FUNDAMENTAL CLASSES IN ALGEBRAIC COBORDISM

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ABSTRACT. We recall our construction of fundamental classes in the algebraic cobordism of smooth quasi-projective varieties, and show by examples that it is not possible to extend this to fundamental classes, functorial for local complete intersection morphisms, for Cohen-Macaulay varieties.

1. INTRODUCTION

Together with Fabien Morel, we have constructed in [2, 3, 4, 5] a theory of *algebraic cobordism*, which gives an algebraic version of the theory of complex cobordism for smooth algebraic varieties over a field k of characteristic zero. This theory extends to a Borel-Moore theory of algebraic cobordism for schemes of finite type over k, which is covariant for projective morphisms and contravariant for local complete intersection morphisms. We denote the Borel-Moore theory as

$$X \mapsto \Omega_*(X).$$

For X an equi-dimensional scheme over k, we set $\Omega^n(X) = \Omega_{\dim_k X - n}(X)$ and extend to X which are locally equi-dimensional by taking the direct sum of the connected components of X. This theory has the following structure (see [4, 5]):

Recall that a *regular imbedding* is a closed imbedding $i : Y \to V$ such that the ideal sheaf of $i(Y)$ is locally generated by a regular sequence, and that a morphism $f : Y \to X$ is a *local complete intersection* morphism (l.c.i.-morphism for short) if there is a factorization of f as $f = p \circ i$, where $i : Y \to V$ is a regular embedding and $p : V \to X$ is smooth and quasi-projective.

Let $\mathbf{Sch}_k$ be the category of k-schemes of finite type, $L\mathbf{Sch}_k$ the category of locally equi-dimensional k-schemes of finite type, with morphisms the l.c.i.-morphisms of k-schemes, and $P\mathbf{Sch}_k$ the category of k-schemes of finite type, with morphisms the projective morphisms of

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Dedicated to Professor Hyman Bass on the occasion of his 70th birthday.

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k-schemes. Let $\mathbf{Sm}_k$ denote the category of smooth, quasi-projective k-schemes; as every morphism in $\mathbf{Sm}_k$ is an l.c.i.-morphism, we have the inclusion $\mathbf{Sm}_k \subset L\mathbf{Sch}_k$. We let $\mathbf{GrAb}$ denote the category of graded abelian groups, and $\mathbf{GrRing}$ the category of commutative, graded rings with unit.

(1.1)

(1) Sending X to $\Omega^*(X)$ extends to a functor
$$\Omega^* : L\mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{GrAb}.$$
The restriction of Ω^* to $\mathbf{Sm}_k$ has the structure of a functor $\Omega^* : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{GrRing}$.
(2) Sending X to $\Omega_*(X)$ extends to a functor $\Omega_* : P\mathbf{Sch}_k \to \mathbf{GrAb}$.
(3) For $X \in L\mathbf{Sch}_k$, pull-back by the projection $p : X \times \mathbb{A}^1 \to X$ gives an isomorphism $p^* : \Omega^*(X) \to \Omega^*(X \times \mathbb{A}^1)$.
(4) Let $f : Y \to X$ be an l.c.i.-morphism of finite type k-schemes, and $g : Z \to X$ a projective morphism of finite type k-schemes. Suppose that $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for $i > 0$, and let $W = Y \times_X Z$, $f' : W \to Y$, $g' : W \to Z$ the projections. Then
$$f^* g_* = g'_* f'^*.$$

In particular, for $p : X \to \operatorname{Spec} k$ a local complete intersection scheme over k, one has the *fundamental class* $1_X \in \Omega^0(X)$ defined by
$$1_X := p^*(1),$$
where $1 \in \Omega^0(k)$ is the unit element. If X is in $\mathbf{Sm}_k$, then 1_X is the unit for the ring structure in $\Omega^*(X)$. The fundamental class is evidently functorial for l.c.i.-morphisms of l.c.i.-schemes over k. Additionally, if $q : X \to Y$ is a projective morphism, we may define the class $[X \xrightarrow{q} Y] \in \Omega_{\dim_k X}(Y)$ by
$$[X \xrightarrow{q} Y] := q_*(1_X).$$
For example, if $p : X \to \operatorname{Spec} k$ is projective over k, one has the class $[X] := [X \xrightarrow{p} \operatorname{Spec} k] \in \Omega^{-\dim_k X}(k)$. The classes $[X]$ have the following properties:

(1.2)

(1) Fix $d \in \mathbb{Z}$, $d \geq 0$. The classes $[X] \in \Omega^{-d}(k)$, for X a smooth projective k-scheme of dimension d, generate $\Omega^{-d}(k)$ as an abelian group; $\Omega^d(k) = 0$ for $d > 0$.

(2) Let $P(t_1, \ldots, t_d) \in \mathbb{Z}[t_1, \ldots, t_d]$ be a weighted-homogeneous polynomial (with $\deg t_i = i$) of degree d. Then there is a unique homomorphism
$$P(c_1, \ldots, c_d) : \Omega^{-d}(k) \to \mathbb{Z}$$
with $P(c_1, \ldots, c_d)([X]) = \deg(P(c_1, \ldots, c_d)(\Theta_X))$ for X a smooth projective variety of dimension d with tangent bundle Θ_X.

One can ask if fundamental classes may be defined for a more general class of k-schemes; this question was posed to the author by M. Rost and A. Merkurjev. The main result of this short note is to exhibit examples of reduced projective Cohen-Macaulay schemes for which no fundamental class, functorial for l.c.i.-morphisms, exists.

2. Cones

Let $X \subset \mathbb{P}^n$ be a smooth irreducible projective variety. Embed $\mathbb{P}^n$ in $\mathbb{P}^{n+1}$ as the hyperplane $X_{n+1} = 0$, and form the cone $C(X) \subset \mathbb{P}^{n+1}$ as the join of X with the point $* := (0 : \ldots : 0 : 1)$.

The following result is well-known:

Lemma 2.1. *The cone $C(X)$ is a Cohen-Macaulay scheme if and only if, for all $m \in \mathbb{Z}$,*

(1) *The map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to H^0(X, \mathcal{O}_{\mathbb{P}^n}(m))$ is surjective, and*
(2) *$H^i(X, \mathcal{O}_X(m)) = 0$ for $1 \leq i < \dim_k X$.*

Proof. Let $U \subset C(X)$ be the affine cone $U := C(X) \setminus \mathbb{P}^n$, and V the open subscheme $U \setminus \{*\}$. As $C(X) \setminus \{*\}$ is smooth, $C(X)$ is Cohen-Macaulay if and only if the local ring $\mathcal{O}_{U,*}$ is Cohen-Macaulay.

By [1, Theorem 3.8, pg. 44], $\mathcal{O}_{U,*}$ is Cohen-Macaulay if and only if the local cohomology groups $H^i_*(U, \mathcal{O}_U)$ vanish for $0 \leq i < \dim_k U$. We have the long exact sequence
$$\ldots \to H^{i-1}(U, \mathcal{O}_U) \to H^{i-1}(V, \mathcal{O}_V) \to H^i_*(U, \mathcal{O}_U) \to H^i(U, \mathcal{O}_U) \to \ldots$$
Since U is affine, the cohomology $H^i(U, \mathcal{O}_U)$ vanishes for $i > 0$ (see [7]). Thus, we have the exact sequence
$$0 \to H^0_*(U, \mathcal{O}_U) \to H^0(U, \mathcal{O}_U) \to H^0(V, \mathcal{O}_V) \to H^1_*(U, \mathcal{O}_U) \to 0$$
and the isomorphisms
$$H^i(V, \mathcal{O}_V) \cong H^{i+1}_*(U, \mathcal{O}_U); \quad i \geq 1.$$
In particular, since U is reduced, we have $H^0_*(U, \mathcal{O}_U) = 0$.

Let $L \to X$ be the line bundle with sections $\mathcal{O}_X(1)$. V is isomorphic to the complement of the zero-section of L, making V a $\mathbb{G}_m$-bundle

$f : V \to X$ over X. Since f is an affine morphism, the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q f_*\mathcal{O}_V) \Longrightarrow H^{p+q}(V, \mathcal{O}_V)$$

yields an isomorphism

$$H^i(V, \mathcal{O}_V) \cong H^i(X, f_*\mathcal{O}_V).$$

We have the isomorphism

$$f_*\mathcal{O}_V \cong \oplus_{m\in\mathbb{Z}}\mathcal{O}_X(m),$$

yielding

$$H^i(V, \mathcal{O}_V) \cong \oplus_{m\in\mathbb{Z}}H^i(X, \mathcal{O}(m)).$$

Thus $H^i_*(U, \mathcal{O}_U) = 0$ for $1 < i < \dim_k U$ if and only if $H^i(X, \mathcal{O}(m)) = 0$ for $1 \leq i < \dim_k X$ and for all $m \in \mathbb{Z}$.

To handle $H^1_*(U, \mathcal{O}_U)$, we have the isomorphism

$$H^0(U, \mathcal{O}_U) \cong \oplus_{m\geq 0}\mathrm{Im}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to H^0(X, \mathcal{O}_{\mathbb{P}^n}(m))).$$

Noting that $H^0(X, \mathcal{O}_X(m)) = 0$ for $m < 0$, this yields the exact sequence

$$\oplus_{m\geq 0}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to \oplus_{m\geq 0}H^0(X, \mathcal{O}_{\mathbb{P}^n}(m)) \to H^1_*(U, \mathcal{O}_U) \to 0$$

which shows that $H^1_*(U, \mathcal{O}_U) = 0$ if and only if the condition (1) above is satisfied. This completes the proof. □

Remark 2.2. A smooth embedded subscheme $X \subset \mathbb{P}^n$ satisfying Lemma 2.1 (1) is called *projectively normal*.

Examples 2.3. (1) We have the *rational normal curve* of degree d, $V^1_d \subset \mathbb{P}^d$, defined as the embedding of $\mathbb{P}^1$ via the global sections of $\mathcal{O}_{\mathbb{P}^1}(d)$. We thus have the isomorphism

$$\mathcal{O}_{V^1_d}(m) \cong \mathcal{O}_{\mathbb{P}^1}(md)$$

for all m. Since the global sections of $\mathcal{O}_{\mathbb{P}^1}(md)$ are the homogeneous forms in X_0, X_1 of degree md, the product map

$$H^0(V^1_d, \mathcal{O}_{V^1_d}(1))^{\otimes m} \to H^0(V^1_d, \mathcal{O}_{V^1_d}(m))$$

is surjective for all $m \geq 1$. Thus the condition (1) of Lemma 2.1 is satisfied; the condition (2) is trivially satisfied by reasons of dimension.

(2) More generally, one can consider the *Veronese embedding*

$$\mathbb{P}^n \xrightarrow{\sim} V^n_d \subset \mathbb{P}^N;\ N = \binom{n+d}{d} - 1$$

of $\mathbb{P}^n$, defined by the global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$. As in example (1), V^n_d is projectively normal, and as $H^i(\mathbb{P}^n, \mathcal{O}(m)) = 0$ for $0 < i < n$ and all m (see e.g. [7]), $C(V^n_d)$ is Cohen-Macaulay.

(3) Finally, one can apply the Segre embedding to the product

$$V^{n_1}_{d_1} \times \ldots \times V^{n_m}_{d_m} \subset \mathbb{P}^{N_1} \times \ldots \times \mathbb{P}^{N_m}$$

giving the subvariety

$$\prod_{i=1}^{m} \mathbb{P}^{n_i} \xrightarrow{\sim} V^{n_1,\ldots,n_m}_{d_1,\ldots,d_m} \subset \mathbb{P}^M;\ M = \prod_{i=1}^{m} \binom{n_i + d_i}{d_i} - 1,$$

as the embedding of $\prod_{i=1}^m \mathbb{P}^{n_i}$ by the global sections of the external tensor product bundle

$$\mathcal{O}(d_1, \ldots, d_m) := p_1^*\mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \otimes \ldots \otimes p_m^*\mathcal{O}_{\mathbb{P}^{n_m}}(d_m).$$

The Künneth formula for cohomology and the computation of (2) shows that $V^{n_1,\ldots,n_m}_{d_1,\ldots,d_m}$ satisfies the conditions (1) and (2) of Lemma 2.1, and thus $C(V^{n_1,\ldots,n_m}_{d_1,\ldots,d_m})$ is Cohen-Macaulay.

3. The example

We consider the following three smooth projective varieties:

1. The rational normal curve $V^1_4 \subset \mathbb{P}^4$, $V^1_4 \cong \mathbb{P}^1$.
2. The Veronese surface $V^2_2 \subset \mathbb{P}^5$, $V^2_2 \cong \mathbb{P}^2$,
3. The bi-Veronese surface $V^{1,1}_{1,2} \subset \mathbb{P}^5$, $V^{1,1}_{1,2} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 3.1. *Let H be a smooth hyperplane section of V^2_2 and let H' be a smooth hyperplane section of $V^{1,1}_{1,2}$. Considering H and H' as subvarieties of $\mathbb{P}^4$, we have (up to projective linear transformation) $H = V^1_4$, $H' = V^1_4$.*

Proof. H is given as the smooth divisor of a section of the embedding line bundle $\mathcal{O}_{\mathbb{P}^2}(2)$ of V^2_2, and the inclusion $H \subset \mathbb{P}^4$ is given by the global sections of the restriction of $\mathcal{O}_{\mathbb{P}^2}(2)$ to H. Thus, H is a conic in $\mathbb{P}^2$, hence $H \cong \mathbb{P}^1$, and

$$\mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{O}_H \cong \mathcal{O}_H(H \cdot H) \cong \mathcal{O}_{\mathbb{P}^1}(4).$$

Noting that an embedding $\mathbb{P}^1 \to \mathbb{P}^d$ by the global sections of $\mathcal{O}_{\mathbb{P}^1}(d)$ is determined by a choice of basis for $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ proves the assertion for H.

For H', the embedding bundle for $V^{1,1}_{1,2}$ is $\mathcal{O}(1, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, H' is the smooth divisor of a section of $\mathcal{O}(1, 2)$. In particular $H' \cdot (\mathbb{P}^1 \times t)$ has degree 1 for all t, and $H' \cdot H'$ has degree 4. Therefore, projection onto the second factor $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ gives an isomorphism $H' \cong \mathbb{P}^1$, and we have

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2) \otimes \mathcal{O}_{H'} \cong \mathcal{O}_{H'}(H' \cdot H') \cong \mathcal{O}_{\mathbb{P}^1}(4).$$

The same argument as above gives the assertion for H'. □

Let $\mathcal{C} \subset L\mathbf{Sch}_k$ be the full sub-category with objects the smooth quasi-projective k-schemes, together with the k-schemes $C(V_4^1)$, $C(V_2^2)$ and $C(V_{1,2}^{1,1})$. The restriction of Ω^* to $\mathcal{C}^{\rm op}$ gives us functor

$$\Omega^* : \mathcal{C}^{\rm op} \to \mathbf{GrAb}.$$

Now suppose that there are classes $1_X \in \Omega^0(X)$ for $X = C(V_4^1)$, $X = C(V_2^2)$ and $X = C(V_{1,2}^{1,1})$ such that the assignment $X \mapsto 1_X$ is functorial in $X \in \mathcal{C}$. We will show that this implies that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ have the same Chern numbers, hence is not possible.

By Lemma 2.1, $C(V_4^1)$, $C(V_2^2)$ and $C(V_{1,2}^{1,1})$ are Cohen-Macaulay schemes; in particular, a hyperplane section of $C(V_2^2)$ or $C(V_{1,2}^{1,1})$ which is generically reduced is reduced (*cf.* [6, Theorem 30(16.D), pg. 107]). Thus, by Lemma 3.1, a general hyperplane section of $C(V_2^2)$ containing the vertex $*$ is isomorphic to $C(V_4^1)$, and similarly for a general hyperplane section of $C(V_{1,2}^{1,1})$ containing $*$. This yields l.c.i.-closed embeddings

$$i_1 : C(V_4^1) \to C(V_2^2),$$
$$j_1 : C(V_4^1) \to C(V_{1,2}^{1,1}).$$

Similarly, cutting with a hyperplane not containing $*$ gives l.c.i.-closed embeddings

$$i_2 : V_2^2 \to C(V_2^2),$$
$$j_2 : V_{1,2}^{1,1} \to C(V_{1,2}^{1,1}).$$

Lemma 3.2. *We have identities*

$$i_{1*} \circ i_1^* = i_{2*} \circ i_2^* : \Omega^*(C(V_2^2)) \to \Omega^{*+1}(C(V_2^2))$$

and

$$j_{1*} \circ j_1^* = j_{2*} \circ j_2^* : \Omega^*(C(V_{1,2}^{1,1})) \to \Omega^{*+1}(C(V_{1,2}^{1,1})).$$

Proof. We give the proof for $X = C(V_2^2) \subset \mathbb{P}^5$; the proof for $C(V_{1,2}^{1,1})$ is the same.

Let s_1 and s_2 be sections of $\mathcal{O}_X(1)$ with $i_1(C(V_4^1))$ defined by s_1 and $i_2(V_2^2)$ defined by s_2. Let $i : Y \to X \times \mathbb{A}^1$ be the subscheme of $X \times \mathbb{A}^1$

defined by $ts_1+(1-t)s_2$, where t is the standard parameter on $\mathbb{A}^1$. By the homotopy property (1.1)(3) for Ω^*, the sections $\iota_0, \iota_1 : X \to X\times\mathbb{A}^1$,

$$\iota_0(x) = x \times 0,\ \ \iota_1(x) = x \times 1,$$

induce isomorphisms

$$\iota_0^*, \iota_1^* : \Omega^*(X \times \mathbb{A}^1) \to \Omega^*(X)$$

inverse to $p^* : \Omega^*(X) \to \Omega^*(X \times \mathbb{A}^1)$, where $p : X \times \mathbb{A}^1 \to X$ is the projection.

Let $\bar{\iota_1} : C(V_4^1) \to Y$ be the inclusion as the fiber over $1 \in \mathbb{A}^1$. Using the compatibility of pull-back and push-forward in cartesian squares (1.1)(4), we have

$$\iota_1^* \circ i_* \circ i^* \circ p^* = i_{1*} \circ \bar{\iota_1}^* \circ i^* \circ p^* = i_{1*} \circ i_1^*.$$

Similarly

$$\iota_0^* \circ i_* \circ i^* \circ p^* = i_{2*} \circ i_2^*.$$

This gives the identity

$$\begin{aligned} i_{1*} \circ i_1^* &= \iota_1^* \circ i_* \circ i^* \circ p^* \\ &= \iota_0^* \circ i_* \circ i^* \circ p^* \\ &= i_{2*} \circ i_2^*. \end{aligned}$$

□

We apply Lemma 3.2 to the fundamental classes, giving the identities

$$\begin{aligned} i_{1*}(1_{C(V_4^1)}) &= i_{1*} \circ i_1^*(1_{C(V_2^2)}) \\ &= i_{2*} \circ i_2^*(1_{C(V_2^2)}) \\ &= i_{2*}(1_{V_2^2}) \end{aligned}$$

in $\Omega_*(C(V_2^2))$. Similarly,

$$j_{1*}(1_{C(V_4^1)}) = j_{2*}(1_{V_{1,2}^{1,1}})$$

in $\Omega_*(C(V_{1,2}^{1,1}))$. Pushing forward to $\operatorname{Spec} k$, we find

$$[\mathbb{P}^1 \times \mathbb{P}^1] = [V_{1,2}^{1,1}] = [C(V_4^1)] = [V_2^2] = [\mathbb{P}^2]$$

in $\Omega^{-2}(k)$. Applying the Chern number homomorphism (1.2)

$$c_2 : \Omega^{-2}(k) \to \mathbb{Z}$$

gives

$$c_2(\mathbb{P}^1 \times \mathbb{P}^1) = c_2(\mathbb{P}^2);$$

since $c_2(\mathbb{P}^2) = 3$ and $c_2(\mathbb{P}^1 \times \mathbb{P}^1) = 4$, this yields a contradiction.

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