ON A SPECTRAL SEQUENCE FOR EQUIVARIANT K-THEORY

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1. Introduction

There are many different approaches to a theory of equivariant algebraic K-theory; for simplicity, we restrict our discussion to the case of a finite group G acting on a scheme X. One can use algebraic versions of the Borel construction, taking an sequence of finite dimensional approximations U_n to EG and then defining $K_*^G(X)$ as a limit of the $K_*((U_n \times X)/G)$. This approach gives a different answer depending on the underlying topology chosen; for the Zariski topology, one may use the standard simplicial model of EG whereas for the étale topology it is necessary to use other models. This latter choice gives for example an equivariant K_0-group closely related to the equivariant Chow groups defined by Totaro [Tot99] and studied by Edidin-Graham.[EG98, EG00], Pandharipande [Pan98] and many others. One can also modify this approach by using the homotopy fixed-point spectrum $K((U_n \times X)^G$ instead of the K-theory of the quotient.

Another approach is to take the K-theory spectrum of the exact category of locally free G-sheaves on X, or the G-theory analog using

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coherent G-sheaves. We will henceforth denote these two theories by $K_*(G, X)$ and $G_*(G, X)$. These theories differ in general from the ones described above, mainly due to the lack of étale descent for algebraic K-theory.

It is possible to give an allied cycle theory by using the orbit category of G, as in topological Bredon (co)homology. This approach has been used by Joshua [Jos05], and has yielded for example Riemann-Roch type results for $G_*(G, X)$.

We will use a somewhat different approach. We feed the equivariant G-theory spectra $G(G, X)$ into the *homotopy coniveau* machine developed in [Lev01, Lev03]. What comes out is a theory of algebraic cycles with an interesting "local coefficient" system, together with an associated theory of higher Chow groups and most importantly, a spectral sequence

$$E^1_{p,q} = CH_p(G, X, p+q) \Longrightarrow G_{p+q}(G, X). \tag{1}$$

In the case $G = \{\mathrm{id}\}$, this reduces to the Bloch-Lichtenbaum/Friedlander-Suslin spectral sequence [BL95, FS00].

The local coefficient system is easy to describe: Let Y be a finite type k-scheme with a G-action, $Z \subset Y$ an irreducible closed subset with generic point z. Let $G_z \subset G$ be the isotropy group of z. Then G_z acts on the residue field $\kappa(z)$, and we define the "coefficient group" of Z to be $K_0(\kappa(z), G_z)$. The dimension p equivariant cycle group of Y is then

$$z_p(G, Y) := \oplus_{[z]\in(Y_{(p)})/G} K_0(G_z, \kappa(z)).$$

To explain: $Y_{(p)}$ is the set of points $z \in Y$ with closure $\bar{z}$ having dimension p over k. G acts on $Y_{(p)}$ and z is a representative of the orbit $[z] \in (Y_{(p)})/G$.

Now if X is a finite type k-scheme with G-action, taking the dimension $p+r$ points of $X \times \Delta^r$ in "good position" with respect to the faces of Δ^r, as in Bloch's construction of his cycle complexes [Blo86], and using our coefficient system $[z] \mapsto K_0(G_z, \kappa(z))$, one defines the Bredon-type equivariant cycle complex $z_p(G, X, *)$ and equivariant higher Chow groups $CH_p(G, X, *)$. The arguments of [Lev01, Lev03] go through without much change to yield the spectral sequence (1).

As alluded to above, one can view our construction as giving a Bredon-type motivic Borel-Moore homology theory for the quotient stack $[X/G]$. It seems reasonable that one could extend the construction of the cycle theory and the spectral sequence to more general stacks, either by removing the restriction that G is finite (e.g. allowing G to be a linear algebraic group) or considering stacks other than quotient stacks. It would also be interesting to see if Joshua's construction, applied to the Friedlander-Suslin-Voevodsky cycle complexes, yield the same motivic Borel-Moore homology groups as ours.

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2. Equivariant G and K-theory

Let G be a finite group and X be a (left) G-scheme. A coherent G-module $\mathcal{F}$ on X is a coherent $\mathcal{O}_X$-module together with an action of G on this module which is compatible with the action of G on X. I.e. for each $g \in G$ we have a morphism $\phi_g : g^*\mathcal{F} \to \mathcal{F}$ and for $g, h \in G$ we have $\phi_{g \cdot h} = \phi_h \circ h^*\phi_g$. We denote by $\mathcal{M}_{G,X}$ the abelian category of coherent G-modules and by $\mathcal{P}_{G,X}$ the exact subcategory of $\mathcal{M}_{G,X}$ of those G-modules which are locally free as $\mathcal{O}_X$-modules. We further denote by $G(G, X)$ and $K(G, X)$ the K-theory spectrum of $\mathcal{M}_{G,X}$ resp. $\mathcal{P}_{G,X}$. Then $G(G, X)$ is contravariant with respect to flat G-morphisms in X and covariant with respect to projective G-morphisms in X. $K(G, X)$ is contravariant with respect to any G-morphism.

If $X = \operatorname{Spec} R$ for a commutative ring R the (left) action of G on X induces a (right) action of G on the ring R. We set $R^{tw}[G] := \oplus_{g \in G} Re_g$. By

$$(r_g \cdot e_g)(r_h \cdot e_h) := r_g \cdot (r_h \cdot g^{-1})e_{(g \cdot h)}$$

for all $g, h \in G$ and $r_g, r_h \in R$ we define a ring structure on $R^{tw}[G]$ and we call $R^{tw}[G]$ the twisted group ring of R.

Lemma 2.1. *Let $X = \operatorname{Spec} R$ be a noetherian affine G-scheme. Then the category of finitely generated $R^{tw}[G]$-modules is equivalent to $\mathcal{M}_{G,X}$.*

Proof. For a R-module M and $g \in G$ let ${}_gM$ be the R-module which is M as abelian group and where the R-module structure is defined by $r \cdot_{{}_AM} m := (r \cdot g^{-1}) \cdot m$ for all $r \in R$ and $m \in M$. A G-module on $\operatorname{Spec} R$ is the same as an R-module together with R-module homomorphisms $\phi_g : {}_AM \to M$ for all $g \in G$ with $\phi_{(gh)} = \phi_h \circ \phi_g$ for all $g, h \in G$. Then M becomes a left $R^{tw}[G]$-module if we set $e_g \cdot m := \phi_{g^{-1}}(m)$. Conversely if M is a $R^{tw}[G]$-module we define $\phi_g : gM \to M$ by $\phi_g(m) := e_{g^{-1}} \cdot m$ for all $g \in G$ and get in this way a G-module on $\operatorname{Spec} R$. Obviously the two functors are inverse to other. Because G is finite and R noetherian the Lemma follows. $\square$

Lemma 2.2. *Let R be commutative noetherian ring with $\frac{1}{\sharp G} \in R$. Then the category of finitely generated projective $R^{tw}[G]$-modules is equivalent to $\mathcal{P}_{G,\operatorname{Spec} R}$.*

Proof. If $\sharp G$ is invertible in R then each $R^{tw}[G]$-module which is projective as R-module is also projective as $R^{tw}[G]$-module. So the lemma follows from the lemma above. $\square$

Now let k be a fixed commutative ring with $\frac{1}{\sharp G} \in k$. We denote by **k-alg** the category of commutative k-algebras. For a functor F : **k-alg** $\to$ **Ab** we define following Vorst [Vor79]

$$\begin{array}{rccc} NF: & \textbf{k-alg} & \to & \textbf{Ab} \\ & R & \mapsto & ker(F(R[T]) \to F(R)), \end{array}$$

where the last map is induced by $R[T] \to R, T \mapsto 0$. For $q > 1$ we define inductively $N^q F := N(N^{q-1}F)$.

For $f \in R$ the morphism $R[X] \to R[X], X \mapsto f \cdot X$ induces a group endomorphism $NF(R) \to NF(R)$. So $NF(R)$ becomes a $\mathbb{Z}[T]$ module. We denote by $NF(R)_{[f]}$ the $\mathbb{Z}[T, T^{-1}]$ module $\mathbb{Z}[T, T^{-1}] \otimes_{\mathbb{Z}[T]} NF(R)$. With these notations Vorst proves the following theorem.

Theorem 2.3. *Let $R \in$ **k-alg** and let $r_1, \ldots, r_n$ be element of R which generates the unit ideal. Suppose further that the map*

$$NF(R[T]_{r_{i_0},\ldots,\hat{r_{i_j}},\ldots,r_{i_p}})_{[r_{i_j}]} \to NF(R[T]_{r_{i_0},\ldots,r_{i_p}})$$

is an isomorphism, for each set of indexes $1 \leq i_0 < \cdots < i_p \leq n$. Then the canonical morphism

$$\epsilon : NF(R) \to \oplus_{j=1}^{n} NF(R_{r_j})$$

is injective.

Proof. Compare [Vor79][Theorem 1.2] or [Lev94][Lemma 1.1]. $\square$

Now let $X = \operatorname{Spec} C$ be an affine G-scheme over k. We consider for each $p \in \mathbb{N}$ the functor

$$\begin{array}{rccc} K_p(G, X \otimes_k -): & \textbf{k-alg} & \to & \textbf{Ab} \\ & R & \mapsto & K_p(G, X \otimes_k R), \end{array}$$

where G acts trivially on R. We say that $\operatorname{Spec}(R)$ is $K_p(G, X \otimes_k -)$-regular if $N^q K_p(G, X \otimes_k R) = 0$ for all $q > 0$.

Lemma 2.4. *Let $R \in$ **k-alg** and $f \in R$. Suppose that there is an $g \in R$ such that $fg = 0$ and $f + g$ is a non-zero divisor. Then the natural map*

$$N^q K_p(G, X \otimes_k R)_{[f]} \to N^q K_p(G, X \otimes_k R_f)$$

is an isomorphism.

Proof. By Lemma 2.2 we have the identification

$$K_p(G, X \otimes_k R) = K_p((C \otimes_k R)^{tw}[G]).$$

Furthermore the element $f \in C \otimes_k R$ lies in the center of the ring. Therefore one can proof the lemma in the same way as Lemma 1.4 in [Vor79]. □

Theorem 2.5. *Let $R \in$ **k-alg** be reduced and $f_1, \dots, f_n \in R$ such that $(f_1, \dots, f_n) = R$ as ideals and such that $\operatorname{Spec}(R_{f_i})$ is $K_p(G, X \otimes_k -)$-regular for $i = 1, \dots, n$. Then $\operatorname{Spec} R$ is also $K_p(G, X \otimes_k -)$-regular.*

Proof. This follow from theorem 2.3 and lemma 2.4. □

In the last section of this paper we need to consider the equivariant K-theory of some singular schemes. This can be partially understood by comparing with multi-relative K-theory and KH-theory, which we now recall.

For an G-invariant subscheme Y of a scheme X we define the relative equivariant K-theory spectrum as

$$K(G, X; Y) := \operatorname{hofib}(K(G, X) \to K(G, Y)).$$

More generally, for a family $\{Y_1, \dots, Y_n\}$ of G-invariant subschemes we inductively define

$$K(G, X; Y_1, \dots, Y_n) := \\ \operatorname{hofib}[K(G, X; Y_1, \dots, Y_{n-1}) \to K(G, Y_n; Y_1 \cap Y_n, \dots, Y_{n-1} \cap Y_n)].$$

It can easily be seen that this definition is independent of the order of $\{Y_1, \dots, Y_n\}$.

For an affine G-scheme $X = \operatorname{Spec} R$ we denote by

$$KH(G, X) := KH(R^{tw}[G])$$

where on the right hand side KH denotes the homotopy K-theory of Weibel for (not necessarily commutative!) rings (compare [Wei89]).

If $\{Y_1, \dots, Y_n\}$ is a family of G-invariant affine subschemes of the affine G-scheme X we define as above $KH(G, X; Y_1, \dots, Y_n)$.

There is a natural transformation of functors from rings to spectra

$$K(-) \to KH(-);$$

this induces a natural transformation of functors from rings with a G action to spectra

$$K(G, -) \to KH(G, -).$$

Since $K(G, T) \to KH(G, T)$ is a weak equivalence for T regular, the map

$$K(G, X; Y_1, \ldots, Y_n) \to KH(G, X; Y_1, \ldots, Y_n)$$

is a weak equivalence if X and all the intersections $Y_{i_1} \cap \ldots \cap Y_{i_s}$ are regular.

Also, if $Y = \cup_{i=1}^n Y_i$, then it is easy to see that

$$K(GX; Y) = K(G, X; Y, Y) = \ldots = K(G, X; Y, \ldots, Y).$$

and similarly for $KH(G, -)$. The inclusions $Y_i \to Y$ thus induce the maps

$$\alpha : K(G, X; Y) = K(G, X; Y, \ldots, Y) \to K(G, X; Y_1, \ldots, Y_n)$$

and

$$\beta : KH(G, X; Y) = KH(G, X; Y, \ldots, Y) \to KH(G, X; Y_1, \ldots, Y_n).$$

By from [Wei89, th. 1.3,Cor. 2.2], β is a weak equivalence.

For the reader's convenience, we include a proof of the following elementary fact:

Lemma 2.6. *Let S be a ring (not necessarily commutative). Let $I_1, I_2 \subset S$ be two-sided ideals, and let $S_i := S/I_jS$, $j = 1, 2$ and $S_{12} := S/(I_1 + I_2)S$. Let $\pi_i : S \to S_i$, $\pi_{12,i} : S_i \to S_{12}$ be the quotient maps and let*

$$S_1 \times^{S_{12}} S_2 = \{(s_1, s_2) \in S_1 \times S_2 \mid \pi_{12,1}(s_1) = \pi_{12,2}(s_2)\}.$$

Suppose that the map

$$\pi : S \to S_1 \times_{S_{12}} S_2$$
$$\pi(s) := (\pi_1(s), \pi_2(s))$$

is an isomorphism. Suppose in addition that the surjection $S_2 \to S_{12}$ is split by a ring homorphism $\sigma_2 : S_{12} \to S_2$. Then the sequence

$$0 \to K_1(S) \xrightarrow{(\pi_{1*}, \pi_{2*})} K_1(S_1) \times K_1(S_2) \xrightarrow{\pi_{12,1*} - \pi_{12,2*}} K_1(S_{12}) \to 0$$

is exact.

Proof. We first show that (π_{1*}, π_{2*}) is injective. If $\alpha \in \mathrm{GL}_N(S)$ goes to zero in $K_1(S_1) \times K_1(S_2)$ then we can write

$$\pi_1(\alpha) = \prod_{r=1}^{n} e_{i_r,j_r}^{\lambda_r} \in \mathrm{GL}_M(S_1),$$
$$\pi_2(\alpha) = \prod_{r=1}^{n} e_{i_r,j_r}^{\mu_r} \in \mathrm{GL}_M(S_2)$$

for some $M \geq N$ and for some $\lambda_r \in S_1$, $\mu_r \in S_2$ (we may take some λ_r or μ_s zero if need be). Letting $\bar{\lambda}_r = \pi_{12,1}(\lambda_r)$ and $\bar{\mu}_r = \pi_{12,1}(\mu_r)$, we have the relation

$$\prod_{r=n}^{1} e_{i_r,j_r}^{-\bar{\mu}_r} \prod_{r=1}^{n} e_{i_r,j_r}^{\bar{\lambda}_r} = 1$$

in $\mathrm{GL}_M(S_{12})$. Thus the element of the Steinberg group $St_M(S_{12})$ given by

$$\prod_{r=n}^{1} x_{i_r,j_r}^{-\bar{\mu}_r} \prod_{r=1}^{n} x_{i_r,j_r}^{\bar{\lambda}_r}$$

defines an element $x \in K_2(S_{12})$. Lift x to the element

$$x_2 := \sigma_*(x) = \prod_{r=n}^{1} x_{i_r,j_r}^{-\tilde{\mu}_r} \prod_{r=1}^{n} x_{i_r,j_r}^{\tilde{\lambda}_r}$$

in $K_2(S_2)$; here $\tilde{\lambda}_r = \sigma(\bar{\lambda}_r)$ and similarly for $\tilde{\mu}_r$.

Lift each $\bar{\mu}_r$ to an element $\rho_r \in S_1$. Then

$$\pi_1(\alpha) = \prod_{r=1}^{n} e_{i_r,j_r}^{\rho_r} \prod_{r=n}^{1} e_{i_r,j_r}^{-\rho_r} \prod_{r=1}^{n} e_{i_r,j_r}^{\lambda_r}$$
$$\pi_2(\alpha) = \prod_{r=1}^{n} e_{i_r,j_r}^{\mu_r} \prod_{r=n}^{1} e_{i_r,j_r}^{-\tilde{\mu}_r} \prod_{r=1}^{n} e_{i_r,j_r}^{\tilde{\lambda}_r}$$

Clearly the pairs (ρ_r, μ_r), $(-\rho_r, -\tilde{\mu}_r)$, $(\lambda_r \tilde{\lambda}_r)$ define elements a_r, b_r and c_r in $S_1 \times^{S_{12}} S_2 = S$ and we have

$$\alpha = \prod_{r=1}^{n} e_{i_r,j_r}^{a_r} \prod_{r=n}^{1} e_{i_r,j_r}^{b_r} \prod_{r=1}^{n} e_{i_r,j_r}^{c_r},$$

as desired.

The exactness at $K_1(S_1) \times K_1(S_2)$ is easier: if $\pi_{12,1*}(\alpha_1) = \pi_{12,2*}(\alpha_2)$ in $K_1(S_{12})$ for $\alpha_i \in \mathrm{GL}_{N_i}(S_i)$, $i = 1, 2$, then there is an element $e \in E_N(S_{12})$ with

$$\pi_{12,1*}(\alpha_1) = \pi_{12,2*}(\alpha_2)e$$

in $\mathrm{GL}_N(S_{12})$ for some $N \geq N_1, N_2$. We can lift e to an $e_2 \in E_N(S_2)$; replacing α_2 with $\alpha_2 e_2$, we may assume that $N_1 = N_2 = N$ and

$$\pi_{12,1*}(\alpha_1) = \pi_{12,2*}(\alpha_2)$$

in $\mathrm{GL}_N(S_{12})$. Since $S = S_1 \times^{S_{12}} S_2$, there is a unique $\alpha \in \mathrm{GL}_N(S)$ with $\pi_i(\alpha) = \alpha_i$, $i = 1, 2$.

The exactness at $K_1(S_{12})$ follows by using the splitting σ_* to $\pi_{12,2*}$. $\square$

Let

$$\Delta^n := \mathrm{Spec}\, k[t_0, \ldots, t_n]/\sum_i t_i - 1.$$

Let $\partial_i\Delta^n$ be the closed subscheme of Δ^n defined by $t_i = 0$ and $\partial\Delta^n = \cup_{i=0}^n \partial_i\Delta^n$, . For a k-scheme with G action X, we have the product schemes $X \times \Delta^n$, $X \times \partial\Delta^n$, with G acting by the identity on Δ^n, $\partial\Delta^n$, and by the given action on X.

Lemma 2.7. *Let k be a field, G a finite group with $\frac{1}{\sharp G} \in k$, X a regular affine G-scheme over k, and let U be a G-stable affine open subscheme of $X \times \partial\Delta^n$. Then U is $K_p(G, -)$-regular for all $p \leq 1$.*

Proof. It suffices to show that U is K_1-regular: we have for each p and q the fundamental exact sequence

$$\begin{aligned} 0 \to N^q K_p(G, U) & \\ &\to N^q K_p(G, U \times_k \mathbb{A}^1) \oplus N^q K_p(G, U \times_k \mathbb{A}^1) \\ &\qquad \to N^q K_p(G, U \times_k \mathbb{G}_m) \to N^q K_{p-1}(G, U) \to 0 \end{aligned}$$

where G acts trivially on $\mathbb{A}^1$, $\mathbb{G}_m$. Clearly, if U is $K_p(G, -)$ regular, so is $U \times \mathbb{A}^1$; by Lemma 2.4, this implies $U \times \mathbb{G}_m$ is also $K_p(G, -)$ regular. The exact sequence shows that U is $K_{p-1}(G, -)$ regular.

Let $\partial_{\leq m}\mathbb{A}^n$ be the closed subscheme of $\mathrm{Spec}\, k[x_1, \ldots, x_n]$ defined by $\prod_{i=1}^m x_i = 0$. Note that each point of U has an open neighborhood isomorphic (as a G-scheme) to an open neighborhood of $X \times \partial_{\leq m}\mathbb{A}^n$ for some m. By Lemma 2.4 and Theorem 2.5, it suffices to show that $X \times \partial_{\leq m}\mathbb{A}^n$ is $K_1(G, -)$ regular. We prove this by induction on n and m, the case of arbitrary n and $m = 1$ following from the smoothness of $\partial_{\leq 1}\mathbb{A}^n = \mathbb{A}^{n-1}$ over k.

Write $R_{n,m} = k[x_1, \ldots, x_n]/\prod_{i=1}^m x_i$. Note that

$$\partial_{\leq m}\mathbb{A}^n = \partial_{\leq m-1}\mathbb{A}^n \cup_{\partial_{\leq m-1}\mathbb{A}^{n-1}} \mathbb{A}^{n-1}.$$

and the inclusion $\partial_{\leq m-1}\mathbb{A}^{n-1} \to \partial_{\leq m-1}\mathbb{A}^n$ is split by the projection

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n),$$

Thus, for each affine G-scheme $Y = \operatorname{Spec} A$ over k, we have the ring

$$S := A \otimes_k R_{n,m}$$

with two-sided ideals $I_1 = (x_1)$, $I_2 = (x_m)$ which satisfy the hypotheses of Lemma 2.6. Note also that $S/I_1 = A \otimes_k R_{n-1,1}$ and $S/I_2 = A \otimes_k R_{n-1,m-1}$, so the sequence

$$0 \to K_1(G, Y \times \partial_{\leq m}\mathbb{A}^n) \to K_1(G, Y \times \partial_{\leq m-1}\mathbb{A}^n) \oplus K_1(G, Y \times \mathbb{A}^{n-1}) \to K_1(G, Y \times \partial_{\leq m-1}\mathbb{A}^{n-1}) \to 0$$

is exact. Taking $Y = X \times \mathbb{A}^q$ and using our induction hypothesis shows that $X \times \partial_{\leq m}\mathbb{A}^n$ is K_1-regular. □

Proposition 2.8. *Let k be a field, G a finite group with $\frac{1}{\sharp G} \in k$, X a regular affine k-scheme with a G-action, and U a G-stable affine open subscheme of $X \times \Delta^n$. Let $\partial U := X \times \partial\Delta^n \cap U$, $\partial_i U := U \cap X \times \partial_i \Delta^n$. Then the map*

$$\alpha : K_0(G, U; \partial U) \to K_0(G, U; \partial U_0, \dots, \partial U_n)$$

is an isomorphism.

Proof. Both U and all the intersections $\partial U_{i_1} \cap \ldots \cap \partial U_{i_s}$ are regular, so

$$K(G, U; \partial U_0, \dots, \partial U_n) \to KH(G, U; \partial U_0, \dots, \partial U_n)$$

is a weak equivalence. Since

$$\beta : KH(G, U; \partial U) \to KH(G, U; \partial U_0, \dots, \partial U_n)$$

is a weak equivalence, it suffices to show that

$$K_0(G, U; \partial U) \to KH_0(G, U; \partial U)$$

is a weak equivalence.

We have the commutative diagram, with rows homotopy fiber sequences:

$$\begin{array}{ccccc} K(G,U;\partial U) & \longrightarrow & K(G,U) & \longrightarrow & K(G,\partial U) \\ \downarrow & & \downarrow & & \downarrow \\ KH(G,U;\partial U) & \longrightarrow & KH(G,U) & \longrightarrow & KH(G,\partial U) \end{array}$$

and

$$K_0(G, U) \to KH_0(G, U)$$
$$K_1(G, U) \to KH_1(G, U)$$

are isomorphisms, so it suffices to show that the natural maps

$$K_0(G, \partial U) \to KH_0(G, \partial U)$$
$$K_1(G, \partial U) \to KH_1(G, \partial U)$$

are isomorphisms. This follows from the spectral sequence ([Wei89][th. 1.3])

$$E^1_{p,q} = N^p K_q(G, -) \Rightarrow KH_{p+q}(G, -)$$

and the $K_p(G, -)$ regularity of ∂U for $p \leq 1$ (Lemma 2.7). □

3. The equivariant homotopy coniveau tower

In this section, we define G-equivariant versions of the homotopy coniveau tower defined in [Lev01].

Let G be a finite group and T be a G-scheme. For a closed G-stable subset $W \subset T$ we define

$$G^W(G, T) := \mathrm{hofib}(G(G, T) \to G(G, T \setminus W)).$$

The localization sequence for equivariant G-theory gives us the canonical isomorphism

$$G(G, W) \to G^W(G, T).$$

We have as well the K-theory version:

$$K^W(G, T) := \mathrm{hofib}(K(G, T) \to K(G, T \setminus W)).$$

and the canonical map

$$K^W(G, T) \to G^W(G, T)$$

which is a weak equivalence in case T is regular.

We proceed to discuss the functoriality of these constructions. There is always a technical problem occuring at this point, in that the pull-back morphisms on the various exact categories involved are not strictly functorial, but only functorial up to a natural isomorphism satisfying a cocycle condition. Quillen has explained in [Qui73] how to rectify this situation by replacing the category of locally free coherent sheaves on a scheme T with an equivalent category, for which there exist strictly functorial pull-back maps. We will use this construction throughout, suppressing its explicit mention, so that the K-theory and G-theory spectra become strictly functorial constructions.

Let $f : T' \to T$ be a G-equivariant morphism of G-schemes, $W' \subset T'$ and $W \subset T$ G-stable subsets with $f^{-1}(W) \subset W'$. The commutative diagram

$$\begin{array}{ccc} K(G,T) & \xrightarrow{j^*} & K(G,T\setminus W) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ K(G,T') & \xrightarrow[j^*]{} & K(G,T'\setminus W') \end{array}$$

defines the pull-back map $f^* : K^W(G,T) \to K^{W'}(G,T')$, satisfying the functoriality $(fg)^* = g^*f^*$. The same holds for G-theory if f is flat; in fact, if we require that f factors as $p \circ i$, with $p : \mathbb{P}^N \times T$ the projection and $i : T' \to \mathbb{P}^N \times T$ a G-equivariant regular embedding into an open subscheme U of $\mathbb{P}^N \times T$, then it is not hard to show that $f^* : G^W(G,T) \to G^{W'}(G,T')$ exists in this case as well. Indeed, p is smooth, so it suffices to define i^*, and for this, the condition on i and Quillen's resolution theorem show that the inclusion of the i-flat G-coherent sheafs on $\mathbb{P}^N \times T$, $\mathcal{M}_{G,\mathbb{P}^N\times T,i}$, into $\mathcal{M}_{G,\mathbb{P}^N\times T}$ induces a weak equivalence on the K-theory spectra. See [Lev01] for details in the non-equivariant case; the arguments for the equivariant case are exactly the same. A morphism f which admits a factorization $p \circ i$ as above is called an l.c.i.-morphism.

If $T = T'$ and $f = \mathrm{id}$, then id^* defines the functorial push-forward map

$$i_{W',W*} : G^W(G,T) \to G^{W'}(G,T)$$

for $W \subset W'$ G-stable closed subsets of T.

We recall the cosimplicial k-scheme Δ^*, $n \mapsto \Delta^n$; the cosimplicial structure is defined by sending an order-preserving map $g : \{0,\ldots,n\} \to \{0,\ldots,m\}$ to the map $g : \Delta^n \to \Delta^m$ with

$$g^*(t_i) := \sum_{j\in g^{-1}(i)} t_j.$$

Definition 3.1. (1) For X of finite type over k, set

$$S^{G,X}_{(p)}(r) := \left\{ W \subset X \times \Delta^r | \begin{array}{c} W \text{ is a closed } G\text{-stable subset} \\ \text{and for all faces } F \subset \Delta^r \text{ we have} \\ \dim W \cap X \times F \le p + \dim F. \end{array} \right\}$$

(2) For X finite type and locally equi-dimensional over k,

$$S^{(p)}_{G,X}(r) := \left\{ W \subset X \times \Delta^r | \begin{array}{c} W \text{ is a closed } G\text{-stable subset} \\ \text{and for all faces } F \subset \Delta^r \text{ we have} \\ \mathrm{codim}_{X\times F} W \cap X \times F \ge p \end{array} \right\}$$

The maps $r \mapsto S^{(p)}_{G,X}(r)$, $r \mapsto S^{G,X}_{(p)}(r)$ define simplicial sets; if X has pure dimension d over k, we have

$$S^{(p)}_{G,X}(r) = S^{G,X}_{(d-p)}(r).$$

We have natural inclusions $S^{(p+1)}_{G,X}(r) \subset S^{(p)}_{G,X}(r)$, $S^{G,X}_{(p)}(r) \subset S^{G,X}_{(p+1)}(r)$.

Definition 3.2. Let X be a G-scheme of finite type over k. Let

$$G_{(p)}(G, X, r) := \operatorname*{hocolim}_{W \in S^{G,X}_{(p)}(r)} G^W(G, X \times \Delta^r).$$

If X is smooth over k, let

$$K^{(p)}(G, X, r) := \operatorname*{hocolim}_{W \in S^{(p)}_{G,X}(r)} K^W(G, X \times \Delta^r).$$

Since Δ^n is smooth over k, the structure morphisms in the cosimplicial scheme $X \times \Delta^*$ are all l.c.i.-morphisms. Thus, the simplicial structure of $S^{G,X}_{(p)}(\_)$ gives us the simplicial spectrum $G_{(p)}(G, X, \_)$. In case X is locally equi-dimensional over k we have as well the simplicial spectrum $G^{(p)}(G, X, \_)$ and if X is smooth over k, the simplicial spectrum $K^{(p)}(G, X, \_)$ with term-wise weak equivalence

$$K^{(p)}(G, X, \_) \to G^{(p)}(G, X, \_).$$

We let $\dim X$ denote the maximum of $\dim X_i$ over the irreducible components X_i of X. The inclusion $S^{G,X}_{(p)}(r) \subset S^{G,X}_{(p+1)}(r)$ gives rise to the *homotopy coniveau tower* of simplicial spectra

$$\cdots \to G_{(p)}(G, X, \_) \to G_{(p+1)}(G, X, \_) \to \cdots \to G_{(\dim X)}(G, X, \_) \tag{2}$$

We denote the layers of this tower by

$$G_{(p/p-1)}(G, X, \_) := \operatorname{hocofib}(G_{(p-1)}(G, X, \_) \to G_{(p)}(G, X, \_)).$$

Remark 3.3. Since $X \times \Delta^r$ is the final element of $S^{G,X}_{(\dim X)}(r)$, the canonical map $G(X \times \Delta^r) \to G_{(\dim X)}(X, r)$ is a weak equivalence. By homotopy invariance (compare [Tho87][Cor. 4.2] and [Tho87][Th. 5.7]), the map of the constant simplicial spectrum $G(G, X)$ to $G_{(\dim X)}(G, X, \_)$ induced by the identity $G(G, X) = G_{(\dim X)}(X, 0)$ is a weak equivalence.

With that notation we have the following proposition.

Proposition 3.4. *There is a strongly convergent spectral sequence*

$$E_1^{p,q} = \pi_{-p-q}(G_{(p/p-1)}(G, X, \_)) \Rightarrow G_{-p-q}(G, X).$$

Proof. The spectral sequence is the spectral sequence of the homotopy coniveau tower (2) together with the identitfication $G_n(G,X) \cong \pi_n G_{(\dim X)}(G,X,\_)$ given by Remark (3.3). Since

$$G_{(p)}(G,X,r) = 0$$

for all $r < -p$ and each $G_{(p)}(G,X,r)$ is -1-connected, $G_{(p)}(G,X,\_)$ is $-p-1$-connected, and hence the canonical map

$$G_{(\dim X)}(G,X,\_) \to \mathrm{hocofib}[G_{(p)}(G,X,\_) \to G_{(\dim X)}(G,X,\_)]$$

is $-p-1$-connected, whence the convergence. □

In case X is smooth over k, we have the homotopy coniveau tower for equivariant K-theory

$$\cdots \to K^{(p+1)}(G,X,\_) \to K^{(p)}(G,X,\_) \to \cdots \to K^{(0)}(G,X,\_) \tag{3}$$

with layers

$$K^{(p/p+1)}(G,X,\_) := \mathrm{hocofib}(K^{(p+1)}(G,X,\_) \to K^{(p)}(G,X,\_)).$$

and the strongly convergent spectral sequence

$$E_1^{p,q} = \pi_{-p-q}(K^{(p/p+1)}(G,X,\_)) \Rightarrow K_{-p-q}(G,X).$$

4. Local coefficients and the cycle class map

In this section, we define the equivariant cycle complex of Bredon type, $z_q(X,G,*)$, and use the homology of $z_q(X,G,*)$ to define the equivariant higher Chow groups of Bredon type.

4.1. **The local coefficients.** For a k-scheme T, we let $T_{(p)}$ denote the set of points $t \in T$ whose closure $\bar{t}$ have $\dim_k t = p$.

Let G and X as the the last section. For any smooth k-scheme Y we consider the G-scheme $X \times Y$, where G acts trivially on the second component. Because G is finite G acts also on the point $(X\times Y)_{(p)}$ of dimension p on $X \times Y$. For an orbit $[x] \in (X\times Y)_{(p)}/G$ of the G-set $(X\times Y)_{(p)}$, take a representative $x \in (X \times Y)_{(p)}$ and let $G_x \subset G$ denote the isotropy subgroup for x. We have the "local coefficient":

$$\begin{aligned} \operatorname*{colim}_{U \subset \overline{G\cdot x}} G_0(G,U) &= G_0(G, \coprod_{y\in G\cdot x} \mathrm{Spec}(\kappa(y)) \\ &= G_0(G_x, \mathrm{Spec}(\kappa(x))) \\ &= K_0(G_x, \mathrm{Spec}(\kappa(x))). \end{aligned}$$

Here the colimit on the left hand side is taken over all open G-invariant subsets U.

We have the following functorial properties for the local coefficients.

Proposition 4.1. *Let $[x] \in (X \times Y)_{(p)}/G$ and let $f : Y' \to Y$ be a morphism in* ***Sm****$/k$ of pure codimension q. Suppose that*

$$dim(1_X \times f)^{-1}(\overline{G \cdot x}) = p - q.$$

Then there is a well defined pull back morphism

$$f^* : K_0(G_x, \operatorname{Spec} \kappa(x)) \to \bigoplus_{y \in (X \times Y')_{(p-d)}/G, f(y) \in G \cdot x} K_0(G_y, \operatorname{Spec} \kappa(y))$$

with$(fg)^* = g^* f^*$ when all three maps are defined.

Proof. The proof is a modification of Quillen's proof of Gersten's conjecture [Qui73]. Let $\mathcal{M}^1(x)$ be the category of coherent G-sheaves on $\overline{G \cdot x}$ whose support contains no generic point, and set $G_n^{(1)}(G, \overline{G \cdot x}) := K_n(\mathcal{M}^1(x))$. Consider the localization sequence

$$\ldots \to G_0^{(1)}(G, \overline{G \cdot x}) \to G_0(G, \overline{G \cdot x}) \to G_0(G, G \cdot x) \to 0.$$

Since $1 \times f$ has finite Tor-dimension, we have the well-defined and functorial map

$$(1 \times f)^* : G_0(G, \overline{G \cdot x}) \to G_0(G, (1 \times f)^{-1}(\overline{G \cdot x}));$$

it clearly suffices to show that the composition

$$G_0^{(1)}(G, \overline{G \cdot x}) \to G_0(G, \overline{G \cdot x}) \xrightarrow{(1 \times f)^*} G_0(G, (1 \times f)^{-1}(\overline{G \cdot x})) \to G_0(G, G \cdot y)$$

is the zero map.

If $f : Y' \to Y$ is smooth, and $\mathcal{F}$ is in $\mathcal{M}^1(x)$, then $(1 \times f)^* \mathcal{F}$ is in $\mathcal{M}^1(y)$, whence the result in this case. Also, if k is finite, the standard trick of passing to infinite extensions $k \to k_1$, $k \to k_2$, with k_i a union of extensions of degree ℓ_i^r for distinct primes ℓ_1, ℓ_2, and using the fact that $p_* \circ p^* = \times m$ on $G_*(G, W)$ for $p : T \to W$ a finite étale G-map of degree m, reduces us to the case of an infinite base-field k.

We may replace Y with any open subscheme containing $p_2(G \cdot x)$; as this is a finite set of points of Y, we may assume that Y is affine; similarly, we may assume that Y' is affine. Thus we can factor f as a closed embedding $i : Y' \to \mathbb{A}^N \times Y$ followed by the smooth projection $\mathbb{A}^N \times Y \to Y$. This reduces us to the case of a closed embedding of affine schemes in $\mathbf{Sm}/k$. By a similar argument, we may assume that X is affine.

If $i : Y' \to Y$ is such a closed embedding, of codimension say q then we can factor i as a sequence of d codimension one closed embeddings

$$Y' = Y_0 \to Y_1 \to \ldots \to Y_{q-1} \to Y_q = Y$$

with each Y_i smooth in a neighborhood of $p_2(G \cdot y)$. By replacing Y with a suitable neighborhood of $p_2(G \cdot y)$, we may assume we have a factorization as above with each Y_i smooth over k. This reduces us to the case of a codimension one closed embedding of smooth affine k-schemes.

Let F be a G-stable closed subset of $\overline{G \cdot x}$ disjoint from $G \cdot x$ and let $\mathcal{F}$ be a coherent G-sheaf supported on F. Write F as a union $F = F_0 \cup F_1$, where F_0 is the union of the irreducible components of F which are disjoint from $G \cdot y$, and F_1 is the union of the remaining components. The localization properties of $G_*(G, -)$ yield a Mayer-Vietoris exact sequence

$$\ldots \to G_0(G, F_0 \cap F_1) \to G_0(G, F_0) \oplus G_0(G, F_1) \to G_0(G, F) \to 0$$

so we can write $[\mathcal{F}] = i_{0*}(x_0) + i_{1*}(x_1)$ for elements $x_j \in G_0(G, F_j)$, where $i_j : F_j \to F$ are the inclusions. Since it is clear that $(1 \times f)^* i_{0*}(x_0)$ goes to zero in $G_0(G, G \cdot y)$, we may assume that $F = F_1$, i.e. that $F \subset \overline{G \cdot y}$.

Suppose $\dim_k Y = n + 1$, $\dim_k Y' = n$. Let $S := p_2(G \cdot y) \subset Y'$. By *loc. cit.* there is a morphism $\pi : Y \to \mathbb{A}^n_k$ such that

(1) The restriction of π to $\bar{\pi} : Y' \to \mathbb{A}^n$ is finite.
(2) π is smooth on a neighborhood of S in Y.

Form the diagram

$$\begin{array}{ccc} Y' \times_{\mathbb{A}^n} Y & \xrightarrow{q_2} & Y \\ {\scriptstyle q_1}\downarrow\uparrow{\scriptstyle s} & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow[\bar{\pi}]{} & \mathbb{A}^n \end{array} \tag{4}$$

where s is the section to p_1 induced by the inclusion $Y' \to Y$. q_2 is finite. Since π is smooth near S, $s(Y')$ is a Cartier divisor in a neighborhood of $q_2^{-1}(S)$. Since q_2 is finite, there is a neighborhood V of S in Y such that $s(Y)$ is principal on $V' := q_2^{-1}(V)$; let t be a defining equation.

Taking the product of (4) with X gives us the diagram of G-schemes

$$\begin{array}{ccc} X \times Y' \times_{\mathbb{A}^n} Y & \xrightarrow{q_2} & X \times Y \\ {\scriptstyle q_1}\downarrow\uparrow{\scriptstyle s} & & \downarrow{\scriptstyle \pi} \\ X \times Y' & \xrightarrow[\bar{\pi}]{} & X \times \mathbb{A}^n \end{array}$$

where we omit the $1_X \times -$ on the morphisms. $T := p_2^*(t)$ is thus a defining equation for $s(X \times Y')$ over $X \times V'$.

Now set $U := X \times V$, $U' := X \times V'$, $D := s^{-1}(s(X \times Y') \cap U') \subset X \times Y'$, and let $j : U' \to X \times Y' \times_{\mathbb{A}^n} Y$ be the inclusion. We have the commutative diagram

$$\begin{array}{ccc} U' & \xrightarrow{q_2'} & U \\ & \nwarrow^{s} & \uparrow{\scriptstyle i} \\ & & D \end{array}$$

with $s(D)$ defined on U' by T. Let $\mathcal{G}$ be the restriction of $\mathcal{F}$ to D. We have the exact sequence

$$0 \to q'_{2*}j^*q_1^*\mathcal{F} \xrightarrow{\times T} q'_{2*}j^*q_1^*\mathcal{F} \to i_*\mathcal{G} \to 0;$$

pulling back to D and noting that $i^*q'_{2*}j^*q_1^*\mathcal{F}$ is supported in $D \cap \overline{G \cdot y}$ gives the identity in $G_0(G, D \cap \overline{G \cdot y})$

$$i^*[i_*\mathcal{G}] = i^*[q'_{2*}j^*q_1^*\mathcal{F}] - i^*[q'_{2*}j^*q_1^*\mathcal{F}] = 0.$$

Restricting to $G \cdot y$ completes the proof. □

We consider the simplicial set

$$X^G_{(p)}(r) := \{[x] \in (X \times \Delta^r)_{(p+r)}/G | \overline{G \cdot x} \in S^{G,X}_{(p)}(r)\}.$$

and set

$$z_p(G, X, r) := \bigoplus_{[x] \in X^G_{(p)}(r)} K_0(G_x, \mathrm{Spec}(\kappa(x))).$$

By Proposition 4.1, the cosimplicial structure on $r \mapsto \Delta^r$ makes $r \mapsto z_p(G, X, r)$ into a simplicial abelian group, denoted $z_p(G, X, \_)$.

Definition 4.2. Let X be a finite type k-scheme with a G-action for a finite group G. The *equivariant cycle complex of Bredon type*, $z_p(X, G, *)$, is the complex associated to the simplicial abelian group $z_p(G, X, \_)$. Define the *equivariant higher Chow groups of Bredon type* by

$$CH_p(G, X, r) := \pi_r(z_p(G, X, \_)) = H_p(z_p(X, G, *)).$$

If X is locally equi-dimensional over k, we may index by codimension, giving us the simplicial abelian group $z^p(G, X, \_)$, the complex $z^p(G, X, *)$ and the codimension p equivariant higher Chow groups $CH^p(X, G, r)$.

4.2. **Functorialities.** Let $\rho : L \to F$ be a finite extension of commutative noetherian rings with G-action (compatible via ρ). ρ induces the exact functor ρ^* (restriction of scalars) from the category of finitely generated $F^{tw}[G]$-modules to finitely generated $L^{tw}[G]$-modules and thereby the map on G-theories $\rho^* : G(G, L) \to G(G, F)$. For a k-algebra homomorphism $\phi : A \to B$, we have the natural isomorphism of functors

$$(\rho \otimes \mathrm{id})^* \circ (\mathrm{id} \otimes \phi)_* \cong (\mathrm{id} \otimes \phi)_* \circ (\rho \otimes \mathrm{id})^*.$$

Thus, if ϕ has finite Tor-dimension, the diagram

$$\begin{array}{ccc} G(G, L \otimes_k B) & \xrightarrow{(\mathrm{id} \otimes \phi)_*} & G(G, L \otimes_k A) \\ {\scriptstyle (\rho\otimes\mathrm{id})^*}\downarrow & & \downarrow{\scriptstyle (\rho\otimes\mathrm{id})^*} \\ G(G, F \otimes_k B) & \xrightarrow[(\mathrm{id} \otimes \phi)_*]{} & G(G, F \otimes_k A) \end{array}$$

commutes.

If $f : Y \to X$ is a proper G-equivariant morphism of k-schemes with G-action, we define the push-forward morphism

$$f_*(r) : z_p(Y, G, r) \to z_p(X, G, r),$$

by

$$[\alpha \in K_0(\kappa(Z), G_z)] \mapsto [(f \times \mathrm{id})^*(\alpha) \in K_0(\kappa((f \times \mathrm{id})(Z)), G_{(f \times \mathrm{id})(z)})]$$

if $Z \to (f \times \mathrm{id})(Z)$ is generically finite, and sending α to zero if not. By the commutativity of the above diagram, the maps $f_*(r)$ extend to the map of simplicial abelian groups

$$f_* : z_p(Y, G, \_) \to z_p(X, G, \_),$$

with $(fg)_* = f_* \circ g_*$ for proper composable morphisms f, g.

Similarly, given a flat G-equivariant morphism $f : Y \to X$ of relative dimension d, we have the pullback map

$$f^* : z_p(X, G, \_) \to z_{p+d}(Y, G, \_)$$

with $(fg)^* = g^* f^*$, and we have the compatibility

$$g^* f_* = f'_* g'^*$$

in G-equivariant cartesian squares

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & X \end{array}$$

with f proper and g flat.

Remark 4.3. Relying on the localization property of the $z_p(G, X, \_)$, one can extend the contravariant functoriality of $Y \mapsto z^p(G, X \times_k Y, \_)$ from flat morphisms to arbitrary morphisms of smooth k-schemes $Y' \to Y$; we will explain this in another paper. However, there seems to be no good pull-back $f^* : z^p(G, X, \_) \to z^p(G, Y, \_)$, compatible with pull-back on equivariant G-theory, for arbitrary G-morphisms $f : Y \to X$, even if both X and Y are smooth over k. We give an example of this phenomenon.

Let k be a field, $Z := \operatorname{Spec} k[x, y]$, $G := \mathbb{Z}/2$ and let G act on Z via

$$(1 \bmod 2\mathbb{Z}) \cdot (x, y) := (-x, -y).$$

Let X be the line $y = 0$, Y the line $x = 0$ and w be the intersection of X and Y, i. e. $w = \{(0, 0)\}$. Let y be the generic point of Y.

We get the following diagram

$$\begin{array}{ccc} K^Y(G, Z) & \longrightarrow & K_0(G_y, \kappa(y)) \\ \big\downarrow & & \\ K^w(G, X) & \longrightarrow & K_0(G_w, \kappa(w)) \end{array}$$

The horizontal arrows are surjective by localization and the bottom horizontal arrow is an isomorphism. The homotopy property for equivariant K-theory implies that the vertical arrow is also an isomorphism, again by the localization theorem.

Since $G_w = G$ acts trivially on $\kappa(w)$, $K_0(G_w, \kappa(w)))$ is isomorphic to the group ring $\mathbb{Z}[\mathbb{Z}/2]$. However, since $G = G_y$ acts non-trivially on $\kappa(y)$, we have

$$K_0(G_y, \kappa(y)) \cong K_0(\kappa(y)^G) \cong \mathbb{Z}.$$

Thus, there is no arrow on the right which makes this diagram commutative.

This is related to the fact that the ring struction on $K_0(X, G)$ (given by tensor product over $\mathcal{O}_X$) does not in general respect the topological filtration, and so one should not expect the equivariant Chow groups $CH^*(G, X, *)$ to have a ring structure. To see an example of this, consider the case $K_0(G, X)$, with X and G as above. It follows from the homotopy property and localization that the image of

$$K_0^w(G, X) \to K_0(G, X)$$

is the augmentation ideal of the group ring $\mathbb{Z}[\mathbb{Z}/2]$. But if σ denotes the non-identity element of $\mathbb{Z}/2$, we have $(1 - \sigma)^2 = 2(1 - \sigma)$. Thus, letting $\mathbb{Z}_\sigma$ denote the sign representation of $\mathbb{Z}/2$,

$$(\mathbb{Z} - \mathbb{Z}_\sigma)^{\otimes 2} \cong 2(\mathbb{Z} - \mathbb{Z}_\sigma).$$

This shows that $F^1_{top} \cdot F^1_{top} \neq 0$, although $F^2_{top} = 0$.

This last point arose during discussion of the first-named author with W. Niziol, while she was visiting Northeastern University.

4.3. **The cycle map.** For each $W \in S^{G,X}_{(p)}(r)$ we have a canonical morphism

$$\pi_0(G^W(G, X \times \Delta^r)) \to \bigoplus_{[x]\in X^G_{(p)}(r), x\in W} K_0(G_x, \mathrm{Spec}(\kappa(x))).$$

This defines a morphism

$$\pi_0(G_{(p)}(G, X, r)) \to \bigoplus_{[x]\in X^G_{(p)}(r)} K_0(G_x, \mathrm{Spec}(\kappa(x))),$$

which factors through the surjection

$$\pi_0(G_{(p)}(G, X, r)) \to \pi_0(G_{(p/p-1)}(G, X, r)).$$

We consider $\pi_0(G_{(p/p-1)}(G, X, n))$ as spectrum by using the associated Eilenberg-Maclane spectra. Noting that $G_{(p/p-1)}(G, X, n)$ is -1-connected for each n, we have the canonical maps of spectra

$$G_{(p/p-1)}(G, X, n) \to \pi_0(G_{(p/p-1)}(G, X, n)),$$

which yield the map of simplicial spectra

$$G_{(p/p-1)}(G, X, -) \to \pi_0(G_{(p/p-1)}(G, X, -)).$$

We similarly consider $z_p(G, X, \_)$ as a simplicial spectrum. Taking the composition of the maps described above yields the *cycle map*

$$cl_p : G_{(p/p-1)}(G, X, \_) \to z_p(G, X, \_),$$

defined as a map of simplicial spectra.

Now we can formulate our main result.

Theorem 4.4. *Let X be a scheme of finite type over a field k with an action of a finite group G. Suppose that $\frac{1}{\sharp G} \in k$. Then the cycle map*

$$cl_p : G_{(p/p-1)}(G, X, \_) \to z_p(G, X, \_)$$

is a weak equivalence for all p.

This and Proposition 3.4 gives

Corollary 4.5. *There is a strongly convergent spectral sequence*

$$E_1^{p,q} = CH_p(G, X, -p-q) \Rightarrow G_{-p-q}(G, X).$$

For the proof we first reduce in the next section via localization techniques to the case where X is a point. In the last section we discuss the case of a point.

Remark 4.6. It is easy to see that the cycle map and the spectral sequence are natural with respect to proper pushforward and flat pull-back.

5. Localization and reduction to the point

The main result of this section is

Theorem 5.1. *Let G be a finite group acting on a finite type k-scheme X, $i : W \to X$ a G-stable closed subscheme with open complement $j : U \to X$. Then for each p, the sequence of simplicial spectra*

$$G_{(p)}(G, W, \_) \xrightarrow{i_*} G_{(p)}(G, X, \_) \xrightarrow{j^*} G_{(p)}(G, U, \_)$$

is a weak homotopy fiber sequence, and the sequence of complexes

$$z_p(G, W, *) \xrightarrow{i_*} z_p(G, X, *) \xrightarrow{j^*} z_p(G, U, *)$$

is isomorphic to a cone sequence in the derived category, i.e., the induced map

$$(j^*, 0) : cone(i_*) \to z_p(G, U, *)$$

is a quasi-isomorphism.

Proof. We first consider the G-theory sequence. The proof is the same as the proof of the analogous localization theorem in the non-equivariant case [Lev01, Cor. 8.2], replacing the spectra $G_{(p)}(?, \_)$ with the G-equivariant versions $G_{(p)}(G, ?, \_)$ throughout. For the readers convenience, we give a sketch of the argument.

Let $S^{G,U^X}_{(p)}(r)$ be the subset of $S^{G,U}_{(p)}(r)$ consisting of those $W \subset U \times \Delta^r$ with closure $\overline{W} \subset X \times \Delta^r$ in $S^{G,U}_{(p)}(r)$. These form a simplicial subset of $r \mapsto S^{G,U}_{(p)}(r)$; let

$$G_{(p)}(G, U^X, r) := \operatorname*{hocolim}_{W \in S^{G,U^X}_{(p)}(r)} G^W(G, X \times \Delta^r),$$

forming the simplicial spectrum $G_{(p)}(G, U^X, \_)$. The inclusions $S^{G,U^X}_{(p)}(r) \subset S^{G,U}_{(p)}(r)$ induce the map of simplicial spectra

$$\iota : G_{(p)}(G, U^X, \_) \to G_{(p)}(G, U, \_),$$

the localization sequence of G-equivariant G-theory gives the weak homotopy fiber sequence

$$G_{(p)}(G, W, r) \xrightarrow{i_*} G_{(p)}(G, X, r) \xrightarrow{j^*} G_{(p)}(G, U^X, r)$$

for each r, and thus the weak homotopy fiber sequence

$$G_{(p)}(G, W, \_) \xrightarrow{i_*} G_{(p)}(G, X, \_) \xrightarrow{j^*} G_{(p)}(G, U^X, \_).$$

Thus, we must show that ι is a weak equivalence. Since both simplicial spectra are -1-connected, the Hurewicz theorem tells us that it suffices to show that ι is a homology isomorphism.

For a spectrum E, let $\mathbb{Z}E$ be a functorial model for a chain complex spectrum with homology groups the homology of E.

For $W \subset U \times \Delta^r$, we have the complex

$$\mathbb{Z}G^W(G, U \times \Delta^r) := \operatorname{cone}[\mathbb{Z}G(G, U \times \Delta^r) \to \mathbb{Z}G(G, U \times \Delta^r \setminus W][-1]$$

representing the homology of $G^W(G, U \times \Delta^r)$. Taking the limit of $W \in S^{G,U}_{(p)}(r)$ or in $S^{G,U^X}_{(p)}(r)$ gives us the complexes $\mathbb{Z}G_{(p)}(G, U, r)$ and $\mathbb{Z}G_{(p)}(G, U^X, r)$ computing the homology of $G_{(p)}(G, U, r)$ and $G_{(p)}(G, U^X, r)$.

For $W \subset U \times \Delta^r$, let $W_n \subset U \times \Delta^n$ be the union of $(\mathrm{id} \times g)^{-1}(W)$, as $g : \Delta^n \to \Delta^r$ runs over structure morphisms for the cosimplicial scheme Δ. Using the usual alternating sum of the pullback by coface maps $\mathrm{id} \times \delta^r_i : U \times \Delta^r \to U \times \Delta^{r+1}$, we form the double complex $n \mapsto \mathbb{Z}G^{W_n}(G, U \times \Delta^n)$ and denote the associated total complex by $\mathbb{Z}G^W(\mathbb{Z}U \times \Delta^*)$. Thus the limit of the complexes $\mathbb{Z}G^W(\mathbb{Z}U \times \Delta^*)$ over $W \in S^{G,U}_{(p)}(r)$ or in $S^{G,U^X}_{(p)}(r)$, $r = 1, 2, \ldots$, computes the homology of $G_{(p)}(G, U, \_)$ and $G_{(p)}(G, U^X, \_)$. We denote the limits of these complexes by $\mathbb{Z}G_{(p)}(G, U)^*$ and $\mathbb{Z}G_{(p)}(G, U^X)^*$, respectively. It thus suffices to show that

$$\iota_{\mathbb{Z}} : \mathbb{Z}G_{(p)}(G, U^X)^* \to \mathbb{Z}G_{(p)}(G, U)^*$$

is a quasi-isomorphism.

For $W \in S^{G,U}_{(p)}(r)$, $W' \in S^{G,U^X}_{(p)}(r)$, let

$$\iota_W : \mathbb{Z}G^W(G, U \times \Delta^*) \to \mathbb{Z}G_{(p)}(G, U)^*$$

and

$$\iota^X_{W'} : \mathbb{Z}G^W(G, U \times \Delta^*) \to \mathbb{Z}G_{(p)}(G, U^X)^*$$

be the canonical maps.

Next, we construct another pair of complexes which approximate $\mathbb{Z}G_{(p)}(G, U)^*$ and $\mathbb{Z}G_{(p)}(G, U^X)^*$. For this, fix an integer $N \geq 0$. Let $\partial\Delta^N_i \subset \Delta^N$ be the subscheme defined by $t_i = 0$; for $I \subset \{0, \ldots, N\}$ let $\partial\Delta^N_I$ be the face $\cap_{i \in I} \partial\Delta^N_i$. For $I \supset J$, let $i_{J,I} : \Delta^N_I \to \Delta^N_J$ be the inclusion.

Form the complex $(\Delta^N, \partial\Delta^N)^*$ be the complex which is $\oplus_{I,\ |I|=n}$ in degree $-n$, and with differential

$$d^{-n} : (\Delta^N, \partial\Delta^N)^{-n} \to (\Delta^N, \partial\Delta^N)^{-n+1}$$

given by $d^{-n} := \prod_{I,\ |I|=n} d_I^{-n}$, where

$$d_I^{-n} : \partial\Delta_I^N \to \oplus_{J,\ |J|=n-1} \partial\Delta_J^N$$

is the sum

$$d_I^{-n} := \sum_{j=1}^{n} i_{I\setminus\{i_j\},I},$$

where $I = (i_1, \ldots, i_n)$, $i_1 < \ldots < i_n$.

The identity map on Δ^N extends to a map of complexes

$$\Phi^N : \mathbb{Z}\Delta^* \to (\Delta^N, \partial\Delta^N)[-N];$$

the maps in degree $r < N$ are all $\pm\,\mathrm{id}_{\Delta^r}$. We can take the product of this construction with U, giving us the complex $U \times (\Delta^N, \partial\Delta^N)$ and the map of complexes

$$\Phi^N : U \times \mathbb{Z}\Delta^* \to U \times (\Delta^N, \partial\Delta^N)[-N]$$

For $W \in S^{G,U}_{(p)}(N)$, form the complex $\mathbb{Z}G^W(G, U \times (\Delta^N, \partial\Delta^N)$ by taking $\oplus_{I,|I|=n}\mathbb{Z}G^{W_{N-n}}(G, U \times \Delta^{N-n})$ in degree $-n$, using the differentials in $U \times (\Delta^N, \partial\Delta^N)$ to form a double complex and then taking the total complex. We thus have the map of complexes

$$\Phi^{N*}_W : \mathbb{Z}G^W(G, U \times (\Delta^N, \partial\Delta^N))[-N] \to \mathbb{Z}G^W(G, U \times \Delta^*).$$

One shows that Φ^{N*}_W induces a homology isomorphism in degrees $< N$.

Take $W \in S^{G,U}_{(p)}(N)$. The main construction of [Lev01] gives a map of complexes

$$\Psi_W : U \times \mathbb{Z}\Delta^* \to U \times (\Delta^N, \partial\Delta^N)[-N]$$

and a degree -1 map

$$H_W : U \times \mathbb{Z}\Delta^* \to U \times (\Delta^N, \partial\Delta^N)[-N]$$

with the following properties:

(1) $dH_W = \Psi_W - \Phi^N$.
(2) Write Ψ_W as a sum

$$\Phi_W = \sum_{i=0}^{N} \sum_{I,j\ |I|=i} n_{ij}\psi_{ijI}$$

with $\psi_{ijI} : \Delta^{N-i} \to \partial\Delta_I^N = \Delta^{N-i}$ maps in $\mathbf{Sm}/k$. Then $\psi_{ijI}^{-1}(W_{N-i})$ is in $S^{G,U^X}_{(p)}(N-i)$.

(3) Write H_W as a sum

$$H_W = \sum_{i=0}^{N} \sum_{I,j\ |I|=i} n_{ij} H_{ijI}$$

with $H_{ijI} : \Delta^{N-i+1} \to \partial\Delta_I^N = \Delta^{N-i}$ maps in $\mathbf{Sm}/k$. Then $H_{ijI}^{-1}(W_{N-i})$ is in $S_{(p)}^{G,U}(N-i+1)$. If $W' \subset W_{N-i}$ is in $S_{(p)}^{G,U^X}(N-i)$, then $H_{ijI}^{-1}(W')$ is in $S_{(p)}^{G,U^X}(N-i+1)$.

Thus Ψ_W induces the map of complexes

$$\Psi_W^* : \mathbb{Z}G^W(G, U \times (\Delta^N, \partial\Delta^N))[-N] \to \mathbb{Z}G_{(p)}(G, U^X)^*$$

and H_W gives a degree 1 map

$$H_W^* : \mathbb{Z}G^W(G, U \times (\Delta^N, \partial\Delta^N))[-N] \to \mathbb{Z}G_{(p)}(G, U)^*$$

with

$$dH_W^* = \iota_{\mathbb{Z}} \circ \Psi_W^* - \iota_W \circ \Phi_W^{N*}$$

Furthermore, if $W' \subset W$ is in $S_{(p)}^{G,U^X}(N)$, then H_W gives a degree 1 map

$$H_W^{X*} : \mathbb{Z}G^{W'}(G, U \times (\Delta^N, \partial\Delta^N))[-N] \to \mathbb{Z}G_{(p)}(G, U^X)^*$$

with

$$dH_W^{X*} = \Psi_W^* - \iota_{W'}^X \circ \Phi_{W'}^{N*}.$$

Since Φ_W^{N*} is a homology isomorphism in degrees $< N$ and $\mathbb{Z}G_{(p)}(G,U)^*$ and $\mathbb{Z}G_{(p)}(G,U^X)^*$ are the limits of $\mathbb{Z}G^W(G, U\times\Delta^*)$ and $\mathbb{Z}G^{W'}(G, U\times\Delta^*)$, respectively, this suffices to prove that $\iota_{\mathbb{Z}}$ is a quasi-isomorphism, completing the proof.

For the sequence of cycle complexes, let $z_p(G, U^X, r)$ be the subgroup of $z_p(G, U, r)$ generated by the irreducible codimension p closed subsets $W \subset U \times \Delta^r$ with $W \in S_{(p)}^{G,U^X}(r)$. This forms the subcomplex $z_p(G, U^X, *)$ of $z_p(G, U, *)$ and gives us the term-wise exact sequence of complexes

$$0 \to z_p(G, W, *) \xrightarrow{i_*} z_p(G, X, *) \xrightarrow{j^*} z_p(G, U^X, *) \to 0.$$

Thus, we need to show that the inclusion

$$z_p(G, U^X, *) \to z_p(G, U, *)$$

is a quasi-isomorphism. The proof is now exactly the same as the case of G-theory, except that we can avoid the use of the Hurewicz theorem by working directly with the complexes $z_p(G, ?, *)$ instead of passing to complexes representing the homology of the simplicial abelian group $z_p(G, ?, \_)$. □

Corollary 5.2. *With the hypotheses and notations as in Theorem 5.1, the sequence*

$$G_{(p/p-1)}(G, W, \_) \xrightarrow{i_*} G_{(p/p-1)}(G, X, \_) \xrightarrow{j^*} G_{(p/p-1)}(G, U, \_)$$

is a weak homotopy fiber sequence for all p.

Proof. This follows directly from Theorem 5.1, the naturality of i_* and j^* with respect to change of p, and the Quetzlcoatl lemma. □

Corollary 5.3. *Suppose that Theorem 4.4 is true for all fields k with $\frac{1}{\sharp G} \in k$ and with $X = \operatorname{Spec} K$, where K is a finite extension of k with a G-action such that G acts trivially on k. Then Theorem 4.4 is true for all all fields k with $\frac{1}{\sharp G} \in k$.*

Proof. We prove Theorem 4.4 by induction on $\dim_k X$, the case of dimension 0 being true by hypothesis.

We have already remarked that the cycle map

$$cl_p : G_{(p/p-1)}(G, X, \_) \to z_p(G, X, \_)$$

is natural with respect to proper pushforward and pullback with respect to flat maps. Thus, for each closed subset $W \subset X$, we have the commutative diagram

$$\begin{array}{ccccc} G_{(p/p-1)}(G, W, \_) & \xrightarrow{i_*} & G_{(p/p-1)}(G, X, \_) & \xrightarrow{j^*} & G_{(p/p-1)}(G, U, \_) \\ \downarrow{\scriptstyle cl_p^W} & & \downarrow{\scriptstyle cl_p^X} & & \downarrow{\scriptstyle cl_p^U} \\ z_p(G, W, \_) & \xrightarrow[i_*]{} & z_p(G, X, \_) & \xrightarrow[j^*]{} & z_p(G, U, \_) \end{array}$$

By Theorem 5.1 and Corollary 5.1 the rows are weak homotopy fiber sequences; by our induction hypothesis, cl_p^W is a weak equivalence, so cl_p^X is a weak equivalence if and only if cl_p^U is. Taking the limit over all open dense $U \subset X$ reduces us to showing that

$$cl_p^{\kappa(X)} : G_{(p)}(G, \kappa(X), \_) \to z_p(G, \kappa(X), \_)$$

is a weak equivalence. Breaking up $\kappa(X)$ into a product of fields reduces us to the case of irreducible X.

Note that $G_{(p)}(G, \kappa(X), \_)$, $z_p(G, \kappa(X), \_)$ and cl_p to not depend on the choice of "constants" $k \subset \kappa(X)$, so we may replace k with the invariant subfield $\kappa(X)^G$. Since the extension $\kappa(X)^G \subset \kappa(X)$ is finite, $cl_p^{\kappa(X)}$ is a weak equivalence by hypothesis, completing the proof. □

6. The case of the point

Now we consider the case of a point. Let $X = \mathrm{Spec}(K)$, $K \supset k$ a finite field extension of k with a G-action, with $\frac{1}{\sharp G} \in K$ such that G acts trivially on k. Let $\hat{k} := K^G$ be the subfield of K which is fixed under the operation of G. Then the field extension $K/\hat{k}$ is finite and Galois; as in the proof of Corollary 5.3, we may replace k with $\hat{k}$. Changing notation, we assume that k is the fixed subfield of K under G . We consider the following functor:

$$\begin{array}{cccc} E: & \mathbf{Sm}/k^{op} & \rightarrow & \mathbf{Spt} \\ & X & \mapsto & K(G, X \otimes_k K) \end{array}$$

First of all we want to show that this functor satisfies the axioms of [Lev03]. We first recall some notations from [Lev03].

Let $v(n) := \{v_0(n), \dots, v_n(n)\}$ be the vertices of Δ^n, where $v_i(n)$ is the point $t_j = 0$, $j \neq i$, $t_i = 1$. For a field F, let $\mathcal{O}_{\Delta^n_F, v(n)}$ be the semi-local ring of $v(n)$ in Δ^n_F and set $\Delta^n_{0,F} := \mathrm{Spec}\, \mathcal{O}_{\Delta^n_F, v(n)}$. The $\Delta^n_{0,F}$ form a cosimplicial subscheme of $n \mapsto \Delta^n_F$. We extend this notation to F a product of fields in the evident manner.

We let $\partial_i \Delta^n_{0,F} := \partial_i \Delta^n_F \cap \Delta^n_{0,F}$, i.e. $\partial_i \Delta^n_{0,F}$ is the closed subscheme of $\partial \Delta^n_{0,F}$ defined by $t_i = 0$, and set $\partial \Delta^n_{0,F} := \cup_{i=0}^n \partial_i \Delta^n_{0,F}$. We let $\partial_* \Delta^n_{0,F}$ denote the set of components of $\partial \Delta^n_{0,F}$,

$$\partial_* \Delta^n_{0,F} := \{\partial_0 \Delta^n_{0,F}, \dots, \partial_n \Delta^n_{0,F}\}$$

and denote, e.g., the relative K-theory $K(\Delta^n_{0,F}; \partial_0 \Delta^n_{0,F}, \dots, \partial_n \Delta^n_{0,F})$ by $K(\Delta^n_{0,F}; \partial_* \Delta^n_{0,F})$.

From [Lev03, Section 5], we have the notion of a *well-connected* functor $E : \mathbf{Sm}/k^{\mathrm{op}} \rightarrow \mathbf{Spt}$. From the definition [Lev03, Definition 5.1.1] and [Lev03, Proposition 5.3.3], E is well-connected if E satisfies:

(1) Homotopy invariance:
For each $X \in \mathbf{Sm}/k$ the map $p^* : E(X) \rightarrow E(\mathbb{A}^1 \times X)$ is an weak equivalence.
(2) Nisnevich excision:
Let $f : X' \rightarrow X$ be an étale morphism in Sm/k, and let $W \subset X$ be a closed subset. Suppose that f restricts to an isomorphism $f^{-1}(W) \rightarrow W$. Then

$$f^* : E^W(X) \rightarrow E^{f^{-1}(W)}(X')$$

is a weak equivalence.
(3) Delooping.
For a functor $F : \mathbf{Sm}/k^{op} \rightarrow \mathbf{Spt}$, define $\Omega_T F : \mathbf{Sm}/k^{op} \rightarrow \mathbf{Spt}$

by

$$\Omega_T E(X) := E^{X\times 0}(X \times \mathbb{P}^1).$$

Then there is a functor $E_2 : \mathbf{Sm}/k^{op} \to \mathbf{Spt}$ satisfying (1) and (2) and a natural equivalence

$$\sigma : E \to \Omega_T(E_2).$$

(4) Well-connectedness (1).
For a closed W in a smooth X the spectrum $E^W(X)$ is -1 connected.

(5) Well connectedness (2).
For all $n \geq 1$, $d \geq 0$, and all finite generated fields F over k,

$$\pi_0[(\Omega_T^d E)(\Delta^n_{0,F}, \partial_* \Delta^n_{0,F})] = 0.$$

Proposition 6.1. *Suppose the functor* $E : \boldsymbol{Sm}/k^{op} \to \boldsymbol{Spt}$*,*

$$E(X) = K(G, X \otimes_k K),$$

is well-connected for all fields K with G-action, such that $\frac{1}{\sharp G} \in K$ and $k = K^G$. Then for all $X \in \boldsymbol{Sm}_k$, the cycle map

$$cl_p : G_{(p/p-1)}(G, X \otimes_k K, \_) \to z_p(G, X \otimes_k K, \_)$$

is a weak equivalence for all p.

Proof. We use the notation of [Lev03, Section 4], except that we index with respect to dimension rather than codimension to maintain the conventions used here.

Since E is well-connected, it follows from [Lev03, Corollary 4.3.2] that for $X \in \mathbf{Sm}/k$, the simplicial spectrum $E_{(p/p-1)}(X, \_) := G_{(p/p-1)}(G, X \otimes_k K, \_)$ is weakly equivalent to to the simplicial spectrum $E^{\text{s.l.}}_{(p/p-1)}(X, \_)$. In addition, there is for each n a weak equivalence

$$cl^{\text{s.l.}}_{p,n} : E^{\text{s.l.}}_{(p/p-1)}(X, n) \to z_p(G, X, n)$$

(more precisely, to the Eilenberg-Maclane spectrum associated to the abelian group $z_p(G, X, n)$).

The argument of [Lev03, Theorem 5.4.1], repeated word for word, shows that $cl^{\text{s.l.}}_{p,n}$ induces a weak equivalence of simplicial spectra

$$cl^{\text{s.l.}}_p : E^{\text{s.l.}}_{(p/p-1)}(X, n) \to z_p(G, X, n)$$

and that composing $cl^{\text{s.l.}}_p$ with the weak equivalence $G_{(p/p-1)}(G, X \otimes_k K, \_) \to E^{\text{s.l.}}_{(p/p-1)}(X, \_)$ yields the map cl_p (in the stable homotopy category). □

Thus, with the help of Corollary 5.3, to finish the proof of Theorem 4.4 we need only show:

Proposition 6.2. *The functor* $E : \boldsymbol{Sm}/k^{op} \to \boldsymbol{Spt}$*,*

$$E(X) = K(G, X \otimes_k K),$$

is well-connected.

Proof. We need to show that E satisfies:

(i) Homotopy invariance.
This follows immediately from the homotopy invariance for equivariant K-theory.

(ii) Nisnevich excision.
This follows from the fact that the equivariant localization theorem implies $E^W(X) = G(G, W \otimes_k K)$.

(iii) Delooping.
By equivariant localization we have

$$E^{X\times 0}(X \times \mathbb{P}^1) = E(X).$$

So we can take $E_2 = E$.

(iv) Well-connectedness (1).
Since the equivariant G-theory spectrum is -1-connected, $E^W(X)$ is -1 connected by the localization property as in (ii).

(v) Well connectedness (2).

To prove (v), we first note that, since $k \to K$ is finite and separable, $F \otimes_k K$ is a finite product of fields and we have

$$\Delta^n_{0,F} \times_k K \cong \Delta^n_{0,F\otimes_k K}$$
$$\partial_i \Delta^n_{0,F} \times_k K \cong \partial_i \Delta^n_{0,F\otimes_k K}$$

Thus (v) follows from

Lemma 6.3. *Let* F *be a finitely generated field extension of* k*. Then*

$$K_0(G, \Delta^n_{0,F\otimes_k K}; \partial_* \Delta^n_{0,F\otimes_k K}) = 0.$$

Proof. Let $A = F \otimes_k K$. G acts transitively on the irreducible components of $\operatorname{Spec} A$. Fix one component $x := \operatorname{Spec} \kappa(x)$, and let G_x be the isotropy group of x. Then

$$K_0(G, \Delta^n_{0,F\otimes_k K}; \partial\Delta^n_{0,F\otimes_k K,*}) = K_0(G_x, \Delta^n_{0,\kappa(x)}; \partial\Delta^n_{0,\kappa(x),*}).$$

In addition, letting $k_x \subset K$ be the fixed field of G_x, we have

$$F \otimes_{k_x} K = \kappa(x).$$

Thus, changing notation, we may assume that $F \otimes_k K$ is a field L and F is the G-fixed subfield of L.

We note that $\Delta^n_{0,L}$ is an intersection of G-stable affine open subschemes U of Δ^n_L. Thus by Proposition 2.8, the map

$$\alpha : K_0(G, \Delta^n_{0,L}; \partial\Delta^n_{0,L}) \to K_0(G, \Delta^n_{0,L}; \partial_*\Delta^n_{0,L})$$

is an isomorphism.

We have the exact sequence

$$K_1(G, \Delta^n_{0,L}) \to K_1(G, \partial\Delta^n_{0,L}) \to K_0(G, \Delta^n_{0,L}; \partial\Delta^n_{0,L}) \to K_0(G, \Delta^n_{0,L}) \to K_0(G, \partial\Delta^n_{0,L})$$

Let $R := \mathcal{O}_{\Delta^n_{0,F}, v(n)}$, $\bar{R} := R/(\prod_{i=0}^n t_i)$. Since $F \to L$ is finite, we have

$$\Delta^n_{0,L} = \operatorname{Spec} R \otimes_F L$$
$$\partial\Delta^n_{0,L} = \operatorname{Spec} \bar{R} \otimes_F L$$

Also, by Lemma 2.1

$$K(G, \Delta^n_{0,L}) = K(R \otimes_F L^{tw}[G])$$
$$K(G, \partial\Delta^n_{0,L}) = K(\bar{R} \otimes_F L^{tw}[G])$$

By the lemma below, $R\otimes_F L^{tw}[G]$ and $\bar{R}\otimes_F L^{tw}[G]$ are semi-local rings, and the surjection $R \otimes_F L^{tw}[G] \to \bar{R} \otimes_F L^{tw}[G]$ induces a bijection on the (finite) sets of maximal two-sided ideals. It follows easily from this that $K_1(R \otimes_F L^{tw}[G]) \to K_1(\bar{R} \otimes_F L^{tw}[G])$ is surjective.

Using the notations of the lemma below, we have

$$R \otimes_F L^{tw}[G] = \prod_{i=1}^r M_{n_i}(R \otimes_F D_i)$$
$$\bar{R} \otimes_F L^{tw}[G] = \prod_{i=1}^r M_{n_i}(\bar{R} \otimes_F D_i).$$

In addition, there is a finite separable field extension $F \subset F_i$ such that $R\otimes_F D_i$ is an Azumaya algebra over $R\otimes_F F_i$ and $\bar{R}\otimes_F D_i$ is an Azumaya algebra over $\bar{R} \otimes_F F_i$. Since $R \otimes_F F_i$ is integral and semi-local, this implies that each projective module over $R \otimes_F D_i$ is free; by Morita equivalence we have

$$K_0(M_{n_i}(R \otimes_F D_i)) = \mathbb{Z},$$

generated by the class of $(R\otimes_F D_i)^{n_i}$. This easily implies that the map

$$K_0(M_{n_i}(R \otimes_F D_i)) \to K_0(M_{n_i}(\bar{R} \otimes_F D_i))$$

is injective, hence $K_0(R \otimes_F L^{tw}[G]) \to K_0(\bar{R} \otimes_F L^{tw}[G])$ is injective.

Thus $K_0(G, \Delta^n_{0,L}; \partial\Delta^n_{0,L}) = 0$, completing the proof. □

Lemma 6.4. *Let L be a field with an action of a finite group G, $F \subset L$ the fixed subfield of L. Suppose that $\frac{1}{\sharp G} \in F$. Let R be a noetherian commutative reduced semi-local F-algebra with the property that for each maximal ideal $m \subset R$ we have $R/m \simeq F$. Then*

$$R \otimes_F L^{tw}[G] = \prod_{i=1}^{r} M_{n_i}(R \otimes_F D_i)$$

where each D_i is a central division algebra over F_i for some finite separable field extensions $F \subset F_i$, and M_{n_i} denotes the n_i by n_i matrix algebra. In addition, the D_i, F_i, n_i and the integer r are independent of R, and the center of $M_{n_i}(R \otimes_F D_i)$ is $R \otimes_F F_i$.

Proof. Since $\frac{1}{\sharp G} \in F$, $L^{tw}[G]$ is a semi-simple, separable algeba over F containing F in its center. Thus, we have $L^{tw}[G] = \prod_{i=1}^{n} M_{n_i}(D_i)$ where D_i are division rings containing F in their centers, $n_i \in \mathbb{N}$. Let F_i be the center of D_i. Since $F \to L^{tw}[G]$ is a finite separable extension, each F_i is finite and separable over F.

We claim that $R \otimes_F F_i$ is the center of $R \otimes_F D_i$. Indeed, the center $\mathcal{Z}_i$ of $R \otimes_F D_i$ is a finitely generated projective R-module containing $R \otimes_F F_i$. By our assumption on the maximal ideals of R, we have $R/m \otimes_R \mathcal{Z}_i = F_i = R/m \otimes_R R \otimes_F F_i$ for all maximal ideals $m \subset R$, so by Nakayama's lemma, $R \otimes_F F_i = \mathcal{Z}_i$. □

This completes the proof of the proposition, and the proof of Theorem 4.4 □

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