CONVERGENCE OF VOEVODSKY'S SLICE TOWER

MARC LEVINE

Abstract. We consider Voevodsky's slice tower for a finite spectrum $\mathcal{E}$ in the motivic stable homotopy category over a perfect field k. In case k has finite cohomological dimension (in characteristic two, we also require that k is infinite), we show that the slice tower converges, in that the induced filtration on the bi-graded homotopy sheaves $\Pi_{a,b}f_n\mathcal{E}$ is finite, exhaustive and separated at each stalk. This partially verifies a conjecture of Voevodsky.

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Introduction

Let k be a perfect field. We continue our investigation, begun in [9], of the slice filtration on the bi-graded homotopy sheaves $\Pi_{*,*}(\mathcal{E})$ for objects $\mathcal{E}$ in the motivic stable homotopy category $\mathcal{SH}(k)$.

Let $\mathcal{SH}(k)$ denote Voevodsky's motivic stable homotopy category of T-spectra over k and let $\Sigma^n_T\mathcal{SH}^{eff}(k) \subset \mathcal{SH}(k)$ be the localizing subcategory generated by objects $\Sigma^m_T\Sigma^\infty_T X_+$, with $X \in \mathbf{Sm}/k$ and $m \geq n$. The inclusion $i_n : \Sigma^n_T\mathcal{SH}^{eff}(k) \to \mathcal{SH}(k)$ admits a right adjoint $r_n : \mathcal{SH}(k) \to \Sigma^n_T\mathcal{SH}^{eff}(k)$ (*cf.* [20, 21] or [16, 17]). Letting $f_n = i_n \circ r_n$, we have the tower of exact endofunctors of $\mathcal{SH}(k)$

$$\ldots \to f_{n+1} \to f_n \to \ldots \to \mathrm{id}.$$

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Applying this to an object $\mathcal{E}$ of $\mathcal{SH}(k)$ yields the *Tate-Postnikov tower* (or slice tower)

$$\ldots \to f_{n+1}\mathcal{E} \to f_n\mathcal{E} \to \ldots \to \mathcal{E}.$$

For integers a, b, we have the stable homotopy sheaf $\Pi_{a,b}(\mathcal{E})$, defined as the Nisnevich sheaf associated to the presheaf

$$U \in \mathbf{Sm}/k \mapsto [\Sigma^a_{S^1}\Sigma^b_{\mathbb{G}_m} U_+, \mathcal{E}]_{\mathcal{SH}(k)}$$

(note that the indexing is not the standard one). Let

$$\mathrm{Fil}^n_{\mathrm{Tate}}\,\Pi_{a,b}(\mathcal{E}) := im(\Pi_{a,b} f_n\mathcal{E} \to \Pi_{a.b}\mathcal{E}).$$

For k a perfect field of characteristic $\neq 2$, Morel [13, theorem 6.4.1] has given a natural isomorphism of $\Pi_{0,p}(\mathbb{S}_k)$ with the *Milnor-Witt sheaf* $\mathcal{K}^{MW}_{-p}$; this is a certain sheaf on $\mathbf{Sm}/k$ with value on each field F over k given by the Milnor-Witt group $K^{MW}_{-p}(F)$ (for details see [10, 13]). For $p = 0$, and F a field containing k, $K^{MW}_0(F)$ is canonically isomorphic to the Grothendieck-Witt group of quadratic forms $\mathrm{GW}(F)$. Let $I(F) \subset \mathrm{GW}(F)$ be the augmentation ideal; the multiplication in the Milnor-Witt ring makes $K^{MW}_{-p}(F)$ a $\mathrm{GW}(F)$-module.

More generally, Morel has constructed a natural isomorphism

$$\Pi_{0,p}(\Sigma^q_{\mathbb{G}_m}\mathbb{S}_k) \cong \mathcal{K}^{MW}_{q-p}$$

Our main result of *loc. cit.* is

Theorem 1 ([9, theorem 1])**.** *Let F be a perfect field of characteristic $\neq 2$. Then*

$$\mathrm{Fil}^n_{Tate}\,\Pi_{0,p}(\Sigma^q_{\mathbb{G}_m}\mathbb{S}_k)(F) := I(F)^M K^{MW}_{q-p}(F) \subset K^{MW}_{q-p}(F) = \Pi_{0,p}(\Sigma^q_{\mathbb{G}_m}\mathbb{S}_k)(F)$$

where $M = 0$ if $n \le p$ or $n \le q$, and $M = \min(n-p, n-q)$ if $n \ge p$ and $n \ge q$.

Let $\mathcal{SH}_{\mathrm{fin}}(k) \subset \mathcal{SH}(k)$ be the thick subcategory of $\mathcal{SH}(k)$. generated by the objects $\Sigma^n_T\Sigma^\infty_T X_+$, with X smooth and projective over k, $n \in \mathbb{Z}$.

Voevodsky has stated a conjecture [20, conjecture 13] that for $\mathcal{E} \in \mathcal{SH}_{\mathrm{fin}}(k)$, the Tate-Postnikov tower is convergent in the following sense: for all $a, b, n \in \mathbb{Z}$, one has

$$\cap_m F^m_{\mathrm{Tate}}\Pi_{a,b} f_n\mathcal{E} = 0.$$

Our computation of $F^n_{\mathrm{Tate}}\Pi_{0,p}\Sigma^q_T\mathbb{S}_k$ gives some evidence for this convergence conjecture.

Proposition 2. *Let k be a perfect field with $\mathrm{char}\, k \neq 2$. For all $p, q \ge 0$, and all perfect field extensions F of k, we have*

$$\cap_n F^n_{Tate}\Pi_{0,p}\Sigma^q_T\mathbb{S}_k(F) = 0.$$

Proof. In light of theorem 1, the assertion is that the $I(F)$-adic filtration on $K^{MW}_{q-p}(F)$ is separated. By [12, théorème 5.3], for $m \ge 0$, $K^{MW}_m(F)$ fits into a cartesian square of $\mathrm{GW}(F)$-modules

$$\begin{array}{ccc} K^{MW}_m(F) & \longrightarrow & K^M_m(F) \\ \downarrow & & \downarrow{\scriptstyle Pf} \\ I(F)^m & \xrightarrow[q]{} & I(F)^m/I(F)^{m+1}, \end{array}$$

where $K^M_m(F)$ is the Milnor K-group, q is the quotient map and Pf is the map sending a symbol $\{u_1, \ldots, u_m\}$ to the class of the Pfister form $<<u_1, \ldots, u_m>>$

mod $I(F)^{m+1}$. For $m < 0$, $K_m^{MW}(F)$ is isomorphic to the Witt group of F, $W(F)$, that is, the quotient of $\mathrm{GW}(F)$ by the ideal generated by the hyperbolic form $x^2 - y^2$. Also, the map $\mathrm{GW}(F) \to W(F)$ gives an isomorphism of $I(F)^r$ with its image in $W(F)$ for all $r \geq 1$. Thus

$$K_m^{MW}(F)I(F)^n = \begin{cases} I(F)^n \subset W(F) & \text{for } m < 0, n \geq 0 \\ I(F)^{n+m} \subset \mathrm{GW}(F) & \text{for } m \geq 0, n \geq 1. \end{cases}$$

The fact that $\cap_n I(F)^n = 0$ in $W(F)$ or equivalently in $\mathrm{GW}(F)$ is a theorem of Arason and Pfister [1]. □

Remarks 1. 1. The proof in [12] that $K_m^{MW}(F)$ fits into a cartesian square as above relies on the Milnor conjecture.

2. As pointed out to me by Igor Kriz, Voevodsky's conjecture [*loc. cit.*] asserting the convergence of the slice tower for all $\mathcal{E} \in \mathcal{SH}_{\mathrm{fin}}(k)$ is false. In fact, take $\mathcal{E}$ to be the Moore spectrum $\mathbb{S}_k/\ell$ for some prime $\ell \neq 2$. Since $\Pi_{a,q}\mathbb{S}_k = 0$ for $a < 0$, proposition 5.7 below shows that $\Pi_{a,q}f_n\mathbb{S}_k = 0$ for $a < 0$, and thus we have the right exact sequence for all $n \geq 0$

$$\Pi_{0,0}f_n\mathbb{S}_k \xrightarrow{\times \ell} \Pi_{0,0}f_n\mathbb{S}_k \to \Pi_{0,0}f_n\mathcal{E} \to 0.$$

In particular, we have

$$F^n_{\mathrm{Tate}}\Pi_{0,0}\mathcal{E}(k) = im\left(F^n_{\mathrm{Tate}}\Pi_{0,0}\mathbb{S}_k(k) \to \Pi_{0,0}\mathbb{S}_k(k)/\ell\right) = im\left(I(k)^n \to \mathrm{GW}(k)/\ell\right).$$

Take $k = \mathbb{R}$. Then $\mathrm{GW}(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$, with virtual rank and virtual index giving the two factors. The augmentation ideal $I(\mathbb{R})$ is thus isomorphic to $\mathbb{Z}$ via the index and it is not hard to see that $I(\mathbb{R})^n = (2^{n-1}) \subset \mathbb{Z} = I(\mathbb{R})$. Thus $\Pi_{0,0}\mathcal{E} = \mathbb{Z}/\ell \oplus \mathbb{Z}/\ell$ and the filtration $F^n_{\mathrm{Tate}}\Pi_{0,0}\mathcal{E}$ is constant, equal to $\mathbb{Z}/\ell = I(\mathbb{R})/\ell$, and is therefore not separated.

The convergence property is thus not a "triangulated" one in general, and therefore seems to be quite subtle. However, if the I-adic filtration on $\mathrm{GW}(F)$ is finite (possibly of varying length depending on F) for all finitely generated F over k, then our computations (at least in characteristic zero) show that the filtration $F^*_{\mathrm{Tate}}\Pi_{0,p}\Sigma^\infty_T\mathbb{G}_m^{\wedge q}$ is at least locally finite, and thus has better triangulated properties; in particular, for $\ell \neq 2$,

$$\Pi_{0,0}(\mathbb{S}_k/\ell) = \mathbb{Z}/\ell,\ F^n_{\mathrm{Tate}}\Pi_{0,0}(\mathbb{S}_k/\ell) = 0 \text{ for } n > 0,$$

as the augmentation ideal in $\mathrm{GW}(F)$ is purely two-primary torsion, and thus $\mathcal{I}\Pi_{0,0}\mathbb{S}_k/\ell = 0$. One can therefore ask if Voevodsky's convergence conjecture is true if one assumes the finiteness of the $I(F)$-adic filtration on $\mathrm{GW}(F)$ for all finitely generated fields F over k.

In this regard, our main theorem of this paper is a partial answer to the convergence question.

Theorem 3. *Let k be a perfect field of finite torsion-cohomologcal dimension and let p denote the exponential characteristic (i.e. $p = \mathrm{char}\, k$ if $\mathrm{char}\, k > 0$, $p = 1$ if $\mathrm{char}\, k = 0$). If $\mathrm{char}\, k = 2$, we suppose in addition that k is an infinite field. Take $\mathcal{E} \in \mathcal{SH}_{\mathrm{fin}}(k)$ and take $x \in X \in \mathbf{Sm}/k$ with X irreducible. Let $d = \dim_k X$. Then for every $r, q \in \mathbb{Z}$, there is an integer N (depending on $\mathcal{E}$, r and d) such that*

$$(\mathrm{Fil}^n_{Tate}\,\Pi_{r,q}\mathcal{E})_x[1/p] = 0$$

for all $n \geq N+q$. In particular, if F is a field extension of k of finite transcendence degree d over k, then $\mathrm{Fil}^n_{Tate}\Pi_{r,q}\mathcal{E}(F)[1/p] = 0$ for all $n \geq N+q$.

For a more detailed statement, we refer the reader to theorem 6.3.

Remarks 2. 1. We expect that the technical condition, that F needs to be infinite in the case of characteristic 2, is unnecessary.
2. The proof of theorem 3 relies on the Bloch-Kato conjecture.
3. Even after this result, the question remains: what is a reasonable convergence conjecture for a wider class of fields than those of finite cohomological dimension. A reasonable class is those having finite virtual cohomological dimension, e.g. $\mathbb{R}$. As we have seen that the slice tower in this case is not in general convergent, one needs to modify the conjecture. We suggest the following formulation:

Conjecture 4. *Let k be a perfect field of finite virtual cohomological dimension and of characteristic $\neq 2$. Then the 2-completed slice tower is convergent.*

The paper is organized as follows: We set the notation in §1 and recall some basic facts about the slice tower and a model for its terms (the homotopy coniveau tower) in §2. In §3 we use the simplicial nature of the homotopy coniveau tower to analyze the terms in the slice tower. This leads to the main inductive step in our argument (lemma 3.5), and isolates the particular piece that we need to study. This is taken up in §4, where we more precisely identify this piece in terms of a $\mathcal{K}^{MW}_*$-module structure on the bi-graded homotopy sheaves. We use results of Cisinski-Deglise on the relation of $\mathcal{SH}(k)$ with $DM(k)$ in §5 to get information on this $\mathcal{K}^{MW}_*$-module structure, which allows us move the induction forward with proposition 5.9. In the next to last section 6, we assemble all the pieces and prove our main result. The final section §7 collects some results on norm maps for finite field extensions that are used throughout the paper.

1. Background and notation

Unless we specify otherwise, k will be a fixed perfect base field, without restriction on the characteristic. For details on the following constructions, we refer the reader to [4, 5, 6, 10, 11, 13, 14].

We write $[n] := \{0, \ldots, n\}$ (including $[-1] = \emptyset$) and let Δ be the category with objects $[n]$, $n = 0, 1, \ldots$, and morphisms $[n] \to [m]$ the order-preserving maps of sets. Given a category $\mathcal{C}$, the category of simplicial objects in $\mathcal{C}$ is as usual the category of functors $\Delta^{\mathrm{op}} \to \mathcal{C}$.

$\mathbf{Spc}$ will denote the category of simplicial sets, $\mathbf{Spc}_\bullet$ the category of pointed simplicial sets, $\mathcal{H} := \mathbf{Spc}[WE^{-1}]$ the classical unstable homotopy category and $\mathcal{H}_\bullet := \mathbf{Spc}_\bullet[WE^{-1}]$ the pointed version. We denote the suspension operator $-\wedge S^1$ by Σ_s. $\mathbf{Spt}$ is the category of suspension spectra and $\mathcal{SH} := \mathbf{Spt}[WE^{-1}]$ the classical stable homotopy category.

The motivic versions are as follows: $\mathbf{Sm}/k$ is the category of smooth finite type k-schemes. $\mathbf{Spc}(k)$ is the category of $\mathbf{Spc}$-valued presheaves on $\mathbf{Sm}/k$, $\mathbf{Spc}_\bullet(k)$ the $\mathbf{Spc}_\bullet$-valued presheaves, and $\mathbf{Spt}_{S^1}(k)$ the $\mathbf{Spt}$-valued presheaves. These all come with "motivic" model structures as simplicial model categories (see for example [6]); we denote the corresponding homotopy categories by $\mathcal{H}(k)$, $\mathcal{H}_\bullet(k)$ and $\mathcal{SH}_{S^1}(k)$, respectively. Sending $X \in \mathbf{Sm}/k$ to the sheaf of sets on $\mathbf{Sm}/k$ represented by X (also denoted X) gives an embedding of $\mathbf{Sm}/k$ to $\mathbf{Spc}(k)$; we have the similarly defined embedding of the category of smooth pointed schemes over k into $\mathbf{Spc}_\bullet(k)$. The categories $\mathbf{Spc}(k)$ and $\mathbf{Spc}_\bullet(k)$ are equipped with an internal Hom, denoted $\mathcal{H}om$.

Let $\mathbb{G}_m$ be the pointed k-scheme $(\mathbb{A}^1 \setminus 0, 1)$. In $\mathcal{H}_\bullet(k)$ we have the objects $S^{a+b,b} := \Sigma_s^a \mathbb{G}_m^{\wedge b}$, for $b \geq 1$, $S^{n,0} := S^n = \Sigma_s^n \operatorname{Spec} k_+$. If X is a scheme with a k-point x, we write (X, x) for the corresponding object in $\mathbf{Spc}_\bullet(k)$ or $\mathcal{H}_\bullet(k)$. For a cofibration $\mathcal{Y} \to \mathcal{X}$ in $\mathbf{Spc}(k)$, we usually give the quotient $\mathcal{X}/\mathcal{Y}$ the canonical base-point $\mathcal{Y}/\mathcal{Y}$, but on occasion, we will give $\mathcal{X}/\mathcal{Y}$ a base-point coming from a point $x \in \mathcal{X}(k)$; we write this as $(\mathcal{X}/\mathcal{Y}, x)$.

We let $T := \mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\})$ and let $\mathbf{Spt}_T(k)$ denote the category of T-spectra, i.e., spectra in $\mathbf{Spc}_\bullet(k)$ with respect to the T-suspension functor $\Sigma_T := - \wedge T$. $\mathbf{Spt}_T(k)$ has a motivic model structure (see [6]) and $\mathcal{SH}(k)$ is the homotopy category. We can also form the category of spectra in $\mathbf{Spt}_{S^1}(k)$ with respect to Σ_T; with an appropriate model structure the resulting homotopy category is equivalent to $\mathcal{SH}(k)$. We will ignore the subtleties of this distinction and simply identify the two homotopy categories.

Both $\mathcal{SH}_{S^1}(k)$ and $\mathcal{SH}(k)$ are triangulated categories with suspension functor Σ_s. We have the triangle of *infinite suspension functors* Σ^∞ and their right adjoints Ω^∞

$$\begin{array}{ccc} \mathcal{H}_\bullet(k) & \xrightarrow{\Sigma_s^\infty} & \mathcal{SH}_{S^1}(k) \\ & \searrow^{\Sigma_T^\infty} & \downarrow{\scriptstyle \Sigma_T^\infty} \\ & & \mathcal{SH}(k) \end{array} \qquad \begin{array}{ccc} \mathcal{H}_\bullet(k) & \xleftarrow{\Omega_s^\infty} & \mathcal{SH}_{S^1}(k) \\ & \nwarrow^{\Omega_T^\infty} & \uparrow{\scriptstyle \Omega_T^\infty} \\ & & \mathcal{SH}(k) \end{array}$$

both commutative up to natural isomorphism. These are all left, resp. right derived versions of Quillen adjoint pairs of functors on the underlying model categories. We note that the suspension functor $\Sigma_{\mathbb{G}_m}$ is invertible on $\mathcal{SH}(k)$.

For $\mathcal{X} \in \mathcal{H}_\bullet(k)$, we have the bi-graded homotopy sheaf $\Pi_{a,b}\mathcal{X}$, defined for $a, b \geq 0$, as the Nisnevich sheaf associated to the presheaf on $\mathbf{Sm}/k$

$$U \mapsto \operatorname{Hom}_{\mathcal{H}_\bullet(k)}(\Sigma_s^a \Sigma_{\mathbb{G}_m}^b U_+, \mathcal{X});$$

note the perhaps non-standard indexing. These extend in the usual way to bi-graded homotopy sheaves $\Pi_{a,b}E$ for $E \in \mathcal{SH}_{S^1}(k)$, $b \geq 0$, $a \in \mathbb{Z}$, and $\Pi_{a,b}\mathcal{E}$ for $\mathcal{E} \in \mathcal{SH}(k)$, $a, b \in \mathbb{Z}$, by taking the Nisnevich sheaf associated to

$$U \mapsto \operatorname{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma_s^a \Sigma_{\mathbb{G}_m}^b \Sigma_s^\infty U_+, E) \text{ or } U \mapsto \operatorname{Hom}_{\mathcal{SH}(k)}(\Sigma_s^a \Sigma_{\mathbb{G}_m}^b \Sigma_T^\infty U_+, \mathcal{E}),$$

as the case may be. We write π_n for $\Pi_{n,0}$; for e.g. $E \in \mathbf{Spt}_{S^1}(k)$ fibrant, $\pi_n E$ is the Nisnevich sheaf associated to the presheaf $U \mapsto \pi_n(E(U))$.

For F a finitely generated field extension of k, we may view $\operatorname{Spec} F$ as the generic point of some $X \in \mathbf{Sm}/k$. Thus, for a Nisnevich sheaf $\mathcal{S}$ on $\mathbf{Sm}/k$, we may define $\mathcal{S}(F)$ as the stalk of $\mathcal{S}$ at $\operatorname{Spec} F \in X$. For an arbitrary field extension F of k (not necessarily finitely generated over k), we define $\mathcal{S}(F)$ as the colimit over $\mathcal{S}(F_\alpha)$, as F_α runs over subfields of F containing k and finitely generated over k. For a finitely generated field F over k, we consider objects such as $\operatorname{Spec} F$, or $\mathbb{A}^n_F$ as pro-objects in $\mathbf{Spc}(k)$ by the usual system of finite-type models; the same holds for related objects such as $\operatorname{Spec} F_+$ in $\mathcal{H}_\bullet(k)$ or $\mathcal{SH}_{S^1}(k)$, etc. We extend this to arbitrary field extensions of k by taking the system of finitely generated subfields. We will usually not explicitly insert the "pro-" in the text, but all such objects, as well as morphisms and isomorphisms between them, should be so understood.

2. The homotopy coniveau tower

Our computations rely heavily on our model for the Tate-Postnikov tower in $\mathcal{SH}_{S^1}(k)$, which we briefly recall (for details, we refer the reader to [7]). We start by recalling the Tate-Postnikov tower in $\mathcal{SH}_{S^1}(k)$ and introducing some notation.

Fix a perfect base-field k. Let

$$\Sigma_T : \mathcal{SH}_{S^1}(k) \to \mathcal{SH}_{S^1}(k)$$

be the T-suspension functor. For $n \geq 0$, we let $\Sigma^n_T\mathcal{SH}_{S^1}(k)$ be the localizing subcategory of $\mathcal{SH}_{S^1}(k)$ generated by infinite suspension spectra of the form $\Sigma^m_T\Sigma^\infty_s X_+$, with $X \in \mathbf{Sm}/k$ and $m \geq n$. We note that $\Sigma^0_T\mathcal{SH}_{S^1}(k) = \mathcal{SH}_{S^1}(k)$. The inclusion functor $i_n : \Sigma^n_T\mathcal{SH}_{S^1}(k) \to \mathcal{SH}_{S^1}(k)$ admits, by results of Neeman [15], a right adjoint r_n; define the functor $f_n : \mathcal{SH}_{S^1}(k) \to \mathcal{SH}_{S^1}(k)$ by $f_n := i_n \circ r_n$. The unit for the adjunction gives us the natural morphism

$$\rho_n : f_nE \to E$$

for $E \in \mathcal{SH}_{S^1}(k)$; similarly, the inclusion $\Sigma^m_T\mathcal{SH}_{S^1}(k) \subset \Sigma^n_T\mathcal{SH}_{S^1}(k)$ for $n < m$ gives the natural transformation $f_mE \to f_nE$, forming the *Tate-Postnikov tower*

$$\ldots \to f_{n+1}E \to f_nE \to \ldots \to f_0E = E.$$

We complete $f_{n+1}E \to f_nE$ to a distinguished triangle

$$f_{n+1}E \to f_nE \to s_nE \to f_{n+1}E[1];$$

this turns out to be functorial in E. The object s_nE is the *nth slice* of E.

There is an analogous construction in $\mathcal{SH}(k)$: For $n \in \mathbb{Z}$, let

$$\Sigma^n_T\mathcal{SH}^{eff}(k) \subset \mathcal{SH}(k)$$

be the localizing category generated by the T-suspension spectra $\Sigma^m_T\Sigma^\infty_T X_+$, for $X \in \mathbf{Sm}/k$ and $m \geq n$. As above, the inclusion $i_n : \Sigma^n_T\mathcal{SH}^{eff}(k) \to \mathcal{SH}(k)$ admits a left adjoint r_n, giving us the truncation functor f_n and the Tate-Postnikov tower

$$\ldots \to f_{n+1}\mathcal{E} \to f_n\mathcal{E} \to \ldots \to \mathcal{E}.$$

Note that this tower is in general infinite in both directions. We define the layer $s_n\mathcal{E}$ as above. For integers $N \geq n$, we let $\rho_{n,N} : f_N \to f_n$ and $\rho_n : f_n \to \mathrm{id}$ denote the canonical natural transformations. We mention the following elementary but useful result.

Lemma 2.1. *For integers N, n, the diagram of natural endomorphisms of $\mathcal{SH}(k)$*

$$\begin{array}{ccc} f_n \circ f_N & \xrightarrow{\rho_n(f_N)} & f_N \\ {\scriptstyle f_n(\rho_N)}\downarrow & & \downarrow{\scriptstyle \rho_N} \\ f_n & \xrightarrow[\rho_n]{} & \mathrm{id} \end{array}$$

commutes. Moreover, for $N \geq n$, the map $\rho_n(f_N)$ is a natural isomorphism, and for $N \leq n$, the map $f_n(\rho_N)$ is a natural isomorphism.

In case $N, n \geq 0$, the same holds with $\mathcal{SH}_{S^1}(k)$ replacing $\mathcal{SH}(k)$.

Proof. The first assertion is just the naturality of ρ_n with respect to the morphism $\rho_N : f_N \to \mathrm{id}$.

Suppose $N \geq n$. Then $\Sigma^N_T\mathcal{SH}^{eff}(k) \subset \Sigma^n_T\mathcal{SH}^{eff}(k)$ and thus for all $\mathcal{E} \in \mathcal{SH}(k)$, $\mathrm{id} : f_N\mathcal{E} \to f_N\mathcal{E}$ satisfies the universal property of $\rho_n(f_N\mathcal{E}) : f_n(f_N\mathcal{E}) \to f_N\mathcal{E}$, namely, $f_N\mathcal{E}$ is in $\Sigma^n_T\mathcal{SH}^{eff}(k)$ and id : $f_N\mathcal{E} \to f_N\mathcal{E}$ is universal for maps $T \to f_N\mathcal{E}$ with $T \in \Sigma^n_T\mathcal{SH}^{eff}(k)$.

If $N \leq n$, then for $\mathcal{E} \in \mathcal{SH}(k)$, $f_n(f_N\mathcal{E})$ is in $\Sigma^n_T\mathcal{SH}^{eff}(k)$ and $\rho_n(f_N\mathcal{E}) : f_n(f_N\mathcal{E}) \to f_N\mathcal{E}$ is universal for maps $T \to f_N\mathcal{E}$ with $T \in \Sigma^n_T\mathcal{SH}^{eff}(k)$. Since $\Sigma^n_T\mathcal{SH}^{eff}(k) \subset \Sigma^N_T\mathcal{SH}^{eff}(k)$, the universal property of $\rho_N(\mathcal{E}) : f_N\mathcal{E} \to \mathcal{E}$ shows that $\rho_N \circ \rho_n(f_N\mathcal{E}) : f_n(f_N\mathcal{E}) \to \mathcal{E}$ is universal for maps $T \to \mathcal{E}$ with $T \in \Sigma^n_T\mathcal{SH}^{eff}(k)$, and thus $f_n(\rho_N)$ is an isomorphism.

The proof for $\mathcal{SH}_{S^1}(k)$ is the same. □

Lemma 2.2. *For $n \in \mathbb{Z}$, there is a natural isomorphism*

$$f_n\Omega^\infty_T\mathcal{E} \cong \Omega^\infty_T f_n\mathcal{E}. \tag{2.1}$$

Proof. First suppose that $n \geq 0$. It follows from [7, theorem 7.4.1] that $\Omega^\infty_T f_n\mathcal{E}$ is in $\Sigma^n_T\mathcal{SH}_{S^1}(k)$ and thus we need only show that $\Omega^\infty\rho_n : \Omega^\infty_T f_n\mathcal{E} \to \Omega^\infty_T\mathcal{E}$ satisfies the universal property of $f_n\Omega^\infty_T\mathcal{E} \to \Omega^\infty_T\mathcal{E}$. But for $G \in \mathcal{SH}_{S^1}(k)$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma^n_T G, \Omega^\infty_T f_n\mathcal{E}) &\cong \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_T\Sigma^n_T G, f_n\mathcal{E}) \\
&\cong \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^n_T\Sigma^\infty_T G, f_n\mathcal{E}) \\
&\xrightarrow[\sim]{\rho_{n*}} \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^n_T\Sigma^\infty_T G, \mathcal{E}) \\
&\cong \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_T\Sigma^n_T G, \mathcal{E}) \\
&\cong \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma^n_T G, \Omega^\infty_T\mathcal{E}).
\end{aligned}$$

It is easy to check that this sequence of isomorphisms is induced by $(\Omega^\infty_T\rho_n)_*$.

Now suppose that $n < 0$. Then $f_n\Omega^\infty_T\mathcal{E} \cong f_0\Omega^\infty_T\mathcal{E} \cong \Omega^\infty_T f_0\mathcal{E}$, so it suffices to show that the map $f_0\mathcal{E} \to f_n\mathcal{E}$ induces an isomorphism $\Omega^\infty_T f_0\mathcal{E} \to \Omega^\infty_T f_n\mathcal{E}$. But for $F \in \mathcal{SH}_{S^1}(k)$, $\Sigma^\infty_T F$ is in $\mathcal{SH}^{eff}(k)$ and

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(F, \Omega^\infty_T f_0\mathcal{E}) &\cong \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_T F, f_0\mathcal{E}) \\
&\xrightarrow[\sim]{\rho_{n,0}} \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_T F, f_n\mathcal{E}) \\
&\cong \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(F, \Omega^\infty_T f_n\mathcal{E}).
\end{aligned}$$

□

Furthermore, for $E \in \mathcal{SH}_{S^1}(k)$, we have (by [7, theorem 7.4.2]) the canonical isomorphism

$$\Omega_{\mathbb{G}_m} f_n E = f_{n-1}\Omega_{\mathbb{G}_m} E. \tag{2.2}$$

As $\Omega_{\mathbb{G}_m} : \mathcal{SH}(k) \to \mathcal{SH}(k)$ is an auto-equivalence, and restricts to an equivalence

$$\Omega_{\mathbb{G}_m} : \Sigma^n_T\mathcal{SH}^{eff}(k) \to \Sigma^{n-1}_T\mathcal{SH}^{eff}(k),$$

the analogous identity in $\mathcal{SH}(k)$ holds as well.

Definition 2.3. For $a \in \mathbb{Z}$, $b \geq 0$, $E \in \mathcal{SH}_{S^1}(k)$, define the filtration $F^n_{\mathrm{Tate}}\Pi_{a,b}E$, $n \geq 0$, of $\Pi_{a,b}E$ by

$$F^n_{\mathrm{Tate}}\Pi_{a,b}E := \mathrm{im}(\Pi_{a,b}f_nE \to \Pi_{a,b}E).$$

Similarly, for $\mathcal{E} \in \mathcal{SH}(k)$, $a, b, n \in \mathbb{Z}$, define

$$F^n_{\mathrm{Tate}}\Pi_{a,b}\mathcal{E} := \mathrm{im}(\Pi_{a,b}f_n\mathcal{E} \to \Pi_{a,b}\mathcal{E}).$$

The main object of this paper is to understand $F^n_{\rm Tate}\Pi_{a,b}E$ for suitable E. For later use, we note the following

Lemma 2.4. *1. For $E \in \mathcal{SH}_{S^1}(k)$, $n, p, a, b \in \mathbb{Z}$ with $n, p, b, n-p, b-p \geq 0$, the adjunction isomorphism $\Pi_{a,b}E \cong \Pi_{a,b-p}\Omega^p_{\mathbb{G}_m}E$ induces an isomorphism*

$$F^n_{Tate}\Pi_{a,b}E \cong F^{n-p}_{Tate}\Pi_{a,b-p}\Omega^p_{\mathbb{G}_m}E.$$

Similarly, for $\mathcal{E} \in \mathcal{SH}(k)$, $n, p, a, b \in \mathbb{Z}$, the adjunction isomorphism $\Pi_{a,b}\mathcal{E} \cong \Pi_{a.b-p}\Omega^p_{\mathbb{G}_m}\mathcal{E}$ induces an isomorphism

$$F^n_{Tate}\Pi_{a,b}\mathcal{E} \cong F^{n-p}_{Tate}\Pi_{a,b-p}\Omega^p_{\mathbb{G}_m}\mathcal{E}.$$

2. For $\mathcal{E} \in \mathcal{SH}(k)$, $a, b, n \in \mathbb{Z}$, with $b, n \geq 0$, we have a canonical isomorphism

$$\varphi_{\mathcal{E},a,b,n} : \Pi_{a,b}f_n\mathcal{E} \to \Pi_{a,b}f_n\Omega^\infty_T\mathcal{E},$$

inducing an isomorphism $F^n_{Tate}\Pi_{a,b}\mathcal{E} \cong F^n_{Tate}\Pi_{a,b}\Omega^\infty_T\mathcal{E}$.

Proof. (1) By (2.2), adjunction induces isomorphisms

$$\begin{aligned} F^n_{\rm Tate}\Pi_{a,b}E &:= {\rm im}(\Pi_{a,b}f_nE \to \Pi_{a,b}E) \\ &\cong {\rm im}(\Pi_{a,b-p}\Omega^p_{\mathbb{G}_m}f_nE \to \Pi_{a,b-p}\Omega^p_{\mathbb{G}_m}E) \\ &= {\rm im}(\Pi_{a,b-p}f_{n-p}\Omega^p_{\mathbb{G}_m}E \to \Pi_{a,b-p}\Omega^p_{\mathbb{G}_m}E) \\ &= F^{n-p}_{\rm Tate}\Pi_{a,b-p}\Omega^p_{\mathbb{G}_m}E. \end{aligned}$$

The proof for $\mathcal{E} \in \mathcal{SH}(k)$ is the same.

For (2), the isomorphism $\varphi_{\mathcal{E},a,b,n}$ arises from (2.1) and the adjunction isomorphism

$$\begin{aligned} {\rm Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma^a_s\Sigma^b_{\mathbb{G}_m}\Sigma^\infty_sU_+, f_n\Omega^\infty_T\mathcal{E}) &\cong {\rm Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma^a_s\Sigma^b_{\mathbb{G}_m}\Sigma^\infty_sU_+, \Omega^\infty_Tf_n\mathcal{E}) \\ &\cong {\rm Hom}_{\mathcal{SH}(k)}(\Sigma^a_s\Sigma^b_{\mathbb{G}_m}\Sigma^\infty_TU_+, f_n\mathcal{E}). \end{aligned}$$

□

We now turn to a discussion of our model for $f_nE(X)$, $X \in \mathbf{Sm}/k$. We start with the cosimplicial scheme $n \mapsto \Delta^n$, with Δ^n the *algebraic n-simplex* ${\rm Spec}\, k[t_0, \ldots, t_n]/\sum_i t_i - 1$. The cosimplicial structure is given by sending a map $g : [n] \to [m]$ to the map $g : \Delta^n \to \Delta^m$ determined by

$$g^*(t_i) = \begin{cases} \sum_{j,g(j)=i} t_j & \text{if } g^{-1}(i) \neq \emptyset \\ 0 & \text{else.} \end{cases}$$

A *face* of Δ^m is a closed subscheme F defined by equations $t_{i_1} = \ldots = t_{i_r} = 0$; we let $\partial\Delta^n \subset \Delta^n$ be the closed subscheme defined by $\prod_{i=0}^n t_i = 0$, i.e., $\partial\Delta^n$ is the union of all the proper faces.

Take $X \in \mathbf{Sm}/k$. We let $\mathcal{S}^{(q)}_X(m)$ denote the set of closed subsets $W \subset X \times \Delta^m$ such that ${\rm codim}_{X\times F} W \cap X \times F \geq q$ for all faces $F \subset \Delta^m$ (including $F = \Delta^m$). We make $\mathcal{S}^{(q)}_X(m)$ into a partially ordered set via inclusions of closed subsets. Sending m to $\mathcal{S}^{(q)}_X(m)$ and $g : [n] \to [m]$ to $g^{-1} : \mathcal{S}^{(q)}_X(m) \to \mathcal{S}^{(q)}_X(n)$ gives us the simplicial poset $\mathcal{S}^{(q)}_X$.

Now take $E \in \mathbf{Spt}_{S^1}(k)$. For $X \in \mathbf{Sm}/k$ and closed subset $W \subset X$, we have the spectrum with supports $E^W(X)$ defined as the homotopy fiber of the restriction map $E(X) \to E(X \setminus W)$. This construction is functorial in the pair (X, W), where we define a map $f:(Y,T)\to(X,W)$ as a morphism $f:Y\to X$ in $\mathbf{Sm}/k$ with $f^{-1}(W)\subset T$.

Define

$$E^{(q)}(X,m) := \operatorname*{hocolim}_{W\in\mathcal{S}^{(q)}_X(m)} E^W(X\times\Delta^m). \tag{2.3}$$

The fact that $m\mapsto\mathcal{S}^{(q)}_X(m)$ is a simplicial poset, and $(Y,T)\mapsto E^T(Y)$ is a functor from the category of pairs to spectra shows that $m\mapsto E^{(q)}(X,m)$ defines a simplicial spectrum. We denote the associated total spectrum by $E^{(q)}(X)$.

For $q\geq q'$, the inclusions $\mathcal{S}^{(q)}_X(m)\subset\mathcal{S}^{(q')}_X(m)$ induces a map of simplicial posets $\mathcal{S}^{(q)}_X\subset\mathcal{S}^{(q')}_X$ and thus a morphism of spectra $i_{q',q}:E^{(q)}(X)\to E^{(q')}(X)$. We have as well the natural map

$$\epsilon_X : E(X)\to \mathrm{Tot}(E(X\times\Delta^*)) = E^{(0)}(X),$$

which is a weak equivalence if E is homotopy invariant. Together, this forms the *augmented homotopy coniveau tower*

$$E^{(*)}(X) := \ldots\to E^{(q+1)}(X)\xrightarrow{i_q}E^{(q)}(X)\xrightarrow{i_{q-1}}\ldots E^{(1)}(X)\xrightarrow{i_0}E^{(0)}(X)\xleftarrow{\epsilon_X}E(X)$$

with $i_q:=i_{q,q+1}$. Thus, for homotopy invariant E, we have the homotopy coniveau tower in $\mathcal{SH}$

$$E^{(*)}(X) := \ldots\to E^{(q+1)}(X)\xrightarrow{i_q}E^{(q)}(X)\xrightarrow{i_{q-1}}\ldots E^{(1)}(X)\xrightarrow{i_0}E^{(0)}(X)\cong E(X).$$

Letting $\mathbf{Sm}/\!\!/k$ denote the subcategory of $\mathbf{Sm}/k$ with the same objects and with morphisms the smooth morphisms, it is not hard to see that sending X to $E^{(*)}(X)$ defines a functor from $\mathbf{Sm}/\!\!/k^{\mathrm{op}}$ to augmented towers of spectra.

On the other hand, for $E\in\mathbf{Spt}_{S^1}(k)$, we have the (augmented) Tate-Postnikov tower

$$f_*E := \ldots\to f_{q+1}E\to f_qE\to\ldots\to f_0E\cong E$$

in $\mathcal{SH}_{S^1}(k)$, which we may evaluate at $X\in\mathbf{Sm}/k$, giving the tower $f_*E(X)$ in $\mathcal{SH}$, augmented over $E(X)$.

As a direct consequence of [7, theorem 7.1.1] we have

Theorem 2.5. *Let E be a quasi-fibrant object in $\mathbf{Spt}_{S^1}(k)$ for the $\mathbb{A}^1$-model structure (see e.g.* [6] *or* [13]*), and take $X\in\mathbf{Sm}/k$. Then there is an isomorphism of augmented towers in $\mathcal{SH}$*

$$(f_*E)(X)\cong E^{(*)}(X)$$

over the identity on $E(X)$, which is natural with respect to smooth morphisms in $\mathbf{Sm}/k$.

In particular, we may use the explicit model $E^{(q)}(X)$ to understand $(f_qE)(X)$.

Remark 2.6. For $X,Y\in\mathbf{Sm}/k$ with given k-points $x\in X(k)$, $y\in Y(k)$, we have a natural isomorphism in $\mathcal{SH}_{S^1}(k)$

$$\Sigma^\infty_s(X\wedge Y)\oplus\Sigma^\infty_s(X\vee Y)\cong\Sigma^\infty_s(X\times Y)$$

i.e. $\Sigma^\infty_s(X\wedge Y)$ is a canonically defined summand of $\Sigma^\infty_s(X\times Y)$. In particular for E a quasi-fibrant object of $\mathbf{Spt}_{S^1}(k)$, we have a natural isomorphism in $\mathcal{SH}$

$$\mathcal{H}om(\Sigma^\infty_s(X\wedge Y),E)\cong\mathrm{hofib}\left(E(X\times Y)\to\mathrm{hofib}(E(X)\oplus E(Y)\to E(k))\right)$$

where the maps are induced by the evident restriction maps. In particular, we may define $E(X \wedge Y)$ via the above isomorphism, and our comparison results for Tate-Postnikov tower and homotopy coniveau tower extend to values at smash products of smooth pointed schemes over k.

3. Some spectral sequences

Lemma 3.1. *Let S be a smooth k-scheme, $W \subset S \times \mathbb{A}^1$ a closed subset such that $p : W \to S$ is finite. Let $E \in \mathbf{Spt}(k)$ be weakly fibrant for the $\mathbb{A}^1$-model structure. Then the map induced by the inclusion $i : W \to p^{-1}(p(W))$ induces the zero map*

$$i_* : \pi_*(E^W(S \times \mathbb{A}^1)) \to E^{p^{-1}(p(W))}(S \times \mathbb{A}^1)$$

Proof. We steal a proof of Morel: Let $Z = p(W)$ and let $\bar{p} : S \times \mathbb{P}^1 \to S$ be the projection. Since W is finite over $S \times \mathbb{A}^1$, W is closed in $S \times \mathbb{P}^1$, so we have the following commutative diagram

$$\begin{array}{ccc} \pi_r(E^W(S \times \mathbb{P}^1))) & \xrightarrow{\bar{i}_*} & \pi_r(E^{Z \times \mathbb{P}^1}(S \times \mathbb{P}^1)) \\ \downarrow{\scriptstyle j_0^*} & & \downarrow{\scriptstyle j_0^*} \\ \pi_r(E^W(S \times \mathbb{A}^1))) & \xrightarrow{i_*} & \pi_r(E^{Z \times \mathbb{A}^1}(S \times \mathbb{A}^1)) \end{array} \tag{3.1}$$

where $j_0 : S \times \mathbb{A}^1 \to S \times \mathbb{P}^1$ is the inclusion. On the other hand, let $i_\infty : S \to S \times \mathbb{P}^1$ be the infinity section. Since $W \cap S \times \infty = \emptyset$, the composition

$$\pi_r(E^W(S \times \mathbb{P}^1)) \xrightarrow{\bar{i}_*} \pi_r(E^{Z \times \mathbb{P}^1}(S \times \mathbb{P}^1)) \xrightarrow{i_\infty^*} \pi_r(E^{Z \times \infty}(S \times \infty))$$

is the zero map. Letting $j_\infty : S \times \mathbb{A}^1 \to S \times \mathbb{P}^1$ be the standard open neighborhood of $S \times \infty$ in $S \times \mathbb{P}^1$, the restriction map

$$i_\infty^* : \pi_r(E^{Z \times \mathbb{A}^1 y}(S \times \mathbb{A}^1)) \to \pi_r(E^{Z \times \infty}(S \times \infty))$$

is an isomorphism, hence

$$j_\infty^* \circ \bar{i}_* : \pi_r(E^W(S \times \mathbb{P}^1)) \to \pi_r(E^{Z \times \mathbb{A}^1}(S \times \mathbb{A}^1))$$

is the zero map. Write $j_0, j_\infty : S \times \mathbb{A}^1 \setminus \{0\} \to S \times \mathbb{A}^1$ for the inclusions of $j_0(S \times \mathbb{A}^1) \cap j_\infty(S \times \mathbb{A}^1)$ into $j_0(S \times \mathbb{A}^1)$, $j_\infty(S \times \mathbb{A}^1)$, respectively. Combining (3.1) with the commutativity of the diagram

$$\begin{array}{ccc} \pi_r(E^W(S \times \mathbb{P}^1)) & \xrightarrow{j_{\infty *}} & \pi_r(E^{Z \times \mathbb{A}^1}(S \times \mathbb{A}^1)) \\ \downarrow{\scriptstyle j_0^*} & & \downarrow{\scriptstyle j_0^*} \\ \pi_r(E^W(S \times \mathbb{A}^1)) & \xrightarrow[j_{\infty *}]{} & \pi_r(E^{Z \times \mathbb{A}^1 \setminus \{0\}}(S \times \mathbb{A}^1 \setminus \{0\})) \end{array}$$

we see that

$$j_\infty^* \circ i_* : \pi_r(E^W(S \times \mathbb{A}^1)) \to \pi_r(E^{Z \times \mathbb{A}^1 \setminus \{0\}}(S \times \mathbb{A}^1 \setminus \{0\}))$$

is the zero map. From the long exact localization sequence

$$\ldots \to \pi_r(E^{Z \times 0}(S \times \mathbb{A}^1)) \xrightarrow{i_{0*}} \pi_r(E^{Z \times \mathbb{A}^1}(S \times \mathbb{A}^1)) \\ \xrightarrow{j_\infty^*} \pi_r(E^{Z \times \mathbb{A}^1 \setminus \{0\}}(S \times \mathbb{A}^1 \setminus \{0\})) \to \ldots$$

we see that

$$i_*(\pi_r(E^W(S\times\mathbb{A}^1)))\subset i_{0*}(\pi_r(E^{Z\times 0}(S\times\mathbb{A}^1)))\subset \pi_r(E^{Z\times\mathbb{A}^1}(S\times\mathbb{A}^1)).$$

But

$$i_{0*}:\pi_r(E^{Z\times 0}(S\times\mathbb{A}^1))\to\pi_r(E^{Z\times\mathbb{A}^1}(S\times\mathbb{A}^1))$$

is the zero map, since $i_1^*:\pi_r(E^{Z\times\mathbb{A}^1}(S\times\mathbb{A}^1))\to\pi_r(E^{Z\times 1}(S\times 1))$ is an isomorphism and $i_1^*\circ i_{0*}=0$. □

Lemma 3.2. *Suppose F is infinite. Take $W\in\mathcal{S}_F^{(n)}(p)$ and suppose* $\operatorname{codim}_{\Delta^p_F}(W)>n$. *Then the canonical map $E^W(\Delta^p_F)\to E^{(n)}(\operatorname{Spec}F,p)$ induces the zero map on π_*.*

Proof. We identify Δ^p with $\mathbb{A}^p$ as usual via the barycentric coordinates $t_1,\ldots,t_p$. Suppose W has dimension $d<p-n$. Then $d\le p-1$ and a general linear projection $L:\mathbb{A}^p\to\mathbb{A}^{p-1}$ restricts to W to a finite morphism $W\to\mathbb{A}^{p-1}$. In addition, for L suitably general, $W':=L^{-1}(L(W))$ is in $\mathcal{S}_F^{(n)}(p)$. Letting $i:W\to W'$ be the inclusion, it suffices to show that the map

$$\pi_*E^W(\Delta^p_F)\to pi_*E^{W'}(\Delta^p_F)$$

is the zero map. Via an affine linear change of coordinates on Δ^p, we may identify Δ^p with $\mathbb{A}^{p-1}\times\mathbb{A}^1$ and $L:\mathbb{A}^p\to\mathbb{A}^{p-1}$ with the projection $\mathbb{A}^{p-1}\times\mathbb{A}^1\to\mathbb{A}^{p-1}$. The result thus follows from lemma 3.1. □

Let $(\Delta^p_F,\partial\Delta^p)^{(n)}$ be the set of codimension n points w of Δ^p_F such that $\overline{\{w\}}$ is in $\mathcal{S}_F^{(n)}(p)$.

Lemma 3.3. *Let F be an infinite field. Then the restriction maps*

$$E^W(\Delta^p_F)\to\oplus_{w\in(\Delta^p_F,\partial\Delta^p)^{(n)}\cap W}E^w(\operatorname{Spec}\mathcal{O}_{\Delta^p_F,w})$$

for $W\in\mathcal{S}_F^{(n)}(p)$ defines an injection

$$\pi_r(E^{(n)}(F,p))\to\oplus_{w\in(\Delta^p_F,\partial\Delta^p)^{(n)}}\pi_rE^w(\operatorname{Spec}\mathcal{O}_{\Delta^p_F,w}).$$

for each $r\in\mathbb{Z}$.

Proof. Take $W\in\mathcal{S}_F^{(n)}(p)$. Since Δ^p_F is affine, we can find a $W'\in\mathcal{S}_F^{(n)}(p)$ of pure codimension n with $W'\supset W$: just take a sufficiently general collection of n functions $f_1,\ldots,f_n$ vanishing on W and let W' be the common zero locus of the f_i. Thus the set of pure codimension n subsets W' of Δ^p_F with $W'\in\mathcal{S}_F^{(n)}(p)$ is cofinal in $\mathcal{S}_F^{(n)}(p)$.

Let $W\in\mathcal{S}_F^{(n)}(p)$ have pure codimension n on Δ^p_F and let $W_0\subset W$ be any closed subset. Then W_0 is also in $\mathcal{S}_F^{(n)}(p)$ and we have the long exact localization sequence

$$\ldots\to\pi_rE^{W_0}(\Delta^p_F)\xrightarrow{i_{W_0*}}\pi_rE^W(\Delta^p_F)\to\pi_rE^{W\setminus W_0}(\Delta^p_F\setminus W_0)\to\ldots$$

Let $\mathcal{S}_F^{(n)}(p)_0\subset\mathcal{S}_F^{(n)}(p)$ be the set of all $W_0\in\mathcal{S}_F^{(n)}(p)$ with $\operatorname{codim}_{\Delta^p_F}W_0>n$. Let

$$E^{(n)}(F,p)_0=\operatorname*{hocolim}_{W_0\in\mathcal{S}_F^{(n)}(p)_0}E^{W_0}(\Delta^p_F).$$

Passing to the limit over the above localization sequences gives us the long exact sequence

$$\ldots \to \pi_r E^{(n)}(F,p)_0 \xrightarrow{i_{0*}} \pi_r E^{(n)}(F,p) \to \oplus_{w\in(\Delta^p_F,\partial\Delta^p)^{(n)}} \pi_r E^w((\mathrm{Spec}\,\mathcal{O}_{\Delta^p_F,w})) \to \ldots$$

By lemma 3.2, the map i_{0*} is the zero map, which proves the lemma. □

Let $S : \Delta^{\mathrm{op}} \to \mathbf{Spt}$ be a simplicial spectrum, giving us the spectral sequence

$$E^1_{p,q} = \pi_q S(p) \Longrightarrow \pi_{p+q} S,$$

inducing an increasing filtration $\mathrm{Fil}^{simp}_* \pi_r S$ on $\pi_r S$. We have the q-truncated simplicial spectrum $S_{\le q}$ and $\mathrm{Fil}^{simp}_q \pi_r S$ is just the image of $\pi_r S_{\le q}$ in $\pi_r S$. In particular $\mathrm{Fil}^{simp}_{-1} \pi_r S = 0$ and $\cup_{q=0}^\infty \mathrm{Fil}^{simp}_q \pi_r S = \pi_r S$, so the spectral sequence is weakly convergent, and is strongly convergent if for instance there is an integer n_0 such that $S(p)$ is n_0-connected for all p.

The isomorphism of theorem 2.5 thus gives us the weakly convergent spectral sequence

$$E^1_{p,q}(X,E,n) = \pi_q E^{(n)}(X,p) \Longrightarrow \pi_{p+q} f_n E(X) \tag{3.2}$$

which is strongly convergent if $\pi_q E^{(n)}(X,p) = 0$ for $q < q_0$, independent of p. This gives us the increasing filtration $\mathrm{Fil}^{simp}_* \pi_r f_n E(X)$ of $\pi_r f_n E(X)$ with associated graded

$$\mathrm{gr}^{simp}_p \pi_r f_n E(X) = E^\infty_{p,r-p}.$$

Lemma 3.4. *Suppose that k is infinite and that $\Pi_{a,*}E(K) = 0$ for $a < 0$ and all fields K over k. Let $F \supset k$ be a field extension of k. Then*

1. *$E^1_{p,r-p}(F,E,n) = 0$ for $p > r+n$ and $\mathrm{Fil}^{simp}_{r+n} \pi_r f_n E(F) = \pi_r f_n E(F)$.*
2. *$E^1_{p,q}(F,E,n)$ is isomorphic to a subgroup of $\oplus_{w\in(\Delta^p_F,\partial\Delta^p)^{(n)}} \Pi_{q+n,n} E(k(w))$.*

In particular, the spectral sequence (3.2) *is strongly convergent.*

Proof. Since the spectral sequence is weakly convergent, to prove (1) it suffices to show that $E^1_{p,r-p} = 0$. for $p > r+n$. This follows from (2) as our hypothesis implies that $\Pi_{a,*}E(k(w)) = 0$ for $a < 0$, $w \in \Delta^p_F$.

For (2), lemma 3.3 gives us an inclusion

$$E^1_{p,q} = \pi_q E^{(n)}(F,p) \subset \oplus_{w\in(\Delta^p_F,\partial\Delta^p)^{(n)}} \pi_q E^w(\mathrm{Spec}\,\mathcal{O}_{\Delta^p_F,w}).$$

Take $w \in (\Delta^p_F, \partial\Delta^p)^{(n)}$. By the Morel-Voevodsky purity isomorphism [14, *loc.cit.]*, we have $E^w(\mathrm{Spec}\,\mathcal{O}_{\Delta^p_F,w}) \cong \mathcal{H}om(\Sigma^n_{S^1}\Sigma^n_{\mathbb{G}_m} w_+, E)$, hence

$$\pi_q E^w(\mathrm{Spec}\,\mathcal{O}_{\Delta^p_F,w}) \cong \Pi_{q+n,n} E(k(w)),$$

which proves (2). □

For an integer $N \ge 0$, and $E \in \mathcal{SH}_{S^1}(k)$ and $X \in \mathbf{Sm}/k$, the map $\rho_N(E) : f_N E \to E$ induces the map of simplicial spectra

$$\rho_N(E)^{(n)}(X) : (f_N E)^{(n)}(X,-) \to E^{(n)}(X,-)$$

and thereby a map of spectral sequences

$$\rho_N(E)^{(n)}(X) : E^*_{*,*}(X, f_N E, n) \to E^*_{*,*}(X,E,n)$$

Via our natural isomorphism $E^{(n)}(X,-)\cong f_nE(X)$, the map $\rho_N(E)^{(n)}(X)$ is the same as the map

$$f_n(\rho_N(E)(X) : f_n(f_NE)(X) \to f_nE(X).$$

Using lemma 2.1, we have the natural isomorphism $f_n(f_NE)(X)\cong f_NE(X)$ for $N\ge n$, under which $f_n(\rho_N(E))(X)$ becomes the map $\rho_{n,N}$. Thus, $\rho_N(E)^{(n)}(X)$ is isomorphic to the map $\rho_{n,N}$.

Lemma 3.5. *Let E be in $\mathcal{SH}_{S^1}(k)$ and suppose $\Pi_{a,*}E(K)=0$ for $a<0$ and all fields K over k. Let p be the exponential characteristic of k; in case $p=2$, we suppose that k is infinite. Suppose we have functions*

$$N_0(d,E), N_1(d,E),\ldots,N_{r-1}(d,E); d\ge 0,$$

such that, for each field K of transcendence degree $\le d$ over k, each $j=0,\ldots,r-1$, and all integers $m\ge N_j(d,E)$, $q\ge 0$, $M\ge 0$,

$$F^{m+q}_{Tate}\Pi_{j,q}f_ME(K)[1/p]=0. \tag{3.3}$$

Let F be a field of transcendence degree d over k and fix an integer $n\ge 0$. Let $N=rn+\sum_{j=0}^{r-1}N_j(r-j+d,E)$. Then for all integers $m\ge N$, $q\ge 0$, $M\ge 0$, $n\ge 0$, we have

$$f_n(\rho_m(f_ME))(F)(\pi_r(f_n(f_mf_ME))(F))[1/p]\subset \mathrm{Fil}^{simp}_n\,\pi_r(f_n(f_ME))(F)[1/p].$$

Proof. It clearly suffices to prove the result for $m=N$. We delete the "$[1/p]$" from the notation in the proof, using the convention that we have inverted the exponential characteristic p throughout.

If k is a finite field of odd characteristic, fix a prime ℓ and let k_ℓ be the union of all ℓ-power extensions of k. If we know the result for k_ℓ, then using proposition 7.1(2) for each $k\subset k'\subset k_\ell$, with k' finite over k proves the result for k, after inverting ℓ. Doing the same for some $\ell'\neq\ell$, we reduce to the case of an infinite field k.

Let $n_0=N$, and define n_j, $j=1,\ldots,r$ inductively by $n_{j+1}=n_j-N_j(r-j+d)-n$; this yields $n_r=0$. We factor ρ_N as $\rho_{n_{r-1}}\circ\rho_{n_{r-1},n_{r-2}}\circ\ldots\circ\rho_{n_1,n_0}$. As $\mathrm{Fil}^{simp}_{r+n}\,\pi_r(f_ME)^{(n)}(F,-)=\pi_r(f_ME)^{(n)}(F,-)$ by lemma 3.4, it suffices to prove the following

Claim. *Under the hypotheses of the lemma, let $b\ge a\ge 0$ be integers with $b\ge N_j(r-j+d)+n$. Then*

$$\rho_{a,b}(f_ME)^{(n)}(F)(\mathrm{Fil}^{simp}_{r+n-j}\,\pi_r(f_bf_ME)^{(n)}(X,-))\subset \mathrm{Fil}^{simp}_{r+n-j-1}\,\pi_r(f_af_ME)^{(n)}(X,-).$$

Proof of claim. For this, it suffices to show that $\rho_{a,b}(f_ME)^{(n)}(F)$ induces the zero map on $E^\infty_{r+n-j,j-n}$, and thus it suffices to see that $\rho_{a,b}(f_ME)^{(n)}(F)$ induces the zero map on $E^1_{r+n-j,j-n}$. By lemma 3.4(2),

$$E^1_{r+n-j,j-n}\subset \oplus_{w\in(\Delta_F^{r+n-j},\partial\Delta^p)^{(n)}}\Pi_{j,n}f_bf_ME(k(w)),$$

and $\rho_{a,b}(f_ME)^{(n)}(F)$ on $E^1_{r+n-j,j-n}$ is the map induced by the maps

$$\rho_{a,b}f_ME(k(w)):\Pi_{j,n}f_bf_ME(k(w))\to\Pi_{j,n}f_af_ME(k(w)). \tag{3.4}$$

Note that w has codimension n on Δ_F^{r+n-j}, hence $k(w)$ has transcendence dimension $r-j$ over F and thus has transcendence dimension $r-j+d$ over k. As $b\ge N_j(r-j+d)+n$, our hypothesis (3.3) implies

$$F^b_{\mathrm{Tate}}\Pi_{j,n}f_af_ME(k(w))=0,$$

and thus the map (3.4) is the zero map, as desired. □

This completes the proof of the lemma. □

4. The bottom of the filtration

In this section, k will be a fixed perfect base field; if $\mathrm{char}\, k = 2$, we suppose that k is an infinite field.

Lemma 4.1. *Let E be in $\mathcal{SH}_{S^1}(k)$. Then $\mathrm{Fil}^{simp}_{n-1}\,\pi_r f_n E(F) = 0$ for all fields F over k.*

Proof. For any $X \in \mathbf{Sm}/k$, $\mathrm{Fil}^{simp}_q\,\pi_r E^{(n)}(X,-)$ is by definition the image in $\pi_r E^{(n)}(X,-)$ of $\pi_r E^{(n)}(X, - \le q)$, where $E^{(n)}(X, - \le q)$ is the q-truncated simplicial spectrum associated to $E^{(n)}(X,-)$. For $X = \mathrm{Spec}\, F$, we clearly have $\mathcal{S}^{(n)}_F(p) = \emptyset$ for $p < n$, as Δ^p_F has no closed subsets of codimension $> p$. Thus $E^{(n)}(X, - \le q)$ is the 0-spectrum for $q < n$ and hence $\mathrm{Fil}^{simp}_{n-1}\,\pi_r E^{(n)}(F,-) = 0$. □

To study the first non-zero layer $\mathrm{Fil}^{simp}_n\,\pi_r f_n E(F)$ in $\mathrm{Fil}^{simp}_*\,\pi_r f_n E(F)$, we apply the results of [9]. For this, we need to recall some of these results and constructions.

We let $V_n = (\Delta^1_F \setminus \partial\Delta^1)^n$. The function $-t_1/t_0$ on Δ^1 gives an open immersion $\rho_n : V_n \to \mathbb{A}^n$, identifying V_n with $(\mathbb{A}^1 \setminus \{0,1\})^n$.

Suppose that E is an n-fold T-loop spectrum, that is, there is an object $\omega_T^{-n}E \in \mathbf{Spt}(k)$ and an isomorphism $E \cong \Omega^n_T \omega_T^{-n} E$ in $\mathcal{SH}_{S^1}(k)$. Given an n-fold delooping $\omega_T^{-n}E$ of E , we have explained in [9, §5] how to construct a "transfer map"

$$\mathrm{Tr}^*_{F(w)/F} : \pi_* E(w) \to \pi_* E(F),$$

for each closed point $w \in \mathbb{A}^n_F$, separable over F.

If now E is the zero-space of some T-spectrum $\mathcal{E} \in \mathcal{SH}(k)$, $E = \Omega^\infty_T \mathcal{E}$, then the bi-graded homotopy sheaves $\Pi_{*,*}E$ admit a canonical right action by the bi-graded homotopy sheaves of the the sphere spectrum $\mathbb{S}_k \in \mathcal{SH}(k)$:

$$\Pi_{a,b}E \otimes \Pi_{p,q}(\mathbb{S}_k) \to \Pi_{a+p,b+q}E$$

Morel's theorem [13, theorem 6.4.1] identifying $\Pi_{0,n}(\mathbb{S}_k)$ with the Milnor-Witt sheaf $\mathcal{K}^{MW}_{-n}$ gives the right action

$$\Pi_{a,b}E \otimes \mathcal{K}^{MW}_n \to \Pi_{a,b-n}E.$$

This gives us the filtration $F^n_{MW}\Pi_{a,b}E$ of $\Pi_{a,b}E$, defined by

$$F^n_{MW}\Pi_{a,b}E := \mathrm{im}[\Pi_{a,n}E \otimes \underline{K}^{MW}_{n-b} \to \Pi_{a,b}E]; \quad n \ge 0.$$

Completing with respect to the transfer maps gives us the filtration $F^n_{MW^{Tr}}\Pi_{a,b}E(F)$:

Definition 4.2 (see [9, definition 7.9])**.** Let $E = \Omega^\infty_T \mathcal{E}$ for some $\mathcal{E} \in \mathcal{SH}(k)$, F a field extension of k. Take integers a, b, n with $n, b \ge 0$.

1. Let $F^n_{MW^{Tr}}\Pi_{a,b}E(F)$ denote the subgroup of $\Pi_{a,b}E(F)$ generated by elements of the form

$$Tr_F(w)^*(x); \quad x \in F^n_{MW}\Pi_{a,b}E(F(w))$$

as w runs over closed points of $\mathbb{A}^n_F$, separable over F.

2. Let $[\Pi_{a,b}E\cdot\mathcal{I}^n]^{Tr}(F)$ denote the subgroup of $\Pi_{a,b}E(F)$ generated by elements of the form

$$Tr_F(w)^*(x\cdot y);\quad x\in\Pi_{a,b}E(F(w)), y\in I(F(w))^n,$$

as w runs over closed points of $\mathbb{A}^n_F$, separable over F.

The main result, theorem 7.11 of [9], expresses the "Tate-Postnikov" filtration on $\Pi_{0,q}E$ in terms of the "Milnor-Witt" filtration $F^*_{MW}\Pi_{0,q}E$, under the connectivity assumption $\Pi_{a,*}E = 0$ for $a < 0$. Since we will not assume that $\Pi_{a,*}E = 0$ for $a < r$, but only $\Pi_{a,*}E = 0$ for $a < 0$, we cannot directly apply theorem 7.11 of *loc. cit.* to give information on $\Pi_{r,q}f_n\mathcal{E}$. However, the only use of the assumption $\Pi_{a,*}E = 0$ for $a < 0$ was to show that the map

$$\mathrm{Fil}^{simp}_n\Pi_{0,q}f_nE(F)\to\Pi_{0,q}f_nE(F)$$

is surjective (*cf.* lemma 3.4). If we merely replace $\Pi_{r,q}f_nE(F)$ with the subgroup $\mathrm{Fil}^{simp}_n\Pi_{r,q}f_nE(F)$, the proof of [9, theorem 7.11] goes through without change to prove

Theorem 4.3. *Let $\mathcal{E}$ be in $\mathcal{SH}(k)$ and let $E = \Omega^\infty_T\mathcal{E}$. Suppose that $\Pi_{a,*}E = 0$ for $a < 0$. Let F be a perfect field containing k. Then for all $r\geq 0$, $n\geq 0$, $q\geq 0$, we have an equality of subgroups of $\Pi_{r,q}E(F)$*

$$\rho_n(\mathrm{Fil}^{simp}_n\Pi_{r,q}f_nE(F)) = F^n_{MW^{Tr}}\Pi_{r,q}E(F)$$

5. Comparison with motivic cohomology

We recall that Morel [13, theorem 6.3.3] has defined a ring homomorphism

$$\theta_k : K^{MW}_0(k)\to\mathrm{End}_{\mathcal{SH}(k)}(\mathbb{S}_k)$$

which he shows is an isomorphism if k is a perfect field of characteristic $\neq 2$.

Furthermore, Morel [13, section 6.1] has considered the action of $\mathbb{Z}/2$ on the sphere spectrum $\mathbb{S}_k$ arising from the exchange of factors

$$\tau:\mathbb{P}^1\wedge\mathbb{P}^1\to\mathbb{P}^1\wedge\mathbb{P}^1.$$

We also write τ for the induced autormorphism of the sphere spectrum $\mathbb{S}_k$. Morel identifies the corresponding element of $\mathrm{End}(\mathbb{S}_k)$ as

$$\tau := \theta_k(1+\eta\cdot[-1]).$$

After inverting 2, the action of τ decomposes $\mathbb{S}_k[\frac{1}{2}]$ into its $+1$ and -1 eigenspaces

$$\mathbb{S}_k[\tfrac{1}{2}] = \mathbb{S}^+_k[\tfrac{1}{2}]\oplus\mathbb{S}^-_k[\tfrac{1}{2}];$$

as $\mathbb{S}_k[\frac{1}{2}]$ is the unit in the tensor category $\mathcal{SH}(k)[\frac{1}{2}]$, this induces a decomposition of $\mathcal{SH}(k)[\frac{1}{2}]$ as

$$\mathcal{SH}(k)[\tfrac{1}{2}] = \mathcal{SH}(k)^+\times\mathcal{SH}(k)^-.$$

This extends to a decomposition of $\mathcal{SH}(k)_\mathbb{Q}$ as $\mathcal{SH}(k)^+_\mathbb{Q}\times\mathcal{SH}(k)^-_\mathbb{Q}$. For an object $\mathcal{E}$ of $\mathcal{SH}(k)$, we write the corresponding factors of $\mathcal{E}_\mathbb{Q}$ as $\mathcal{E}^+_\mathbb{Q}$, $\mathcal{E}^-_\mathbb{Q}$.

Lemma 5.1. *Suppose that either*

(1) $\mathrm{char}\, k = 0$ *and k has finite 2-cohomological dimension*
(2) $\mathrm{char}\, k > 0$.

Then $2^{N+1}\cdot\eta = 0$*, where in case (1),* $N = \mathrm{cd}_2\, k$*, and in case (2)* $N = 0$ *if* $\mathrm{char}\, k = 2$ *or if* $\mathrm{char}\, k = p$ *is odd and* $p \equiv 1 \mod 4$*,* $N = 1$ *if* $p \equiv 3 \mod 4$*. Moreover, letting* $I \subset \mathrm{GW}(k)$ *be the augmentation ideal, we have* $I^n = 0$ *for* $n > \mathrm{cd}_2\, k$*, assuming* k *has characteristic* $\neq 2$*.*

Proof. We first assume that k has characteristic 0. Let $I \subset \mathrm{GW}(k)$ be the augmentation ideal. By the Milnor conjecture [19], we have

$$I^n/I^{n+1} \cong H^n_{\text{ét}}(k, \mu_2^{\otimes n})$$

and thus $I^n = I^{n+1}$ for $n \geq N+1$. It also follows that $2 \cdot I^n \subset I^{n+1}$, although this may be proven by much more elementary means (see e.g. [18]). By the theorem of Arason-Pfister [1], $\cap_n I^n = \{0\}$, hence $I^{N+1} = 0$, and thus 2^{N+1} kills the Witt group $W(k)$.

As a $K^{MW}_0(k)$-module, $K^{MW}_{-1}(k)$ is cyclic with generator η. However,

$$K^{MW}_{-1}(k) \cong W(k)$$

[13] and thus $2^{N+1}\eta = 0$.

If k has characteristic different from 2, then the same argument shows that $I^{N+1} = 0$, where $N = \mathrm{cd}_2\, k$.

In case k has characteristic $p > 0$, then as η comes from base extension from the prime field $\mathbb{F}_p$, it suffices to show that $2^{N+1}\eta = 0$ in $\mathcal{SH}(\mathbb{F}_p)$, with N as in the statement of the lemma.

For p odd, we have $\mathrm{GW}(\mathbb{F}_p) \cong K^{MW}_0(\mathbb{F}_p)$, with the one-dimenisonal quadratic form $< u >$ mapping to the element $1 + \eta[u]$. In particular, $1 + \eta[-1]$ corresponds to the form $-x^2$. If $p \equiv 1 \mod 4$, then -1 is a square, hence $-x^2$ and x^2 are isometric forms and $1 + \eta[-1] = 1$ in $K^{MW}_0(\mathbb{F}_p)$. The relation $\eta(2 + \eta[-1]) = 0$ in $K^{MW}_0(\mathbb{F}_p)$ thus simplifies to $2\eta = 0$.

If $p \equiv 3 \mod 4$, then -1 is a sum of two squares, and hence the quadratic form $x^2 + y^2 + z^2$ is isotropic. Thus the Pfister form $x^2 + y^2 + z^2 + w^2$ is also isotropic and hence hyperbolic, and hence the form $-x^2 - y^2 - z^2 - w^2$ is hyperbolic as well. Translating this back to $K^{MW}_0(\mathbb{F}_p)$ gives the relation $4(1 + \eta[-1]) = 2(2 + \eta[-1])$ or $2\eta[-1] = 0$. Combining this with relation $\eta(2 + \eta[-1]) = 0$ yields $4\eta = 0$.

In case k has characteristic 2, $-1 = +1$. The relation $\eta(2+\eta[-1]) = 0$ simplifies to $2\eta = 0$. □

Remark 5.2. Instead of the finiteness of cd_2k, one can require that -1 is a sum of squares in k. Under the isomorphism $K^{MW}_0(k) \cong \mathrm{GW}(k)$ (assuming $\mathrm{char}\, k \neq 2$) the element $1 + \eta[-1]$ goes to the quadratic form $-X^2$. If -1 is a sum of N squares in k, then the n-fold Pfister form $<< -1, \ldots, -1> >= \sum_{i=1}^{2^n} x_i^2$ is isotropic if $2^n > N$, hence hyperbolic. Thus $2^n \cdot (-X^2)$ is hyperbolic, so

$$2^n(1 + \eta[-1]) = 2^{n-1}(2 + \eta[-1])$$

or $2^{n-1}\eta[-1] = 0$. Since $2\eta = -\eta^2[-1]$, this gives $2^n\eta = 0$, without using the Milnor conjecture.

In any case, lemma 5.1 and this remark imply

Lemma 5.3. *Suppose that* k *has finite 2-cohomological dimension or that -1 is a sum of squares in* k *or that* $\mathrm{char}\, k > 0$*. Then* $\mathcal{SH}(k)[\frac{1}{2}] = \mathcal{SH}(k)^+$*.*

Definition 5.4. 1. Let $\mathcal{SH}(k)_{\rm fin}(k)$ be the thick subcategory of $\mathcal{SH}(k)$ generated by objects $\Sigma^n_T\Sigma^\infty_T X_+$ for X a smooth projective k-scheme and $n \in \mathbb{Z}$.
2. Let $\mathcal{SH}(k)_{\rm coh.fin}(k)$ be the full subcategory of $\mathcal{SH}(k)$ with objects those $\mathcal{E}$ such that

i) there is an integer d such that, for $n > d$, $\Pi_{r,n}(\mathcal{E})_\mathbb{Q} = 0$ for all r.
ii) there is an integer c such that $\Pi_{r,q}\mathcal{E} = 0$ for $r < c$, $q \in \mathbb{Z}$.

For $\mathcal{E} \in \mathcal{SH}(k)_{\rm coh.fin}(k)$ we define the *dimension* of $\mathcal{E}$, $d(\mathcal{E})$, to be the smallest integer such that $\Pi_{r,n}(\mathcal{E})_\mathbb{Q} = 0$ for all r and for all $n > d$. We let $c(\mathcal{E})$ be the largest integer c such that $\Pi_{r,*}\mathcal{E} = 0$ for $r \le c$; we say that $\mathcal{E}$ is *topologically* $c(\mathcal{E})$ connected.

Lemma 5.5. *Let $\mathcal{E}$ be in $\mathcal{SH}(k)_{\rm coh.fin}(k)$.*
1. For $U \in \mathbf{Sm}/k$, we have

$$\mathrm{Hom}_{\mathcal{SH}(k)_\mathbb{Q}}(\Sigma^p_s\Sigma^q_{\mathbb{G}_m}\Sigma^\infty_T U_+, \mathcal{E}) = 0$$

for all $p \in \mathbb{Z}$, $q > d(\mathcal{E})$.

2. For all $n \ge d(\mathcal{E})$, $f_n\mathcal{E} \cong 0$ in $\mathcal{SH}(k)_\mathbb{Q}$.

Proof. (1) implies (2), as (1) implies that for $\mathcal{F} \in \Sigma^q_T\mathcal{SH}^{eff}(k)_\mathbb{Q}$,

$$\mathrm{Hom}_{\mathcal{SH}(k)_\mathbb{Q}}(\mathcal{F}, \mathcal{E}) = 0.$$

Since $f_n\mathcal{E} \to \mathcal{E}$ (in $\mathcal{SH}(k)_\mathbb{Q}$) is universal for maps $\mathcal{F} \to \mathcal{E}$ with $\mathcal{F} \in \Sigma^q_T\mathcal{SH}^{eff}(k)_\mathbb{Q}$, it follows that $f_n\mathcal{E}_\mathbb{Q} = 0$.

For (1), we have the Gersten-Quillen spectral sequence

$$E_1^{a,b} = \oplus_{u\in U^{(a)}}\Pi_{-b,q+a}(\mathcal{E})(k(u)) \Longrightarrow \mathrm{Hom}_{\mathcal{SH}(k)_\mathbb{Q}}(\Sigma_s^{-a-b}\Sigma^q_{\mathbb{G}_m}\Sigma^\infty_T U_+, \mathcal{E})$$

Thus, the assumption $q \ge d(\mathcal{E})$ implies that $E_1^{a,b} = 0$ for all a, b, proving (1). □

This yields

Lemma 5.6. *Let $\mathcal{E}$ be in $\mathcal{SH}(k)_{\rm coh.fin}(k)$ and choose an integer n. Then $f_n\mathcal{E}$ is in $\mathcal{SH}(k)_{\rm coh.fin}(k)$, $d(f_n\mathcal{E}) \le d(\mathcal{E})$ and $c(f_n\mathcal{E}) \ge c(\mathcal{E})$.*

Proof. We have already seen in lemma 5.5 that $f_n\mathcal{E}_\mathbb{Q} = 0$ for $n > d(\mathcal{E})$, hence $f_n\mathcal{E}$ is in $\mathcal{SH}(k)_{\rm coh.fin}(k)$ and $d(f_n\mathcal{E}) = -\infty$ for $n > d(\mathcal{E})$. For $n \le d(\mathcal{E})$, take $m \ge d(\mathcal{E}) \ge n$. Then the universal property of $f_n\mathcal{E} \to \mathcal{E}$ implies

$$\rho_n(\mathcal{E}) : \Pi_{r,m}(f_n\mathcal{E}) \to \Pi_{r,m}(\mathcal{E})$$

is an isomorphism, hence $\Pi_{r,m}(f_n\mathcal{E})_\mathbb{Q} = 0$ for $m > d(\mathcal{E})$.

We have shown in [9, proposition 3.2] that, for $E \in \mathcal{SH}_{S^1}(k)$, if E is -1-connected, the f_nE is also -1-connected for all $n \ge 0$. Applying this to $E := \Omega^\infty_T\Sigma_s^{-c}\Sigma^{-q}_{\mathbb{G}_m}\mathcal{E}$ shows that, if $\Pi_{r,q}\mathcal{E} = 0$ for $r \le c$, the same holds for $f_n\mathcal{E}$, for $n \ge q$. For $n \le q$, we already have $\Pi_{r,q}\mathcal{E} = \Pi_{r,q}f_n\mathcal{E}$. □

Proposition 5.7. *Take $X \in \mathbf{Sm}/k$. Then*
1. $\Pi_{r,n}(\Sigma^\infty_T X_+) = 0$ for $r < 0$ and $n \in \mathbb{Z}$
2. Suppose X is smooth and projective over k of dimension d and that k has finite 2-cohomological dimension or $\mathrm{char}\, k > 0$. Then $\Pi_{r,n}(\Sigma^\infty_T X_+)_\mathbb{Q} = 0$ for $n > d$, $r \in \mathbb{Z}$.
3. Suppose k has finite 2-cohomological dimension or $\mathrm{char}\, k > 0$. Then each $\mathcal{E} \in \mathcal{SH}(k)_{\rm fin}(k)$ is in $\mathcal{SH}(k)_{\rm coh.fin}(k)$.

Proof. Noting that

$$\Pi_{r,n}(\Sigma^m_T\mathcal{E}) = \Pi_{r-m,n-m}(\mathcal{E}),$$

we see that (3) follows from (1) and (2).

We first prove (1). We have the S^1 spectrum $\Sigma^\infty_{S^1}X_+ \in \mathcal{SH}_{S^1}(k)$. It suffices to show that $\pi_r(\Omega^a_{\mathbb{G}_m}\Sigma^\infty_{S^1}\Sigma^b_{\mathbb{G}_m}X_+) = 0$ for all $r < 0$, $a, b \geq 0$ and $X \in \mathbf{Sm}/k$. As $\Sigma^\infty_{S^1}\Sigma^b_{\mathbb{G}_m}X_+$ is a summand of $\Sigma^\infty_{S^1}(\mathbb{G}^b_m \times X)_+$, it suffices to handle the case $b = 0$.

We recall, for a presheaf A of abelian groups on $\mathbf{Sm}/k$, we have the presheaf A_{-1} defined by

$$A_{-1}(U) := \ker(i^*_1 : A(U \times \mathbb{G}_m) \to A(U)).$$

For $n > 1$, A_{-n} is defined to be $(A_{-n+1})_{-1}$.

We recall the following result of Morel [13, lemma 4.3.11]: Let E be in $\mathcal{SH}_{S^1}(k)$. Then the natural map

$$\pi_n(\Omega_{\mathbb{G}_m}E) \to (\pi_n(E))_{-1}$$

is an isomorphism. In particular, if E is -1 connected, then so is $\Omega_{\mathbb{G}_m}E$. By Morel's connectedness theorem [13, theorem 4.2.10], $\Sigma^\infty_{S^1}X_+$ is -1 connected, whence (1).

Our assumption on k together with lemma 5.3 shows that $\mathcal{SH}(k(y))_{\mathbb{Q}} = \mathcal{SH}(k(y))^+_{\mathbb{Q}}$. By a result of Morel (proved in detail by Cisinski-Deglise [3, theorem 15.2.13]), we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^a_{S^1}\Sigma^b_{\mathbb{G}_m}\Sigma^\infty_T U_+, \Sigma^\infty_T X_+))_{\mathbb{Q}} &\cong \\ \mathrm{Hom}_{\mathcal{SH}(k)^+_{\mathbb{Q}}}((\Sigma^a_{S^1}\Sigma^b_{\mathbb{G}_m}\Sigma^\infty_T U_+)_{\mathbb{Q}^+}, (\Sigma^\infty_T X_+)_{\mathbb{Q}^+}) \\ &\cong \mathrm{Hom}_{DM(k)_{\mathbb{Q}}}(M(U)(b)_{\mathbb{Q}}[a+b], M(X)_{\mathbb{Q}}),\end{aligned}$$

where $M : \mathbf{Sm}/k \to DM(k)$ is the canonical functor. By duality, we have

$$\begin{aligned}&\mathrm{Hom}_{DM(k)}(M(U)(b)[a+b], M(X)) \\ &\cong \mathrm{Hom}_{DM(k)}(M(U \times X), \mathbb{Z}(d-b)[2d-a-b]) \\ &= H^{2d-a-b}(U \times X, \mathbb{Z}(d-b)).\end{aligned}$$

But for all $Y \in \mathbf{Sm}/k$, $H^p(Y, \mathbb{Z}(q)) = 0$ for $q < 0$, $p \in \mathbb{Z}$. Thus the presheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^a_{S^1}\Sigma^b_{\mathbb{G}_m}\Sigma^\infty_T U_+, \Sigma^\infty_T X_+)_{\mathbb{Q}}$$

is zero for $b > d$, and hence the associated sheaf $\Pi_{a,b}(\Sigma^\infty_T X_+)_{\mathbb{Q}}$ is zero as well. □

Remark 5.8. The statement of result of Cisinski-Deglise [3, theorem 15.2.13] cited above is not the same as cited here; Cisinski-Deglise show that the $\mathbb{A}^1$-derived category over a base-scheme S, with $\mathbb{Q}$-coefficients, $D_{\mathbb{A}^1,\mathbb{Q}}(S)$, has plus part $D_{\mathbb{A}^1,\mathbb{Q}}(S)_+$ equivalent to the category of "Beilinson motives" $DM_B(S)$. For S geometrically unibranch, they show [3, theorem 15.1.4] that $DM_B(S) \cong DM(S)_{\mathbb{Q}}$.

Furthermore, the $\mathbb{Q}$-singular chain complex functor defines an equivalence $C_{*\mathbb{Q}} : \mathcal{SH}_{\mathbb{Q}} \to D(\mathbb{Q})$, where $\mathcal{SH}_{\mathbb{Q}}$ is the $\mathbb{Q}$-localized stable homotopy category, and $D(\mathbb{Q})$ is the derived category of $\mathbb{Q}$-vector spaces. As $C_{*\mathbb{Q}}$ is a well-defined functor on the model category level, the evident extension of the functor $C_{*\mathbb{Q}}$ gives an equivalence of $\mathcal{SH}(S)_{\mathbb{Q}}$ with $D_{\mathbb{A}^1,\mathbb{Q}}(S)$. Putting these equivalences all together gives us the equivalence $\mathcal{SH}(k)^+_{\mathbb{Q}} \cong DM(k)_{\mathbb{Q}}$ we use in the proof of proposition 5.7.

Proposition 5.9. *Let k be a perfect field; if $\operatorname{char} k = 2$, we suppose that k is infinite. Suppose that k has finite Galois cohomological dimension D_k for torsion Galois modules. Let F be a perfect field extension of k of transcendence dimension d over k. Let $E = \Omega^\infty_T \mathcal{E}$ for some $\mathcal{E} \in \mathcal{SH}_{\mathrm{coh.fin}}(k)$ and suppose that $\Pi_{a,*}E = 0$ for $a < 0$. Consider the map*

$$\rho_n(f_M E) : \mathrm{Fil}^{simp}_n \pi_r f_n(f_M E)(F) \to \pi_r f_M E(F).$$

Then, for all $r, M \geq 0$, $\rho_n(f_M E)(\mathrm{Fil}^{simp}_n \pi_r f_n(f_M E)(F)) = 0$ for $n > \max(D_k + d, d(\mathcal{E}))$.

Proof. Let $A = max(D_k + d, d(\mathcal{E}))$. By theorem 4.3, and the definition of the filtration $F^n_{MW^{Tr}} \pi_r f_M E(F)$, it suffices to show that, for every finite extension F' of F, the product

$$\cup : K^{MW}_n(F') \otimes \Pi_{r,n} f_M E(F') \to \Pi_{r,0} f_M E(F')$$

is zero.

By proposition 5.7, $\Pi_{r,n}E_{\mathbb{Q}} = 0$ for $n > d(\mathcal{E})$. We note that $f_M E \cong \Omega^\infty_T f_M \mathcal{E}$ for $M \geq 0$. Thus, by lemma 5.6, $\Pi_{r+n,n} f_M E_{\mathbb{Q}} = 0$ for $n > d(\mathcal{E})$ and the image of $\cup$ is the same as the subgroup generated by the images of the maps

$$\cup_N : K^{MW}_n(F')/N \otimes \Pi_{r,n} f_M E(F')_{N-\mathrm{tor}} \to \pi_r f_M E(F'); \quad N \in \mathbb{N}.$$

By Morel's theorem [12, theorem 5.3], we have a cartesian diagram

$$\begin{array}{ccc} K^{MW}_n(F') & \longrightarrow & K^M(F') \\ \downarrow & & \downarrow \\ I^n & \longrightarrow & I^n/I^{n+1} \end{array}$$

Thus, for $n > D_k$, lemma 5.1 tells us that $K^{MW}_n(F') = K^M_n(F')$. Furthermore, for N prime to the characteristic, the Bloch-Kato conjecture (proved by Voevodsky and Rost, see [22]) gives the isomorphism

$$K^M_n(F')/N \cong H^n_{\text{ét}}(F' \mu_N^{\otimes n}).$$

But F', having transcendence dimension d over k, has torsion-cohomological dimension at most $D_k + d$, and hence $K^M_n(F')/N = 0$ for $n > D_k + d$. For $N = p = \mathrm{char}(k)$, it follows from [2, theorem 2.1] that

$$K^M_n(F')/p = 0,$$

since F' is perfect. Thus, for $n > D_k + d$, $K^{MW}_n(F')/N = 0$, and hence the image of $\cup$ is zero if $n > A$. □

6. The proof of the convergence theorem

Lemma 6.1. *Take $\mathcal{E} \in \mathcal{SH}_{\mathrm{fin}}(k)$. Then there is an integer $n(\mathcal{E})$ such that $f_n(\mathcal{E}) = \mathcal{E}$ for all $n \leq n(\mathcal{E})$.*

Proof. Clearly, it suffices to prove the result for $\mathcal{E} = \Sigma^m_T \Sigma^\infty_T X_+$ with X smooth and projective over k and $m \in \mathbb{Z}$. As $f_n \circ \Sigma^m_T = \Sigma^m_T \circ f_{n-m}$ and $\Pi_{a,b}\Sigma_T \mathcal{E} = \Pi_{a-1,b-1}\mathcal{E}$, we need only prove the result for $\mathcal{E} = \Sigma^\infty_T X_+$. As $\Sigma^\infty_T X_+$ is in $\mathcal{SH}^{eff}(k)$, we have $f_0 \Sigma^\infty_T X_+ = \Sigma^\infty_T X_+$, proving (1).

The fact that $\Pi_{a,b}\Sigma^\infty_T X_+ = 0$ for $a < 0$ is proposition 5.7(1). □

Here is the main step in the proof of theorem 3:

Proposition 6.2. *Let k be a perfect field of finite torsion-cohomologcal dimension and let p be the exponential characteristic of k; if* $\operatorname{char} k = 2$*, we suppose that k is infinite. Let $\mathcal{E}$ be an object in $\mathcal{SH}_{\text{coh.fin}}(k)$ such that $\Pi_{a,*}\mathcal{E} = 0$ for $a < 0$ and $d(\mathcal{E}) \le e$ for some integer e. Let $E = \Omega^\infty_T\mathcal{E}$. Then there is an integer $N(e,r,d) \ge 0$ such that*

$$\mathrm{Fil}^N_{Tate}\,\Pi_{r,q}f_M E(F)[1/p] = 0$$

for all $N \ge N(e,r,d) + q$, all $q \ge 0$, $M \ge 0$ and all fields F of transcendence dimension $\le d$ over k.

Proof. By lemma 5.6, $f_M\mathcal{E}$ is in $\mathcal{SH}_{\text{coh.fin}}(k)$, $f_M E \cong \Omega^\infty_T f_M\mathcal{E}$ and $d(f_M\mathcal{E}) \le d(\mathcal{E})$. Moreover, $\Pi_{a,*}f_M\mathcal{E} = 0$ for $a < 0$ [9, proposition 3.2]. Thus it suffices to prove the result for $M = 0$.

Fix $q \ge 0$ and let $\mathcal{E}' = \Omega^q_T\mathcal{E}$. Then $\Omega^\infty_T\mathcal{E}' \cong \Omega^q_T E$, $d(\mathcal{E}') = d(\mathcal{E}) - q$, $\Pi_{a,b}\Omega^q_T E = \Pi_{a,b+q}E$ and

$$\mathrm{Fil}^{n'-q}_{\text{Tate}}\,\Pi_{r,0}\Omega^q_T E \cong \mathrm{Fil}^{n'}_{\text{Tate}}\,\Pi_{r,q}E.$$

Thus, we may replace $\mathcal{E}$ with $\mathcal{E}'$ and it suffices to prove the result for $q = 0$.

By proposition 7.1, it suffices to prove the result for perfect fields F. We proceed by induction on r. We omit the $[1/p]$ in the notation, using the convention that we invert the exponential characteristic throughout the proof of this proposition.

The case $r = -1$ follows from our hypothesis $\Pi_{a,*}\mathcal{E} = 0$ for $a < 0$, since $\pi_r E = \Pi_{r,0}\mathcal{E}$ for all r. Thus, taking $N(e,-1,d) = 0$ for all d and e settles the case $r = -1$.

Assume we have integers $N_j(d) = N(e,j,d)$ satisfying the conditions of the proposition for $\mathrm{Fil}^n_{\text{Tate}}\,\pi_j f_M E(F)$, with $j = 0,\ldots,r-1$.

By lemma 3.5, it follows that for all $n \ge 0$, all perfect F of transcendence degree $\le d$ over k and all $m \ge nr + \sum_{j=0}^{r-1} N_j(d)$ we have

$$im(\pi_r(f_m f_n E)(F) \to \pi_r f_n E(F)) \subset \mathrm{Fil}^{simp}_n\,\pi_r(f_n E)(F)$$

If we assume that $n \ge \max(D_k + d, d(\mathcal{E})) + 1$, then by proposition 5.9

$$im(\mathrm{Fil}^{simp}_n\,\pi_r(f_n E)(F) \to \pi_r(E)(F)) = 0.$$

Thus, for $n = \max(D_k + d, e) + 1$ and $m \ge nr + \sum_{j=0}^{r-1} N_j(r-j+d)$, we have

$$im(\pi_r(f_m f_n E)(F) \to \pi_r E(F)) = 0.$$

Since $m \ge n$, we have $f_m f_n = f_m$, this implies that

$$F^N_{\text{Tate}}\pi_r E(F) = 0$$

for $N \ge N(e,r,d) := nr + \sum_{j=0}^{r-1} N_j(r-j+d)$, $n = \max(D_k + d, e) + 1$, and the induction goes through. □

Now for the proof of the main theorem. For $x \in X \in \mathbf{Sm}/k$, and $\mathcal{F}$ a sheaf on $\mathbf{Sm}/k_{\text{Nis}}$, we let $\mathcal{F}_x$ as usual denote the Nisnevich stalk of $\mathcal{F}$ at x.

Theorem 6.3. *Let k be a perfect field of finite torsion cohomological dimension and of exponential characteristic p; if* $\operatorname{char} k = 2$*, we suppose that k is infinite. Let $x \in X \in \mathbf{Sm}/k$ and let $d = \dim_k X$. Let $\mathcal{E}$ be in $\mathcal{SH}_{\text{fin}}(k)$. For an integer q, let $m(q) = \min(0,q)$. Then for all $M \ge m(q)$,*

$$(\mathrm{Fil}^n_{Tate}\,\Pi_{r,q}f_M\mathcal{E})_x[1/p] = 0$$

for all $n \geq N(d(\mathcal{E}) - m(q), r - c(\mathcal{E}) - 1, d) + q - m(q)$, where $N = N(e, d, q)$ is the integer given by proposition 6.2.

Proof. As before, we invert p throughout the proof and omit the "$[1/p]$" from the notation.

The proof of [11, lemma 6.1.4] shows that, for $x \in X \in \mathbf{Sm}/k$, $X_x := \operatorname{Spec} \mathcal{O}_{X,x}$ and $U \subset X_x$ open, the map

$$X_x \to X_x/U$$

in $\mathcal{H}(k)$ is equal to the map sending X_x to the base-point of X_x/U. This implies that for any $\mathcal{F} \in \mathcal{SH}(k)$, $a, b \in \mathbb{Z}$, the restriction map

$$[\Sigma^a_{S^1} \Sigma^b_{\mathbb{G}_m} \Sigma^\infty X_{x+}, \mathcal{F}]_{\mathcal{SH}(k)} \to [\Sigma^a_{S^1} \Sigma^b_{\mathbb{G}_m} \Sigma^\infty U+, \mathcal{F}]_{\mathcal{SH}(k)}$$

is injective. Passing to the limit over U, this shows that the restriction map

$$\Pi_{a,b}(\mathcal{F})_x \to \Pi_{a,b}(\mathcal{F})(k(X))$$

is injective. From this it easily follows that the restriction map

$$(\mathrm{Fil}^n_{\mathrm{Tate}} \Pi_{r,q} f_M \mathcal{E})_x \to (\mathrm{Fil}^n_{\mathrm{Tate}} \Pi_{r,q} f_M \mathcal{E})(k(X))$$

is injective.

Thus, it suffices to show that $\mathrm{Fil}^n_{\mathrm{Tate}} \Pi_{r,q} f_M \mathcal{E}(F) = 0$ for all finitely generated fields F over k, and for $n \geq N(d(\mathcal{E}) - m(q), r - c(\mathcal{E}) - 1, d) + q - m(q)$.

As

$$\mathrm{Fil}^n_{\mathrm{Tate}} \Pi_{r+p,q} \Sigma^p_{S^1} \mathcal{E} = \mathrm{Fil}^n_{\mathrm{Tate}} \Pi_{r,q} \mathcal{E}$$

we may replace $\mathcal{E}$ with $\Sigma^{-c(\mathcal{E})-1}_{S^1} \mathcal{E}$ and assume that $\Pi_{a,*}\mathcal{E} = 0$ for $a < 0$. Similarly, we may replace $\mathcal{E}$ with $\Sigma^{-m(q)}_{\mathbb{G}_m} \mathcal{E}$, since

$$\mathrm{Fil}^{n+p}_{\mathrm{Tate}} \Pi_{r,q+p} \Sigma^p_{\mathbb{G}_m} \mathcal{E} = \mathrm{Fil}^n_{\mathrm{Tate}} \Pi_{r,q} \mathcal{E}.$$

Noting that $d(\Sigma^{-m(q)}_{\mathbb{G}_m} \mathcal{E}) = d(\mathcal{E}) - m(q)$ and $f_{M-m(q)}(\Sigma^{-m(q)}_{\mathbb{G}_m} \mathcal{E}) = \Sigma^{-m(q)}_{\mathbb{G}_m} f_M \mathcal{E}$, this reduces us to the case $M, q \geq 0$ and $\Pi_{a,*}\mathcal{E} = 0$ for $a < 0$. Letting $E = \Omega^\infty_T \mathcal{E}$, we have

$$\mathrm{Fil}^n_{\mathrm{Tate}} f_M \Pi_{r,q} \mathcal{E} = \mathrm{Fil}^n_{\mathrm{Tate}} f_M \Pi_{r,q} E$$

for all $M \geq 0$, $q \geq 0$, $n \geq 0$, so the result follows from proposition 6.2. □

7. Norm maps

Suppose our perfect base-field k has characteristic $p > 0$. For an abelian group A, we write A' for $A \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$. Our task in this section is to prove

Proposition 7.1. *1. Suppose that k is a perfect field of characteristic $p > 0$. Take $\mathcal{E} \in \mathcal{SH}(k)$ and let $\alpha : F \to L$ be a purely inseparable extension of finitely generate fields over k. Then the map*

$$\alpha^* : \Pi_{a,b}\mathcal{E}(F)' \to \Pi_{a,b}\mathcal{E}(L)'$$

is injective.

2. Suppose that k is a finite field of characteristic $p \neq 2$. Take $\mathcal{E} \in \mathcal{SH}(k)$ and let $\alpha : k \to k'$ be a finite extension of degree n. Then the kernel of

$$\alpha^* : \Pi_{a,b}\mathcal{E}(k) \to \Pi_{a,b}\mathcal{E}(k')$$

is n-torsion.

As one would expect, we construct a quasi-inverse to α^* by constructing transfers. Take $a \in F^\times \setminus (F^\times)^p$ and consider the extension $F_a := F(a^{1/p})$. We have the corresponding closed point $p(a)$ of $\mathbb{A}^1_F$ defined by the homogeneous ideal $(X^p - a) \subset F[X]$. This gives us the closed point $p(a)$ of $\mathbb{P}^1_F$ via the standard open immersion $x \mapsto (1 : x)$ of $\mathbb{A}^1$ into $\mathbb{P}^1$.

Suppose $F = k(U)$ for some finite type k-scheme U. Since k is perfect, U has a dense open subscheme smooth over k; shrinking U and changing notation, we may assume that U is affine and smooth over k, and also that a is a global unit on U. Let $V \subset \mathbb{A}^1 \times U$ be the closed subscheme defined by $X^p - a$. Then V is reduced and irreducible (since $a \notin (F^\times)^p$) and is finite over U. Shrinking U again, we may assume that V is also smooth and affine over k. Let $I_V \subset k[U][t]$ be the ideal defining V in $\mathbb{A}^1 \times U$.

Using $X^p - a$ as generator for I_V/I_V^2, we have the Morel-Voevodsky purity isomorphism [14, theorem 3.2.23]

$$\mathbb{A}^1 \times U/(\mathbb{A}^1 \times U \setminus V) \cong \mathbb{P}^1_V.$$

Combining with the excision isomorphism $\mathbb{A}^1 \times U/(\mathbb{A}^1 \times U \setminus V) \cong \mathbb{P}^1 \times U/(\mathbb{P}^1 \times U \setminus V)$ and passing to the limit over open subschemes of U gives us the sequence of maps of pro-objects in $\mathcal{H}_\bullet(k)$:

$$(\mathbb{P}^1_F, \infty) \to \mathbb{P}^1_F/\mathbb{P}^1_F \setminus \{p(a)\} \cong (\mathbb{P}^1_{F_a}, \infty);$$

we denote the composition by $\mathrm{Tr}_{p(a)/F}$. Passing to $\mathcal{SH}(k)$ gives us the morphism

$$\mathrm{Tr}_{p(a)/F} : \mathbb{S}_k \wedge \mathrm{Spec}\, F_+ \to \mathbb{S}_k \wedge p(a)_+ = \mathbb{S}_k \wedge F_{a+}$$

where we consider these objects as pro-objects in $\mathcal{SH}(k)$. Composing with the map induced by the structure morphism $\pi_a : p(a) \to \mathrm{Spec}\, F$ gives us the endomorphism $\pi_{a*} \circ \mathrm{Tr}_{p(a)/F}$ of $\mathbb{S}_k \wedge \mathrm{Spec}\, F_+$.

Proposition 7.1(1) is an immediate consequence of

Lemma 7.2. *After inverting p, the the endomorphism*

$$\pi_{a*} \circ \mathrm{Tr}_{p(a)/F} : \mathbb{S}_k \wedge \mathrm{Spec}\, F_+ \to \mathbb{S}_k \wedge \mathrm{Spec}\, F_+$$

is an isomorphism.

Proof. We consider a deformation of $\mathrm{Tr}_{F_a/F}$. Let $P(a) \subset \mathrm{Spec}\, F[t][X]$ be the closed subscheme defined by the ideal $(X^p + t(X+1) + (t-1)a)$. We have

$$d(X^p + t(X+1) + (t-1)a) = (X + a + 1)dt + tdX \in \Omega_{F[t][X]/F}.$$

Thus the singular locus of $P(a)$ (over F) is given by $(X + a + 1 = t = 0) \cap (X^p + t(X+1) + (t-1)a = 0)$; since $a \notin (F^\times)^p$, the singular locus is empty, i.e., $P(a)$ is smooth over F. Using $X^p + t(X+1) + (t-1)a$ as the generator for $I_{P(a)}/I^2_{P(a)}$, we have as above the map

$$\mathrm{Tr}_{P(a)/F[t]} : \mathbb{P}^1_k \wedge \mathrm{Spec}\, F[t]_+ \to \mathbb{P}^1_k \wedge P(a)_+$$

of pro-objects in $\mathcal{H}_\bullet(k)$, and the endomorphism

$$\pi_{P(a)/F[t]*} \circ Tr_{P(a)/F[t]} : \mathbb{S}_k \wedge \mathrm{Spec}\, F[t]_+ \to \mathbb{S}_k \wedge \mathrm{Spec}\, F[t]_+$$

of pro-objects in $\mathcal{SH}(k)$.

Setting $t = 1$ gives us the closed subscheme p_0 of $\operatorname{Spec} F[X]$ defined by the ideal $(X^p + X + 1)$. Clearly $\pi_0 : p_0 \to \operatorname{Spec} F$ is finite and étale of degree p. Using $X^p + X + 1$ as generator of $I_{p_0}/I_{p_0}^2$ gives us the map

$$\mathrm{Tr}_{p_0/F} : (\mathbb{P}^1_k, \infty) \wedge \operatorname{Spec} F_+ \to (\mathbb{P}^1_k, \infty) \wedge p_{0+}$$

of pro-objects in $\mathcal{H}_\bullet(k)$, and the endomorphism

$$\pi_{0*} \circ Tr_{p_0/F} : \mathbb{S}_k \wedge \operatorname{Spec} F_+ \to \mathbb{S}_k \wedge \operatorname{Spec} F_+$$

of pro-objects in $\mathcal{SH}(k)$.

The map $\pi_{P(a)/F[t]*} \circ Tr_{P(a)/F[t]}$ thus gives us an $\mathbb{A}^1$ homotopy between the maps $\pi_{a*} \circ \mathrm{Tr}_{p(a)/F}$ and $\pi_{0*} \circ Tr_{p_0/F}$. Thus, these maps are equal in $\mathcal{SH}(k)$, and it suffices to show that $\pi_{0*} \circ Tr_{p_0/F}$ is an isomorphism after inverting p.

As $X^p + X + 1$ has coefficients in $\mathbb{F}_p$, $\pi_{0*} \circ Tr_{p_0/F}$ arises as base-extension from the similarly defined map

$$\pi_{0*} \circ Tr_{p_0/\mathbb{F}_p} : \mathbb{S}_{\mathbb{F}_p} \to \mathbb{S}_{\mathbb{F}_p}$$

in $\mathcal{SH}(\mathbb{F}_p)$, that is, from the corresponding element $[\pi_{0*} \circ Tr_{p_0/\mathbb{F}_p}] \in [\mathbb{S}_{\mathbb{F}_p}, \mathbb{S}_{\mathbb{F}_p}]_{\mathcal{SH}(\mathbb{F}_p)}$.

Suppose that $p > 2$. By Morel's theorem [13, theorem 6.4.1], $[\mathbb{S}_{\mathbb{F}_p}, \mathbb{S}_{\mathbb{F}_p}]_{\mathcal{SH}(\mathbb{F}_p)} \cong GW(\mathbb{F}_p)$. Since $[\mathbb{F}_p(p_0) : \mathbb{F}_p] = p$, it follows that the image of $\pi_{0*} \circ Tr_{p_0/\mathbb{F}_p} \in [\mathbb{S}_{\mathbb{F}_p}, \mathbb{S}_{\mathbb{F}_p}]_{\mathcal{SH}(\mathbb{F}_p)}$ under the rank homomorphism $\mathrm{GW}(\mathbb{F}_p) \to \mathbb{Z}$ is p. Since the augmentation ideal $I \subset GW(\mathbb{F}_p)$ is nilpotent (in fact $I^2 = 0$), it follows that $\pi_{0*} \circ Tr_{p_0/\mathbb{F}_p}$ is a unit in $\mathrm{GW}(\mathbb{F}_p)[1/p]$, proving the result in case $p > 2$.

Suppose $p = 2$. We consider the deformation $p(t) \subset \operatorname{Spec} \mathbb{F}_2[X][t]$ defined by the ideal $(X^2 + X + 1 - t)$. Then $p(t)$ is smooth over $\mathbb{F}_2$ and gives an $\mathbb{A}^1$ homotopy between the map $\pi_{0*} \circ Tr_{p_0/\mathbb{F}_2}$ $(t = 0)$ and the map $\pi_{p(1)*} \circ Tr_{p(1)/\mathbb{F}_2}$ defined as above using the closed subscheme $p(1)$ of $\operatorname{Spec} F[X]$ given by $X^2 + X = 0$. We note that $p(1) = \{0, 1\}$. In addition, we have $X^2 + X \equiv X \mod m_0^2$ and $X^2 + X \equiv X - 1 \mod m_1^2$, where m_0, m_1 are the respective maximal ideals of $0, 1$. By [8, lemma 4.1], it follows that

$$Tr_{p(1)/\mathbb{F}_2} : (\mathbb{P}^1_{\mathbb{F}_2}, \infty) \to (\mathbb{P}^1_{p(1))}, \infty) = (\mathbb{P}^1_{\mathbb{F}_2}, \infty) \vee (\mathbb{P}^1_{\mathbb{F}_2}, \infty)$$

is the canonical co-addition on $(\mathbb{P}^1_{\mathbb{F}_2}, \infty)$ in $\mathcal{H}_\bullet(\mathbb{F}_2)$, and thus $\pi_{p(1)*} \circ Tr_{p(1)/\mathbb{F}_2}$ is multiplication by 2. This handles the case $p = 2$ and completes the proof of the lemma □

The proof of proposition 7.1(2) is similar. Write $k' = k(a)$, let $f(x) \in k[x]$ be the minimal polynomial of a and let $p \in \mathbb{A}^1_k$ be the closed point defined by the ideal $(f(x))$. Using $f(x)$ as the generator for I_p/I_p^2, the Morel-Voevodsky purity isomorphism [14, *loc. cit.]* defines the map

$$\mathrm{Tr}_{p/k} : (\mathbb{P}^1_k, \infty) \to (\mathbb{P}^1_k, \infty) \wedge p_+$$

in $\mathcal{H}_\bullet(k)$, and the endomorphism

$$\pi_p \circ \mathrm{Tr}_{p/k} : \mathbb{S}_k \to \mathbb{S}_k$$

in $\mathcal{SH}(k)$. Since the characteristic p is odd, we can use Morel's theorem again and identify $\mathrm{End}(\mathbb{S}_k)$ with $\mathrm{GW}(k)$. Under this identification, $\pi_p \circ Tr_{p/k}$ gives us an element $\alpha \in \mathrm{GW}(k)$, mapping to the degree n under the rank homomorphism $\mathrm{GW}(k) \to \mathbb{Z}$. Since the augmentation ideal $I \subset \mathrm{GW}(k)$ is nilpotent, this implies that there is an element $\beta \in \mathrm{GW}(k)$ with $\beta \cdot \alpha = n$. Viewing β as an endomorphism of $\mathbb{S}_k$, this gives $\pi_p \circ (\mathrm{Tr}_{p/k} \circ \beta) = n \times \mathrm{id}$. As $\alpha^* = \pi_p^*$ on $\Pi_{a,b}\mathcal{E}(k)$, this implies $n \cdot \ker \alpha^* = 0$, as desired.

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Universität Duisburg-Essen, Fakultät Mathematik, Campus Essen, 45117 Essen, Germany

E-mail address: marc.levine@uni-due.de