P}^1) = 4.$$
Consequence for Gromov-Witten theory

The formalism of Gromov-Witten theory can be extended to a cobordism valued version, at least if the relevant moduli stack is an $\text{Lci}$ stack.

One needs a theory of algebraic cobordism for (Deligne-Mumford) stacks: one can make a cheap version with $\mathbb{Q}$-coefficients by the universal twisting of $\text{CH}_*(\mathbb{Q})$.

BUT: there may be problems in defining the virtual fundamental class for a perfect deformation theory if the intrinsic normal cone of the moduli stack is not $\text{Lci}$.
Part B: Cobordism motives
Outline: Part B

- Motives over an O.C.T.
- Cobordism motives
- Motivic computations
- Algebraic cobordism of Pfister quadrics
Motives over an O.C.T.

We follow the discussion of Nenashev-Zainoulline.
$A$-correspondences

**Definition** $A^*$ an O.C.T. on $\text{Sm}/k$. $X$, $Y$ smooth projective $k$-varieties. Set

$$\text{Cor}_A^0(X,Y) := A^{\dim Y}(X \times Y).$$

$\text{Cor}_A^0$ is the category with

**objects:** smooth projective $k$-varieties $\text{SmProj}/k$,

**morphisms:**

$$\text{Hom}_{\text{Cor}_A^0}(X,Y) := \text{Cor}_A^0(X,Y)$$

and **composition law:**

$$\gamma_{Y,Z} \circ \gamma_{X,Y} := p_{X,Z}^*(p_{X,Y}^*(\gamma_{X,Y}) \cdot p_{Y,Z}^*(\gamma_{Y,Z}))$$
• Cor\(_A^0\) is a tensor category: \(X \oplus Y = X \amalg Y\) and \(X \otimes Y := X \times Y\).

• Sending \(f : X \to Y\) to the “graph”

\[
\Gamma_f := (\text{id}_X, f)_* (1_X) \in A^{\dim Y}(X \times Y)
\]

gives the functor

\[
m_A : \text{SmProj}/k \to \text{Cor}_A^0.
\]

**Definition** \(\mathcal{M}_A^{\text{eff}}\) is the pseudo-abelian hull of Cor\(_A^0\): Obects are pairs \((X, \alpha)\), \(\alpha \in \text{End}_{\text{Cor}_A^0}(X)\), \(\alpha^2 = \alpha\).

\[
\text{Hom}_{\mathcal{M}_A^{\text{eff}}}((X, \alpha), (Y, \beta)) := \beta \text{Hom}_{\text{Cor}_A^0}(X, Y) \alpha
\]

with the evident composition.
**Definition**  Let \( \text{Cor}_A^*(X, Y) := A^{\dim Y + *}(X \times Y) \).

\( \widetilde{\text{Cor}}_A \) is the category with objects pairs \((X, n)\), \(X\) a smooth projective \(k\)-variety \(n \in \mathbb{Z}\), morphisms

\[
\text{Hom}_{\widetilde{\text{Cor}}_A}(((X, n), (Y, m))) := \text{Cor}_A^{m-n}(X, Y)
\]

\( \text{Cor}_A \) is the additive category generated by \( \widetilde{\text{Cor}}_A \) and \( \mathcal{M}_A \) is the pseudo-abelian hull of \( \text{Cor}_A \).

For \( M = (X, \alpha) \in \mathcal{M}_A^{\text{eff}} \), write \( M(m) := ((X, \alpha), m) \).

\( \alpha \in \text{Cor}_A^n(X, Y) \) acts as a homomorphism

\[
\alpha_* : A^*(X) \to A^{*+n}(Y).
\]

We have \( t \alpha \in \text{Cor}^{n+\dim X - \dim Y}(Y, X) \); set

\[
\alpha^* := t \alpha_* : A^*(Y) \to A^{*+\dim X - \dim Y + n}(X).
\]
• We have $m_A : \text{SmProj}/k \to \text{Cor}_A$.
• $\text{Cor}_A$ is a tensor category, $1 = m_A(\text{Spec } k)$ and
\[
\text{Hom}_{\text{Cor}_A}(1(n), m_A(X)) = A_n(X).
\]
• Sending $X$ to $(X, 0)$ defines tensor functors
\[
\begin{align*}
\text{Cor}_A^0 & \to \text{Cor}_A \\
\mathcal{M}_A^\text{eff} & \to \mathcal{M}_A
\end{align*}
\]
• A natural transformation of O.C.T.’s on $\text{Sm}/k$, $\vartheta : A \to B$, induces tensor functors
\[
\begin{align*}
\vartheta_* : \text{Cor}_A^0 & \to \text{Cor}_B^0 \\
\vartheta_* : \mathcal{M}_A^\text{eff} & \to \mathcal{M}_B^\text{eff} \\
\vartheta_* : \mathcal{M}_A & \to \mathcal{M}_B
\end{align*}
\]
We add the ground field $k$ to the notation when necessary: \( \text{Cor}^0_A(k), \text{M}_A(k) \), etc.

If $R$ is a commutative ring, set

\[
\text{Cor}^0_{A,R} := \text{Cor}^0_A \otimes R \\
\text{Cor}_{A,R} := \text{Cor}_A \otimes R
\]

\( \text{M}^{\text{eff}}_{A,R} \) and \( \text{M}_{A,R} \) are the respective pseudo-abelian hulls.
Examples

(1) For $A^* = \text{CH}^*$, we have the well-known categories:

$\text{Cor}^0_{\text{CH}}(k)$ is the category of correspondences mod rational equivalence, $\mathcal{M}^\text{eff}_{\text{CH}}(k)$ is the category of effective Chow motives, $\mathcal{M}_{\text{CH}}(k)$ is the category of Chow motives (all over $k$).

(2) For $A^* = \Omega^*$, we call $\text{Cor}^0_{\Omega}(k)$ the category of cobordism correspondences, $\mathcal{M}^\text{eff}_{\Omega}(k)$ the category of effective cobordism motives, $\mathcal{M}_{\Omega}(k)$ the category of cobordism motives (over $k$).

(3) We can also take e.g. $A^* = K_0[\beta, \beta^{-1}]$; we write $\text{Cor}^0_{K_0}$, $\mathcal{M}^\text{eff}_{K_0}$, etc.
Cobordism motives

Vishik-Yagita have considered the category $\mathcal{M}_{\Omega}^{\text{eff}}(k)$ and discussed its relation with Chow motives.
Remarks

(1) Since $\Omega^*$ is universal, there are canonical functors

$$\vartheta^A_* : \text{Cor}_\Omega^0(k) \to \text{Cor}_A^0(k)$$
$$\vartheta^A_* : \mathcal{M}_{\Omega}^{\text{eff}}(k) \to \mathcal{M}_{A}^{\text{eff}}(k)$$

etc. Thus, identities in $\mathcal{M}_{\Omega}^{\text{eff}}(k)$ or $\mathcal{M}_{\Omega}(k)$ yield identities in $\mathcal{M}_{A}^{\text{eff}}(k)$ or $\mathcal{M}_{A}(k)$ for all O.C.T.'s $A$ on $\text{Sm}/k$.

(2) $\Omega^* \otimes \mathbb{Q}$ is isomorphic to the “universal twist” of $\text{CH}^* \otimes \mathbb{L} \otimes \mathbb{Q}$, so one can hope to understand $\mathcal{M}_{\Omega,Q}$ by modifying $\mathcal{M}_{\text{CH},L \otimes \mathbb{Q}}$ by a twisting construction, i.e., a deformation of the composition law. We will see that $\mathcal{M}_{\Omega,Q}$ is NOT equivalent to $\mathcal{M}_{\text{CH},L \otimes \mathbb{Q}}$.

(3) The work of Vishik-Yagita allows one to lift identities in $\mathcal{M}_{\text{CH}}^{\text{eff}}(k)$ or $\mathcal{M}_{\text{CH}}(k)$ to $\mathcal{M}_{\Omega}^{\text{eff}}(k)$ or $\mathcal{M}_{\Omega}(k)$
**Example** [The Lefschetz motive in $\mathcal{M}_{}^{\text{eff}}$] Let’s compare $\text{End}_{\text{Cor}^0_\Omega}(\mathbb{P}^1)$ with $\text{End}_{\text{Cor}^0_{\text{CH}}}(\mathbb{P}^1)$

$$\Omega^1(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}[0 \times \mathbb{P}^1] \oplus \mathbb{Z}[\mathbb{P}^1 \times 0] \oplus \mathbb{Z}[\mathbb{P}^1] \times [(0,0)]$$

Set: $\alpha = [0 \times \mathbb{P}^1]; \beta = [\mathbb{P}^1 \times 0]; \gamma = [\mathbb{P}^1] \cdot [(0,0)]$.

$\text{Cor}^0_\Omega \to \text{Cor}^0_{\text{CH}}$ just sends $\gamma$ to zero. We have the composition laws:

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<th>$\text{End}<em>{\text{Cor}^0</em>{\text{CH}}}(\mathbb{P}^1)$</th>
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So

\[ \text{End}_{\text{Cor}_0^0}(\mathbb{P}^1) \to \text{End}_{\text{Cor}_{CH}^0}(\mathbb{P}^1) = \mathbb{Z} \times \mathbb{Z} \]

is a non-commutative extension with square-zero kernel \((\gamma)\). Hence:

- The Lefschetz Chow motive \(L := (\mathbb{P}^1, \alpha)\) lifts to “the Lefschetz \(\Omega\)-motive”

\[ L_{\Omega}(\lambda) := (\mathbb{P}^1, \alpha + \lambda \gamma) \]

for any choice of \(\lambda \in \mathbb{Z}\). Since \([\Delta_{\mathbb{P}^1}] = \alpha + \beta - \gamma:\)

\[(\mathbb{P}^1, \text{id}) = (\mathbb{P}^1, \alpha + \lambda \gamma) \oplus (\mathbb{P}^1, \beta - (1 + \lambda) \gamma) \cong L_{\Omega}(\lambda) \oplus 1.\]

Also: \(L_{\Omega}(\lambda) \cong L_{\Omega}(\lambda')\) for all \(\lambda, \lambda'\), but are not equal as summands of \((\mathbb{P}^1, \text{id})\).
Remark  We have seen that

$$\text{End}_{\mathcal{M}_\Omega}(m_\Omega(\mathbb{P}^1)) \rightarrow \text{End}_{\mathcal{M}_{\text{CH}}}(m_{\text{CH}}(\mathbb{P}^1))$$

is surjective with kernel a square zero ideal. A similar computation shows that

$$\text{End}_{\mathcal{M}_\Omega}(m_\Omega(\mathbb{P}^n)) \rightarrow \text{End}_{\mathcal{M}_{\text{CH}}}(m_{\text{CH}}(\mathbb{P}^n)) = \prod_{i=1}^{n} \mathbb{Z}$$

is surjective for all $n$ with $\ker^{n+1} = 0$, but $\ker^n \neq 0$. 
Definition Let $A$ be an O.C.T. on $\text{Sm}/k$, $\vartheta_A : \mathcal{M}_\Omega^\text{eff}(k) \to \mathcal{M}_A^\text{eff}(k)$ the canonical functor. Define the \textit{Lefschetz A-motive}

$$L_A := \vartheta_A(L_\Omega).$$

Proposition For $M = (X, \alpha)$, $N = (Y, \beta)$ in $\mathcal{M}_A^\text{eff}(k)$,

$$\text{Hom}_{\mathcal{M}_A^\text{eff}}(M \otimes L_A^\otimes m, N \otimes L_A^\otimes n)$$

$$= (\text{id} \times \alpha)^* (\beta \times \text{id})_* A^{\dim Y - m + n}(X \times Y).$$

Hence

Theorem The inclusion functor $\mathcal{M}_A^\text{eff}(k) \to \mathcal{M}_A(k)$ identifies $\mathcal{M}_A(k)$ with the localization of $\mathcal{M}_A^\text{eff}(k)$ with respect to $- \otimes L_A$. Also

$$(X, \alpha)(m) := (X, \alpha, m) \cong (X, \alpha) \otimes L_A^\otimes m.$$
The nilpotence theorem

**Theorem**  Take $X, Y \in \text{SmProj}/k$. Then

$$\vartheta_{\text{CH}} : \text{Cor}_\Omega^0(X, Y) \to \text{Cor}_\text{CH}^0(X, Y)$$

is surjective. If $X = Y$, then the kernel $\ker(X)$ of $\vartheta_{\text{CH}}$ is nilpotent.

**Proof.** Surjectivity: Since $\text{CH}^* = \Omega^* \otimes_{\mathbb{Z}} \mathbb{Z}$, $\Omega^*(T) \to \text{CH}^*(T)$ is surjective for all $T \in \text{Sm}/k$. 
Nilpotence of the kernel: \( \mathbb{CH}^* = \Omega^* \otimes \mathbb{L} \mathbb{Z} = \Omega^* \otimes (\mathbb{L}/\mathbb{L}^*<0) \implies \)

\[
ker(X) = \sum_{n>0} \mathbb{L}^{-n} \Omega^{\dim X+n} (X \times X).
\]

Composition is \( \mathbb{L} \)-linear, hence operates as:

\[
\mathbb{L}^{-n} \Omega^{\dim X+n} (X \times X) \otimes \mathbb{L}^{-m} \Omega^{\dim X+m} (X \times X) \text{ } \circ \text{ } \mathbb{L}^{-n-m} \Omega^{\dim X+n+m} (X \times X).
\]

Also: \( \Omega^d(T) = 0 \) for \( d > \dim T \). Thus

\[
ker(X)^{\circ \dim X+1} = 0.
\]
Proposition (Vishik-Yagita)

(1) For $X \in \text{SmProj}/k$, each idempotent in $\text{Cor}_{\text{CH}}^0(X,X)$ lifts to an idempotent in $\text{Cor}_{\Omega}^0(X,X)$.

(2) For $M, N$ in $\mathcal{M}^\text{eff}_\Omega(k)$, each isomorphism $f : \vartheta_{\text{CH}}(M) \to \vartheta_{\text{CH}}(N)$ lifts to an isomorphism $\tilde{f} : M \to N$.

Theorem (Isomorphism) $\vartheta_{\text{CH}} : \mathcal{M}^\text{eff}_\Omega(k) \to \mathcal{M}^\text{eff}_{\text{CH}}(k)$ and $\vartheta_{\text{CH}} : \mathcal{M}_\Omega(k) \to \mathcal{M}_{\text{CH}}(k)$ both induce bijections on the set of isomorphism classes of objects.

Proof. For $\mathcal{M}^\text{eff}$, this follows from the proposition. For $\mathcal{M}$, this follows by localization.

Note. These result are also valid for motives with $R$-coefficients, $R$ a commutative ring.
Motivic computations
Elementary computations

- $m_A(\mathbb{P}^n) \cong \bigoplus_{i=0}^{n} L_A^{\otimes i} \cong \bigoplus_{i=0}^{n} 1_A(i)$.
- Let $E \to B$ be a vector bundle of rank $n + 1$, $\mathbb{P}(E) \to B$ the projective-space bundle. Then
  $$m_A(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{n} m_A(B)(i).$$
- Let $\mu : X_F \to X$ be the blow-up of $X$ along a codimension $d$ closed subscheme $F$. Then
  $$m_A(X_F) \cong m_A(X) \oplus \bigoplus_{i=1}^{d-1} m_A(F)(i).$$

So:

$$A^*(X_F) \cong A^*(X) \oplus \bigoplus_{i=1}^{d-1} A^*-d+i(F').$$

Indeed, we have all these isomorphisms in $\mathcal{M}_{\text{CH}}$, hence in $\mathcal{M}_{\Omega}$ by the isomorphism theorem, and thus in $\mathcal{M}_A$ by applying $\vartheta_A$. 
Cellular varieties

**Definition** $X \in \text{SmProj}/k$ is called *cellular* if there is a filtration by closed subsets

$$X = X^0 \supset X^1 \supset \ldots \supset X^d \supset X^{d+1} = \emptyset; \ d = \dim X,$$

such that $\text{codim}_X X^i \geq i$ and either $X^i \setminus X^{i+1} \cong \bigsqcup_{i=1}^{n_i} \mathbb{A}^{d-i}$ or $X^i = X^{i+1}$. If $\overline{X}_k$ is cellular, call $X$ *geometrically cellular*.

- For $X$ cellular as above, we have

$$m_A(X) \cong \bigoplus_{i=0}^{d} 1_A(i)^{n_i}$$

because we have this isomorphism for $A = \text{CH}$.

**Examples** Projective spaces and Grassmannians are cellular. A smooth quadric over $k$ is geometrically cellular.
**Quadratic forms**

First some elementary facts about quadratic forms:

- Each quadratic form over $k$ can be diagonalized. If $q = \sum_{i=1}^{n} a_i x_i^2$, let $Q_q \subset \mathbb{P}^{n-1}$ be the quadric $q = 0$. The dimension of $q$ is $n$.

- For $q_1 = \sum_{i=1}^{n} a_i x_i^2$, $q_2 = \sum_{j=1}^{m} b_j y_j^2$, we have the orthogonal sum
  \[
  q_1 \perp q_2 := \sum_{i=1}^{n} a_i x_i^2 + \sum_{j=1}^{m} b_j y_j^2
  \]
  and tensor product
  \[
  q_1 \otimes q_2 := \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j z_{i,j}^2.
  \]
Pfister forms and Pfister quadrics

• For \( a \in k^\times \) we have \( \langle \langle a \rangle \rangle := x^2 - a y^2 \) and for \( a_1, \ldots, a_n \in k^\times \) the \( n \)-fold Pfister form

\[
\alpha := \langle \langle a_1, \ldots, a_n \rangle \rangle := \langle \langle a_1 \rangle \rangle \otimes \ldots \otimes \langle \langle a_n \rangle \rangle
\]

The quadric \( Q_\alpha \subset \mathbb{P}^{2^n-1} \) is the associated Pfister quadric.

• The isomorphism class of \( \alpha = \langle \langle a_1, \ldots, a_n \rangle \rangle \) depends only on the symbol

\[
\{a_1, \ldots, a_n\} \in k_n(k) := K_n^M(k)/2.
\]

\( \langle \langle a_1, \ldots, a_n \rangle \rangle \) is isomorphic to a hyperbolic form if and only if \( Q_\alpha \) is isotropic, i.e. \( \langle \langle a_1, \ldots, a_n \rangle \rangle = 0 \) has a non-trivial solution in \( k \).
The Rost motive

Proposition (Rost) (1) Let $\alpha = \langle a_1, \ldots, a_n \rangle$ and let $Q_\alpha$ be the associated Pfister quadric. Then there is a motive $M_\alpha \in M_{\text{eff}}^{\text{ch}}(k)$ with

$$m_{\text{ch}}(Q_\alpha) \cong M_\alpha \otimes m_{\text{ch}}(\mathbb{P}^{2n-1}-1).$$

(2) Let $\overline{k}$ be the algebraic closure. There are maps

$$L \otimes \mathbb{P}^{2n-1}-1 \rightarrow M_\alpha \rightarrow 1$$

which induce

$$M_{\alpha\overline{k}} \cong 1 \oplus L \otimes \mathbb{P}^{2n-1}-1 \text{ in } M_{\text{eff}}^{\text{ch}}(\overline{k}).$$

$M_\alpha$ is the Rost motive.
The Rost cobordism-motive

Applying the Vishik-Yagita bijection, there is a unique (up to isomorphism) cobordism motive

\[ M^\Omega_\alpha \in \mathcal{M}^\text{eff}_\Omega(k) \]

with \( \vartheta_{\text{CH}}(M^\Omega_\alpha) \cong M_\alpha \). In addition:

1. \( m_\Omega(Q_\alpha) \cong M^\Omega_\alpha \otimes m_\Omega(\mathbb{P}^{2n-1}-1) \).

2. There are maps \( L^\otimes_{\Omega} 2^{n-1}-1 \rightarrow M^\Omega_\alpha \rightarrow 1 \) which induce \( M^\Omega_{\alpha k} \cong 1 \oplus L^\otimes_{\Omega} 2^{n-1}-1 \) in \( \mathcal{M}^\text{eff}_\Omega(k) \).
Algebraic cobordism of Pfister quadrics

Vishik-Yagita use the Rost cobordism motive to compute $\Omega^*(Q_\alpha)$. The computation is in two parts:

1. Compute the image of base-change $\Omega^*(Q_\alpha) \to \Omega^*(Q_{\alpha\bar{k}})$. $\Omega^*(Q_{\alpha\bar{k}})$ is easy because $Q_{\alpha\bar{k}}$ is cellular.

2. Show that $\Omega^*(Q_\alpha) \to \Omega^*(Q_{\alpha\bar{k}})$ is injective.
Structure of $\mathbb{L}$

We need some information on $\mathbb{L}$ to state the main result.

Recall the Conner-Floyd Chern classes $c_I$ and the Landweber-Novikov operations $s_I$. Let $\bar{s}_I(x)$ be the image of $s_I(x)$ in $\text{CH}^*$. For $X \in \text{SmProj}/k$ of dimension $|I|$

$$\bar{s}_I([X]) = \deg c_I(-T_X) \in \mathbb{Z} = \text{CH}^0(k).$$

Since the $\bar{s}_I$ are indexed by the monomials in $t_1, t_2, \ldots$, $\deg t_i = i$, we have

$$\bar{s} : \Omega^*(k) = \mathbb{L}^* \to \mathbb{Z}[t]$$

with $\bar{s}([X]) = \sum_I \bar{s}_I(X)t^I = \sum_I c(-T_X)t^I$. 
**Theorem (Quillen)** \( \bar{s} : \Omega^*(k) = \mathbb{L}^* \to \mathbb{Z}[t] \) is an injective ring homomorphism with image of finite index in each degree.

**Definition** \( I(p) \subset \mathbb{L}^* \) is the prime ideal

\[
I(p) := \bar{s}^{-1}(p\mathbb{Z}[t]).
\]

\( I(p, n) \subset I(p) \) is the sub-ideal generated by elements of degree \( \leq p^n - 1 \).

In words: \( I(p) \subset \mathbb{L} \) is the ideal generated by \([X], X \in \text{SmProj}/k\) all of whose Chern numbers \( \deg c_I(-T_X) \) are divisible by \( p \).

**Note.** The fact that \( s_{2^n-1}(Q_{2^n-1}) \equiv 1 \mod 2 \) for \( Q_{2^n-1} \) a quadric of dimension \( 2^n - 1 \) implies that \( I(2, r) \) is the ideal generated by the classes \([Q_{2^n-1}], 0 \leq 2^n - 1 \leq r ([Q_0] = 2 \in \mathbb{L}^0) \).
The main theorem
Fix \( \alpha := \langle \langle a_1, \ldots, a_n \rangle \rangle \), \( Q_\alpha \subset \mathbb{P}^{2^n-1} \) the associated Pfister quadric.

Let \( h_i^\Omega \in \Omega^i(Q_\alpha \bar{k}) \) be the class of a codimension \( i \) linear section,

Let \( \ell_i^\Omega \in \Omega_i(Q_\alpha \bar{k}) \) be the class of a linear \( \mathbb{P}^i \subset Q_\alpha \bar{k} \).

\( h^i, \ell_i \): the images of \( h^i_\Omega \) and \( \ell_i^\Omega \) in \( \text{CH}^i, \text{CH}_i \).

Since \( Q_\alpha \bar{k} \) is cellular
\[
\Omega^*(Q_\alpha \bar{k}) = \bigoplus_{i=0}^{2^n-1-1} \mathbb{L} \cdot h_i^\Omega \bigoplus \mathbb{L} \cdot \ell_i^\Omega.
\]

Theorem  The base-change map \( p^* : \Omega^*(Q_\alpha) \to \Omega^*(Q_\alpha \bar{k}) \) is injective and the image of \( p^* \) is
\[
\bigoplus_{i=0}^{2^n-1-1} \mathbb{L} \cdot h_i^\Omega \bigoplus I(2, n-2) \cdot \ell_i^\Omega.
\]
**Idea of proof:**

Use the isomorphisms

\[ m_\Omega(Q_\alpha) \cong M^\Omega_\alpha \otimes m_\Omega(\mathbb{P}^{2^{n-1}-1}), \quad M^\Omega_\alpha \cong 1 \oplus L_\Omega^{2^{n-1}-1} \]

to show that the image of base-change is \( \oplus_{i=0}^{2^{n-1}-1} \mathbb{L} \cdot h^i_\Omega \oplus J \cdot \ell_i^\Omega \) for some ideal \( J \subset \mathbb{L} \).

A result of Rost on \( M_\alpha^{\text{CH}} \) plus Vishik-Yagita lifting shows that

\[ M^\Omega_\alpha \oplus ? = m_\Omega(P_\alpha), \]

\( P_\alpha \): a linear section of \( Q_\alpha \) of dimension \( 2^{n-1} - 1 \).

The “small” dimension (\( \leq 2^{n-1} - 1 \)) of \( P_\alpha \) allows one to show that \( J = I(2, n - 2) \).

The injectivity is handled by the fact that \( P_\alpha \) splits \( M_\alpha \).