SIX LECTURES ON MOTIVES

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Preface

These lecture notes are taken from my lecture series in the Asian-French summer school on motives and related topics. My goal in my lectures was two-fold: to give first of all a sketch of Voevodsky's foundational construction of the triangulated category of motives and its basic properties, and then to give an idea of some of the applications and wider vistas this construction has made possible. In doing this, wanted also to point out some of the origins of this theory, coming from both the categorical side involving aspects of sheaf theory and triangulated categories, as well as the input from algebraic geometry, mainly through algebraic cycles. This latter aspect lead me to devote an entire lecture to so-called moving lemmas, as I felt this subject captured much of the geometric side of the theory. I also reviewed much of the necessary material about triangulated categories and sheaves on a Grothendieck site, with the intention of making the discussion as accessible as possible.

I chose mixed Tate motives for illustrating applications,. This subject touches on a broad range of subjects, including the theory of t-structures, Tannakian categories, rational homotopy theory, Grothendieck-Teichmüller theory, moduli of curves, polylogarithms and multiple zeta values. For this reason, I felt that an overview would be of interest to a fairly wide audience. I have also included two lectures on the extension of motives given by the motivic stable homotopy category of Morel-Voevodsky, giving a sketch of the construction as well as a discussion of the motivic Postnikov tower.

Many thanks are due to Joël Riou, whose careful reading and thoughtful comments allowed me to correct quite a few errors, as well as greatly improving the exposition. Other than this, I have made only minor changes and additions to my original lectures in these notes; I hope this will transmit the informal nature of the lectures to the reader. The rewriting of these notes let me recall how much I enjoyed the summer school at the I.H.E.S and gives me the opportunity of thanking most heartily the organizers of summer school, Jean-Marc Fontaine and Jean-Benoit Bost, for putting together a truly worthwhile conference.

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Lecture 1. Triangulated categories of motives

Our goal in this lecture is to construct a *category of motives* that should capture the fundamental properties and structures of a reasonable cohomology theory on smooth varieties over a field k. To guide the construction, we ask the rather vague question: what kind of structures does "cohomology" have? At the very least, one should have

(1) Pull-back maps $f^* : H^*(Y) \to H^*(X)$ maps $f : X \to Y$.
(2) Products $H^*(X) \times H^*(Y) \to H^*(X \times Y)$
(3) Some long exact sequences: for example, Mayer-Vietoris for (Zariski) open covers.
(4) Some isomorphisms, for example $H^*(X) \cong H^*(X \times \mathbb{A}^1)$.

Next, what categorical constructions will lead to all these structures? First of all, there is an algebraic machinery for generating long exact sequences and imposing isomorphisms, namely the machinery of triangulated categories. This structure is a result of axiomatizing the basic example of the derived category of an abelian category. For example, if one considers the abelian category Shv_T of sheaves of abelian groups on a topological space T, then the sheaf cohomology $H^*(T, A)$ with coefficients in an abelian group A is given as the Ext-group

$$H^n(T, A) \cong \mathrm{Ext}^n_{Shv_T}(\mathbb{Z}_T, A_T),$$

where $\mathbb{Z}_T$, A_T are the constant sheaves with value $\mathbb{Z}, A$. In the derived category $D(Shv_T)$, one has the canonical isomorphism

$$\mathrm{Ext}^n(\mathbb{Z}_T, A_T) \cong \mathrm{Hom}_{D(Shv_T)}(\mathbb{Z}_T, A_T[n]).$$

All the well-known long exact sequences for cohomology, such as the Mayer-Vietoris sequence, or the Bockstein sequence, arise from the long exact sequence machinery encoded in the triangulated category $D(Shv_T)$. In a general triangulated category D, one can define the cohomology of an object X with values in another object A by

$$H^n(X, A) := \mathrm{Hom}_D(X, A[n]);$$

we shall see how the triangulated structure in D gives rise to lots of long exact sequence. This formal view of cohomology has proven extremely valuable in many areas of mathematics.

The product in cohomology comes from a tensor structure in the triangulated category D, namely a bi-functor

$$\otimes_D : D \times D \to D$$

with certain exactness properties. If our coefficent group A has a multiplication $A \otimes A \to A$ and our object X has a "diagonal" $\delta : X \to X \otimes X$, then our formal cohomology becomes a ring via

$$\mathrm{Hom}_D(X, A[n]) \otimes_{\mathbb{Z}} \mathrm{Hom}_D(X, A[m]) \xrightarrow{\otimes_D} \mathrm{Hom}_D(X \otimes X, A[n+m]) \xrightarrow{\delta^*} \mathrm{Hom}_D(X, A[n+m]).$$

We need some geometric input to feed this machine, coming from algebraic cycles. Finally, to understand what comes out of this construction, we need the homological algebra of sheaf theory. Schematically, we have the following picture:

$$\begin{tikzcd}
\text{Triangulated tensor categories} \arrow[drr] & & \\
\text{Algebraic cycles} \arrow[rr] & & \text{Motives} \\
\text{Sheaf theory} \arrow[urr] & &
\end{tikzcd}$$

1. Triangulated categories

1.1. Translations and triangles.

Definition 1.1. A *translation* on an additive category $\mathcal{A}$ is an equivalence $T : \mathcal{A} \to \mathcal{A}$. We write $X[1] := T(X)$. An additive functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories with translation is *graded* if $F(X[1]) = (FX)[1]$ and similarly for morphisms.

Let $\mathcal{A}$ be an additive category with translation. A *triangle* (X, Y, Z, a, b, c) in $\mathcal{A}$ is a sequence of maps

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1].$$

A morphism of triangles

$$(f, g, h) : (X, Y, Z, a, b, c) \to (X', Y', Z', a', b', c')$$

is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, "f"'] & Y \arrow[r, "b"] \arrow[d, "g"'] & Z \arrow[r, "c"] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"'] \\
X' \arrow[r, "a'"'] & Y' \arrow[r, "b'"'] & Z' \arrow[r, "c'"'] & X'[1].
\end{tikzcd}$$

1.2. Triangulated categories.

Verdier [61] has defined a *triangulated category* as an additive category $\mathcal{A}$ with translation, together with a collection $\mathcal{E}$ of triangles, called the *distinguished triangles* of $\mathcal{A}$, which satisfy

TR1 $\mathcal{E}$ is closed under isomorphism of triangles.
$A \xrightarrow{\mathrm{id}} A \to 0 \to A[1]$ is distinguished.
Each morphism $X \xrightarrow{u} Y$ extends to a distinguished triangle

$$X \xrightarrow{u} Y \to Z \to X[1]$$

TR2 $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished $\Leftrightarrow$ $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished.

TR3 Given a commutative diagram with distinguished rows

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & X[1] \\
X \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z \arrow[r, "w'"'] & X[1]
\end{tikzcd}$$

there exists a morphism $h : Z \to Z'$ such that (f, g, h) is a morphism of triangles:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & X[1] \arrow[d, "f[1]"] \\
X \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z \arrow[r, "w'"'] & X][1]
\end{tikzcd}$$

TR4 If we have three distinguished triangles $(X, Y, Z', u, i, *)$, $(Y, Z, X', v, *, j)$, and $(X, Z, Y', w, *, *)$, with $w = v \circ u$, then there are morphisms $f : Z' \to Y'$, $g : Y' \to X'$ such that

- (id_X, v, f) is a morphism of triangles
- (u, id_Z, g) is a morphism of triangles
- $(Z', Y', X', f, g, i[1] \circ j)$ is a distinguished triangle.

A graded functor $F : \mathcal{A} \to \mathcal{B}$ of triangulated categories is called *exact* if F takes distinguished triangles in $\mathcal{A}$ to distinguished triangles in $\mathcal{B}$.

Remark 1.2. Suppose $(\mathcal{A}, T, \mathcal{E})$ satisfies (TR1), (TR2) and (TR3). If (X, Y, Z, a, b, c) is in $\mathcal{E}$, and A is an object of $\mathcal{A}$, then the sequences

$$\dots \xrightarrow{c[-1]_*} \mathrm{Hom}_{\mathcal{A}}(A, X) \xrightarrow{a_*} \mathrm{Hom}_{\mathcal{A}}(A, Y) \xrightarrow{b_*}$$
$$\mathrm{Hom}_{\mathcal{A}}(A, Z) \xrightarrow{c_*} \mathrm{Hom}_{\mathcal{A}}(A, X[1]) \xrightarrow{a[1]_*} \dots$$

and

$$\dots \xrightarrow{a[1]^*} \mathrm{Hom}_{\mathcal{A}}(X[1], A) \xrightarrow{c^*} \mathrm{Hom}_{\mathcal{A}}(Z, A) \xrightarrow{b^*}$$
$$\mathrm{Hom}_{\mathcal{A}}(Y, A) \xrightarrow{a^*} \mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{c[-1]^*} \dots$$

are exact. This yields:

- (five-lemma): If (f, g, h) is a morphism of triangles in $\mathcal{E}$, and if two of f, g, h are isomorphisms, then so is the third.
- If (X, Y, Z, a, b, c) and (X, Y, Z', a, b', c') are two triangles in $\mathcal{E}$, there is an isomorphism $h : Z \to Z'$ such that

  $$(\mathrm{id}_X, \mathrm{id}_Y, h) : (X, Y, Z, a, b, c) \to (X, Y, Z', a, b', c')$$

  is an isomorphism of triangles.

If (TR4) holds as well, then one has "Mayer-Vietoris"-type distinguised triangles: Given a commutative diagram with distinguished rows

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, equal] & Y \arrow[r, "b"] \arrow[d, "f"] & Z \arrow[r, "c"] & X[1] \arrow[d, equal] \\ X \arrow[r, "a'"'] & Y' \arrow[r, "b'"'] & Z' \arrow[r, "c'"'] & X[1] \end{tikzcd}$$

there exists a morphism $g : Z \to Z'$ so that

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, equal] & Y \arrow[r, "b"] \arrow[d, "f"] & Z \arrow[r, "c"] \arrow[d, "g"] & X[1] \arrow[d, equal] \\ X \arrow[r, "a'"'] & Y' \arrow[r, "b'"'] & Z' \arrow[r, "c'"'] & X[1] \end{tikzcd}$$

is a map of triangles and the sequence

$$Y \xrightarrow{(f,-b)} Y' \oplus Z \xrightarrow{b'+g} Z' \xrightarrow{a[1]\circ c'} Y[1]$$

is distinguished (see [49, lemma 1.4.3]).

Shortly speaking: A triangulated category is a machine for generating natural long exact sequences.

1.3. **An example: the homotopy category of an additive category.** Let $\mathcal{A}$ be an additive category, $C^?(\mathcal{A})$ the category of cohomological complexes (with boundedness condition $? = \emptyset, +, -, b$). Recall that a morphism of complexes $f : A \to B$ is a collection of maps $f^n : A^n \to B^n$ such that

$$d_B^n f^n = f^{n+1} d_A^n.$$

For a complex (A, d_A), let $A[1]$ be the complex

$$A[1]^n := A^{n+1}; \quad d_{A[1]}^n := -d_A^{n+1}.$$

For a map of complexes $f : A \to B$, we have the *cone sequence*

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} A[1]$$

where $\mathrm{Cone}(f) := A^{n+1} \oplus B^n$ with differential

$$d(a, b) := (-d_A(a), f(a) + d_B(b))$$

i and p are the evident inclusions and projections.

Recall that two maps of complexes $f, g : A \to B$ are *homotopic* if there are maps $h^n : A^n \to B^{n-1}$ with

$$f^n - g^n = d_B^{n-1} h^n + h^{n+1} d_A^n$$

The homotopy relation is preserved by left and right composition, so we may form the *homotopy category* $K^?(\mathcal{A})$ with the same objects as $C^?(\mathcal{A})$ and with morphisms the homotopy classes of morphisms in $C^?(\mathcal{A})$:

$$\mathrm{Hom}_{K^?(\mathcal{A})}(A, B) := \mathrm{Hom}_{C^?(\mathcal{A})}(A, B)/htpy.$$

We make $K^?(\mathcal{A})$ a triangulated category by declaring a triangle to be distinguished if it is isomorphic to the image of a cone sequence.

1.4. **Tensor structure.** A *tensor structure* on an additive category $\mathcal{A}$ is given by a bi-functor

$$\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

which is associative, commutative and unital. Strictly speaking, this means one has a unit object $\mathbb{1}$ and natural isomorphisms

- *associativity constraints* $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$
- *commutativity constraints* $\tau_{X,Y} : X \otimes Y \to Y \otimes X$
- *unit constraints* $\mu_{X,\ell}\mathbb{1} \otimes X \to X$, $\mu_{X,r} : X \otimes \mathbb{1} \to X$

which make a short list of diagrams commute (see e.g. Mac Lane [42, chapter VII] for details).

Concretely: for $X, Y \in \mathcal{A}$, we have $X \otimes Y \in \mathcal{A}$. For $f : X_1 \to X_2$, $g : Y_1 \to Y_2$, we have

$$f \otimes g : X_1 \otimes Y_1 \to X_2 \otimes Y_2,$$

bilinear in f and g and respecting composition:

$$(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g').$$

If $\mathcal{A}$ has a translation $X \mapsto TX =: X[1]$, we say the tensor structure is graded if

(1) $T \circ (- \otimes -) = T(-) \otimes -$, $T^2(-) \otimes - = - \otimes T^2(-)$ as functors $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, i.e. $TX \otimes Y = T(X \otimes Y)$, $T^2 X \otimes Y = X \otimes T^2 Y$, and similarly for morphisms,
(2) The natural isomorphism $\tau_{X,Y} : X \otimes TY \to TX \otimes Y$

$$X \otimes TY \cong TY \otimes X = T(Y \otimes X) \cong T(X \otimes Y) = TX \otimes Y$$

satisfies: $T(\tau_{X,TY})\tau_{X,TY} : X \otimes T^2 Y \to T^2(X \otimes Y) = X \otimes T^2 Y$ is the identity.

Definition 1.3. Suppose $\mathcal{A}$ is both a triangulated category and a tensor category (with tensor operation $\otimes$) such that $\otimes$ is graded.

Suppose that, for each distinguished triangle (X, Y, Z, a, b, c), and each $W \in \mathcal{A}$, the sequence

$$X \otimes W \xrightarrow{a \otimes \mathrm{id}_W} Y \otimes W \xrightarrow{b \otimes \mathrm{id}_W} Z \otimes W \xrightarrow{c \otimes \mathrm{id}_W} X[1] \otimes W = (X \otimes W)[1]$$

is a distinguished triangle. Then $\mathcal{A}$ is a *triangulated tensor category.*

Example 1.4. If $\mathcal{A}$ is a tensor category, then $K^?(\mathcal{A})$ inherits a tensor structure, by the usual tensor product of complexes, and becomes a triangulated tensor category. (For $? = \emptyset$, $\mathcal{A}$ must admit infinite direct sums for the tensor structure to be defined).

Giving a tensor structure on an additive category induces a natural external product on the Hom-groups; in a triangulated tensor category D the operation $- \otimes f$ is compatible with the long exact sequences arising from Hom into or Hom from a distinguished triangle. For instance, given a distinguished triangle $X \to Y \to Z \to X[1]$, objects T, T' and W and morphism $f : W \to T'$, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}(Y,T) \arrow[r] \arrow[d, "-\otimes f"'] & \mathrm{Hom}(Z,T) \arrow[r] \arrow[d, "-\otimes f"'] & \mathrm{Hom}(X[1],T) \arrow[r] \arrow[d, "-\otimes f"'] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}(Y \otimes W, T \otimes T') \arrow[r] & \mathrm{Hom}(Z \otimes W, T \otimes T') \arrow[r] & \mathrm{Hom}((X \otimes W)[1], T \otimes T') \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

1.5. **Thick subcategories.** The process of Verdier localization allows one to build new triangulated categories from old ones by inverting a chosen set of morphisms or equivalently, killing a chosen set of objects.

Definition 1.5. A full triangulated subcategory $\mathcal{B}$ of a triangulated category $\mathcal{A}$ is *thick* if $\mathcal{B}$ is closed under taking direct summands.

If $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$, the set of morphisms $s : X \to Y$ in $\mathcal{A}$ which fit into a distinguished triangle $X \xrightarrow{s} Y \to Z \to X[1]$ with Z in $\mathcal{B}$ forms a *saturated multiplicative system* of morphisms.

The intersection of thick subcategories of $\mathcal{A}$ is a thick subcategory of $\mathcal{A}$. So, for each set $\mathcal{T}$ of objects of $\mathcal{A}$, there is a smallest thick subcategory $\mathcal{B}$ containing $\mathcal{T}$, called the thick subcategory *generated* by $\mathcal{T}$.

Remark 1.6. The original definition (Verdier) of a thick subcategory had the condition:

Let $X \xrightarrow{f} Y \to Z \to X[1]$ be a distinguished triangle in $\mathcal{A}$, with Z in $\mathcal{B}$. If f factors as $X \xrightarrow{f_1} B' \xrightarrow{f_2} Y$ with B' in $\mathcal{B}$, then X and Y are in $\mathcal{B}$.

This is equivalent to the condition given above, that $\mathcal{B}$ is closed under direct summands in $\mathcal{A}$ (*cf.* Rickard [54]).

1.6. **Localization of triangulated categories.** Let $\mathcal{B}$ be a thick subcategory of a triangulated category $\mathcal{A}$ and let $\mathcal{S}$ be the saturated multiplicative system of map $A \xrightarrow{s} B$ with "cone" in $\mathcal{B}$.

Form the category $\mathcal{A}[\mathcal{S}^{-1}] = \mathcal{A}/\mathcal{B}$ with the same objects as $\mathcal{A}$, with

$$\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X, Y) = \varinjlim_{s:X'\to X\in\mathcal{S}} \mathrm{Hom}_{\mathcal{A}}(X', Y).$$

Composition of diagrams

$$\begin{tikzcd}
 & Y' \arrow[r, "g"] \arrow[d, "t"] & Z \\
X' \arrow[r, "f"'] \arrow[d, "s"'] & Y & \\
X & &
\end{tikzcd}$$

is defined by filling in the middle

$$\begin{tikzcd}
X'' \arrow[r, "f'"] \arrow[d, "s'"'] & Y' \arrow[r, "g"] \arrow[d, "t"] & Z \\
X' \arrow[r, "f"'] \arrow[d, "s"'] & Y & \\
X. & &
\end{tikzcd}$$

One can describe $\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X, Y)$ by a calculus of left fractions as well, i.e.,

$$\mathrm{Hom}_{\mathcal{A}[\mathcal{S}^{-1}]}(X, Y) = \varinjlim_{s:Y\to Y'\in\mathcal{S}} \mathrm{Hom}_{\mathcal{A}}(X, Y').$$

Let $Q_{\mathcal{B}} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the canonical functor.

Theorem 1.7 (Verdier). *(i) $\mathcal{A}/\mathcal{B}$ is a triangulated category, where a triangle T in $\mathcal{A}/\mathcal{B}$ is distinguished if T is isomorphic to the image under $Q_{\mathcal{B}}$ of a distinguished triangle in $\mathcal{A}$.*

(ii) The functor $Q_{\mathcal{B}}$ is universal for exact functors $F : \mathcal{A} \to \mathcal{C}$ such that $F(B)$ is isomorphic to 0 for all B in $\mathcal{B}$. $Q_{\mathcal{B}}$ is also universal for exact functors $F : \mathcal{A} \to \mathcal{C}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$.

(iii) $\mathcal{S}$ is equal to the collection of maps in $\mathcal{A}$ which become isomorphisms in $\mathcal{A}/\mathcal{B}$ and $\mathcal{B}$ is the subcategory of objects of $\mathcal{A}$ which becomes isomorphic to zero in $\mathcal{A}/\mathcal{B}$.

Remark 1.8. If $\mathcal{A}$ admits some infinite direct sums, it is sometimes better to preserve this property. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called *localizing* if $\mathcal{B}$ is thick and is closed under all direct sums which exist in $\mathcal{A}$.

For instance, if $\mathcal{A}$ admits arbitrary direct sums and $\mathcal{B}$ is a localizing subcategory, then $\mathcal{A}/\mathcal{B}$ also admits arbitrary direct sums.

Localization with respect to localizing subcategories has been studied by Thomason [59] and by Ne'eman [49], among others.

In concrete situations, one starts with a triangulated category $\mathcal{D}$, such as the homotopy category $\mathcal{K}(\mathcal{A})$ of an additive category $\mathcal{A}$. One can then choose a collection of morphisms in $\mathcal{D}$ that one would like to invert. Verdier's localization theorem is applied with the thick subcategory $\mathcal{B} \subset \mathcal{D}$ being the smallest thick subcategory containing the "cones" of the chosen morphisms. The localization $\mathcal{D} \to \mathcal{D}/\mathcal{B}$ is then the universal construction for inverting the chosen morphisms in the setting of triangulated categories.

Remark 1.9 (Localization of triangulated tensor categories). If $\mathcal{A}$ is a triangulated tensor category, and $\mathcal{B}$ a thick subcategory, call $\mathcal{B}$ a *thick tensor subcategory* if A in $\mathcal{A}$ and B in $\mathcal{B}$ implies that $A \otimes B$ and $B \otimes A$ are in $\mathcal{B}$.

The quotient $Q_{\mathcal{B}} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ of $\mathcal{A}$ by a thick tensor subcategory inherits the tensor structure, and the distinguished triangles are preserved by tensor product with an object.

Example 1.10 (The derived category). The classic example is the *derived category* $D^?(\mathcal{A})$ of an abelian category $\mathcal{A}$. $\mathcal{D}^?(\mathcal{A})$ is the localization of the homotopy category $K^?(\mathcal{A})$ with respect to the multiplicative system of *quasi-isomorphisms* $f : A \to B$, i.e., f which induce isomorphisms $H^n(f) : H^n(A) \to H^n(B)$ for all n.

The derived category is the natural place to perform homological algebra in $\mathcal{A}$. For instance, for M, N are in $\mathcal{A}$, there is a natural isomorphism

$$\mathrm{Ext}^n_{\mathcal{A}}(M, N) \cong \mathrm{Hom}_{D^?(\mathcal{A})}(M, N[n]).$$

If $\mathcal{A}$ is an abelian tensor category, then $D^-(\mathcal{A})$ inherits a tensor structure $\otimes^L$ if each object A of $\mathcal{A}$ admits a surjection $P \to A$ where P is *flat*, i.e. $M \mapsto M \otimes P$ is an exact functor on $\mathcal{A}$. If each A admits a finite flat (right) resolution, then $D^b(\mathcal{A})$ has a tensor structure $\otimes^L$ as well. The tensor structure $\otimes^L$ is given by forming for each $A \in K^?(\mathcal{A})$ a quasi-isomorphism $P \to A$ with P a complex of flat objects in $\mathcal{A}$, and defining

$$A \otimes^L B := \mathrm{Tot}(P \otimes B).$$

2. Geometric motives

Voevodsky constructs a number of categories: the category of geometric motives $DM_{\mathrm{gm}}(k)$ with its effective subcategory $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, as well as a sheaf-theoretic construction DM^{eff}_{-}, containing $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as a full dense subcategory. In contrast to almost all other constructions of categories of motives, these are based on *homology* rather than cohomology as the starting point, in particular, the motives functor from $\mathbf{Sm}/k$ to these categories is covariant.

In this section, we discuss the construction of the "geometric" categories $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ and $DM_{\mathrm{gm}}(k)$; we take k to be a perfect field.

2.1. Algebraic cycles and rational equivalence.

Definition 2.1. For $X \in \mathbf{Sch}_k$, let $z_r(X)$ be the free abelian group on the integral closed subschemes of X of dimension r over k We write $z_*(X)$ for the graded group $\oplus_r z_r(X)$. An element $Z = \sum_i n_i Z_i$ of $z(X)$ is an *algebraic cycle on X*.

•*Associated cycle and support.* Let $W \subset X$ be a closed subscheme of pure dimension n over k, with irreducible components $W_1, \ldots, W_r$. The cycle associated to W is

$$|W| := \sum_{i=1}^{r} [\ell_{\mathcal{O}_{X,W_i}}(\mathcal{O}_{W,W_i})] \cdot W_i$$

where $\ell_{\mathcal{O}_{X,W_i}}$ is the length as an $\mathcal{O}_{X,W_i}$ module. For example, if W is reduced, then $|W| = \sum_i W_i$. In the other direction, if $Z = \sum_i n_i Z_i$ is a cycle with all $n_i \neq 0$, the *support* of Z is

$$\mathrm{Supp}(Z) := \cup_i Z_i.$$

•*Intersection with a divisor.* Let D be a Cartier divisor on a scheme $X \in \mathbf{Sch}_k$. Let $Z \subset X$ be an integral closed subscheme such that Z is not contained in the support of D. We define the *intersection product* $D \cdot Z$ by

$$D \cdot Z := |D \cap Z|$$

where $D \cap Z$ is the scheme-theoretic intersection (i.e. $D \cap Z := D \times_X Z$). Letting $z_n(X)_D \subset z_n(X)$ be the subgroup of $z_n(X)$ generated by the integral Z of dimension n with $Z \not\subset D$, we extend by linearity to define the operation

$$D \cdot - : z_n(X)_D \to z_{n-1}(D).$$

Definition 2.2. Let X be in $\mathbf{Sch}_k$. Two cycles $Z, Z' \in z_n(X)$ are *rationally equivalent* if there is a cycle $\mathcal{Z} \in z_{n+1}(X \times \mathbb{A}^1)_{X\times 0 + X\times 1}$ with

$$Z - Z' = (X \times 0 - X \times 1) \cdot \mathcal{Z}.$$

To interpret this formula, note that

$$z_{n+1}(X \times \mathbb{A}^1)_{X\times 0+X\times 1} = z_{n+1}(X \times \mathbb{A}^1)_{X\times 0} \cap z_{n+1}(X \times \mathbb{A}^1)_{X\times 1}.$$

Also $(X \times 0) \cdot \mathcal{Z}$ is in $z_n(X \times 0)$, which we identify with $z_n(X)$. Similarly we can view $(X \times 1) \cdot \mathcal{Z}$ as in $z_n(X)$, so the difference $(X \times 0 - X \times 1) \cdot \mathcal{Z}$ is a well-defined cycle on X.

We denote this equivalence relation by $\sim_r$. Set $\mathrm{CH}_n(X) := z_n(X)/\sim_r$, the *Chow group* of dimension n cycles on X modulo rational equivalence.

If X has dimension d over k, set $z^n(X) := z_{d-n}(X)$ and $\mathrm{CH}^n(X) := \mathrm{CH}_{d-n}(X)$.

2.2. **Operations.** These cycle groups admit a number of operations and functorialities:

•*Projective push-forward.* Let $f : Y \to X$ be a projective morphism in $\mathbf{Sch}_k$. For $W \subset Y$ an integral closed subscheme of dimension n, we have the integral closed subscheme $f(W) \subset X$ (closed because f is proper) and the extension of function fields $f^* : k(f(W)) \to k(W)$. Note $\dim_k f(W) \leq n$ and that the field extension $k(W)/k(f(W))$ is finite if and only if $\dim_k f(W) = n$. Define the cycle $f_*(W) \in z_n(X)$ by

$$f_*(W) := \begin{cases} 0 & \text{if } \dim_k f(W) < n \\ [\deg_{k(f(W))} k(W)] \cdot f(W) & \text{if } \dim_k f(W) = n \end{cases}$$

Extend f_* by linearity to $f_* : z_n(Y) \to z_n(X)$.

One shows that push-forward is functorial: $(gf)_* = g_* f_*$ and that this operation descends to $f_* : \mathrm{CH}_n(Y) \to \mathrm{CH}_n(X)$.

•*Pull-back.* Let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$ with X smooth over k. For simplicity, we suppose X and Y are integral. There is a partially defined pull-back morphism $f^* : z^n(X)_f \to z^n(Y)$, where $z^n(X)_f$ is the subgroup of $z^n(X)$ consisting of cycles *in good position* for f. For $W \subset X$ an integral codimension n subscheme, W is in good position if each irreducible component of $f^{-1}(W)$ has codimension n on Y. If Z is an irreducible component of $f^{-1}(W)$ define the multiplicity $m(W, Z; f)$ by Serre's formula

$$m(W, Z; f) := \sum_{i \geq 0} (-1)^i \ell_{\mathcal{O}_{Y,Z}} (\mathrm{Tor}_i^{\mathcal{O}_{X,W}} (\mathcal{O}_W, \mathcal{O}_{Y,Z}));$$

the fact that X is smooth implies that the sum is finite. Define

$$f^*(W) := \sum_Z m(W, Z; f) Z$$

where the sum is over all irreducible components Z of $f^{-1}(W)$. Letting $z^n(X)_f$ be the subgroup of $z^n(X)$ generated by all W in good position for f, we extend by linearity to define the pull-back $f^* : z^n(X)_f \to z^n(Y)$.

The fundamental advantage of the quotient CH^n is that this partially defined pull-back descends to a well-defined pull-back

$$f^* : \mathrm{CH}^n(X) \to \mathrm{CH}^n(Y)$$

which is functorial, $(fg)^* = g^* f^*$, if X and Y are both smooth. The case of X quasi-projective was handled by the geometric methods of Chow et. al., see for example [15]; the general case relies on Fulton's methods [21], or the K-theoretic approach of Quillen-Grayson [23].

•*Products.* For X and Y in $\mathbf{Sch}_k$, we have the external product

$$\boxtimes : z_n(X) \otimes z_m(Y) \to z_{n+m}(X \times_k Y)$$

defined as the $\mathbb{Z}$-linear extension of the operation $(W, W') \mapsto |W \times_k W'|$ for $W \subset X$, $W' \subset Y$ integral of dimensions n, m, respectively. $\boxtimes$ is commutative, associative and unital (with unit the cycle $\mathrm{Spec}\, k \in z_0(\mathrm{Spec}\, k)$). The external product descends to an external product

$$\boxtimes : \mathrm{CH}_n(X) \otimes \mathrm{CH}_m(Y) \to \mathrm{CH}_{n+m}(X \times_k Y).$$

For X smooth, we have the cup product

$$\cup_X : \mathrm{CH}^n(X) \otimes \mathrm{CH}^m(X) \to \mathrm{CH}^{n+m}(X)$$

defined by

$$a \cup_X b := \delta^*(a \boxtimes b)$$

where $\delta : X \to X \times X$ is the diagonal. This product makes the graded group $\mathrm{CH}^*(X) := \oplus_n \mathrm{CH}^n(X)$ into a commutatve graded ring with unit 1_X the class of $[X] \in \mathrm{CH}^0(X)$.

• *Compatibilities.* Beside the functorialities for pull-back and push-forward, we have the following identities and compatibilities of maps on CH^*:

(1) For $f : Y \to X$ a morphism in $\mathbf{Sm}/k$, $f^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$ is a ring homomorphism
(2) Let $f : Y \to X$, $g : Z \to X$ be morphisms in $\mathbf{Sm}/k$ with f projective. Suppose that $W := Y \times_X Z$ is smooth of dimension $\dim Y + \dim Z - \dim X$. Consider the cartesian diagram

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Then f' is projective and $g^* f_* = f'_* g'^*$.
(3) *Projection formula.* Let $f : Y \to X$ be a projective morphism in $\mathbf{Sm}/k$. Then for $a \in \mathrm{CH}^n(X)$, $b \in \mathrm{CH}^m(Y)$, we have

$$f_*(f^*(a) \cup_Y b) = a \cup_X f_*(b).$$

2.3. **Correspondences and Chow motives.** A cycle of dimension $n := \dim X$ on $X \times Y$ defines a *correspondence* from X to Y.

If X, Y and Z are smooth and projective, we can compose correspondences: $\alpha \in \mathrm{CH}_{\dim X}(X \times Y)$, $\beta \in \mathrm{CH}_{\dim Y}(Y \times Z)$:

$$\beta \circ \alpha := p_{XZ*}(p^*_{XY}(\alpha) \cdot p^*_{YZ}(\beta)).$$

Remark 2.3. Let $\mathbf{SmProj}/k$ be the category of smooth projective k-schemes. For $f : X \to Y$ a morphism in $\mathbf{SmProj}/k$, we have the class $[\Gamma_f] \in \mathrm{CH}_{\dim X}(X \times Y)$ of the graph of f. One easily checks that for morphisms $f : X \to Y$, $g : Y \to Z$, we have

$$[\Gamma_g] \circ [\Gamma_f] = [\Gamma_{gf}].$$

Form the category of Chow correspondences $Cor_{\mathrm{CH}}(k)$ with objects $[X]$ for $X \in \mathbf{SmProj}_k$, with morphisms

$$\mathrm{Hom}_{Cor_{\mathrm{CH}}(k)}([X], [Y]) := \mathrm{CH}_{\dim X}(X \times Y)$$

and composition the composition of correspondences defined above.

Before defining the category of effective Chow motives, we make a brief categorical detour. Recall that for an additive category $\mathcal{A}$, we call $\mathcal{A}$ *pseudo-abelian* if for each idempotent endomorphism $\alpha : M \to M$, $\alpha^2 = \alpha$, there exist objects $M_0, M_1 \in \mathcal{A}$ and an isomorphism $\phi : M \to M_0 \oplus M_1$ such that $\phi \circ \alpha \circ \phi^{-1} = 0_{M_0} \oplus \mathrm{id}_{M_1}$. Given an arbitrary additive category $\mathcal{A}$, there is an additive functor $i : \mathcal{A} \to \mathcal{A}^\natural$ to a pseudo-abelian category $\mathcal{A}^\natural$ which is universal for additive functors of $\mathcal{A}$ to pseudo-abelian categories. $\mathcal{A}^\natural$ is constructed as follows: The objects of $\mathcal{A}^\natural$ are pairs (M, α) with $M \in \mathcal{A}$ and $\alpha : M \to M$ an idempotent endomorphism. The morphisms are given by

$$\mathrm{Hom}_{\mathcal{A}^\natural}(M, \alpha), (N, \beta)) := \{\beta \circ f \circ \alpha \mid f \in \mathrm{Hom}_{\mathcal{A}}(M, N)\}$$

with composition $(\gamma \circ g \circ \beta) \circ (\beta \circ f \circ \alpha) := \gamma \circ (g \circ \beta \circ f) \circ \alpha$. The functor i is given by $i(M) := (M, \mathrm{id})$. For a pair M, α, the identity maps on M give an isomorphism in $\mathcal{A}^\natural$

$$(M, \mathrm{id}) \cong (M, 1 - \alpha) \oplus (M, \alpha);$$

it is not hard to extend this to show that $\mathcal{A}^\natural$ is pseudo-abelian.

Definition 2.4. The category $\mathcal{M}^{\mathrm{eff}}(k)$ of *effective homological Chow motives over* k is the pseudo-abelian hull $Cor_{\mathrm{CH}}(k)^\natural$ of $Cor_{\mathrm{CH}}(k)$. Denote the object $([X], \mathrm{id})$ of $\mathcal{M}^{\mathrm{eff}}(k)$ by $m_{\mathrm{CH}}(X)$.

Remark 2.5. Sending $X \in \mathbf{SmProj}/k$ to $m_{\mathrm{CH}}(X)$ and $f : X \to Y$ to $[\Gamma_f] \in \mathrm{CH}_{\dim X}(X \times Y)$ defines a functor

$$m_{\mathrm{CH}} : \mathbf{SmProj}/k \to \mathcal{M}^{\mathrm{eff}}(k).$$

Grothendieck's original construction of Chow motives is different from the construction given here in that the Grothendieck construction uses the correspondence group $Cor^{\mathrm{CH}}(X, Y) := \mathrm{CH}^{\dim X}(X \times Y)$ (with the same composition law) instead of $\mathrm{CH}_{\dim X}(X \times Y)$. As the graph of a morphism $f : X \to Y$ is thus an element $[\Gamma_f] \in Cor^{\mathrm{CH}}(Y, X)$, the analogous construction leads to the category of *effective cohomological motives* $\mathcal{M}_{\mathrm{eff}}(k)$ and a functor

$$m^{\mathrm{CH}} : \mathbf{Sm}/k^{\mathrm{op}} \to \mathcal{M}_{\mathrm{eff}}(k)$$

In the end, the "true" category of Chow motives $\mathcal{M}(k)$, formed from $\mathcal{M}_{\mathrm{eff}}(k)$ by inverting the Lefschetz motive, has a duality operation, so there is no essential difference between $\mathcal{M}(k)$ and $\mathcal{M}(k)^{\mathrm{op}}$. We use the homological formulation to fit with Voevodsky's construction of the triangulated category of effective motives.

2.4. **Some philosophy.** We lose information in the category of motives by making the morphisms the rational equivalence classes of cycles. It would be better to use the cycles themselves, and think of the rational equivalences as *homotopies* of maps. It would also be nice to have objects in our category for each $X \in \mathbf{Sm}/k$ rather than just the smooth projective ones.

It is however not possible to use our composition formula to give a well-defined operation

$$z_{\dim X}(X \times Y) \otimes z_{\dim Y}(Y \times Z) \to z_{\dim X}(X \times Z)$$

since the intersection product is not always defined. Furthermore, if Y is not projective, the projection operation p_{XZ*} is not defined.

2.5. **Finite correspondences.** To solve the problem of the partially defined composition of correspondences and to extend the composition formula to smooth but non-projective schemes, Voevodsky introduces the notion of *finite* correspondences.

Definition 2.6. Let X and Y be in $\mathbf{Sch}_k$. The group $c(X,Y)$ is the subgroup of $z(X\times_k Y)$ generated by integral closed subschemes $W\subset X\times_k Y$ such that

(1) the projection $p_1 : W\to X$ is finite
(2) the image $p_1(W)\subset X$ is an irreducible component of X.

The elements of $c(X,Y)$ are called the *finite correspondences* from X to Y.

The following basic lemma is easy to prove:

Lemma 2.7. *Take X, Y in $\mathbf{Sm}/k$ and Z in $\mathbf{Sch}_k$, $W\in c(X,Y)$, $W'\in c(Y,Z)$. Suppose that X and Y are irreducible. Then each irreducible component C of* $\mathrm{Supp}(W)\times Z\cap X\times \mathrm{Supp}(W')$ *is finite over X and $p_X(C)=X$.*

Thus: for $W\in c(X,Y)$, $W'\in c(Y,Z)$, and $X,Y,Z\in\mathbf{Sm}/k$, we have the composition:

$$W'\circ W := p^S_{XZ*}(p^*_{XY}(W)\cdot p^*_{YZ}(W'));$$

where $S:=\mathrm{Supp}(W)\times Z\cap X\times\mathrm{Supp}(W')$ and $p^S_{XZ}: S\to X\times Z$ is the morphism induced from the projection p_{XZ}. This operation yields an associative bilinear composition law

$$\circ : c(Y,Z)\times c(X,Y)\to c(X,Z).$$

Remark 2.8. In fact, a modification of the formula gives a well-defined composition $\circ : c(Y,Z)\times c(X,Y)\to c(X,Z)$ for $X,Y\in\mathbf{Sm}/k$, $Z\in\mathbf{Sch}_k$.

2.6. **The category of finite correspondences.**

Definition 2.9. The category $\mathrm{Cor}(k)$ is the category with the same objects as $\mathbf{Sm}/k$, with

$$\mathrm{Hom}_{\mathrm{Cor}(k)}(X,Y) := c(X,Y),$$

and with the composition as defined above.

Remarks 2.10. (1) We have the functor $\mathbf{Sm}/k\to\mathrm{Cor}(k)$ sending a morphism $f: X\to Y$ in $\mathbf{Sm}/k$ to the graph $\Gamma_f\subset X\times_k Y$. We write the morphism corresponding to Γ_f as f_*, and the object corresonding to $X\in\mathbf{Sm}/k$ as $[X]$.

(2) The operation $\times_k$ (on smooth k-schemes and on cycles) makes $\mathrm{Cor}(k)$ a tensor category. Thus, the bounded homotopy category $K^b(\mathrm{Cor}(k))$ is a triangulated tensor category.

2.7. **The category of effective geometric motives.**

Definition 2.11. The category $\widehat{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is the localization of $K^b(\mathrm{Cor}(k))$, as a triangulated tensor category, by

• *Homotopy.* For $X\in\mathbf{Sm}/k$, invert $p_* : [X\times\mathbb{A}^1]\to[X]$

• *Mayer-Vietoris.* Let X be in $\mathbf{Sm}/k$. Write X as a union of Zariski open subschemes U,V: $X=U\cup V$. We have the canonical map

$$\mathrm{Cone}([U\cap V]\xrightarrow{(j_{U,U\cap V*},-j_{V,U\cap V*})}[U]\oplus[V])\xrightarrow{(j_{U*}+j_{V*})}[X]$$

since $(j_{U*} + j_{V*}) \circ (j_{U,U\cap V*}, -j_{V,U\cap V*}) = 0$. Invert this map.

The category $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of *effective geometric motives* is the pseudo-abelian hull of $\widehat{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

The morphisms inverted to form $\widehat{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ are closed under $\otimes$, so $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ inherits the tensor structure $\otimes$ from $K^b(\mathrm{Cor}(k))$. By [4], the pseudo-abelian hull of a triangulated category is a triangulated category.

2.8. **The category of geometric motives.** To define the category of geometric motives we invert the Lefschetz motive. For $X \in \mathbf{Sm}_k$, the *reduced motive* is

$$\widetilde{[X]} := \mathrm{Cone}(p_* : [X] \to [\mathrm{Spec}\, k])[-1].$$

Set $\mathbb{Z}(1) := \widetilde{[\mathbb{P}^1]}[-2]$, and set $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ for $n \geq 0$.

Definition 2.12. The category of *geometric motives*, $DM_{\mathrm{gm}}(k)$, is defined by inverting the functor $\otimes\mathbb{Z}(1)$ on $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, i.e., one has objects $X(n)$ for $X \in DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, $n \in \mathbb{Z}$ and

$$\mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(X(n), Y(m)) := \varinjlim_N \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(X \otimes \mathbb{Z}(n+N), Y \otimes \mathbb{Z}(m+N)).$$

Remarks 2.13. (1) Sending X to $X(0)$ and using the canonical map to the limit

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(X, Y) \to \varinjlim_N \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(X \otimes \mathbb{Z}(N), Y \otimes \mathbb{Z}(N))$$

defines the functor $i : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)$. For $n \geq 0$, we have the evident map $i(X \otimes \mathbb{Z}(n)) \to X(n)$, which is an isomorphism

(2) In order that $DM_{\mathrm{gm}}(k)$ be again a triangulated tensor category, it suffices that the commutativity involution $\mathbb{Z}(1) \otimes \mathbb{Z}(1) \to \mathbb{Z}(1) \otimes \mathbb{Z}(1)$ be the identity, which is in fact the case (in fact, by a remark of Voevodsky, it suffices that the cyclic permutation of $\mathbb{Z}(1) \otimes \mathbb{Z}(1) \otimes \mathbb{Z}(1)$ is the identity).

(3) Setting $\mathbb{Z}(n) := \mathbb{1}(n)$ for $n \in \mathbb{Z}$, we have $X(n) \cong X \otimes \mathbb{Z}(n)$ and $\mathbb{Z}(n) \otimes \mathbb{Z}(m) \cong \mathbb{Z}(n+m)$.

(4) We have the functor $M_{\mathrm{gm}} : \mathbf{Sm}/k \to DM^{\mathrm{eff}}_{\mathrm{gm}}$ sending X to the image of $[X]$ and f to the image of the graph Γ_f.

Of course, there arises the question of the behavior of the functor $i : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)$. Here we have the result due to Voevodsky

Theorem 2.14 (Cancellation)**.** *The functor $i : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)$ is a fully faithful embedding.*

The first proof of this result used resolution of singularities, but the later proof does not, and is valid in all characteristics. We will look at the proof of the cancellation theorem in more detail, when we consider various cycle complexes.

3. Sites and sheaves

In order to make computations of the morphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, Voevodsky has introduced a parallel sheaf-theoretic construction, leading to the category $DM^{\mathrm{eff}}_{-}(k)$. We begin our discussion of this approach with a quick review of the theory of sheaves on a Grothendieck site. For a detailed reference, see [1, 2, 3].

3.1. **Presheaves.** A *presheaf* P on a small category $\mathcal{C}$ with values in a category $\mathcal{A}$ is a functor

$$P : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}.$$

Morphisms of presheaves are natural transformations of functors. This defines the category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$, $PreShv^{\mathcal{A}}(\mathcal{C})$.

Remark 3.1. We require $\mathcal{C}$ to be small so that the collection of natural transformations $\vartheta : F \to G$, for presheaves F, G, form a set. It would suffice that $\mathcal{C}$ be essentially small (the collection of isomorphism classes of objects form a set).

3.2. **Structural results.**

Theorem 3.2. *(1) If $\mathcal{A}$ is an abelian category, then so is $PreShv^{\mathcal{A}}(\mathcal{C})$, with kernel and cokernel defined objectwise: For $f : F \to G$,*

$$\begin{aligned} ker(f)(x) &= \ker(f(x) : F(x) \to G(x)); \\ \operatorname{coker}(f)(x) &= \operatorname{coker}(f(x) : F(x) \to G(x)). \end{aligned}$$

(2) For $\mathcal{A} = \mathbf{Ab}$, $PreShv^{\mathbf{Ab}}(\mathcal{C})$ has enough injectives.

The second part is proved by using a result of Grothendieck [24], noting that $PreShv^{\mathbf{Ab}}(\mathcal{C})$ has the set of generators $\{\mathbb{Z}_X \mid X \in \mathcal{C}\}$, where $\mathbb{Z}_X(Y)$ is the free abelian group on $\operatorname{Hom}_{\mathcal{C}}(Y, X)$.

3.3. **Pre-topologies.**

Definition 3.3. Let $\mathcal{C}$ be a category. A *Grothendieck pre-topology* τ on $\mathcal{C}$ is given by: For $X \in \mathcal{C}$ there is a set $\mathrm{Cov}_\tau(X)$ of *covering families* of X: a covering family of X is a set of morphisms $\{f_\alpha : U_\alpha \to X\}$ in $\mathcal{C}$. These satisfy:

A1. $\{\mathrm{id}_X\}$ is in $\mathrm{Cov}_\tau(X)$ for each $X \in \mathcal{C}$.

A2. For $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau(X)$ and $g : Y \to X$ a morphism in $\mathcal{C}$, the fiber products $U_\alpha \times_X Y$ all exist and $\{p_2 : U_\alpha \times_X Y \to Y\}$ is in $\mathrm{Cov}_\tau(Y)$.

A3. If $\{f_\alpha : U_\alpha \to X\}$ is in $\mathrm{Cov}_\tau(X)$ and if $\{g_{\alpha\beta} : V_{\alpha\beta} \to U_\alpha\}$ is in $\mathrm{Cov}_\tau(U_\alpha)$ for each α, then $\{f_\alpha \circ g_{\alpha\beta} : V_{\alpha\beta} \to X\}$ is in $\mathrm{Cov}_\tau(X)$.

Remark 3.4. We will not define the "correct" notion, that of a Grothendieck topology. Suffice it to say that a pre-topology generates a topology (although not every topology arises this way), and that one can define our category of interest, sheaves, directly from a given pre-topology.

A category with a (pre) topology is a *site*.

Example 3.5. If T is a topological space, let $Op(T)$ be the category with objects the open subsets $U \subset T$ and morphisms the inclusions $V \subset U$. Give $Op(T)$ a pre-topology by defining a covering family of $U \subset T$ to be a collection of $V_\alpha \subset U$

such that $U = \cup_\alpha V_\alpha$.

But there are more examples than this!

The main difference between a Grothendieck (pre)-topology and the usual notion coming from classical topology is that we don't require that an "open" of X, $U_\alpha \to X$ be an open subset, but just a morphism in the underlying category. Fiber product $U \times_X V$ replaces intersection $U \cap V$.

3.4. **Sheaves on a site.** For S presheaf of abelian groups on $\mathcal{C}$ and $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau(X)$ for some $X \in \mathcal{C}$, we have the "restriction" morphisms

$$\begin{aligned} &f_\alpha^* : S(X) \to S(U_\alpha) \\ &p_{1,\alpha,\beta}^* : S(U_\alpha) \to S(U_\alpha \times_X U_\beta) \\ &p_{2,\alpha,\beta}^* : S(U_\beta) \to S(U_\alpha \times_X U_\beta). \end{aligned}$$

Taking products, we have the sequence of abelian groups

$$0 \to S(X) \xrightarrow{\prod f_\alpha^*} \prod_\alpha S(U_\alpha) \xrightarrow{\prod p_{1,\alpha,\beta}^* - \prod p_{2,\alpha,\beta}^*} \prod_{\alpha,\beta} S(U_\alpha \times_X U_\beta). \tag{3.4.1}$$

Definition 3.6. A presheaf S is a *sheaf* for τ if for each covering family $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau$, the sequence (3.4.1) is exact. The category $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ of sheaves of abelian groups on $\mathcal{C}$ for τ is the full subcategory of $PreShv^{\mathbf{Ab}}(\mathcal{C})$ with objects the sheaves.

Proposition 3.7. *(1) The inclusion $i : Shv_\tau^{\mathbf{Ab}}(\mathcal{C}) \to PreShv_\tau^{\mathbf{Ab}}(\mathcal{C})$ admits an exact left adjoint: the sheafification functor.*

(2) $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ is an abelian category: For $f : F \to G$, $\ker(f)$ is the presheaf kernel. $\mathrm{coker}(f)$ is the sheafification of the presheaf cokernel.

(3) $Shv_\tau^{\mathbf{Ab}}(\mathcal{C})$ has enough injectives.

Remark 3.8. For the "classical" pre-topology on $Op(T)$, T a topological space, we recover the usual notion of a sheaf.

4. Motivic complexes

Voevodsky's construction of a sheaf-theoretic category of motives mixes together the category of Nisnevich sheaves on $\mathbf{Sm}/k$ with presheaves on the larger category $\mathrm{Cor}(k)$, giving the category of "Nisnevich sheaves with transfers". This is still not the correct category, one needs to form the derived category and then pass to the subcategory of complexes with $\mathbb{A}^1$-invariant cohomology to force $\mathbb{A}^1$-homotopy invariance; the Mayer-Vietoris property is already built in.

4.1. **Nisnevich sheaves with transfers.**

Definition 4.1. Let X be a k-scheme of finite type. A *Nisnevich cover* $\mathcal{U} \to X$ is an étale morphism of finite type such that, for each finitely generated separable field extension F of k, the map on F-valued points $\mathcal{U}(F) \to X(F)$ is surjective.

Using Nisnevich covers as covering families gives us the *small Nisnevich site on* X, X_{Nis}. The *big Nisnevich site over* k is defined similarly, except that now the underlying category is all of $\mathbf{Sm}/k$ and for $X \in \mathbf{Sm}/k$ the covering families of X are the same as for X_{Nis}.

Notation 4.2. $\mathrm{Sh}^{\mathrm{Nis}}(X) :=$ Nisnevich sheaves of abelian groups on X, $\mathrm{Sh}^{\mathrm{Nis}}(k) :=$ Nisnevich sheaves of abelian groups on $\mathbf{Sm}/k$. For a presheaf $\mathcal{F}$ on $\mathbf{Sm}/k$ or X_{Nis}, we let $\mathcal{F}_{\mathrm{Nis}}$ denote the associated sheaf.

For $X \in \mathbf{Sch}_k$, $\mathbb{Z}(X)$ denotes the presheaf of abelian groups on $\mathbf{Sm}/k$ freely generated by $\mathrm{Hom}_{\mathbf{Sch}_k}(-, X)$, $\mathbb{Z}_{\mathrm{Nis}}(X)$ the Nisnevich sheaf. $\mathrm{PreSh}^{\mathrm{Nis}}(\mathbf{Sm}/k)$ has a tensor product:

$$(F \otimes G)(X) := F(X) \otimes_{\mathbb{Z}} G(X)$$

and internal Hom: $\mathcal{H}om(F, G)(X) := \mathrm{Hom}_{\mathrm{PreSh}^{\mathrm{Nis}}(\mathbf{Sm}/k)}(F \otimes \mathbb{Z}(X), G)$.

$\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}/k)$ has the tensor product by sheafifying the presheaf $\otimes$. The internal Hom in $\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}/k)$ is given by $\mathcal{H}om(F, G)(X) := \mathrm{Hom}_{\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}/k)}(F \otimes \mathbb{Z}_{\mathrm{Nis}}(X), G)$.

Definition 4.3. (1) The category $\mathrm{PST}(k)$ of presheaves with transfer is the category of additive presheaves of abelian groups on $\mathrm{Cor}(k)$, i.e., the category of additive functors $F : \mathrm{Cor}(k)^{\mathrm{op}} \to \mathbf{Ab}$.

(2) The category of Nisnevich sheaves with transfer on $\mathbf{Sm}/k$, $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$, is the full subcategory of $\mathrm{PST}(k)$ with objects those F such that, for each $X \in \mathbf{Sm}/k$, the restriction of F to X_{Nis} is a sheaf. We have the sheafification functor $F \mapsto F_{\mathrm{Nis}}$.

Remark 4.4. A PST F is a presheaf on $\mathbf{Sm}/k$ together with *transfer maps*

$$\mathrm{Tr}(a) : F(Y) \to F(X)$$

for every finite correspondence $a \in \mathrm{Cor}(X, Y)$, satisfying:

$\mathrm{Tr}(\Gamma_f) = f^*$, $\mathrm{Tr}(a \circ b) = \mathrm{Tr}(b) \circ \mathrm{Tr}(a)$, $\mathrm{Tr}(a \pm b) = \mathrm{Tr}(a) \pm \mathrm{Tr}(b)$.

Example 4.5. For X in $\mathbf{Sch}_k$, we have the sheaf with transfers $L(X)$ defined by $L(X)(Y) = \mathrm{Cor}(Y, X)$ for $Y \in \mathbf{Sm}/k$. $L(X)$ is the free sheaf with transfers generated by the representable sheaf of sets $\mathrm{Hom}(-, X)$. In particular, we have the canonical isomorphisms $\mathrm{Hom}_{\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))}(L(X), F) = F(X)$. In fact, for $F \in \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ there is a canonical isomorphism

$$\mathrm{Ext}^n_{\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))}(L(X), F) \cong H^n(X_{\mathrm{Nis}}, F).$$

This should remind us of our classical example

$$\mathrm{Ext}^n_{\mathrm{Sh}^{\mathbf{Top}}(X)}(\mathbb{Z}_X, F) \cong H^n(X, F)$$

for F a sheaf of abelian groups on a topological space X, $\mathbb{Z}_X$ the constant sheaf.

4.2. **The tensor structure.** We define a tensor structure on $\mathrm{PST}(k)$:

Set $L(X) \otimes^{tr} L(Y) := L(X \times Y)$. For a general F, we have the canonical surjection

$$\mathcal{L}_0(F) := \oplus_{X, s \in F(X)} L(X) \to F$$

Applying $\mathcal{L}_0$ to the kernel of this map and iterating gives us the *canonical left resolution* $\mathcal{L}(F) \to F$ in $\mathrm{PST}(k)$:

$$\ldots \to \mathcal{L}_1(F) \to \mathcal{L}_0(F) \to F.$$

Define

$$F \otimes G := H_0^{\mathrm{PST}(k)}(\mathcal{L}(F) \otimes^{tr} \mathcal{L}(G)).$$

Taking the associated Nisnevich sheaf gives a tensor structure on $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$. We will use the same notation for the sheaf tensor product; the meaning will be clear from the context.

There is an internal Hom in $\mathrm{PST}(k)$ and in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ with

$$\mathcal{H}om(L(X), G)(Y) = G(X \times Y)$$

and extended to $\mathcal{H}om(F, G)$ by

$$\mathcal{H}om(F, G) := \ker[\mathcal{H}om(\mathcal{L}_0(F), G) \to \mathcal{L}_1(F), G)]$$

Since each $\mathcal{L}_i(F)$ is a direct sum of representable presheaves $L(X_\alpha)$, this is well-defined. The internal Hom satisfies the usual adjunction with respect to tensor product:

$$\mathrm{Hom}_{\mathrm{PST}(k)}(F \otimes^{tr} G, H) \cong \mathrm{Hom}_{\mathrm{PST}(k)}(F, \mathcal{H}om(G, H))$$

and similarly in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$.

In the usual manner, $\otimes^{tr}$ extends to a tensor structure on $C^-(\mathrm{PST}(k))$ and $C^-\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$; the internal Hom $\mathcal{H}om(F, G)$ extends as well, at least for F a bounded complex. The unit for $\otimes^{tr}$ (for presheaves and sheaves) is $L(\mathrm{Spec}\, k)$.

4.3. **Homotopy invariant presheaves.**

Definition 4.6. Let F be a presheaf of abelian groups on $\mathbf{Sm}/k$. We call F *homotopy invariant* if for all $X \in \mathbf{Sm}/k$, the map

$$p^* : F(X) \to F(X \times \mathbb{A}^1)$$

is an isomorphism. We call F *strictly homotopy invariant* if for all $q \geq 0$, the cohomology presheaf $X \mapsto H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$ is homotopy invariant.

Theorem 4.7 (PST [63, chapter 3, thm. 4.27, thm. 5.7])**.** *Let F be a homotopy invariant PST on $\mathbf{Sm}/k$. Then*

(1) The cohomology presheaves $X \mapsto H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$ are PST's.
(2) F_{Nis} is strictly homotopy invariant.
(3) $F_{\mathrm{Zar}} = F_{\mathrm{Nis}}$ and $H^q(X_{\mathrm{Zar}}, F_{\mathrm{Zar}}) = H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$.

Remarks 4.8. (1) uses the fact that for finite map $Z \to X$ with X Hensel local and Z irreducible, Z is also Hensel local. (2) and (3) rely on Voevodsky's generalization of Quillen's proof of Gersten's conjecture, viewed as a "moving lemma using transfers":

Lemma 4.9 (Voevodsky's moving lemma [63, chapter 3, lemma 4.17])**.** *Let X be in $\mathbf{Sm}/k$, S a finite set of points of X, $j_U : U \to X$ an open subscheme. Then* *there is an open neighborhood $j_V : V \to X$ of S in X and a finite correspondence $a \in \mathrm{Cor}(V, U)$ such that, for all homotopy invariant PST's F, the diagram*

$$\begin{tikzcd} F(X) \arrow[r, "j_U^*"] \arrow[d, "j_V^*"'] & F(U) \arrow[dl, "\mathrm{Tr}(a)"] \\ F(V) & \end{tikzcd}$$

commutes.

One consequence of the moving lemma is:

(1) If X is semi-local, then $F(X) \to F(U)$ is a split injection.

Variations on this construction prove:

(2) If X is semi-local and smooth then $F(X) = F_{\mathrm{Zar}}(X)$ and $H^n(X_{\mathrm{Zar}}, F_{\mathrm{Zar}}) = 0$ for $n > 0$.

(3) If U is an open subset of $\mathbb{A}^1_k$, then $F_{\mathrm{Zar}}(U) = F(U)$ and $H^n(U, F_{\mathrm{Zar}}) = 0$ for $n > 0$.

(4) If $j : U \to X$ has complement a smooth k-scheme $i : Z \to X$, then

$$\mathit{coker}\left[F_{|X_{\mathrm{Zar}}} \to j_* F_{|U_{\mathrm{Zar}}}\right]$$

(as a sheaf on Z_{Zar}) depends only on the Nisnevich neighborhood of Z in X.

(1)-(4) together with some cohomological techniques prove the theorem.

4.4. **The category of motivic complexes.**

Definition 4.10. Inside the derived category $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$, we have the full subcategory $DM_-^{\mathrm{eff}}(k)$ consisting of complexes whose cohomology sheaves are homotopy invariant.

Proposition 4.11. *$DM_-^{\mathrm{eff}}(k)$ is a triangulated subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$.*

This follows from

Lemma 4.12. *Let $HI(k) \subset \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ be the full subcategory of homotopy invariant sheaves. Then $HI(k)$ is an abelian subcategory of $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$, closed under extensions in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$.*

Proof. Proof of the lemma. Given $f : F \to G$ in $HI(k)$, $\mathit{ker}(f)$ is the presheaf kernel, hence in $HI(k)$. The presheaf $\mathit{coker}(f)$ is homotopy invariant, so by the PST theorem $\mathit{coker}(f)_{\mathrm{Nis}}$ is homotopy invariant. Given $0 \to A \to E \to B \to 0$ exact in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ with $A, B \in HI(k)$, consider $p : X \times \mathbb{A}^1 \to X$. The PST theorem implies $R^1 p_* A = 0$, so

$$0 \to p_* A \to p_* E \to p_* B \to 0$$

is exact as sheaves on X. Thus $p_* E = E$, so E is homotopy invariant. $\square$

5. The localization theorem

To understand $DM_-^{\mathrm{eff}}(k)$ better, we describe it as a localization of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ rather than a subcategory.

5.1. **The Suslin complex.** Let $\Delta^n := \mathrm{Spec}\, k[t_0, \dots, t_n]/\sum_{i=0}^n t_i - 1$. $n \mapsto \Delta^n$ defines the cosimplicial k-scheme Δ^* with coface map

$$\delta_i^n : \Delta^n \to \Delta^{n+1}$$

the map

$$\delta_i^n(x_0, \dots, x_n) := (x_0, \dots, x_{i-1}, 0, x_i \dots, x_n).$$

Definition 5.1. Let F be a presheaf (of abelian groups) on $\mathbf{Sm}/k$. Define the presheaf $C_n(F)$ by

$$C_n(F)(X) := F(X \times \Delta^n)$$

The *Suslin complex* $C_*(F)$ is the complex with differential

$$d_n := \sum_i (-1)^i \delta_i^* : C_n(F) \to C_{n-1}(F).$$

For $X \in \mathbf{Sm}/k$, let $C_*(X)$ be the complex of sheaves

$$C_n(X)(U) := \mathrm{Cor}(U \times \Delta^n, X).$$

Clearly $C_*(X) = C_*(L(X))$.

Remarks 5.2. (1) If F is a PST, resp. sheaf with transfers on $\mathbf{Sm}/k$, then $C_*(F)$ is a complex of PST's, resp. sheaves with transfers.

(2) The homology presheaves $h_i(F) := \mathcal{H}^{-i}(C_*(F))$ are homotopy invariant. Thus, by Voevodsky's PST theorem, the associated Nisnevich sheaves $h_i^{\mathrm{Nis}}(F)$ are strictly homotopy invariant for F a PST. We thus have the functor

$$C_* : \mathrm{PST}(k) \to \mathcal{DM}_-^{\mathrm{eff}}(k).$$

factoring through the canonical functor $\mathrm{PST}(k) \to \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$.
(3) By taking the total complex of the evident double complex, we extend the definition of $C_*(F)$ to bounded above complexes. In particular, we have the functor

$$C_* : C^-(\mathrm{PST}(k)) \to C^-(\mathrm{PST}(k)),$$

which factors through the functor $C^-(\mathrm{PST}(k)) \to C^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$. As in (2), the image of $C_*(F)$ in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ lands in $\mathcal{DM}_-^{\mathrm{eff}}(k)$, giving us the exact functor

$$C_* : K^-(\mathrm{PST}(k)) \to \mathcal{DM}_-^{\mathrm{eff}}(k),$$

5.2. **$\mathbb{A}^1$-homotopy.** The inclusions $i_0, i_1 : \mathrm{Spec}\, k \to \mathbb{A}^1$ give maps of PST's $i_0, i_1 : \mathbb{Z} = L(\mathrm{Spec}\, k) \to L(\mathbb{A}^1)$.

Definition 5.3. Let F and G be in $C^-(\mathrm{PST}(k))$. Two maps $f, g : F \to G$ are $\mathbb{A}^1$-*homotopic* if there is a map

$$h : F \otimes^{tr} L(\mathbb{A}^1) \to G$$

with $f = h \circ (\mathrm{id} \otimes i_0)$, $g = h \circ (\mathrm{id} \otimes i_1)$; we write this equivalence relation as $f \sim_{\mathbb{A}^1} g$. A map $f : F \to G$ in in $C^-(\mathrm{PST}(k))$ is an $\mathbb{A}^1$-*homotopy equivalence* if there is a map $g : G \to F$ such that $fg \sim_{\mathbb{A}^1} \mathrm{id}_G$, $gf \sim_{\mathbb{A}^1} \mathrm{id}_F$.

Example 5.4. Take $X \in \mathbf{Sm}/k$. Then the map $L(X \times \mathbb{A}^1) \to L(X)$ induced by the projection $p : X \times \mathbb{A}^1 \to X$ is an $\mathbb{A}^1$-homotopy equivalence. The homotopy inverse is induced for example by the zero-section $i_0 : X \to X \times \mathbb{A}^1$; as $p \circ i_0 = \mathrm{id}_X$, we need only show that $L(i_0 \circ p) : L(X \times \mathbb{A}^1) \to L(X \times \mathbb{A}^1)$ is $\mathbb{A}^1$-homotopic to the identity.

As $L(X \times \mathbb{A}^1) \otimes^{tr} L(\mathbb{A}^1) = L(X \times \mathbb{A}^1 \times \mathbb{A}^1)$, we have the map

$$L(\mathrm{id}_X \times \mu) : L(X \times \mathbb{A}^1) \otimes^{tr} L(\mathbb{A}^1) \to L(X \times \mathbb{A}^1); \quad \mu(x, y) = xy,$$

which does the job.

Example 5.5. We have the dual to the last example: For $F \in C^-(\mathrm{PST}(k))$, we have the PST $F^{\mathbb{A}^1} := \mathcal{H}om(L(\mathbb{A}^1), F)$; the projection $p : \mathbb{A}^1 \to \mathrm{Spec}\, k$ and inclusion $i_0 : \mathrm{Spec}\, k \to \mathbb{A}^1$ induces the maps $p^* : F \to F^{\mathbb{A}^1}$ and $i_0^* : F^{\mathbb{A}^1} \to F$ with $i_0^* \circ p^* = \mathrm{id}$. To get an $\mathbb{A}^1$-homotopy between $p^* \circ i_0^*$ and id, we need a map $h : F^{\mathbb{A}^1} \otimes^{tr} L(\mathbb{A}^1) \to F^{\mathbb{A}^1}$, or by adjunction, a map

$$\begin{aligned} H : \mathcal{H}om(L(\mathbb{A}^1), F) \to &\mathcal{H}om(L(\mathbb{A}^1), \mathcal{H}om(L(\mathbb{A}^1), F)) \\ &= \mathcal{H}om(L(\mathbb{A}^1) \otimes^{tr} L(\mathbb{A}^1), F) \\ &= \mathcal{H}om(L(\mathbb{A}^1 \times \mathbb{A}^1), F) \end{aligned}$$

with $H \circ (\mathrm{id} \otimes i_1)^* = \mathrm{id}$ and $H \circ (\mathrm{id} \times i_0)^* = p^* \circ i_0^*$. As above, the map $L(\mu)^*$ with

$$\mu : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$$

the multiplication map is what we need.

This implies that, for $F \in C^-(\mathrm{PST}(k))$, $p^* : F \to C_n(F)$ is an $\mathbb{A}^1$-homotopy equivalence: $n = 1$ is the crucial case since $C_1(C_{n-1}(F)) = C_n(F)$, and we have $C_1(F) = F^{\mathbb{A}^1}$.

We need extensions of $\mathbb{A}^1$-homotopy equivalence which behaves well in the triangulated categories $D^-(\mathrm{PST}(k))$ and $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$. For this, let $\mathcal{A}_{\mathbb{A}^1-WE}$ be the smallest localizing subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ containing the cone of each $\mathbb{A}^1$-homotopy equivalence $f : F \to G$ in $C^-(\mathrm{PST}(k))$. Similarly, let $\mathcal{T}_{\mathbb{A}^1-WE}$ be the smallest localizing subcategory of $D^-(\mathrm{PST}(k))$ containing the cone of each $\mathbb{A}^1$-homotopy equivalence $f : F \to G$ in $C^-(\mathrm{PST}(k))$.

Definition 5.6. A map $f : F \to G$ in $C^-(\mathrm{PST}(k))$ is an $\mathbb{A}^1$-*weak equivalence* if the cone of f maps to $\mathcal{T}_{\mathbb{A}^1-WE}$ under the canonical functor

$$C^-(\mathrm{PST}(k)) \to D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))).$$

Call $f : F \to G$ a *strict* $\mathbb{A}^1$-*weak equivalence* if the cone of f maps to $\mathcal{A}_{\mathbb{A}^1-WE}$ under the canonical functor

$$C^-(\mathrm{PST}(k)) \to D^-(\mathrm{PST}(k)).$$

Clearly a strict $\mathbb{A}^1$-weak equivalence is an $\mathbb{A}^1$-weak equivalence.

Remark 5.7. (1) The term $\mathbb{A}^1$-weak equivalence is borrowed from a slightly different setting, which uses a model category structure on the larger category $C(\mathrm{PST}(k))$.

(2) Let

$$\begin{tikzcd}
F_0 \arrow[r, "i_0"] \arrow[d, "f_0"'] & \cdots \arrow[r, "i_{n-1}"] & F_n \arrow[r, "i_n"] \arrow[d, "f_n"] & \cdots \\
G_0 \arrow[r, "i'_0"] & \cdots \arrow[r, "i'_{n-1}"] & G_n \arrow[r, "i'_n"] & \cdots
\end{tikzcd}$$

be a commutative diagram in $C^-(\mathrm{PST}(k))$. Suppose that the infinite direct sums $\oplus_{n=0}^{\infty} F_n$, $\oplus_{n=0}^{\infty} G_n$ both exist in $C^-(\mathrm{PST}(k))$. Then the colimits $F := \mathrm{colim}_n F_n$, $G := \mathrm{colim}_n G_n$ exist in $C^-(\mathrm{PST}(k))$, as

$$F \cong \mathrm{coker}[\oplus_{n=0}^{\infty} F_n \xrightarrow{\mathrm{id}-\iota_F} \oplus_{n=0}^{\infty} F_n]$$

where ι_F is the sum of the maps $i_n : F_n \to F_{n+1}$, and similarly for G.

If all the maps $f_n : F_n \to G_n$ are strict $\mathbb{A}^1$-weak equivalences, then so is the induced map $f : F \to G$. Indeed, $\mathrm{id} - \iota_F$ is a monomorphism, hence

$$\oplus_{n=0}^{\infty} F_n \xrightarrow{\mathrm{id}-\iota_F} \oplus_{n=0}^{\infty} F_n \to F$$

extends to distinguished triangle in $D^-(\mathrm{PST}(k))$, and similarly for G. As $\mathcal{T}_{\mathbb{A}^1-WE}$ is localizing, it follows that f is a strict $\mathbb{A}^1$-weak equivalence.

In particular, let $f : F \to G$ be a map in $C^-(\mathrm{PST}(k))$. Suppose that in each degree n, the map $f^n : F^n \to G^n$ is a strict $\mathbb{A}^1$-weak equivalence. Then f is a strict $\mathbb{A}^1$-weak equivalence. Indeed, let $\sigma_n(-)$ denote the stupid truncation functor

$$(\sigma_{\geq n} F)^m : \begin{cases} F^m & \text{for } m \geq n \\ 0 & \text{for } m < n, \end{cases}$$

and let $\sigma_{\geq n} f : \sigma_{\geq n} F \to \sigma_{\geq n} G$ be the map induced by f. Then as $\mathcal{T}_{\mathbb{A}^1-WE}$ is a full triangulated subcategory of $D^-(\mathrm{PST}(k))$, we see that $\sigma_{\geq n} f$ is a strict $\mathbb{A}^1$-weak equivalence for all n; as $F \cong \mathrm{colim}_{n \to -\infty} \sigma_{\geq n} F$, and similarly for G, we may pass to the limit to conclude that f is a strict $\mathbb{A}^1$-weak equivalence.

The comments in (2) all hold for $\mathbb{A}^1$-weak equivalences, replacing $D^-(\mathrm{PST}(k))$ with $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ and $\mathcal{T}_{\mathbb{A}^1-WE}$ with $\mathcal{A}_{\mathbb{A}^1-WE}$.

Remark 5.8. $\mathcal{A}_{\mathbb{A}^1-WE}$ is in fact the smallest localizing subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ containing all the complexes $L(X \times \mathbb{A}^1) \xrightarrow{p_*} L(X)$. Indeed, let $\mathcal{A}$ denote this *a priori* smaller localizing subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$. Using the canonical left resolution $\mathcal{L}(F) \to F$, we see that $\mathcal{A}$ contains all cones of the maps $F \otimes^{tr} L(\mathbb{A}^1) \to F$. Thus, in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))/\mathcal{A}$, all $\mathbb{A}^1$-homotopy equivalences become isomorphisms, hence $\mathcal{A}$ contains the generators of $\mathcal{A}_{\mathbb{A}^1-WE}$. The analogous statement for $\mathcal{T}_{\mathbb{A}^1-WE} \subset D^-(\mathrm{PST}(k))$ is proved the same way.

Lemma 5.9. *For $F \in C^-(\mathrm{PST}(k))$, the inclusion $F = C_0(F) \to C_*(F)$ is a strict $\mathbb{A}^1$-weak equivalence.*

Proof. Let F_* be the "constant" complex, $F_n := F$, $d_n : F_n \to F_{n-1}$ is 0 for n odd, id for n even. The evident map $F \to F_*$ (with F supported in degree 0) is a chain homotopy equivalence. The canonical map $F_n \to C_n(F)$ is an $\mathbb{A}^1$-homotopy equivalence by example 5.5, which by remark 5.7(2) suffices to prove the lemma. □

5.3. **The localization theorem.** We can now prove the main result describing $DM_-^{\text{eff}}(k)$ as a localization of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$.

Theorem 5.10. *The functor $C_*: C^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))) \to DM_-^{\text{eff}}(k)$ descends to an exact functor*

$$\mathbf{R}C_*: D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))) \to DM_-^{\text{eff}}(k),$$

left adjoint to the inclusion $DM_-^{\text{eff}}(k) \to D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$. $\mathbf{R}C_*$ identifies $DM_-^{\text{eff}}(k)$ with the localization $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))/\mathcal{A}_{\mathbb{A}^1-WE}$. In addition $\mathcal{A}_{\mathbb{A}^1-WE}$ is the localizing subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ generated by complexes

$$L(X \times \mathbb{A}^1) \xrightarrow{L(p_1)} L(X); \quad X \in \mathbf{Sm}/k.$$

Proof. It suffices to prove:

1. For each $F \in C^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$, the map $F \to C_*(F)$ is an isomorphism in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))/\mathcal{A}_{\mathbb{A}^1-WE}$.
2. For each $T \in DM_-^{\text{eff}}(k)$, $B \in \mathcal{A}_{\mathbb{A}^1-WE}$, $\mathrm{Hom}(B,T) = 0$.

Indeed: (1) implies $DM_-^{\text{eff}}(k) \to D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))/\mathcal{A}_{\mathbb{A}^1-WE}$ is surjective on isomorphism classes. (2) implies $DM_-^{\text{eff}}(k) \to D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))/\mathcal{A}_{\mathbb{A}^1-WE}$ is fully faithful, hence an equivalence. (1) again implies the composition

$$D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))) \to D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))/\mathcal{A}_{\mathbb{A}^1-WE} \to DM_-^{\text{eff}}(k)$$

sends F to $C_*(F)$.

To prove (2): We have noted in remark 5.8 that $\mathcal{A}_{\mathbb{A}^1-WE}$ is the localizing subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ generated by the complexes $I(X) := L(X \times \mathbb{A}^1)) \xrightarrow{L(p_1)} L(X)$. But

$$\begin{aligned}\mathrm{Hom}_{D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))}(L(Y),T) &\cong \mathrm{Hom}_{D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}/k))}(\mathbb{Z}_{\mathrm{Nis}}(Y),T)\\ &\cong \mathbb{H}^0(Y_{\mathrm{Nis}},T)\end{aligned}$$

for $T \in D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$. Also, for T is in $DM_-^{\text{eff}}(k)$,

$$\mathbb{H}^*(X_{\mathrm{Nis}},T) \cong \mathbb{H}^*(X \times \mathbb{A}^1_{\mathrm{Nis}},T)$$

so for T is in $DM_-^{\text{eff}}(k)$, $\mathrm{Hom}_{D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))}(I(X),T) = 0$.

To prove (1): This follows from lemma 5.9. $\square$

5.4. $\mathbb{A}^1$**-weak equivalences and Ext.** As immediate consequence of the localization theorem, we see that for $G \in HI(k)$, $\mathrm{Ext}^n(-,G)$ transforms $\mathbb{A}^1$-homotopy equivalences into isomorphisms. In the derived category, this yields:

Proposition 5.11. *Let $f: F' \to F$ be an $\mathbb{A}^1$-weak equivalence in $C^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$. Then*

$$\mathrm{Hom}_{D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))}(F,G) \xrightarrow{f^*} \mathrm{Hom}_{D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))}(F',G)$$

is an isomorphism for all $G \in DM_-^{\text{eff}}(k)$, in particular, if $G = G'[n]$ for some $G' \in HI(k)$.

Proof. The localization theorem together with remark 5.8 identifies $DM_-^{\text{eff}}(k)$ with $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))/\mathcal{A}_{\mathbb{A}^1-WE}$. In particular

$$\mathrm{Hom}_{D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))}(\mathrm{Cone}(f),G) = 0.$$

The proposition follows from this and the exact sequence

$$\mathrm{Hom}(\mathrm{Cone}(f), G) \to \mathrm{Hom}(F, G) \xrightarrow{f^*} \mathrm{Hom}(F', G) \to \mathrm{Hom}(\mathrm{Cone}(f), G[1])$$

□

Theorem 5.12 (Nisnevich-acyclicity)**.** *For $G \in HI(k)$, F a PST, we have*

$$\mathrm{Ext}^n_{\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))}(F_{\mathrm{Nis}}, G) \cong \mathrm{Hom}_{D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))}(C_*(F)_{\mathrm{Nis}}, G[n])$$

for all n. In addition:

$$\begin{aligned} &\mathrm{Ext}^n_{\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))}(F_{\mathrm{Nis}}, G) = 0 \textit{ for } 0 \le i \le n \textit{ and all } G \in HI(k) \\ &\iff h_i^{\mathrm{Nis}}(F) = 0 \textit{ for } 0 \le i \le n \\ &\iff h_i^{\mathrm{Zar}}(F) = 0 \textit{ for } 0 \le i \le n. \end{aligned}$$

Proof. The $\mathbb{A}^1$-weak equivalence $F \to C_*(F)$ induces an $\mathbb{A}^1$-weak equivalence $F_{\mathrm{Nis}} \to C_*(F)_{\mathrm{Nis}}$; this, together with proposition 5.11 proves the first assertion.

For the series of equivalences, as F is a PST, then $h_i^{\mathrm{Nis}}(F) = h_i^{\mathrm{Zar}}(F)$ by Voevodsky's PST theorem. If $h_n^{\mathrm{Nis}}(F) \neq 0$, but $h_i^{\mathrm{Nis}}(F) = 0$ for $i < n$, taking $G = h_n^{\mathrm{Nis}}(F)$ we have the canonical non-zero map $C_*(F)_{\mathrm{Nis}} \to G[n]$ in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$. □

The Nisnevich-acyclicity theorem is often applied to yield the following statement:

Corollary 5.13. *Let F be a PST such that $F_{\mathrm{Nis}} = 0$. Then $C_*(F)_{\mathrm{Zar}}$ is acyclic, i.e., $h_i^{\mathrm{Zar}}(F) = 0$ for all i.*

One has the following extension

Corollary 5.14. *Take F in $C^-(\mathrm{PST}(k))$ such that F_{Nis} is acyclic. Then $C_*(F)_{\mathrm{Zar}}$ is acyclic.*

Proof. Using the canonical truncations, we can assume that F is bounded; after shifting F we may assume that $F^n = 0$ for $n < 0$. The result follows easily from corollary 5.13 and a devissage using the distinguished triangle in $D^-(\mathrm{PST}(k))$

$$H^0(F) \to F \to \tau_{\ge 1}F \to H^0(F)[1].$$

□

6. The embedding theorem

We will use the sheaf-theoretic constructions to study the geometric category $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$. This relies on constructing a full embedding $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM^{\mathrm{eff}}_-(k)$.

Start with the functor

$$L : \mathrm{Cor}(k) \to \mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$$

sending X to the representable sheaf $L(X)$. L extends to the homotopy category of bounded complexes:

$$L : K^b(\mathrm{Cor}(k)) \to D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))).$$

Theorem 6.1. *There is a commutative diagram of exact tensor functors*

$$\begin{array}{ccc} K^b(\mathrm{Cor}(k)) & \xrightarrow{L} & D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))) \\ \downarrow & & \downarrow \mathbf{R}C_* \\ DM^{\mathrm{eff}}_{\mathrm{gm}}(k) & \xrightarrow[i]{} & DM^{\mathrm{eff}}_-(k) \end{array}$$

such that

1. i *is a full embedding with dense image.*
2. $\mathbf{R}C_*(L(X)) \cong C_*(X)$.

Corollary 6.2. *For X and $Y \in \mathbf{Sm}/k$,* $\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), M_{\mathrm{gm}}(X)[n]) \cong \mathbb{H}^n(Y_{\mathrm{Nis}}, C_*(X)) \cong \mathbb{H}^n(Y_{\mathrm{Zar}}, C_*(X))$.

Proof of the embedding theorem. We already know that $\mathbf{R}C_*(L(X)) \cong C_*(L(X)) = C_*(X)$. To show that $i : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM^{\mathrm{eff}}_-(k)$ exists:

$DM^{\mathrm{eff}}_-(k)$ is already pseudo-abelian. Using the localization theorem, we need to show that the two types of complexes we inverted in $K^b(\mathrm{Cor}(k))$ are already inverted in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))/\mathcal{A}_{\mathbb{A}^1-WE}$.

Type *hom*. $[X \times \mathbb{A}^1] \to [X]$. This goes to $L(X \times \mathbb{A}^1) \to L(X)$, which is a generator in $\mathcal{A}_{\mathbb{A}^1-WE}$.

Type MV. $([U \cap V] \to [U] \oplus [V]) \to [U \cup V]$. The sequence

$$0 \to L(U \cap V) \to L(U) \oplus L(V) \to L(U \cup V) \to 0$$

is exact as Nisnevich sheaves, hence the map is inverted in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$.

To show that i is a full embedding: Replace $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ with the derived presheaf category $D^-(\mathrm{PST}(k))$. Since

$$\mathrm{Hom}_{\mathrm{PST}(k)}(L(X), F) = F(X)$$

it follows that $L(X)$ is projective in $\mathrm{PST}(k)$, and hence Then

$$L : K^b(\mathrm{Cor}(k)) \to D^-(\mathrm{PST}(k))$$

is a full embedding. Note that all the objects $L(X)$ in $D^-(\mathrm{PST}(k))$ are compact, i.e.,

$$\mathrm{colim}_\alpha \mathrm{Hom}_{D^-(\mathrm{PST}(k))}(L(X), F_\alpha) = \mathrm{Hom}_{D^-(\mathrm{PST}(k))}(L(X), \mathrm{colim}_\alpha F_\alpha),$$

since $\mathrm{Hom}_{D^-(\mathrm{PST}(k))}(L(X), F) = H^0(F(X))$. It follows that $L(K)$ is compact for each $K \in K^b(\mathrm{Cor}(k))$.

To pass from presheaves to sheaves, note that the natural map

$$\phi : D^-(\mathrm{PST}(k)) \to D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$$

is the localization with respect to the localizing category $\mathcal{T}_{\mathrm{Nis}}$ consisting of F such that $\phi(F) := F_{\mathrm{Nis}}$ is isomorphic to 0 in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$.

Let $\mathcal{T}$ be the localizing subcategory of $D^-(\mathrm{PST}(k))$ generated by applying L to Type *hom* and Type MV. It follows from results of Ne'eman on compact objects in triangulated categories [49, theorem 4.4.9] that $L^{-1}(\mathcal{T})$ is the thick subcategory generated by cones of maps of Type *hom* and Type *MV*, so it suffices to show that

$$\phi^{-1}(\mathcal{A}_{\mathbb{A}^1-WE}) = \mathcal{T}.$$

Clearly $\phi^{-1}(\mathcal{A}_{\mathbb{A}^1-WE}) \supset \mathcal{T}$; as $\phi \circ L$ applied to the complexes of Type *hom* generates $\mathcal{A}_{\mathbb{A}^1-WE}$, we need to show that $\mathcal{T}_{\mathrm{Nis}} \subset \mathcal{T}$, i.e., if $F_{\mathrm{Nis}} \cong 0$ in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$, then F is in $\mathcal{T}$.

So, take $F \in D^-(\mathrm{PST}(k))$ with $F_{\mathrm{Nis}} \cong 0$ in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$, and let $H^i : \mathbf{Sm}/k \to \mathbf{Ab}$ be the functor

$$H^i(X) := \mathrm{Hom}_{D^-(\mathrm{PST}(k))/\mathcal{T}}(L(X), F[i]).$$

It suffices to show that all $H^i = 0$, since the $L(X)$ generate.

We first claim it that suffices to show that $H^i_{\mathrm{Zar}} = 0$ for all i. Since $\mathcal{T}$ contains Type *MV* sequences, $\{H^i\}$ give long exact Mayer-Vietoris sequences. Applying the theorem of Brown-Gersten [14, theorem 1'], we have strongly convergent local-to-global spectral sequence

$$E_2^{p,q} = H^p(X, H^q_{\mathrm{Zar}}) \Longrightarrow H^{p+q}(X),$$

which verifies our claim.

Since the H^i are PST's and are homotopy invariant (Type *hom*), we have $H^i_{\mathrm{Zar}} = H^i_{\mathrm{Nis}}$. So we need to show that $H^i_{\mathrm{Nis}} = 0$.

Represent a morphism $\phi : L(X) \to F[i]$ by

$$\begin{tikzcd} L(X) \arrow[dr, "f"'] & & F[i] \arrow[dl, "g"] \\ & K & \end{tikzcd}$$

with $\mathrm{Cone}(g) \in \mathcal{T}$. By lemma 5.9, $K \to C_*(K)$ has cone in $\mathcal{T}$, so we may replace K with $C_*(K)$. But by corollary 5.14, $F_{\mathrm{Nis}} \cong 0$ in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ implies that $C_*(F)_{\mathrm{Nis}} \cong 0$ in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$. Furthermore, as $F[i] \to K$ has cone in $\mathcal{T}$, $C_*(F)_{\mathrm{Nis}}[i] \to C_*(K)_{\mathrm{Nis}}$ has cone that goes to zero in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))/\mathcal{A}_{\mathbb{A}^1-WE}$. But $C_*(F)_{\mathrm{Nis}}[i]$ and $C_*(K)_{\mathrm{Nis}}$ are in $\mathcal{DM}^{\mathrm{eff}}_-(k)$, hence $C_*(K)_{\mathrm{Nis}} \cong C_*(F)_{\mathrm{Nis}}[i] \cong 0$ in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$, using the localization theorem again. Therefore, there is a Nisnevich cover $\mathcal{U} \to X$ with composition

$$L(\mathcal{U}) \to L(X) \to K \to C_*(K)$$

being zero. Thus ϕ goes to 0 in $H^i(\mathcal{U})$, hence in $H^i_{\mathrm{Nis}}(X)$.

To show that i has dense image: This uses the canonical left resolution $\mathcal{L}(F) \to F$ to show that every object is expressible in terms of direct sums of representable objects. □

6.1. **Suslin homology.** One can use the Suslin complex to define a purely algebraic version of singular homology. In fact, this was Suslin's original construction, which was later generalized to the setting of sheaves with transfers.

Definition 6.3. For $X \in \mathbf{Sm}/k$, define the *Suslin homology* of X as

$$H_i^{\mathrm{Sus}}(X) := H_i(C_*(X)(\mathrm{Spec}\, k)).$$

Theorem 6.4. *For $X \in \mathbf{Sm}/k$, there is a canonical isomorphism*

$$H_i^{\mathrm{Sus}}(X) \cong \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(\mathbb{Z}[i], M_{\mathrm{gm}}(X)).$$

Proof. Here $\mathbb{Z} := M_{\mathrm{gm}}(\mathrm{Spec}\, k)$. This follows directly from the embedding theorem. $\square$

Corollary 6.5. *Let U, V be open subschemes of $X \in \mathbf{Sm}/k$. Then there is a long exact Mayer-Vietoris sequence*

$$\begin{aligned}\ldots \to H_{n+1}^{\mathrm{Sus}}(U \cup V) \to H_n^{\mathrm{Sus}}(U \cap V)&\\ \to H_n^{\mathrm{Sus}}(U) \oplus H_n^{\mathrm{Sus}}(V) &\to H_n^{\mathrm{Sus}}(U \cup V) \to \ldots\end{aligned}$$

Proof. By the previous theorem, we have

$$H_n^{\mathrm{Sus}}(Y) = \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(\mathbb{Z}[n], M_{\mathrm{gm}}(Y)).$$

for all $Y \in \mathbf{Sm}/k$, $n \in \mathbb{Z}$. Also,

$$M_{\mathrm{gm}}(U \cap V) \to M_{\mathrm{gm}}(U) \oplus M_{\mathrm{gm}}(V) \to M_{\mathrm{gm}}(U \cup V)$$

extends to a distinguished triangle in $DM_-^{\mathrm{eff}}(k)$, so the result follows by applying $\mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(\mathbb{Z}, -)$ to this distinguished triangle. $\square$

Lecture 2. Motives and cycle complexes

In this second lecture, we examine the category $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ in more detail. The basic structures we expect from cohomology will all be reflected in structures in the category $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$; the examination of these structures will occupy the first part of the lecture. The second part is concerned with giving in some sense a computation of the cohomology theory arising from the category $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$. The computation is achieved through the use of various cycle complexes, which we describe in the second half; the proofs of some of the basic results on cycle complexes will be put off until Lecture 4. The cycle complexes and their properties are used to prove the cancellation theorem: that $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)$ is an embedding. We conclude with a discusion of duality in $DM_{\mathrm{gm}}(k)$.

The material in this lecture is taken mainly from [63].

7. Basic structures in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$

We discuss motivic cohomology, the projective bundle formula and the Gysin isomorphism, realizing these as morphisms and isomorphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}$.

7.1. **Motivic cohomology.** We have already seen Suslin homology as

$$H_i^{\mathrm{Sus}}(X) := H_i(C_*(X)(\operatorname{Spec} k)) \cong \operatorname{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(\mathbb{Z}[i], M_{\mathrm{gm}}(X)).$$

The twisted contravariant version is *motivic cohomology*:

Definition 7.1. For $X \in \mathbf{Sm}/k$, $q \geq 0$, set

$$H^p(X, \mathbb{Z}(q)) := \operatorname{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(X), \mathbb{Z}(q)[p]).$$

Motivic cohomology has products: Define the cup product

$$H^p(X, \mathbb{Z}(q)) \otimes H^{p'}(X, \mathbb{Z}(q')) \to H^{p+p'}(X, \mathbb{Z}(q+q'))$$

by sending $a \otimes b$ to

$$M_{\mathrm{gm}}(X) \xrightarrow{\delta} M_{\mathrm{gm}}(X) \otimes M_{\mathrm{gm}}(X) \xrightarrow{a \otimes b} \mathbb{Z}(q)[p] \otimes \mathbb{Z}(q')[p'] \cong \mathbb{Z}(q+q')[p+p'].$$

This makes $\oplus_{p,q} H^p(X, \mathbb{Z}(q))$ a graded commutative ring with unit 1 the map $M_{\mathrm{gm}}(X) \to \mathbb{Z}$ induced by $p_X : X \to \operatorname{Spec} k$.

7.2. **Homotopy and Mayer-Vietoris.** Applying $\operatorname{Hom}_{DM_{\mathrm{gm}}}(-, \mathbb{Z}(q)[p])$ to the isomorphism $p_* : M_{\mathrm{gm}}(X \times \mathbb{A}^1) \to M_{\mathrm{gm}}(X)$ gives the *homotopy property* for $H^*(-, \mathbb{Z}(*))$:

$$p^* : H^p(X, \mathbb{Z}(q)) \xrightarrow{\sim} H^p(X \times \mathbb{A}^1, \mathbb{Z}(q)).$$

For $U, V \subset X$ open subschemes, applying $\operatorname{Hom}_{DM_{\mathrm{gm}}}(-, \mathbb{Z}(q)[p])$ to the distinguished triangle

$$M_{\mathrm{gm}}(U \cap V) \to M_{\mathrm{gm}}(U) \oplus M_{\mathrm{gm}}(V) \to M_{\mathrm{gm}}(U \cup V) \to M_{\mathrm{gm}}(U \cap V)[1]$$

gives the *Mayer-Vietoris exact sequence* for $H^*(-, \mathbb{Z}(*))$:

$$\begin{aligned} \ldots \to H^{p-1}(U \cap V, \mathbb{Z}(q)) &\to H^p(U \cup V, \mathbb{Z}(q)) \\ &\to H^p(U, \mathbb{Z}(q)) \oplus H^p(V, \mathbb{Z}(q)) \to H^p(U \cap V, \mathbb{Z}(q)) \to \ldots \end{aligned}$$

7.3. **Weight one motivic cohomology.** The motivic cohomology in weight 1, $H^*(-,\mathbb{Z}(1)$ recovers some well-known invariants.

$\mathbb{Z}(1)[2]$ is the reduced motive of $\mathbb{P}^1$, and $M_{\mathrm{gm}}(\mathbb{P}^1)$ is represented in DM^{eff}_- by the Suslin complex $C_*(\mathbb{P}^1)$. The homology sheaves of $C_*(\mathbb{P}^1)$ and $C_*(\operatorname{Spec} k)$ are given by:

Lemma 7.2. *$h_0^{\mathrm{Zar}}(\mathbb{P}^1)=\mathbb{Z}$, $h_1^{\mathrm{Zar}}(\mathbb{P}^1)=\mathbb{G}_m$ and $h_n^{\mathrm{Zar}}(\mathbb{P}^1)=0$ for $n\geq 2$.*
$h_0^{\mathrm{Zar}}(\operatorname{Spec} k)=\mathbb{Z}$, $h_n^{\mathrm{Zar}}(\operatorname{Spec} k)=0$ for $n\geq 1$.

For $\operatorname{Spec} k$, this is an easy computation, we will prove this for $\mathbb{P}^1$ later.

This computation implies that $\mathbb{Z}(1)\cong\mathbb{G}_m[-1]$ in $DM^{\mathrm{eff}}_-(k)$. Indeed:

$$\mathbb{Z}(1)[2]\cong \operatorname{Cone}(C_*(\mathbb{P}^1)\to C_*(\operatorname{Spec} k))[-1]\cong h_1^{\mathrm{Zar}}(\mathbb{P}^1)[1]=\mathbb{G}_m[1]$$

This yields:

Proposition 7.3. *For $X\in\mathbf{Sm}/k$, we have*

$$H^n(X,\mathbb{Z}(1))=\begin{cases} H^0_{\mathrm{Zar}}(X,\mathcal{O}_X^*) & \text{for } n=1\\ \operatorname{Pic}(X):=H^1_{\mathrm{Zar}}(X,\mathcal{O}_X^*) & \text{for } n=2\\ 0 & \text{else.}\end{cases}$$

Proof. Since $\mathbb{Z}(1)\cong\mathbb{G}_m[-1]$ in $DM^{\mathrm{eff}}_-(k)$, the embedding theorem of Lecture 1 gives:

$$\begin{aligned}\operatorname{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}}(M_{\mathrm{gm}}(X),\mathbb{Z}(1)[n])&\cong\mathbb{H}^n_{\mathrm{Nis}}(X,\mathbb{Z}(1))\\ &\cong\mathbb{H}^n_{\mathrm{Zar}}(X,\mathbb{Z}(1))\cong H^{n-1}_{\mathrm{Zar}}(X,\mathbb{G}_m).\end{aligned}$$

□

7.4. **Chern classes of line bundles.**

Definition 7.4. Let $L\to X$ be a line bundle on $X\in\mathbf{Sm}_k$. We let $c_1(L)\in H^2(X,\mathbb{Z}(1))$ be the element corresponding to $[L]\in H^1_{\mathrm{Zar}}(X,\mathcal{O}_X^*)$.

7.5. **Projective bundle formula.** Let $E\to X$ be a rank $n+1$ vector bundle over $X\in\mathbf{Sm}/k$, $q:\mathbb{P}(E)\to X$ the resulting $\mathbb{P}^n$ bundle, $\mathcal{O}(1)$ the tautological quotient bundle.

Define $\alpha_j: M_{\mathrm{gm}}(\mathbb{P}(E))\to M_{\mathrm{gm}}(X)(j)[2j]$ by

$$M_{\mathrm{gm}}(\mathbb{P}(E))\xrightarrow{\delta} M_{\mathrm{gm}}(\mathbb{P}(E))\otimes M_{\mathrm{gm}}(\mathbb{P}(E))\xrightarrow{q\otimes c_1(\mathcal{O}(1))^j} M_{\mathrm{gm}}(X)(j)[2j]$$

Theorem 7.5 (Projective bundle formula)**.** *The map*

$$\alpha_E:=\oplus_{j=0}^n\alpha_j: M_{\mathrm{gm}}(\mathbb{P}(E))\to\oplus_{j=0}^n M_{\mathrm{gm}}(X)(j)[2j]$$

is an isomorphism.

The proof of the projective bundle formula requires the following computation:

Lemma 7.6. *There is a canonical isomorphism*

$$M_{\mathrm{gm}}(\mathbb{A}^n\setminus 0)\to\mathbb{Z}(n)[2n-1]\oplus\mathbb{Z}.$$

Proof. For $n=1$, we have $M_{\mathrm{gm}}(\mathbb{P}^1)=\mathbb{Z}\oplus\mathbb{Z}(1)[2]$, by definition of $\mathbb{Z}(1)$. The Mayer-Vietoris distinguished triangle

$$M_{\mathrm{gm}}(\mathbb{A}^1\setminus 0)\to M_{\mathrm{gm}}(\mathbb{A}^1)\oplus M_{\mathrm{gm}}(\mathbb{A}^1)\to M_{\mathrm{gm}}(\mathbb{P}^1)\to M_{\mathrm{gm}}(\mathbb{A}^1\setminus 0)[1]$$

defines an isomorphism $t : M_{\mathrm{gm}}(\mathbb{A}^1 \setminus 0) \to \mathbb{Z}(1)[1] \oplus \mathbb{Z}$.

For general n, write $\mathbb{A}^n \setminus 0 = \mathbb{A}^n \setminus \mathbb{A}^{n-1} \cup \mathbb{A}^n \setminus \mathbb{A}^1$. By induction, Mayer-Vietoris and homotopy invariance, this gives the distinguished triangle

$$\begin{aligned}(\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z})\\ \to (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \oplus (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z}) \to M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0)\\ \to (\mathbb{Z}(1)[1] \oplus \mathbb{Z}) \otimes (\mathbb{Z}(n-1)[2n-3] \oplus \mathbb{Z})[1]\end{aligned}$$

yielding the result. □

Proof of the projective bundle formula. The map α_E is natural in X, E. Mayer-Vietoris reduces to the case of a trivial bundle, then to the case $X = \operatorname{Spec} k$, so we need to prove:

Lemma 7.7. *$\oplus_{j=0}^n \alpha_j : M_{\mathrm{gm}}(\mathbb{P}^n) \to \oplus_{j=0}^n \mathbb{Z}(j)[2j]$ is an isomorphism.*

Proof. Write $\mathbb{P}^n = \mathbb{A}^n \cup (\mathbb{P}^n \setminus 0)$. $M_{\mathrm{gm}}(\mathbb{A}^n) = \mathbb{Z}$. $\mathbb{P}^n \setminus 0$ is an $\mathbb{A}^1$ bundle over $\mathbb{P}^{n-1}$, so induction gives

$$M_{\mathrm{gm}}(\mathbb{P}^n \setminus 0) = \oplus_{j=0}^{n-1} \mathbb{Z}(j)[2j].$$

By our computation, we have $M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0) = \mathbb{Z}(n)[2n-1] \oplus \mathbb{Z}$.

The Mayer-Vietoris distinguished triangle

$$M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0) \to M_{\mathrm{gm}}(\mathbb{A}^n) \oplus M_{\mathrm{gm}}(\mathbb{P}^n \setminus 0) \to M_{\mathrm{gm}}(\mathbb{P}^n) \to M_{\mathrm{gm}}(\mathbb{A}^n \setminus 0)[1]$$

gives the result. □

□

7.6. **The Gysin isomorphism.**

Definition 7.8. For $i : Z \to X$ a closed embedding in $\mathbf{Sm}/k$, let $M_{\mathrm{gm}}(X/X \setminus Z) \in DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ be the image in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ of the complex $[X \setminus Z] \xrightarrow{j} [X]$, with $[X]$ in degree 0.

Remark 7.9. The Mayer-Vietoris property for $M_{\mathrm{gm}}(-)$ yields a Zariski excision property: If Z is closed in U, with U an open subscheme of X, then

$$M_{\mathrm{gm}}(U/U \setminus Z) \to M_{\mathrm{gm}}(X/X \setminus Z)$$

is an isomorphism.

In fact, Voevodsky's moving lemma shows that $M_{\mathrm{gm}}(X/X \setminus Z)$ depends only on the Nisnevich neighborhood of Z in X: this is the *Nisnevich excision* property.

Theorem 7.10 (Gysin isomorphism)**.** *Let $i : Z \to X$ be a closed embedding in $\mathbf{Sm}/k$ of codimension n. Then there is a natural isomorphism in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$*

$$M_{\mathrm{gm}}(X/X \setminus Z) \cong M_{\mathrm{gm}}(Z)(n)[2n].$$

We first prove

Lemma 7.11. *Let $E \to Z$ be a vector bundle of rank n with zero section s. Then $M_{\mathrm{gm}}(E/E \setminus s(Z)) \cong M_{\mathrm{gm}}(Z)(n)[2n]$.*

Proof. Since $M_{\mathrm{gm}}(E) \to M_{\mathrm{gm}}(Z)$ is an isomorphism by homotopy, we need to show

$$M_{\mathrm{gm}}(E \setminus s(Z)) \cong M_{\mathrm{gm}}(Z) \oplus M_{\mathrm{gm}}(Z)(n)[2n-1].$$

Let $\mathbb{P} := \mathbb{P}(E \oplus \mathcal{O}_Z)$, and write $\mathbb{P} = E \cup (\mathbb{P} \setminus s(Z))$. Mayer-Vietoris gives the distinguished triangle

$$M_{\mathrm{gm}}(E \setminus s(Z)) \to M_{\mathrm{gm}}(E) \oplus M_{\mathrm{gm}}(\mathbb{P} \setminus s(Z)) \to M_{\mathrm{gm}}(\mathbb{P}) \to M_{\mathrm{gm}}(E \setminus s(Z))[1]$$

Since $\mathbb{P} \setminus s(Z) \to \mathbb{P}(E)$ is an $\mathbb{A}^1$ bundle, the projective bundle formula gives the isomorphism we wanted. □

The general case uses the deformation to the normal bundle to reduce to the case of the zero-section of a vector bundle.

7.7. **Deformation to the normal bundle.** For $i : Z \to X$ a closed immersion in $\mathbf{Sm}/k$, let $p : (X \times \mathbb{A}^1)_{Z \times 0} \to X \times \mathbb{A}^1$ be the blow-up of $X \times \mathbb{A}^1$ along $Z \times 0$. Let $N_{Z/X}$ denote the normal bundle of Z in X. Letting X_Z denote the blow-up of X along Z, the fiber of $(X \times \mathbb{A}^1)_{Z \times 0}$ over $0 \in \mathbb{A}^1$ is the union of X_Z with the exceptional divisor $\mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z)$, with $\mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z)$ and X_Z meeting transversely along $\mathbb{P}(N_{Z/X})$, this latter projective bundle being the exceptional divisor of the blow-up $X_Z \to X$. In addition, this shows that $X_Z \subset (X \times \mathbb{A}^1)_{Z \times 0}$ is the same as the proper transform $p^{-1}[X \times 0]$.

Set

$$Def(i) := (X \times \mathbb{A}^1)_{Z \times 0} \setminus X_Z,$$

As p maps the proper transform $p^{-1}[Z \times \mathbb{A}^1]$ isomorphically onto $Z \times \mathbb{A}^1$ and is disjoint from $X_Z = p^{-1}[X \times 0]$, we have the section $\tilde{i} : Z \times \mathbb{A}^1 \to Def(i)$ to p over $Z \times \mathbb{A}^1$. Let $q : Def(i) \to \mathbb{A}^1$ be the map induced by the projection $X \times \mathbb{A}^1 \to \mathbb{A}^1$.

The fiber $q^{-1}(1)$ is just X, and the fiber $\tilde{i}_1$ is original embedding $i : Z \to X$. More interestingly, the fiber $q^{-1}(0)$ is $\mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z) \setminus \mathbb{P}(N_{Z/X})$, which is just the normal bundle $N_{Z/X}$, and $\tilde{i}_0$ is the zero section $s : Z \to N_{Z/X}$. Thus, the deformation space $Def(i)$ deforms the embedding i to the zero section of the normal bundle.

Lemma 7.12. *The maps*

$$M_{\mathrm{gm}}(N_{Z/X}/(N_{Z/X} \setminus s(Z))) \to M_{\mathrm{gm}}(Def(i) \setminus Z \times \mathbb{A}^1) \leftarrow M_{\mathrm{gm}}(X/(X \setminus Z))$$

are isomorphisms.

Sketch of proof. The proof uses the fact that a closed codimension d embedding $i : Z \to X$ in $\mathbf{Sm}/k$ is, locally in the Nisnevich topology, isomorphic to a closed embedding of the form $Z \xrightarrow{s_0} Z \times \mathbb{A}^d$, with s_0 the 0-section. One needs to make this more explicit, but in the end, using Nisnevich excision, we reduce to this case. But then $\tilde{i}_0 : Z \times \mathbb{A}^1 \to Def(i)$ is is the 0-section $Z \times \mathbb{A}^1 \xrightarrow{s_0 \times \mathrm{id}} Z \times \mathbb{A}^d \times \mathbb{A}^1$ and $\tilde{i}_1 : Z \times \mathbb{A}^1 \to Def(i)$ is the 1-section $Z \times \mathbb{A}^1 \xrightarrow{s_1 \times \mathrm{id}} Z \times \mathbb{A}^d \times \mathbb{A}^1$, whence the result. □

The proof of the Gysin isomorphism theorem is now immediate from the two lemmas:

$$M_{\mathrm{gm}}(X/(X \setminus Z)) \cong M_{\mathrm{gm}}(N_{Z/X}/(N_{Z/X} \setminus s(Z))) \cong M_{\mathrm{gm}}(Z)(n)[2n]$$

7.8. **Gysin distinguished triangle.**

Theorem 7.13. *Let $i : Z \to X$ be a codimension n closed immersion in $\mathbf{Sm}/k$ with open complement $j : U \to X$. There is a canonical distinguished triangle in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$:*

$$M_{\mathrm{gm}}(U) \xrightarrow{j_*} M_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(Z)(n)[2n] \to M_{\mathrm{gm}}(U)[1]$$

Proof. By definition of $M_{\mathrm{gm}}(X/U)$, we have the canonical distinguished triangle in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$:

$$M_{\mathrm{gm}}(U) \xrightarrow{j_*} M_{\mathrm{gm}}(X) \to M_{\mathrm{gm}}(X/U) \to M_{\mathrm{gm}}(U)[1]$$

then insert the Gysin isomorphism $M_{\mathrm{gm}}(X/U) \cong M_{\mathrm{gm}}(Z)(n)[2n]$. □

Applying $\mathrm{Hom}_{DM_{\mathrm{gm}}}(-, \mathbb{Z}(q)[p])$ to the Gysin distinguished triangle yields the long exact Gysin sequence for motivic cohomology:

$$\begin{aligned} \ldots \to H^{p-1}(U, \mathbb{Z}(q)) &\xrightarrow{\partial} H^{p-2n}(Z, \mathbb{Z}(q-n)) \\ &\xrightarrow{i_*} H^p(X, \mathbb{Z}(q)) \xrightarrow{j^*} H^p(U, \mathbb{Z}(q)) \to \ldots \end{aligned}$$

8. Cycle complexes and bivariant cycle cohomology

We introduce various cycle complexes and describe their main properties. We will discuss the proofs of these in detail in Lecture 4. Our ultimate goal is to describe the morphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ using algebraic cycles, more precisely, as the homology of a cycle complex.

8.1. **Bloch's cycle complex.** This is historically the first cycle complex that arose as a candidate for motivic cohomology, constructed by Bloch in [8]

A *face* of $\Delta^n := \mathrm{Spec}\, k[t_0, \ldots, t_n]/\sum_i t_i - 1$ is a closed subset defined by $t_{i_1} = \ldots = t_{i_s} = 0$.

Definition 8.1. Take $X \in \mathbf{Sch}_k$. $z_r(X, n) \subset z_{r+n}(X \times \Delta^n)$ is the subgroup generated by the closed integral $W \subset X \times \Delta^n$ of dimension $r + n$ such that, for each face $F \subset \Delta^n$, either $W \cap X \times F = \emptyset$ or

$$\dim W \cap X \times F = r + \dim F.$$

If X is equi-dimensional over k of dimension d, set

$$z^q(X, n) := z_{d-q}(X, n).$$

Let $\delta^n_i : \Delta^n \to \Delta^{n+1}$ be the inclusion to the face $t_i = 0$. The cycle pull-back δ^{n*}_i is a well-defined map

$$\delta^{n*}_i : z_r(X, n+1) \to z_r(X, n).$$

Definition 8.2. Bloch's cycle complex $z_r(X, *)$ is $z_r(X, n)$ in degree n, with differential

$$d_n := \sum_{i=0}^{n+1} (-1)^i \delta^{n*}_i : z_r(X, n+1) \to z_r(X, n).$$

Bloch's *higher Chow groups* are

$$\mathrm{CH}_r(X, n) := H_n(z_r(X, *)).$$

For X equi-dimensional of dimension d over k, we have the complex $z^q(X, *) := z_{d-q}(X, *)$ and the higher Chow groups $\mathrm{CH}^q(X, n) = \mathrm{CH}_{d-q}(X, n)$.

Remark 8.3. A problem with functoriality: Even for $X \in \mathbf{Sm}/k$, the complex $z^q(X,*)$ is only functorial for *flat* maps, and covariantly functorial for proper maps (with a shift in q). This complex is NOT a complex of PST's.

We will see later that this is corrected in the derived category by a version of Chow's moving lemma.

8.2. **Properties of the higher Chow groups.**
(0) **Products**
There is an external product $z^q(X,*)\otimes z^{q'}(Y,*) \to z^{q+q'}(X\times_k Y,*)$ (in the derived category), induced by taking products of cycles. For X smooth, this induces a cup product, using δ_X^*.

(1) **Homotopy**
$p^* : z_r(X,*) \to z_{r+1}(X\times\mathbb{A}^1,*)$ is a quasi-isomorphism for $X \in \mathbf{Sch}_k$.

(2) **Localization and Mayer-Vietoris**
For $X \in \mathbf{Sch}_k$, let $i : W \to X$ be a closed subset with complement $j : U \to X$. Then

$$z_r(W,*) \xrightarrow{i_*} z_r(X,*) \xrightarrow{j^*} z_r(U,*)$$

canonically extends to a distinguished triangle in $D^-(\mathbf{Ab})$. Similarly, if $X = U\cup V$, U, V open in X, the sequence

$$z_r(X,*) \to z_r(U,*) \oplus z_r(V,*) \to z_r(U\cap V,*)$$

canonically extends to a distinguished triangle in $D^-(\mathbf{Ab})$.

(3) ***K*-theory**
For X regular, there is a functorial *Chern character isomorphism*

$$ch : K_n(X)_\mathbb{Q} \to \oplus_q \mathrm{CH}^q(X,n)_\mathbb{Q}$$

identifying $\mathrm{CH}^q(X,n)_\mathbb{Q}$ with the weight q eigenspace $K_n(X)^{(q)}$ for the Adams operations.

(4) **Classical Chow groups**
$\mathrm{CH}^n(X,0) = \mathrm{CH}^n(X) := z^n(X)/$ rational equivalence.

(5) **Weight one**
For $X \in \mathbf{Sm}/k$, $\mathrm{CH}^1(X,1) = H^0(X,\mathcal{O}_X^*)$, $\mathrm{CH}^1(X,0) = H^1(X,\mathcal{O}_X^*) = \mathrm{Pic}(X)$, $\mathrm{CH}^1(X,n) = 0$ for $n > 1$.

(6) **Chern classes of line bundles**
Using (4) or (5), we have $c_1(L) \in \mathrm{CH}^1(X,0)$ for each line bundle $L \to X$.

(7) **Projective bundle formula**
Let $E \to X$ be a vector bundle of rank $n+1$, $q : \mathbb{P}(E) \to X$ the $\mathbb{P}^n$ bundle of E, $\xi := c_1(\mathcal{O}(1))$. Then $\mathrm{CH}^*(\mathbb{P}(E),*)$ is a free $\mathrm{CH}^*(X,*)$ module with basis $1,\xi,\dots,\xi^n$.

The proofs of the functoriality and localization properties use two different types of *moving lemmas*, to be discussed in Lecture 4.

8.3. **Equi-dimensional cycles.** The use of equi-dimensional cycles allows us to form a cycle complex which is a complex of PSTs.

Definition 8.4. Fix $X \in \mathbf{Sch}_k$. For $U \in \mathbf{Sm}/k$ let $z_r^{\text{equi}}(X)(U) \subset z(X \times U)$ be the subgroup generated by the closed integral $W \subset X \times U$ such that $W \to U$ is dominant and equi-dimensional with fibers of dimension r (or empty) over some component of U.

Remark 8.5. The standard formula for composition of correspondences makes $z_r^{\text{equi}}(X)$ a PST; in fact $z_r^{\text{equi}}(X)$ is a Nisnevich sheaf with transfers.

Definition 8.6. The complex of *equi-dimensional cycles* is

$$z_r^{\text{equi}}(X,*) := C_*(z_r^{\text{equi}}(X))(\operatorname{Spec} k).$$

Explicitly: $z_r^{\text{equi}}(X,n)$ is the subgroup of $z_{r+n}(X \times \Delta^n)$ generated by irreducible W such that $W \to \Delta^n$ is equi-dimensional with fiber dimension r. Thus:

There is a natural inclusion

$$z_r^{\text{equi}}(X,*) \to z_r(X,*).$$

8.4. **The cdh topology.** The equivariant cycle complexes form the basis for a reasonable cohomology theory on $\mathbf{Sm}/k$. This is formed by combining the Suslin complex construction with hypercohomology in a new Grothendieck topology: the cdh topology.

Definition 8.7. The cdh site is given by the pre-topology on $\mathbf{Sch}_k$ with covering families generated by

1. Nisnevich covers

2. Maps of the form $p \amalg i : Y \amalg F \to X$, where $i : F \to X$ is a closed immersion, $p : Y \to X$ is proper, and $p : (Y \setminus p^{-1}F)_{red} \to (X \setminus F)_{red}$ is an isomorphism (the map p is called an *abstract blow-up*).

We call a cdh-cover of the form $Y \amalg F \to X$ as in (2) an *abstract blow-up cover*, a cdh-cover $p : Y \to X$ with p proper is called a *proper* cdh-cover.

The inclusion $\pi : \mathbf{Sm}/k \to \mathbf{Sch}_k$ induced the restriction functor

$$\pi_* : \mathrm{Sh}^{\mathrm{cdh}}(\mathbf{Sch}_k) \to \mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}/k),$$

which has the left adjoint

$$\pi^* : \mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}/k) \to \mathrm{Sh}^{\mathrm{cdh}}(\mathbf{Sch}_k).$$

Explicitly, $\pi^*\mathcal{F}$ is the cdh-sheafification of the presheaf

$$U \in \mathbf{Sch}_k \mapsto \operatorname{colim}_{U \to V \in U\backslash \mathbf{Sm}/k} \mathcal{F}(V)$$

where $U\backslash\mathbf{Sm}/k$ is the category of morphisms $f : U \to V$ in $\mathbf{Sch}_k$, $V \in \mathbf{Sm}/k$. For F a presheaf on $\mathbf{Sm}/k$, we define the cdh-sheafification F_{cdh} as $\pi^*(F_{\mathrm{Nis}})$.

Remark 8.8. If k admits resolution of singularities (for finite type k-schemes and for abstract blow-ups to smooth k-schemes), then each cdh cover admits a refinement consisting of *smooth* k-schemes; in particular, a cdh-sheaf on $\mathbf{Sch}_k$ is determined by its restriction to $\mathbf{Sm}/k$.

At first, it seems difficult to get a hold on covers in the cdh topology, as a general cover is formed by a sequence of maps, alternating between blow-up covers and Nisnevich covers. In order to use the cdh topology effectively, we need a basic result:

Lemma 8.9. *Let X be in $\mathbf{Sch}/k$. The* cdh*-covers of the form $U \to X$ which factor as $U \xrightarrow{q} Y \xrightarrow{p} X$, with $q : U \to Y$ a Nisnevich cover and $p : Y \to X$ is a proper* cdh*-cover are cofinal among all* cdh *covers of X. In addition, we may take $p : Y \to X$ so that Y is reduced and has an open and closed subscheme Y' such that $Y' \to X_{red}$ is an abstract blow-up.*

Proof. This follows from [43, example 12.24, example 12.25 and proposition 12.28] and is hinted at in [63, lemma 3.3, chap. 4]. We give a sketch here.

The main point is that given a Nisnevich cover $V \to X$ and a blow-up cover $V' \amalg F' \to V$, one can refine the composition $V' \amalg F' \to X$ to a composition of a blow-up cover $X' \amalg F \to X$ with a Nisnevich cover $V'' \to X' \amalg F$. For this, one uses "platification by abstract blow-ups" [25] to find an abstract blow up $X' \to X$ such that $V \times_X X' \to X$ factors through $V' \to X$. We then take $V'' := V \times_X (X' \amalg F)$.

Using this, one can rearrange an arbitrary sequence of maps alternating between Nisnevich covers and blow-up covers to one of the desired form, proving the first assertion; we may also replace Y with the disjoint union of the reduced, irreducible components of Y, so we may assume Y is reduced and each connected component of Y is irreducible.

For the second assertion, note that every cdh-cover $U \to X$ has the property that each point $x \in X$ lifts to a point $u \in U$ with $k(x) \cong k(u)$. Thus, if we have a proper cdh cover $Y \to X$, we may lift each generic point of X to a point of Y and take the closure, giving us a closed subscheme Y' of Y which maps properly and birationally to X_{red}. As we have assumed Y is a disjoint union of its irreducible components, Y' is both open and closed in Y, completing the proof. □

8.5. **Bivariant cycle cohomology.**

Definition 8.10. Take $X, Y \in \mathbf{Sch}_k$. The *bivariant cycle cohomology* of Y with coefficients in cycles on X are

$$A_{r,i}(Y, X) := \mathbb{H}^{-i}(Y_{\mathrm{cdh}}, C_*(z_r^{\mathrm{equi}}(X))_{\mathrm{cdh}}).$$

$A_{r,i}(Y, X)$ is contravariant in Y for arbitrary maps, and covariant in X for proper maps. We have the natural map

$$h_i(z_r^{\mathrm{equi}}(X))(Y) := H_i(C_*(z_r^{\mathrm{equi}}(X))(Y)) \to A_{r,i}(Y, X).$$

8.6. **Mayer-Vietoris and blow-up sequences.** Since Zariski open covers and abstract blow-ups are covering families in the cdh topology, we have a Mayer-Vietoris sequence for $U, V \subset Y$:

$$\ldots \to A_{r,i}(U \cup V, X) \to A_{r,i}(U, X) \oplus A_{r,i}(V, X) \\ \to A_{r,i}(U \cap V, X) \to A_{r,i-1}(U \cup V, X) \to \ldots$$

and for $p \amalg i : Y' \amalg F \to Y$:

$$\ldots \to A_{r,i}(Y, X) \to A_{r,i}(Y', X) \oplus A_{r,i}(F, X) \\ \to A_{r,i}(p^{-1}(F), X) \to A_{r,i-1}(Y, X) \to \ldots$$

Additional properties of $A_{r,i}$ require some fundamental results on the behavior of homotopy invariant PST's with respect to cdh-sheafification. Additionally, we will need some purely geometric results comparing different cycle complexes. These two types of results are:

1. Acyclicity theorems. We have already seen the Nisnevich acyclicity theorem, one version of which states that for a homotopy invariant PST F with $F_{\mathrm{Nis}} = 0$, the Suslin complex $C_*(F)_{\mathrm{Zar}}$ is acyclic (corollary 5.13).

We will also need the cdh version: Assume that k admits resolution of singularities. For F a homotopy invariant PST with $F_{\mathrm{cdh}} = 0$, the Suslin complex $C_*(F)_{\mathrm{Zar}}$ is acyclic.

These two results construct for us distinguished triangles by applying $C_*(-)_{\mathrm{Zar}}$ to a sequence of homotopy invariant PST's, which becomes short exact after cdh-sheafification.

Using a hypercovering argument, they also show that cdh, Nis and Zar cohomology of a homotopy invariant PST all agree on smooth varieties.

We will prove the cdh acyclicity theorem and derive its important consequences later on in this lecture (see theorem 9.4). The second type of result we need is

2. Moving lemmas. The bivariant cohomology $A_{r,i}$ is defined using cdh-hypercohomology of z_r^{equi}, so comparing z_r^{equi} with other complexes, such as Bloch's cycle complex, leads to identification of $A_{r,i}$ with cdh-hypercohomology of the other complexes. The comparision of z_r^{equi} with other complexes is based partly on geometric constructions. These moving lemmas will be discussed in Lecture 4.

8.7. **Homotopy.** Bivariant cycle homolopy is homotopy invariant:

Proposition 8.11. *Suppose k admits resolution of singularities. Then the pull-back map*

$$p^* : A_{r,i}(Y, X) \to A_{r,i}(Y \times \mathbb{A}^1, X)$$

is an isomorphism.

Proof. Using hypercovers and resolution of singularities, we reduce to the case of smooth Y. As mentioned above, the cdh-acyclicity theorem changes the cdh hypercohomology defining $A_{r,i}$ to Nisnevich hypercohomology. Since the homology presheaves of $C_*(z_r^{\mathrm{equi}}(X))$ are homotopy invariant PST's, we can use Voevodsky's PST theorem to conclude the homotopy invariance. □

8.8. **The geometric comparison theorem.**

Theorem 8.12 (Geometric comparison)**.** *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$. Then the natural map $z^{\mathrm{equi}}(X, *) \to z_r(X, *)$ is a quasi-isomorphism.*

The proof of this result is based on *Suslin's moving lemma.* We will discuss the details of the proof in Lecture 4.

8.9. **The geometric duality theorem.** Let $z_r^{\mathrm{equi}}(U,X) := \mathcal{H}om(L(U), z_r^{\mathrm{equi}}(X))$. Explicitly:

$$z_r^{\mathrm{equi}}(U,X)(V) = z_r^{\mathrm{equi}}(X)(U \times V).$$

We have the inclusion $z_r^{\mathrm{equi}}(U,X) \to z_{r+\dim U}^{\mathrm{equi}}(X \times U)$.

Theorem 8.13 (Geometric duality)**.** *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$, $U \in \mathbf{Sm}/k$, quasi-projective of dimension n. Then the inclusion $z_r^{\mathrm{equi}}(U,X) \to z_{r+n}^{\mathrm{equi}}(X \times U)$ induces a quasi-isomorphism of complexes on $\mathbf{Sm}/k_{\mathrm{Zar}}$:*

$$C_*(z_r^{\mathrm{equi}}(U,X))_{\mathrm{Zar}} \to C_*(z_{r+n}^{\mathrm{equi}}(X \times U))_{\mathrm{Zar}}$$

The proof for U and X smooth and projective uses the Friedlander-Lawson moving lemma. The extension to U smooth quasi-projective, and X general uses the cdh-acyclicity theorem. We will discuss the details of the proof in Lecture 4.

8.10. **The** cdh **comparison and duality theorems.**

Theorem 8.14 (cdh comparison)**.** *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$. Then for U smooth and quasi-projective, the natural map*

$$h_i(z_r^{\mathrm{equi}}(X))(U) \to A_{r,i}(U,X)$$

is an isomorphism.

Theorem 8.15 (cdh duality)**.** *Suppose k admits resolution of singularities. Take $X, Y \in \mathbf{Sch}_k$, $U \in \mathbf{Sm}/k$ of dimension n. There is a canonical isomorphism*

$$A_{r,i}(Y \times U, X) \to A_{r+n,i}(Y, X \times U).$$

To prove the cdh comparison theorem, first use the cdh-acyclicity theorem to identify

$$\mathbb{H}^{-i}_{\mathrm{Zar}}(U, z_r^{\mathrm{equi}}(X)) \xrightarrow{\sim} \mathbb{H}^{-i}_{\mathrm{cdh}}(U, z_r^{\mathrm{equi}}(X)) =: A_{r,i}(U,X)$$

Next, if V_1, V_2 are Zariski open in U, use the geometric duality theorem to identify the Mayer-Vietoris sequence

$$C_*(z_r^{\mathrm{equi}}(X))(V_1 \cup V_2) \to C_*(z_r^{\mathrm{equi}}(X))(V_1) \oplus C_*(z_r^{\mathrm{equi}}(X))(V_2) \to C_*(z_r^{\mathrm{equi}}(X))(V_1 \cap V_2)$$

with

(8.10.1)

$$0 \to C_*(z_{r+n}^{\mathrm{equi}}(X \times (V_1 \cup V_2)) \to C_*(z_{r+n}^{\mathrm{equi}}(X \times V_1)) \oplus C_*(z_{r+n}^{\mathrm{equi}}(X \times V_2)) \to C_*(z_{r+n}^{\mathrm{equi}}(X \times (V_1 \cap V_2))$$

evaluated at $\operatorname{Spec} k$, $n = \dim_k U$.

But the presheaf sequence (8.10.1) is exact, and $coker_{\mathrm{cdh}} = 0$. The cdh-acyclicity theorem thus gives us the distinguished triangle

$$C_*(z_{r+n}^{\mathrm{equi}}(X \times (V_1 \cup V_2))_{\mathrm{Zar}} \to C_*(z_{r+n}^{\mathrm{equi}}(X \times V_1))_{\mathrm{Zar}} \oplus C_*(z_{r+n}^{\mathrm{equi}}(X \times V_2))_{\mathrm{Zar}} \to C_*(z_{r+n}^{\mathrm{equi}}(X \times (V_1 \cap V_2))_{\mathrm{Zar}} \to$$

Evaluating at $\operatorname{Spec} k$, we find that our original Mayer-Vietoris sequence for $C_*(z_r^{\mathrm{equi}}(X))$ was in fact a distinguished triangle.

Via the Brown-Gersten theorem [14, theorem 1'] (see also [60, exercise 2.5]) the Mayer-Vietoris property for $C_*(z_r^{\mathrm{equi}}(X))$ implies that the canonical map

$$h_i(C_*(z_r^{\mathrm{equi}}(X)))(U) \to \mathbb{H}^{-i}_{\mathrm{Zar}}(U, C_*(z_r^{\mathrm{equi}}(X)))$$

is an isomorphism.

To prove the cdh-duality theorem,

$$A_{r,i}(Y \times U, X) \cong A_{r+n,i}(Y, X \times U).$$

write $U = \cup_i U_i$ as a union of quasi-projective open subschemes, and let $z_r^{\mathrm{equi}}(\mathcal{U}, X)$ be the Čech complex formed from the $z_r^{\mathrm{equi}}(U_i, X)$.

The duality quasi-iso's $z_r^{\mathrm{equi}}(U_i, X) \to z_{r+n}^{\mathrm{equi}}(X \times U_i)$ and a cdh-acyclicity argument as in the proof of the comparison theorem give the quasi-iso

$$C_*(z_r^{\mathrm{equi}}(\mathcal{U}, X))_{\mathrm{cdh}} \to C_*(z_{r+n}^{\mathrm{equi}}(X \times U))_{\mathrm{cdh}}.$$

Thus, we need only check that the map

$$\mathbb{H}^{-i}_{\mathrm{cdh}}(Y, C_*(z_r^{\mathrm{equi}}(\mathcal{U}, X))_{\mathrm{cdh}}) \to \mathbb{H}^{-i}_{\mathrm{cdh}}(Y \times U, C_*(z_r^{\mathrm{equi}}(X))_{\mathrm{cdh}}) = A_{r,i}(Y \times U)$$

is an isomorphism. This assertion is cdh-local on Y, so we can assume Y is smooth and quasi-projective and reduces us to the case U quasi-projective.

In this case, we can pass from cdh-hypercohomology to Zariski hypercohomology, and then reduce to showing

$$\mathbb{H}^{-i}_{\mathrm{Zar}}(Y, C_*(z_r^{\mathrm{equi}}(U, X))_{\mathrm{Zar}}) \to \mathbb{H}^{-i}_{\mathrm{Zar}}(Y \times U, C_*(z_r^{\mathrm{equi}}(X))_{\mathrm{Zar}})$$

is an isomorphism. It follows from the geometric comparison and duality theorems that both sides are $h_i(z_r^{\mathrm{equi}}(X))(Y \times U)$.

8.11. **The** cdh**-descent theorem.**

Theorem 8.16 (cdh-descent)**.** *Suppose k admits resolution of singularities. Take $Y \in \mathbf{Sch}_k$.*
(1) Let $U \cup V = X$ be a Zariski open cover of $X \in \mathbf{Sch}_k$. There is a long exact sequence

$$\ldots \to A_{r,i}(Y, U \cap V) \to A_{r,i}(Y, U) \oplus A_{r,i}(Y, V) \to A_{r,i}(Y, X) \to A_{r,i-1}(Y, U \cap V) \to \ldots$$

(2) Let $Z \subset X$ be a closed subset. There is a long exact sequence

$$\ldots \to A_{r,i}(Y, Z) \to A_{r,i}(Y, X) \to A_{r,i}(Y, X \setminus Z) \to A_{r,i-1}(Y, Z) \to \ldots$$

(3) Let $p \amalg i : X' \amalg F \to X$ be an abstract blow-up. There is a long exact sequence

$$\ldots \to A_{r,i}(Y, p^{-1}(F)) \to A_{r,i}(Y, X') \oplus A_{r,i}(Y, F) \to A_{r,i}(Y, X) \to A_{r,i-1}(Y, p^{-1}(F)) \to \ldots$$

Proof. For (1) and (3), the analogous properties are obvious in the "first variable", so the theorem follows from duality.

For (2), the presheaf sequence

$$0 \to z_r^{\mathrm{equi}}(Z) \to z_r^{\mathrm{equi}}(X) \to z_r^{\mathrm{equi}}(X \setminus U)$$

is exact and $\mathit{coker}_{\mathrm{cdh}} = 0$. The cdh-acyclicity theorem says that applying $C_*(-)_{\mathrm{cdh}}$ to the above sequence yields a distinguished triangle. □

Corollary 8.17 (Duality). *For $X, Y \in \mathbf{Sch}_k$, $n = \dim Y$ we have a canonical isomorphism*

$$\mathrm{CH}_{r+n}(X \times Y, i) \cong A_{r,i}(Y, X)$$

Proof. For $U \in \mathbf{Sm}/k$, quasi-projective, we have the quasi-isomorphisms

$$C_*(z^{\mathrm{equi}}_{r+n}(X \times U))(\mathrm{Spec}\, k) = z^{\mathrm{equi}}_{r+n}(X \times U, *) \to z_{r+n}(X \times U, *)$$

$$C_*(z^{\mathrm{equi}}_r(U, X))(\mathrm{Spec}\, k) \to C_*(z^{\mathrm{equi}}_{r+n}(X \times U))(\mathrm{Spec}\, k)$$

and the isomorphisms

$$A_{r,i}(U, X) \to A_{r+n,i}(\mathrm{Spec}\, k, X \times U) \leftarrow h_i(z^{\mathrm{equi}}_{r+n}(X \times U))(\mathrm{Spec}\, k)$$

This gives the isomorphism

$$\mathrm{CH}_{r+n}(X \times U, i) \to A_{r,i}(U, X).$$

One checks this map is natural with respect to the localization sequences for $\mathrm{CH}_{r+n}(X \times -, i)$ and $A_{r,i}(-, X)$.

Given $Y \in \mathbf{Sch}_k$, there is a filtration by closed subsets

$$\emptyset = Y_{-1} \subset Y_0 \subset \ldots \subset Y_m = Y$$

with $Y_i \setminus Y_{i-1} \in \mathbf{Sm}/k$ and quasi-projective (k is perfect), so this extends the result from $U \in \mathbf{Sm}/k$, quasi-projective, to $Y \in \mathbf{Sch}_k$. □

Corollary 8.18. *Suppose k admits resolution of singularities. For $X, Y \in \mathbf{Sch}_k$ we have*
(1) (homotopy) *The projection $p : X \times \mathbb{A}^1 \to X$ induces an isomorphism $p^* : A_{r,i}(Y, X) \to A_{r+1,i}(Y, X \times \mathbb{A}^1)$.*

(2) (suspension) *The maps $i_0 : X \to X \times \mathbb{P}^1$, $p : X \times \mathbb{P}^1 \to X$ induce an isomorphism*

$$A_{r,i}(Y, X) \oplus A_{r-1,i}(Y, X) \xrightarrow{i_* + p^*} A_{r,i}(Y, X \times \mathbb{P}^1)$$

(3)(cosuspension) *There is a canonical isomorphism*

$$A_{r,i}(Y \times \mathbb{P}^1, X) \cong A_{r,i}(Y, X) \oplus A_{r+1,i}(Y, X)$$

(4) (localization) *Let $i : Z \to U$ be a codimension n closed embedding in $\mathbf{Sm}/k$. Then there is a long exact sequence*

$$\ldots \to A_{r+n,i}(Z, X) \to A_{r,i}(U, X) \xrightarrow{j^*} A_{r,i}(U \setminus Z, X) \to A_{r+n,i-1}(Z, X) \to \ldots$$

Proof. These all follow from the corresponding properties of $\mathrm{CH}^*(-, *)$ and the duality corollary:

(1) from homotopy
(2) and (3) from the projective bundle formula
(4) from the localization sequence. □

9. Motives of schemes of finite type

We can use the cdh-site to define both the motive of a scheme of finite type, as well as the motive with compact supports. Taking motivic cohomology, this gives a good definition of motivic cohomology and motivic cohomology with compact supports for an arbitrary finite type k-scheme.

9.1. **Motives and motives with compact support.** Recall: For $X \in \mathbf{Sch}_k$, $L(X)$ is the PST with $L(X)(U)$ the free abelian group on irreducible $W \subset U \times X$ such that $W \to U$ is finite, and surjective onto some component of U. We have $C_*(X) := C_*(L(X))$.

Definition 9.1. *$X \in \mathbf{Sch}_k$, $L^c(X)$ is the PST with $L^c(X)(U)$ the free abelian group on irreducible $W \subset U \times X$ such that $W \to U$ is quasi-finite, and dominant onto some component of U.*

We let $C^c_*(X) := C_*(L^c(X))$.

This gives us functors

$$C_* : \mathbf{Sch}_k \to DM^{\mathrm{eff}}_-(k)$$
$$C^c_* : \mathbf{Sch}'_k \to DM^{\mathrm{eff}}_-(k)$$

where $\mathbf{Sch}'_k \subset \mathbf{Sch}_k$ is the subcategory with the same objects as $\mathbf{Sch}_k$, but with only the *proper* morphisms.

Before we study C^c_*, we state and prove the cdh-acyclicity theorem.

9.2. **The** cdh**-acyclicity theorem.** There is a fundamental extension of the Nisnevich-acyclicity theorem 5.12, 5.14 to the cdh topology:

Theorem 9.2 (cdh-acyclicity)**.** *Suppose that k admits resolution of singularities. Take $G \in HI(k)$, F a presheaf on $\mathbf{Sm}/k$. If $F_{\mathrm{cdh}} = 0$, then for all n*

$$\mathrm{Ext}^n_{\mathrm{Nis}}(F_{\mathrm{Nis}}, G) = 0.$$

The crucial fact needed for the proof is:

Lemma 9.3. *Let $i : Z \to X$ be a closed immersion in $\mathbf{Sm}/k$, $p : X_Z \to X$ the blow-up of X along Z, $G \in HI(k)$. Then for all $i \geq 0$*

$$\mathrm{Ext}^i_{\mathrm{Nis}}(\mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(X_Z) \to \mathbb{Z}_{\mathrm{Nis}}(X)), G) = 0.$$

Proof of the lemma. Recall from Lecture 1 that for $T \to Y$ a closed embedding in $\mathbf{Sm}/k$ with complement $j : U \to Y$, the sheaf on T_{Zar}: $\mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(U) \to \mathbb{Z}_{\mathrm{Nis}}(X))$ depends only on the local Nisnevich neighborhoods of T in Y. This reduces the Lemma to the case: $i_0 : Z \to Z \times \mathbb{A}^d$.

In this case, $p^{-1}(Z) = \mathbb{P}^{d-1}$ and the horizontal maps in

$$\begin{array}{ccc} p^{-1}(Z) & \longrightarrow & (Z \times \mathbb{A}^d)_Z \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle p} \\ Z & \longrightarrow & Z \times \mathbb{A}^d \end{array}$$

induce $\mathbb{A}^1$-homotopy equivalences

$$\mathbb{Z}_{\mathrm{Nis}}(p^{-1}(Z)) \to \mathbb{Z}_{\mathrm{Nis}}((Z \times \mathbb{A}^d)_Z);\ \mathbb{Z}_{\mathrm{Nis}}(Z) \to \mathbb{Z}_{\mathrm{Nis}}(Z \times \mathbb{A}^d).$$

One easily checks that $\ker \mathbb{Z}_{\mathrm{Nis}}(\bar{p}) \to \ker \mathbb{Z}_{\mathrm{Nis}}(p)$ is an isomorphism, whence the lemma. □

Proof of the theorem. We proceed by induction on $j \geq 0$, assuming that for all pre-sheaves F,

$$F_{\mathrm{cdh}} = 0 \Longrightarrow \mathrm{Ext}^i_{\mathrm{Nis}}(F_{\mathrm{Nis}}, G) = 0 \text{ for } 0 \leq i < j,$$

the case $j = 0$ being trivially true.

We first (partially) extend lemma 9.3 to maps $p : X' \to X$ which factor as a sequence of blow-ups with smooth center. More precisely, we will use our induction hypothesis to show that

$$\mathrm{Ext}^i_{\mathrm{Nis}}(\mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(X') \to \mathbb{Z}_{\mathrm{Nis}}(X)), G) = 0 \text{ for } 0 \le i \le j.$$

For this, we use induction on the number n of blow-ups (with smooth center) needed to form the map p; the case $n = 1$ follows from lemma 9.3.

We note that the sheaves $\mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p))$ has cdh-sheafification zero. Indeed, it suffices to see $\mathbb{Z}_{\mathrm{cdh}}(p)$ is surjective for $p : Y' \to Y$ the blow-up of $Y \in \mathbf{Sm}/k$ along a smooth center $F \subset Y$. Clearly $\mathbb{Z}_{\mathrm{cdh}}(Y') \oplus \mathbb{Z}_{\mathrm{cdh}}(F) \to \mathbb{Z}_{\mathrm{cdh}}(Y)$ is surjective, since $Y' \amalg F \to Y$ is a cdh-cover. Letting $E \to F$ be the exceptional divisor, it suffices to show that $\mathbb{Z}_{\mathrm{cdh}}(E) \to \mathbb{Z}_{\mathrm{cdh}}(F)$ is surjective. But already $\mathbb{Z}_{\mathrm{Zar}}(E) \to \mathbb{Z}_{\mathrm{Zar}}(F)$ is surjective, since $E \to F$ is a Zariski locally trivial projective space bundle.

Factor p as $p_0 \circ p_1$, where p_0 is a single blow-up and p_1 is a sequence of of $n - 1$ blow-ups. We have the exact sequence

$$0 \to \Phi \to \mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p_1)) \to \mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p)) \to \mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p_0)) \to 0.$$

We have already seen that $\mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p_\epsilon))_{\mathrm{cdh}} = 0$ for $\epsilon = 0, 1, \emptyset$, so $\Phi_{\mathrm{cdh}} = 0$ as well. Break the sequence up into two short exact sequences

$$0 \to \Phi \to \mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p_1)) \to \Phi_1 \to 0; \quad 0 \to \Phi_1 \to \mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p)) \to \mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p_0)) \to 0.$$

By our induction on n, we have

$$\mathrm{Ext}^i_{\mathrm{Nis}}(\mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p_\epsilon)), G) = 0 \text{ for } 0 \le i \le j, \epsilon = 0, 1.$$

By our induction on j, we have

$$\mathrm{Ext}^i_{\mathrm{Nis}}(\Phi_{\mathrm{Nis}}, G) = 0 \text{ for } 0 \le i \le j - 1,$$

and thus, using the first short exact sequence

$$\mathrm{Ext}^i_{\mathrm{Nis}}(\Phi'_{\mathrm{Nis}}, G) = 0 \text{ for } 0 \le i \le j.$$

The second short exact sequence continues the induction to p.

We now proceed to the induction in j. As the statement of the theorem only depends on F_{Nis}, we may assume that F is a Nisnevich sheaf. Suppose $F_{\mathrm{cdh}} = 0$.

Take the canonical epimorphism

$$\phi : \oplus_{X, s \in F(X)} \mathbb{Z}_{\mathrm{Nis}}(X) \to F,$$

with $X \in \mathbf{Sm}/k$. Since $F_{\mathrm{cdh}} = 0$ there is for each (X, s) a cdh-cover $\pi_s : V \to X$ such that $\pi_s^*(s) = 0$. Using lemma 8.9, we see there is an abstract blow-up $r_s : Y' \to X$ (with Y' reduced) and a Nisnevich cover $q_s : V' \to Y'$ such that $(r_s q_s)^*(s) = 0$. But F is a Nisnevich sheaf, hence $r_s^*(s) = 0$. Using resolution of singularities, we may dominate $Y' \to X$ with a sequence of blow-ups with smooth center $p_s : X'_s \to X$, so $p_s^*(s) = 0$ as well.

Then ϕ factors through $\Psi := \oplus_s \mathrm{coker}(\mathbb{Z}_{\mathrm{Nis}}(p_s))$, giving the exact sequence

$$0 \to \Psi_0 \to \Psi \to F \to 0. \tag{9.2.1}$$

Then, as we have shown, $\mathrm{Ext}^i_{\mathrm{Nis}}(\Psi, G) = 0$ for all $i \le j$.

Since $\Psi_{\mathrm{cdh}} = 0$, $\Psi_{0\mathrm{cdh}} = 0$ as well. As $\mathrm{Ext}^i_{\mathrm{Nis}}(\Psi, G) = 0$ for all $i \le j$, it follows from the exact sequence (9.2.1) that $\mathrm{Ext}^0_{\mathrm{Nis}}(F_{\mathrm{Nis}}, G) = 0$ and

$$\mathrm{Ext}^i_{\mathrm{Nis}}(F, G) = \mathrm{Ext}^{i-1}_{\mathrm{Nis}}(\Psi_0, G)$$

for $1 \leq i \leq j$. The induction hypothesis for Ψ_0 thus carries the induction forward for F. ☐

Corollary 9.4 (cdh-acyclicity)**.** *Suppose that k admits resolution of singularities.*
0. For $U \in \mathbf{Sm}/k$, G a homotopy invariant PST,

$$H^i_{\mathrm{cdh}}(U, G_{\mathrm{cdh}}) \cong H^i_{\mathrm{Nis}}(U, G_{\mathrm{Nis}}) \cong H^i_{\mathrm{Zar}}(U, G_{\mathrm{Zar}}).$$

Take $F \in \mathrm{PST}(k)$.
1. For $U \in \mathbf{Sm}/k$, $\mathbb{H}^i_{\mathrm{cdh}}(U, C_*(F)_{\mathrm{cdh}}) \cong \mathbb{H}^i_{\mathrm{Nis}}(U, C_*(F)_{\mathrm{Nis}}) \cong \mathbb{H}^i_{\mathrm{Zar}}(U, C_*(F)_{\mathrm{Zar}})$.

2. If $F_{\mathrm{cdh}} = 0$, then $C_*(F)_{\mathrm{Zar}}$ is acyclic.

3. For $X \in \mathbf{Sch}_k$, $p^* : \mathbb{H}^i_{\mathrm{cdh}}(X, C_*(F)_{\mathrm{cdh}}) \to \mathbb{H}^i_{\mathrm{cdh}}(X \times \mathbb{A}^1, C_*(F)_{\mathrm{cdh}})$ is an isomorphism.

4. For $X \in \mathbf{Sch}_k$ there are canonical isomorphisms

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_-(k)}(C_*(X), C_*(F)[i]) \cong \mathbb{H}^i_{\mathrm{cdh}}(X, C_*(F)_{\mathrm{cdh}})$$

Proof. (0) $\implies$ (1) since $H^i(C_*(F))$ is homotopy invariant. For (0), we know $H^i_{\mathrm{Nis}}(U, G_{\mathrm{Nis}}) \cong H^i_{\mathrm{Zar}}(U, G_{\mathrm{Zar}})$ from Lecture 1.

We go from $H^i_{\mathrm{Nis}}(U, G_{\mathrm{Nis}})$ to $H^i_{\mathrm{cdh}}(U, G_{\mathrm{Nis}})$ by replacing U with a limit of cdh-hypercovers $\mathcal{U}$. But $(\mathrm{coker}\ \mathbb{Z}_{\mathrm{Nis}}(\mathcal{U}) \to \mathbb{Z}_{\mathrm{Nis}}(U))_{\mathrm{cdh}} = 0$, so by the theorem

$$\begin{aligned} H^i_{\mathrm{cdh}}(U, G_{\mathrm{cdh}}) &= \lim_{\mathcal{U}} \mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}/k))}(\mathbb{Z}_{\mathrm{Nis}}(\mathcal{U}), G[i]) \\ &= \mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}/k))}(\mathbb{Z}_{\mathrm{Nis}}(U), G[i]) \\ &= H^i_{\mathrm{Nis}}(U, G_{\mathrm{Nis}}) \end{aligned}$$

For (2): Suffices to show $h_i^{\mathrm{Nis}}(F) = 0$ for all i. If not, there is a non-trivial map $C_*(F)_{\mathrm{Nis}} \to h_n^{\mathrm{Nis}}(F)[n]$ in $D(\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}/k))$. But from Lecture 1:

$$\mathrm{Hom}(C_*(F)_{\mathrm{Nis}}, h_n^{\mathrm{Nis}}(F)[n]) \cong \mathrm{Ext}^n_{\mathrm{Nis}}(F_{\mathrm{Nis}}, h_n^{\mathrm{Nis}}(F))$$

which is 0 if $F_{\mathrm{cdh}} = 0$.

(3) and (4) are just like (1): use cdh-hypercovers to reduce from cdh to Nis. (3) reduces to the homotopy invariance of $H^*_{\mathrm{Nis}}(-, h_i(F)_{\mathrm{Nis}})$ and (4) reduces to the isomorphism in the Nisnevich acyclicity theorem 5.12 from Lecture 1:

$$\mathrm{Ext}^n_{\mathrm{Nis}}(L(X), G) \cong \mathrm{Hom}_{D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathbf{Sm}/k))}(C_*(X), G[n])$$

for $G \in HI(k)$. ☐

9.3. **Fundamental exact sequences.**

Theorem 9.5 (Mayer-Vietoris)**.** *Let $U \cup V = X$ be an open cover of $X \in \mathbf{Sch}_k$. There is a canonical distinguished triangle in $DM^{\mathrm{eff}}_-(k)$:*

$$C_*(U \cap V) \to C_*(U) \oplus C_*(V) \to C_*(X) \to C_*(U \cap V)[1]$$

giving a long exact sequence in Suslin homology

$$\ldots \to H^{\mathrm{Sus}}_{n+1}(X) \to H^{\mathrm{Sus}}_n(U \cap V) \to H^{\mathrm{Sus}}_n(U) \oplus H^{\mathrm{Sus}}_n(V) \to H^{\mathrm{Sus}}_n(X) \to \ldots$$

Proof. This uses the Nisnevic-acyclicity theorem 5.14, noting that

$$0 \to L(U \cap V) \to L(U) \oplus L(V) \to L(X) \to \mathit{coker} \to 0$$

is exact (as presheaves) and $\mathit{coker}_{\mathrm{Nis}} = 0$. ☐

Theorem 9.6 (Blow-up)**.** *Suppose k admits resolution of singularities. Let $p \amalg i : Y \amalg F \to X$ be an abstract blow-up. There is a canonical distinguished triangle in $DM^{\mathrm{eff}}_-(k)$:*

$$C_*(p^{-1}(Z)) \to C_*(Y) \oplus C_*(F) \to C_*(X) \to C_*(p^{-1}(F))[1]$$

Theorem 9.7 (Localization)**.** *Suppose k admits resolution of singularities. Let $i : W \to X$ be a closed immersion in $\mathbf{Sch}_k$ with open complement $j : U \to X$. There is a distinguished triangle in $DM^{\mathrm{eff}}_-(k)$:*

$$C^c_*(W) \to C^c_*(X) \to C^*_c(U) \to C^c_*(W)[1]$$

Proof of the blow-up theorem. The sequence

$$0 \to L(p^{-1}(Z)) \to L(Y) \oplus L(Z) \to L(X) \to coker \to 0$$

is exact, with $coker_{\mathrm{cdh}} = 0$. Then apply (2) of the cdh-acyclicity corollary 9.4 □

The proof of the localization theorem is the same, since

$$0 \to L^c(W) \to L^c(X) \to L^c(U) \to coker \to 0$$

is exact, with $coker_{\mathrm{cdh}} = 0$.

Corollary 9.8. *Suppose k admits resolution of singularities. Then both C_* and C^c_* factor canonically through the embedding $i : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM^{\mathrm{eff}}_-(k)$.*

Proof. For C_*: If Y is in $\mathbf{Sm}/k$, then $C_*(Y) = i(M_{\mathrm{gm}}(Y))$. For arbitrary X, $C_*(X_{\mathrm{red}}) = C_*(X)$, so can assume $X = X_{\mathrm{red}}$. Then use resolution of singularities, induction on dimension and the blow-up distinguished triangle.

For C^c_*, we proceed as above: k is perfect, so for $X \in \mathbf{Sch}_k$ reduced, there is a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_n = X$$

with $X_i \setminus X_{i-1} \in \mathbf{Sm}/k$. Then use the localization distinguished triangle. □

Definition 9.9. Suppose k admits resolution of singularities. Let

$$M_{\mathrm{gm}} : \mathbf{Sch}_k \to DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$$
$$M^c_{\mathrm{gm}} : \mathbf{Sch}'_k \to DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$$

be the functors with $i \circ M_{\mathrm{gm}} = C_*$, $i \circ M^c_{\mathrm{gm}} = C^c_*$, as given by corollary 9.8. $M_{\mathrm{gm}}(X)$ is the *motive of* X, $M^c_{\mathrm{gm}}(X)$ is the *motive of X with compact supports.*

Remarks 9.10. (1) Suppose X is proper over k. Then $L^c(X) = L(X)$, hence $M^c_{\mathrm{gm}}(X) = M_{\mathrm{gm}}(X)$.

(2) $M^c_{\mathrm{gm}}(\mathbb{A}^n) \cong \mathbb{Z}(n)[2n]$. This follows from the localization distinguished triangle

$$M_{\mathrm{gm}}(\mathbb{P}^{n-1}) \to M_{\mathrm{gm}}(\mathbb{P}^n) \to M^c_{\mathrm{gm}}(\mathbb{A}^n) \to M_{\mathrm{gm}}(\mathbb{P}^{n-1})[1]$$

and the projective bundle formula.

(3) Changing C_* to M_{gm} and C^c_* to M^c_{gm}, we have Mayer-Vietoris, blow-up and localization distinguished triangles in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

10. Morphisms and cycles

We describe how morphisms in $DM^{\text{eff}}_{\text{gm}}(k)$ can be realized as algebraic cycles. This corresponds to Hanamura's construction of motives via "higher correspondences" (see [27]). We assume throughout that k admits resolution of singularities.

10.1. **Bivariant cycle cohomology reappears.** The cdh-acyclicity theorem relates the bivariant cycle cohomology (and hence higher Chow groups) to the morphisms in $DM^{\text{eff}}_{\text{gm}}(k)$.

Theorem 10.1. *For $X, Y \in \mathbf{Sch}_k$ $r \geq 0$, $i \in \mathbb{Z}$, there is a canonical isomorphism*

$$\mathrm{Hom}_{DM^{\text{eff}}_{\text{gm}}(k)}(M_{\text{gm}}(Y)(r)[2r+i], M^c_{\text{gm}}(X)) \cong A_{r,i}(Y, X).$$

Proof. For $r = 0$, the cdh acyclicity theorem 9.4(4) gives an isomorphism

$$\mathrm{Hom}_{\mathcal{DM}^{\text{eff}}_{-}(k)}(C_*(Y)[i], C^c_*(X)) \cong \mathbb{H}^{-i}_{\text{cdh}}(Y, C_*(z^{\text{equi}}_0(X))) = A_{0,i}(Y, X).$$

Thus

$$\mathrm{Hom}_{\mathcal{DM}^{\text{eff}}_{-}(k)}(C_*(Y \times (\mathbb{P}^1)^r)[i], C^c_*(X)) \cong A_{0,i}(Y \times (\mathbb{P}^1)^r, X).$$

By the cosuspension isomorphism (corollary 8.18(3)) $A_{r,i}(Y, X)$ is a summand of $A_{0,i}(Y \times (\mathbb{P}^1)^r, X)$; by the definition of $\mathbb{Z}(1)$, $M_{\text{gm}}(Y)(r)[2r]$ is a summand of $M_{\text{gm}}(Y \times (\mathbb{P}^1)^r)$. One checks the two summands match up. □

10.2. **Chow motives.**

Corollary 10.2. *Sending a smooth projective variety X of dimension n to $M_{\text{gm}}(X)$ extends to a full embedding $M^{\text{eff}}_{\text{CH}}(k) \to DM^{\text{eff}}_{\text{gm}}(k)$, $M^{\text{eff}}_{\text{CH}}(k) :=$ effective homological Chow motives.*

Proof. For X and Y smooth and projective

$$\begin{aligned}\mathrm{Hom}_{DM^{\text{eff}}_{\text{gm}}(k)}(M_{\text{gm}}(Y), M_{\text{gm}}(X)) &= A_{0,0}(Y, X)\\ &\cong A_{\dim Y,0}(\mathrm{Spec}\, k, Y \times X)\\ &\cong \mathrm{CH}_{\dim Y}(Y \times X)\\ &= \mathrm{Hom}_{M^{\text{eff}}_{\text{CH}}(k)}(Y, X).\end{aligned}$$

□

The cancellation theorem 2.14 thus gives a full embedding

$$M_{\text{gm}} : M_{\text{CH}}(k) \to DM_{\text{gm}}(k).$$

10.3. **The higher Chow groups reappear.**

Corollary 10.3. *For $Y \in \mathbf{Sch}_k$, equi-dimensional over k, $i \geq 0$, $j \in \mathbb{Z}$, $\mathrm{CH}^i(Y, j) \cong \mathrm{Hom}_{DM^{\text{eff}}_{\text{gm}}(k)}(M_{\text{gm}}(Y), \mathbb{Z}(i)[2i-j])$. That is*

$$\mathrm{CH}^i(Y, j) \cong H^{2i-j}(Y, \mathbb{Z}(i)).$$

Proof. Take $i \geq 0$. Then $M^c_{\text{gm}}(\mathbb{A}^i) \cong \mathbb{Z}(i)[2i]$ and

$$\begin{aligned}A_{0,j}(Y, \mathbb{A}^i) &\cong A_{\dim Y,j}(\mathrm{Spec}\, k, Y \times \mathbb{A}^i)\\ &= \mathrm{CH}_{\dim Y}(Y \times \mathbb{A}^i, j)\\ &= \mathrm{CH}^i(Y \times \mathbb{A}^i, j)\\ &\cong \mathrm{CH}^i(Y, j)\end{aligned}$$

□

Remark 10.4. Combining the Chern character isomorphism

$$ch : K_j(Y)^{(i)} \xrightarrow{\sim} CH^i(Y, j)_{\mathbb{Q}}$$

(for $Y \in \mathbf{Sm}/k$) with our isomorphism $\mathrm{CH}^i(Y, j) \cong H^{2i-j}(Y, \mathbb{Z}(i))$ identifies rational motivic cohomology with weight-graded K-theory:

$$H^{2i-j}(Y, \mathbb{Q}(i)) \cong K_j(Y)^{(i)}.$$

Thus motivic cohomology gives an integral version of weight-graded K-theory, in accordance with conjectures of Beilinson on mixed motives.

The computation (theorem 10.1) of morphisms in $\mathcal{DM}_-^{\mathrm{eff}}(k)$ as bivariant cycle cohomology leads to a proof of the cancellation theorem 2.14 and other related results. We recall the statement:

Corollary 10.5 (cancellation)**.** *For $A, B \in DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ the map*

$$- \otimes \mathrm{id} : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A(1), B(1))$$

is an isomorphism. Thus

$$DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \to DM_{\mathrm{gm}}(k)$$

is a full embedding.

Corollary 10.6. *For $Y \in \mathbf{Sch}_k$, $n, i \in \mathbb{Z}$, set*

$$H^n(Y, \mathbb{Z}(i)) := \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(M_{\mathrm{gm}}(Y), \mathbb{Z}(i)[n]).$$

Then $H^n(Y, \mathbb{Z}(i)) = 0$ for $i < 0$ and for $n > 2i$.

Proof of the cancellation theorem and corollary. The Gysin distinguished triangle for for M_{gm} shows that $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ is generated by $M_{\mathrm{gm}}(X)$, X smooth and projective. So, we may assume $A = M_{\mathrm{gm}}(Y)[i]$, $B = M_{\mathrm{gm}}(X)$, X and Y smooth and projective, $i \in \mathbb{Z}$.

Then $M_{\mathrm{gm}}(X) = M_{\mathrm{gm}}^c(X)$ and $M_{\mathrm{gm}}(X)(1)[2] = M_{\mathrm{gm}}^c(X \times \mathbb{A}^1)$. Thus:

$$\begin{aligned}\mathrm{Hom}(M_{\mathrm{gm}}(Y)(1)[i], M_{\mathrm{gm}}(X)(1)) &\cong A_{1,i}(Y, X \times \mathbb{A}^1)\\ &\cong A_{0,i}(Y, X)\\ &\cong \mathrm{Hom}(M_{\mathrm{gm}}(Y)[i], M_{\mathrm{gm}}(X))\end{aligned}$$

For the corollary 10.6, supposes $i < 0$. Quasi-invertibility implies

$$\begin{aligned}H^{2i-j}(Y, \mathbb{Z}(i)) &= \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(Y)(-i)[j-2i], \mathbb{Z})\\ &= A_{-i,j}(Y, \mathrm{Spec}\, k)\\ &= A_{\dim Y - i, j}(\mathrm{Spec}\, k, Y)\\ &= H^{-j}(C_*(z_{\dim Y - i}^{\mathrm{equi}}(Y))(\mathrm{Spec}\, k)).\end{aligned}$$

Since $\dim Y - i > \dim Y$, $z_{\dim Y - i}^{\mathrm{equi}}(Y) = 0$.

If $i \geq 0$ but $n > 2i$, then $H^n(Y, \mathbb{Z}(i)) = \mathrm{CH}^i(Y, 2i - n) = 0$. □

11. Duality

We describe the duality involution

$$^{*} : DM_{\rm gm}(k)^{\rm op} \to DM_{\rm gm}(k),$$

assuming k admits resolution of singularities. We use Voevodsky's method via the internal Hom functor in $\mathcal{DM}^{\rm eff}_{-}(k)$. One can also construct the duality involution in $\mathcal{DM}_{\rm gm}(k)$ via the purely geometric construction in the category of Chow motives $\mathcal{M}_{\rm CH}(k)$ and the embeding $M_{\rm gm} : M_{\rm CH}(k) \to DM_{\rm gm}(k)$ described in §10.2. We use the internal Hom as this gives in addition a formula for the truncation functors for the "motivic weight filtration" on the triangulated category of mixed Tate motives, described in §12.3.

11.1. **Internal Hom.** Recall the internal Hom in PST(k) defined in Lecture 1:

$$\mathcal{H}om(F, G)(U) := \mathrm{Hom}(F \otimes L(U), G)$$

We have:

$$\begin{aligned}
&\mathcal{H}om(L(X), G)(U) = G(U \times X);\\
&\mathcal{H}om(F, G) := H^0_{\rm Nis}(\mathcal{H}om(\mathcal{L}(F), G))\\
&\mathrm{Hom}(F, \mathcal{H}om(G, H)) \cong \mathrm{Hom}(F \otimes G, H).
\end{aligned}$$

Also if G is a Nisnevich sheaf with transfers (resp. homotopy invariant), then so is $\mathcal{H}om(F, G)$.

Now take $A, B \in C^{-}(\mathrm{Sh}_{\rm Nis}(\mathrm{Cor}(k))$. We have the internal Hom complex $\mathcal{H}om(A, B)$ in $C(\mathrm{Sh}_{\rm Nis}(\mathrm{Cor}(k))$, with

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathcal{H}om(B, C)).$$

This gives the right-derived version $R\mathcal{H}om(A, B)$ in $D(\mathrm{Sh}_{\rm Nis}(\mathrm{Cor}(k))$ with a similar adjunction property.

Proposition 11.1. *If A is in the image of $DM^{\rm eff}_{\rm gm}(k)$ and B is in $DM^{\rm eff}_{-}(k)$, then $R\mathcal{H}om(A, B)$ is in $DM^{\rm eff}_{-}(k)$.*

Proof. For $A = C_*(X)$, the Nisnevich acyclicity theorem 5.12 implies that the cohomology sheaf $\mathcal{H}^n(R\mathcal{H}om(A, B))$ is the sheaf $R^n p_{X*}(B)$ associated to the presheaf $U \mapsto \mathbb{H}^n_{\rm Zar}(U \times X, B)$. Since a homotopy invariant sheaf with transfers is strictly homotopy invariant, the $\mathcal{H}^n(R\mathcal{H}om(A, B))$ are homotopy invariant. This passes to $R\mathcal{H}om(A, B)$ for arbitrary A.

Since $R^n p_{X*}(B)$ is in $HI(k)$, the restriction map

$$R^n p_{X*}(B)(U) \to R^n p_{X*}(B)(k(U))$$

is injective for each $U \in \mathbf{Sm}/k$, by Voevodsky's moving lemma.

Since $X_{k(U)}$ has Zariski cohomological dimension $\dim X$ and B is bounded above, $R^n p_{X*}(B) = 0$ for $n >> 0$. Thus $R\mathcal{H}om(A, B)$ is in D^{-}, hence in $DM^{\rm eff}_{-}(k)$. □

In general, $R\mathcal{H}om(A, B)$ is not in the image of $DM^{\rm eff}_{\rm gm}$, even if both A and B are so. Our first task is to show that $R\mathcal{H}om(A, B(N))$ *is* in $DM^{\rm eff}_{\rm gm}$ for $N >> 0$ (depending on A and B). Write

$$\mathcal{H}om_{DM^{\rm eff}_{-}}(A, B) := R\mathcal{H}om(i(A), i(B))$$

if A and B are in $DM^{\rm eff}_{\rm gm}$.

11.2. **Duality for smooth proper X.** Let X be smooth and proper of dimension n. Then

$$\mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(X),\mathbb{Z}(n)[2n])\cong \mathrm{CH}^n(X\times X),$$

so we have $\delta : M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(X)\to \mathbb{Z}(n)[2n]$. Applying the adjunction isomorphism

$$\begin{aligned}&\mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(Y)(m),\mathcal{H}om_{DM_-^{\mathrm{eff}}}(M_{\mathrm{gm}}(X),M_{\mathrm{gm}}(Y)(n+m)[2n]))\\ &\quad\cong \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(Y)(m)\otimes M_{\mathrm{gm}}(X),M_{\mathrm{gm}}(Y)(n+m)[2n])\end{aligned}$$

to the morphism

$$\begin{aligned}M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(Y)(m)\otimes M_{\mathrm{gm}}(X)&\xrightarrow{\sim} M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(Y)(m)\\ &\xrightarrow{\delta\otimes\mathrm{id}} M_{\mathrm{gm}}(Y)(n+m)[2n]\end{aligned}$$

gives us the map

$$\delta_*: M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(Y)(m)\to \mathcal{H}om_{DM_-^{\mathrm{eff}}}(M_{\mathrm{gm}}(X),M_{\mathrm{gm}}(Y)(n+m)[2n]).$$

Lemma 11.2. *For Y smooth and projective, the map*

$$M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(Y)(m)\xrightarrow{\delta_*}\mathcal{H}om_{DM_-^{\mathrm{eff}}}(M_{\mathrm{gm}}(X),M_{\mathrm{gm}}(Y)(n+m)[2n])$$

is an isomorphism for all $m\ge 0$

Proof. δ_* induces

$$\begin{aligned}&\mathrm{Hom}(M_{\mathrm{gm}}(U)[i],M_{\mathrm{gm}}(X\times Y)(m))\\ &\quad\to \mathrm{Hom}(M_{\mathrm{gm}}(U)[i],\mathcal{H}om_{DM_-^{\mathrm{eff}}}(M_{\mathrm{gm}}(X),M_{\mathrm{gm}}(Y)(n+m)[2n]))\\ &\quad= \mathrm{Hom}(M_{\mathrm{gm}}(U\times X)[i],M_{\mathrm{gm}}(Y)(n+m)[2n])\end{aligned}$$

The LHS is $A_{0,i+2m}(U,X\times Y\times\mathbb{A}^m)$, the RHS is

$$\begin{aligned}A_{0,i+2m}(U\times X,Y\times\mathbb{A}^{n+m})&\cong A_{n,i+2m}(U,X\times Y\times\mathbb{A}^{n+m})\\ &\cong A_{0,i+2m}(U,X\times Y\times\mathbb{A}^m).\end{aligned}$$

Since $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ is generated by the $M_{\mathrm{gm}}(U)$, this suffices. □

Lemma 11.3. *Take X,Y smooth and projective with $\dim X=n$. Then for all $m\ge n$, the canonical morphism*

$$\begin{aligned}&\mathcal{H}om_{DM_-^{\mathrm{eff}}}(M_{\mathrm{gm}}(X),M_{\mathrm{gm}}(Y)(n))(m-n)\\ &\quad\to \mathcal{H}om_{DM_-^{\mathrm{eff}}}(M_{\mathrm{gm}}(X),M_{\mathrm{gm}}(Y)(m))\end{aligned}$$

is an isomorphism.

Proof. Both sides are $M_{\mathrm{gm}}(X\times Y)(m-n)[-2n]$. □

Proposition 11.4. *Take $A,B\in DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$. There is an $N=N(A,B)$ such that for all $m\ge N$:*

(1) $\mathcal{H}om_{DM_-^{\mathrm{eff}}}(A,B(m))$ is in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$.

(2) $\mathcal{H}om_{DM_-^{\mathrm{eff}}}(A,B(m))(a)\to\mathcal{H}om_{DM_-^{\mathrm{eff}}}(A(n),B(n+m+a))$ is an isomorphism for all $a,n\ge 0$.

Proof. For $A = M_{\mathrm{gm}}(X)$, $B = M_{\mathrm{gm}}(Y)$, X, Y smooth and projective, this follows from the two lemmas and the cancellation theorem. This suffices since these generate $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$. □

Definition 11.5. Take $A, B \in DM_{\mathrm{gm}}$. Define

$$\mathcal{H}om_{DM_{\mathrm{gm}}}(A, B) := \mathcal{H}om_{DM^{\mathrm{eff}}_{-}}(A(n), B(n+N))(-N)$$

where $n >> 0$ is taken so $A(n)$ and $B(n)$ are in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ and $N \geq N(A(n), B(n))$.

It follows from the proposition that $\mathcal{H}om_{DM_{\mathrm{gm}}}(A, B)$ is independent of the choice of n and N.

Set $A^* := \mathcal{H}om_{DM_{\mathrm{gm}}}(A, \mathbb{Z})$.

Theorem 11.6 (Duality)**.** *(1) For $A \in DM_{\mathrm{gm}}(k)$, the canonical map $A \to (A^*)^*$ is an isomorphism.*

(2) For $A, B \in DM_{\mathrm{gm}}(k)$ there are canonical isomorphisms

$$(A \otimes B)^* \cong A^* \otimes B^*$$
$$\mathcal{H}om(A, B) \cong A^* \otimes B$$

(3) For $X \in \mathbf{Sm}/k$ of dimension n there are canonical isomorphisms

$$M_{\mathrm{gm}}(X)^* \cong M^c_{\mathrm{gm}}(X)(-n)[-2n]$$
$$M^c_{\mathrm{gm}}(X)^* \cong M_{\mathrm{gm}}(X)(-n)[-2n]$$

Proof. For (1) and (2), we may take $A = M_{\mathrm{gm}}(X)$, $B = M_{\mathrm{gm}}(Y)$ with X, Y smooth and projective of dimensions n and m. Then we have shown:

$$M_{\mathrm{gm}}(X)^* = M_{\mathrm{gm}}(X)(-n)[-2n]$$

$$\begin{aligned} M_{\mathrm{gm}}(X \times Y)^* &= M_{\mathrm{gm}}(X \times Y)(-n-m)[-2n-2m] \\ &= M_{\mathrm{gm}}(X)^* \otimes M_{\mathrm{gm}}(Y)^* \end{aligned}$$

$$\begin{aligned} \mathcal{H}om(M_{\mathrm{gm}}(X), M_{\mathrm{gm}}(Y)) &= M_{\mathrm{gm}}(X \times Y)(-n)[-2n] \\ &= M_{\mathrm{gm}}(X)^* \otimes M_{\mathrm{gm}}(Y). \end{aligned}$$

For (3): Let $n = \dim X$. For all $U \in \mathbf{Sm}/k$ we have canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}(M_{\mathrm{gm}}(U)[i], &\mathcal{H}om_{DM^{\mathrm{eff}}_{-}}(M_{\mathrm{gm}}(X), \mathbb{Z}(n)[2n])) \\ &= \mathrm{Hom}(M_{\mathrm{gm}}(U \times X)[i], \mathbb{Z}(n)[2n]) \\ &= A_{0,i}(U \times X, \mathbb{A}^n) \\ &= A_{n,i}(U, X \times \mathbb{A}^n) \\ &= A_{0,i}(U, X) \\ &= \mathrm{Hom}(M_{\mathrm{gm}}(U)[i], M^c_{\mathrm{gm}}(X)) \end{aligned}$$

Since the $M_{\mathrm{gm}}(U)[i]$ generate, we have

$$\mathcal{H}om_{DM^{\mathrm{eff}}_{-}}(M_{\mathrm{gm}}(X), \mathbb{Z}(n)[2n]) \cong M^c_{\mathrm{gm}}(X).$$

□

Lecture 3. Mixed Tate motives

The most basic motives are the powers $\mathbb{Z}(n)$ of the Tate motive $\mathbb{Z}(1)$. The full triangulated subcategory $\mathrm{DMT}(k)$ of $DM_{\mathrm{gm}}(k)$ generated by the "pure" Tate motives $\mathbb{Z}(n)$ is the triangulated category of *mixed Tate motives* over k. Even though concrete computations in the full category $DM_{\mathrm{gm}}(k)$ are often very difficult, and many suspected structural properties rely on (up to now) inaccessible conjectures, the mixed Tate category is much more amenable to such analysis. Especially in the case of number fields, there is enough known so that $\mathrm{DMT}(k)_{\mathbb{Q}}$ looks just like a generalization of the classical derived category of sheaves, or perhaps more precisely, like the derived category of the category of representations of an algebraic group G. In this lecture, we discuss these constructions and give some applications, such as relating mixed Tate motives with the action of $\mathrm{Gal}(\mathbb{Q})$ on $\pi_1^{alg}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ and showing that multiple zeta values arise as periods of mixed Tate motives.

12. Mixed Tate motives in $DM_{\mathrm{gm}}(k)$

12.1. Let $\widetilde{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ denote the $\mathbb{Q}$-localization of $DM_{\mathrm{gm}}(k)$, i.e. the objects are the same and

$$\mathrm{Hom}_{\widetilde{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}}(A, B) := \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(A, B) \otimes \mathbb{Q}.$$

We denote the image of the Tate objects $\mathbb{Z}(n)$ in $\widetilde{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ by $\mathbb{Q}(n)$.

Definition 12.1. The triangulated category $\mathrm{DMT}(k)$ of *mixed Tate motives over* k is the full triangulated subcategory of $\widetilde{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ generated by the *Tate motives* $\mathbb{Q}(n)$, $n \in \mathbb{Z}$. $\mathrm{DMT}^{\mathrm{eff}}(k)$ is the full triangulated subcategory generated by the $\mathbb{Q}(n)$, $n \geq 0$.

Here we take "generated" to mean that $\mathrm{DMT}(k)$ is the smallest full triangulated subcategory of $\widetilde{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ containing all the objects $\mathbb{Q}(n)$ and closed under isomorphism in $DM_{\mathrm{gm}}(k)$.

Remarks 12.2. 1. $\mathrm{DMT}(k)$ is a triangulated tensor subcategory of $\mathcal{DM}_{\mathrm{gm}}(k)$.

2. Let $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$ be the pseudo-abelian hull of $\widetilde{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ (a triangulated category by [4]). Then one can show that $\mathrm{DMT}(k)$ is closed under taking summands in $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$, hence is itself pseudo-abelian. We will use $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$ rather than $\widetilde{DM}_{\mathrm{gm}}(k)_{\mathbb{Q}}$ in what follows, although it doesn't really matter.

Examples 12.3. (1) Take $x \in k^\times$. We have the corresponding element

$$[x] \in H^1(k, \mathbb{Q}(1)) = \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(\mathbb{Z}, \mathbb{Z}(1)[1]) \otimes \mathbb{Q}.$$

Fill in $[x] : \mathbb{Z} \to \mathbb{Z}(1)[1]$ to a distinguished triangle $E_x \to \mathbb{Z} \to \mathbb{Z}(1)[1] \to E_x[1]$. Then the image of E_x in $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$ is in $DTM(k)$.

(2) The same construction gives an object E_x for each $x \in H^1(k, \mathbb{Q}(n))$.

12.2. ***t*-structures.** We are interested in finding an additional structure on $\mathrm{DMT}(k)_{\mathbb{Q}}$ which will give us an abelian category. Following Beilinson-Bernstein-Deligne [6], this is given by a *t-structure*.

Let $\mathcal{A}$ be an abelian category. In $D(\mathcal{A})$, one has full subcategories:

$$D_{\leq 0} := \{C \mid H^n(C) = 0 \text{ for } n > 0\},$$
$$D_{\geq 0} := \{C \mid H^n(C) = 0 \text{ for } n < 0\}.$$

These satisfy:

1. $D_{\leq 0}[1] \subset D_{\leq 0}$, $D_{\geq 0}[-1] \subset D_{\geq 0}$.

2. Each $X \in D(\mathcal{A})$ fits in a distinguished triangle $X_{\leq 0} \to X \to X_{>0} \to X_{\leq 1}[1]$ with $X_{\leq 0} \in D_{\leq 0}$, $X_{>0} \in D_{>0} := D_{\geq 0}[-1]$. $X_{\leq 0}$, for instance, is given by the canonical truncation $\tau_{\leq 0}X$, which is the subcomplex of X given by

$$\tau_{\leq 0}X^n := \begin{cases} X^n & \text{for } n < 0 \\ ker(d : X^0 \to X^1) & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases}$$

There is a dual formula for $X_{>0}$ as a quotient of X.

3. $\mathrm{Hom}(D_{\leq 0}, D_{>0}) = 0$.

Also, sending $M \in \mathcal{A}$ to the complex M in degree zero gives an equivalence of $\mathcal{A}$ with $D_{\leq 0} \cap D_{\geq 0}$; the inverse is given by the cohomology functor H^0. Also, a sequence in $\mathcal{A}$ is exact iff it extends to a distinguished triangle in $D(\mathcal{A})$. Finally, if $A \to B \to C \to A[1]$ is a distinguished triangle in $D(\mathcal{A})$ with $A, C \in \mathcal{A}$, then B is in $\mathcal{A}$ ($\mathcal{A}$ is *closed under extensions* in $D(\mathcal{A})$).

Abstracting these constructions and properties leads to

Definition 12.4 ([6])**.** $(D, D_{\leq 0}, D_{\geq 0})$ with D a triangulated category and $D_{\leq 0}$, $D^{\geq 0}$ satisfying (1)-(3) is a *t-structure* on D. The full additive subcategory $D_{\leq 0} \cap D^{\geq 0}$ is the *heart* of the t-structure.

Theorem 12.5 (Beilinson-Bernstein-Deligne)**.** *The heart $\mathcal{A}$ of a t-structure on D is an abelian category, closed under extensions in D, with a sequence in $\mathcal{A}$ exact iff the sequence extends to a distinguished triangle in D.*

Remarks 12.6. (1) D is *not* always a derived category of $\mathcal{A}$!

(2) Set $X_{\leq n} := (X[n]_{\leq 0})[-n]$, $X_{\geq n} := (X[1-n]_{>0})[n-1]$. Sending X to $X_{\leq n}$ or $X_{\geq n}$ define functors

$$\tau_{\leq n} : D \to D_{\leq n} := D_{\leq 0}[n],$$
$$\tau_{\geq n} : D \to D_{\geq n} := D_{\geq 0}[-n],$$

right (resp. left) adjoint to the inclusions $D_{\leq n} \to D$, $D_{\geq n} \to D$. The functor functor $H^0 := \tau_{\leq 0} \circ \tau_{\geq 0} : D \to \mathcal{A}$ is a cohomological functor, with $H^n = \tau_{\leq n} \circ \tau_{\geq n}$. H^0 gives the inverse to the inclusion $\mathcal{A} \to D$.

12.3. The weight filtration.

Definition 12.7. $W_{\leq n}\mathrm{DMT}(k)$ is the full triangulated subcategory of $\mathrm{DMT}(k)$ generated by the Tate objects $\mathbb{Q}(-m)$ with $m \leq n$.

This gives the tower of triangulated subcategories

$$\ldots \subset W_{\leq n}\mathrm{DMT}(k) \subset W_{\leq n+1}\mathrm{DMT}(k) \subset \ldots \subset \mathrm{DMT}(k)$$

Dually, let $W_{>n}\mathrm{DMT}(k)$ be the full triangulated subcategory of $\mathrm{DMT}(k)$ generated by the Tate objects $\mathbb{Q}(-m)$ with $m > n$.

Remarks 12.8. (1) $W_{\leq n}\mathrm{DMT}(k) = \mathrm{DMT}^{\mathrm{eff}}(k) \otimes \mathbb{Q}(-n)$.

(2) The weights in Hodge theory are twice what they are in DMT: the Hodge structure $\mathbb{Q}(-n)$ is pure weight $2n$. We divide by two since all the "Hodge weights" that appear in our construction will be even.

The basic fact on which the construction of the weight filtration rests is:

Lemma 12.9. *Fix $n \geq 0$.*

(1) Take $E \in W_{\leq -n}\mathrm{DMT}(k) \subset \mathrm{DMT}^{\mathrm{eff}}(k)$. Then

$$\mathcal{H}om_{DM_-^{\mathrm{eff}}}(\mathbb{Q}(a+N), E(N))(a) = E$$

for all $a \leq n$, $N \geq 0$.

(2) Take $F \in \mathrm{DMT}^{\mathrm{eff}}(k)$. Then $\mathcal{H}om_{DM_-^{\mathrm{eff}}}(\mathbb{Q}(n), F) = 0$ if and only if F is in $W^{>-n}\mathrm{DMT}(k)$.

(3) $\mathrm{Hom}(W_{\leq -n}\mathrm{DMT}(k), W_{>-n}\mathrm{DMT}(k)) = 0$

Proof. For $U \in \mathbf{Sm}/k$

$$\begin{aligned}\mathrm{Hom}(M_{\mathrm{gm}}(U)[i],&\mathcal{H}om_{DM_-^{\mathrm{eff}}}(\mathbb{Z}(n+N), \mathbb{Z}(m+N)))\\ &= \mathrm{Hom}(M_{\mathrm{gm}}(U)(n+N)[i], \mathbb{Z}(m+N))\\ &= \mathrm{Hom}(M_{\mathrm{gm}}(U)[i], \mathbb{Z}(m-n))\end{aligned}$$

As the latter is zero if $m < n$, the result is true for $E = \mathbb{Z}(m)$, $F = \mathbb{Z}(m)$. This proves (1) and the if part of (2). The only if part of (2) requires a bit more argument.

(3) follows as above from $\mathrm{Hom}(\mathbb{Z}(a), \mathbb{Z}(b)) = 0$ if $b < a$. □

Definition 12.10. For $A \in \mathrm{DMT}(k)$, set

$$W_{\leq n}(A) := \varinjlim_{N} \mathcal{H}om_{DM_-^{\mathrm{eff}}}(\mathbb{Q}(N-n), A(N))(-n)$$

Proposition 12.11. *(1) $A \mapsto W_{\leq n}(A)$ defines an exact functor*

$$W_{\leq n} : \mathrm{DMT}(k) \to W_{\leq n}\mathrm{DMT}(k)$$

right adjoint to the inclusion $W_{\leq n}\mathrm{DMT}(k) \to \mathrm{DMT}(k)$.

(2) For $A \in \mathrm{DMT}(k)$, there is a canonical distinguished triangle

$$W_{\leq n}A \to A \to W_{>n}(A) \to W_{\leq n}A[1]$$

with $W_{>n}(A) \in W^{>n}\mathrm{DMT}(k)$.

(3) $W_{>n}(A) = (W_{\leq -n-1}A^*)^*$.

Proof. This follows from the preceeding lemma. □

Remarks 12.12. (1) $A \mapsto W_{>n}A$ defines an exact functor

$$W_{>n} : \mathrm{DMT}(k) \to W_{>n}\mathrm{DMT}(k)$$

left adjoint to the inclusion.

In fact, for each n $(W_{\le n}\mathrm{DMT}(k), W_{>n}\mathrm{DMT}(k))$ is a t-structure on $\mathrm{DMT}(k)$ with heart $= 0$ and with truncation functors $\tau_{\le 0} = W_{\le n}$, $\tau_{>0} = W_{>n}$.

(2) For $a \le b$, let $W_{[a,b]}\mathrm{DMT}(k)$ be the full triangulated subcategory generated by the $\mathbb{Q}(-n)$, $a \le n \le b$, and set $W_{[a,b]} := W_{\le b}W_{>a-1}$. Then for $a \le c \le b$, we have a canonical distinguished triangle

$$W_{[a,c]}A \to W_{[a,b]}A \to W_{[c+1,b]}A \to W_{[a,c]}A[1]$$

Set $\mathrm{gr}^W_a := W_{[a,a]}$, $\mathrm{gr}^W_a\mathrm{DMT}(k) := W_{[a,a]}\mathrm{DMT}(k)$.

(3) Let $\mathrm{Vec}_{\mathbb{Q}}$ be the category of finite dimensional $\mathbb{Q}$-vector spaces $\mathrm{DMT}(k)_{[a,a]}$ is equivalent to $D^b(\mathrm{Vec}_{\mathbb{Q}})$, since $\mathrm{Hom}(\mathbb{Q}(a)[i], \mathbb{Q}(a)) = 0$ for $i \ne 0$, $= \mathbb{Q} \cdot \mathrm{id}$ for $i = 0$.

This gives us the cohomological functor

$$H^0 \circ \mathrm{gr}^W_a : \mathrm{DMT}(k) \to \mathrm{Vec}_{\mathbb{Q}}.$$

(4) There is a "big category of motivic sheaves" $DM(k)$, which is formed from $DM^{\mathrm{eff}}_-(k)$ by a process of inverting $-\otimes\mathbb{Z}(1)$. Set $W_{\le n}DM(k) := DM^{\mathrm{eff}}_-(k)\otimes\mathbb{Z}(-n)$. The same formula as for $\mathrm{DMT}(k)$:

$$W_{\le n}A := \varinjlim_N \mathcal{H}om(\mathbb{Q}(N-n), A(N))(-n)$$

gives a right adjoint to the inclusion $W_{\le n}DM(k)$. All the properties of the weight filtration on $\mathrm{DMT}(k)$ extend, however, $W_{\le n}$ does not in general restrict to a map $DM_{\mathrm{gm}}(k)_{\mathbb{Q}} \to DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$.

Warning: Under the realization to the derived category of mixed Hodge structures, the "motivic" weight filtration does *not* correspond to the usual weight filtration. It does so on $\mathrm{DMT}(k)$.

12.4. The Beilinson-Soulé vanishing conjectures.

Conjecture 12.13 (Beilinson, Soulé)**.** *Let F be a field. Then $K_{2q-p}(F)^{(q)} = 0$ for $p \le 0$, except for the case $p = q = 0$.*

In terms of motivic cohomology, the conjecture asserts:

$$H^p(F, \mathbb{Q}(q)) = 0 \text{ for } p \le 0, \text{ except for } p = q = 0.$$

Indeed

$$H^p(X, \mathbb{Q}(q)) \cong K_{2q-p}(X)^{(q)}.$$

12.5. The mixed Tate t-structure.

Definition 12.14. Let $\mathrm{DMT}(k)_{\le 0} \subset \mathrm{DMT}(k)$ be the full subcategory of M such that $H^n(\mathrm{gr}^W_a M) = 0$ for all $n > 0$ and all a. Let $\mathrm{DMT}(k)_{\ge 0} \subset \mathrm{DMT}(k)$ be the full subcategory of M such that $H^n(\mathrm{gr}^W_a M) = 0$ for all $n < 0$ and all a.

Proposition 12.15. *Suppose the B-S vanishing conjectures are true for k. Then* $(\mathrm{DMT}(k)_{\leq 0}, \mathrm{DMT}(k)_{\geq 0})$ *is a t-structure on* $\mathrm{DMT}(k)$. *In addition, the heart*

$$\mathrm{MT}(k) := \mathrm{DMT}(k)_{\leq 0} \cap \mathrm{DMT}(k)_{\geq 0}$$

contains the Tate motives $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, *and is equal to the smallest abelian subcateogry of* $\mathrm{MT}(k)$ *containing the* $\mathbb{Q}(n)$ *and closed under extensions.*

Remarks 12.16. (1) $\mathrm{MT}(k)$ is the *category of mixed Tate motves over* k.

(2) $\mathrm{MT}(k)$ is a tensor abelian category. In addition, the duality on $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$ has $\mathbb{Q}(n)^\vee = \mathbb{Q}(-n)$, so restricts to a duality on $\mathrm{DMT}(k)$ and on $\mathrm{MT}(k)$. This makes $\mathrm{MT}(k)$ into a *rigid* tensor category (more about this in the next section).

(3) The weight filtration functors $W_{\leq n}$ define a functorial exact *weight filtration*

$$0 = W_{\leq m+1}M \subset W_{\leq m}M \subset \ldots \subset W_{\leq n-1}M \subset W_{\leq n}M = M$$

for each $M \in \mathrm{MT}(k)$, with $\mathrm{gr}^W_a M$ an object in $\mathrm{gr}^W_a \mathrm{MT}(k)$. This category is equivalent to the category of finite dimensional $\mathbb{Q}$-vector spaces, with $\mathbb{Q}$ corresponding to $\mathbb{Q}(-a)$.

(4) The B-S vanishing conjectures are true for k a finite field, a number field, a curve over a finite field or a field of rational functions in 1 variable over any of these.

13. The motivic Hopf algebra and Lie algebra

The theory of Tannakian categories (see [57, 19]), applied to the category $\mathrm{MT}(k)$, allows us to view $\mathrm{MT}(k)$ as the category of representations of an algebraic group in finite dimensional vector spaces. In fact, one can explicitly construct the group (or rather its Hopf algebra of functions) directly from algebraic cycles. For $k = \mathbb{Q}$, one can "motivate" the so-called multiple zeta values by interpreting them as periods arising from this motivic Hopf algebra.

13.1. Tannakian formalism. Let $G = \operatorname{Spec} A$ be an affine group-scheme over a field k. This means that A is a *Hopf algebra* over k: the group multiplication gives the *comultiplication* $\delta : A \to A \otimes A$, the identity $e : \operatorname{Spec} k \to G$ gives the *co-unit* $\epsilon : A \to k$ and the inverse gives the *antipode* $\iota : A \to A$.

Let $\mathrm{Rep}_k(G)$ denote the category of algebraic representations of G in finite dimensional k-vector spaces. $\mathrm{Rep}_k(G)$ is an abelian tensor category, with a forgetful functor ω_0 to finite dimension k-vector spaces Vec_k.

$\mathrm{Rep}_k(G)$ is in fact a *rigid* tensor category: there is a perfect duality $A \mapsto A^*$ and $A^* \otimes B$ is an internal Hom in $\mathrm{Rep}_k(G)$. The functor ω_0 is a faithful exact rigid tensor functor.

Suppose A is a union of sub-algebras A_n, such that the product on G defines a pro-group scheme on G with respect to the maps $G \to \operatorname{Spec} A_n$. Then we modify the definition of $\mathrm{Rep}_k(G)$ to give us the category of continuous representations of the pro-group scheme G, that is, representations

$$\rho : G \to \mathrm{GL}(V)$$

for some finite dimensional k-vector space V such that ρ factors through $G \to \operatorname{Spec} A_n$ for some n.

The Tannakian formalism gives an inverse to this construction.

Definition 13.1. Let $\mathcal{A}$ be a k-linear abelian rigid tensor category. A *fiber functor* is a faithful exact rigid tensor functor $\omega : \mathcal{A} \to \mathrm{Vec}_k$. A (neutral) *Tannakian category over* k is an pair $\mathcal{A}, \omega$.

The *Tannaka group* of ω is the pro-algebraic group over k, $\mathrm{Aut}_r^{\otimes}(\omega)$, of rigid tensor automorphisms of ω.

Explicitly, for each k-algebra R, $g \in \mathrm{Aut}_r^{\otimes}(\omega)(R)$ is a collection of automorphisms

$$g(x) : \omega(x) \otimes_k R \to \omega(x) \otimes_k R; \quad x \in \mathcal{A},$$

such that

$$(\omega(f) \otimes \mathrm{id}) \circ g(x) = g(y) \circ (\omega(f) \otimes \mathrm{id})$$

for each morphism $f : x \to y$ in $\mathcal{A}$. In addition, $g(x \otimes y) = g(x) \otimes g(y)$ and $g(x^\vee) = {}^t g(x)^{-1}$ for all $x, y \in \mathcal{A}$. The "pro-structure" is given by the restriction maps of $\mathrm{Aut}_r^{\otimes}(\omega)(R)$ to collections $g(x)$ for a fixed finite collection of objects $x \in \mathcal{A}$.

The main theorem of Tannakian categories is:

Theorem 13.2 ([19])**.** *Let $(\mathcal{A}, \omega)$ be a neutral Tannakian category over k. Let $G := \mathrm{Aut}_r^{\otimes}(\omega)$. Then there is an equivalence $\mathcal{A} \sim \mathrm{Rep}_k(G)$ transforming ω to ω_0.*

13.2. $\mathrm{MT}(k)$ **as a Tannakian category.** Let k be a field satisfying the B-S vanishing conjectures. The functor $A \mapsto \mathrm{gr}_*^W A := \oplus_n \mathrm{gr}_n^W A$ defines a fiber functor

$$\mathrm{gr}_*^W : \mathrm{MT}(k) \to \mathrm{Vec}_{\mathbb{Q}}$$

so by the main theorem of Tannakian categories there is a *motivic Galois group* $\mathrm{Gal}_{\mathrm{mot}}(k)$ whose $\mathbb{Q}$-representations is $\mathrm{MT}(k)$: the Tannaka group of $(\mathrm{MT}(k), \mathrm{gr}_*^W)$. In fact:

Lemma 13.3. $\mathrm{Gal}_{\mathrm{mot}}(k)$ *is a split extension*

$$1 \to U_{\mathrm{mot}}(k) \to \mathrm{Gal}_{\mathrm{mot}}(k) \to \mathbb{G}_m \to 1$$

with $U_{\mathrm{mot}}(k)$ a unipotent pro-group-scheme. The sequence is a sequence of pro-group schemes over $\mathbb{Q}$.

Proof. $\mathbb{G}_m$ is the Tannaka group of graded $\mathbb{Q}$-vector spaces, $\mathrm{GrVec}_{\mathbb{Q}}$ (where t acts by $\times t^n$ on a vector space concentrated in degree n). Sending $\oplus_n V_n$ to $\oplus_n V_n \otimes_{\mathbb{Q}} \mathbb{Q}(-n)$ defines the rigid tensor functor $\mathrm{GrVec}_{\mathbb{Q}} \to \mathrm{MT}(k)$, hence a map of Tannaka groups $\pi : \mathrm{Gal}_{\mathrm{mot}}(k) \to \mathbb{G}_m$.

Considering $\mathrm{gr}_*^W A$ as a graded $\mathbb{Q}$-vector space defines the rigid tensor functor $\mathrm{MT}(k) \to \mathrm{GrVec}_{\mathbb{Q}}$, hence a right inverse $s : \mathbb{G}_m \to \mathrm{Gal}_{\mathrm{mot}}(k)$ to π. This gives us the split exact sequence

$$1 \to U_{\mathrm{mot}}(k) \to \mathrm{Gal}_{\mathrm{mot}}(k) \xrightarrow{\pi} \mathbb{G}_m \to 1$$

so we need only check that $U_{\mathrm{mot}}(k)$ is unipotent.

Let $\phi : \mathrm{gr}_*^W \to \mathrm{gr}_*^W$ be an automorphism that restricts to the identity $\mathrm{gr}_n^W \to \mathrm{gr}_n^W$ for each n. For each $A \in \mathrm{MT}(k)$ we have the weight filtration

$$0 = W_{\le M+1} A \subset W_{\le M} A \subset \ldots \subset W_{\le N-1} A \subset W_{\le N} A = A$$

Since gr_*^W is exact and ϕ is natural, ϕ must preserve the "weight filtration" of $\mathrm{gr}_*^W A$:

$$W_{\le n}(\mathrm{gr}_*^W A) := \oplus_{m \le n} \mathrm{gr}_m^W A.$$

Thus the (a,b) component $\phi_{a,b} : \mathrm{gr}^W_a A \to \mathrm{gr}^W_b A$ is zero if $b > a$ and the identity if $b = a$, so ϕ is unipotent. □

13.3. The motivic Lie algebra. Still assuming the B-S vanishing conjectures, the unipotent algebraic group $U_{\mathrm{mot}}(k)$ is determined by its Lie algebra $\mathrm{Lie}_{\mathrm{mot}}(k)$, called the *motivic Lie algebra* over k. $\mathrm{Lie}_{\mathrm{mot}}(k)$ is determined by its $\mathbb{Q}$-points, so we will think of $\mathrm{Lie}_{\mathrm{mot}}(k)$ as a pro-$\mathbb{Q}$-Lie algebra.

The $\mathbb{G}_m$-action defined via the splitting of

$$1 \to U_{\mathrm{mot}}(k) \to \mathrm{Gal}_{\mathrm{mot}}(k) \to \mathbb{G}_m \to 1$$

makes $\mathrm{Lie}_{\mathrm{mot}}(k)$ a graded Lie algebra, concentrated in negative degrees.

Our main goal is to understand $\mathrm{Lie}_{\mathrm{mot}}(k)$ or $U_{\mathrm{mot}}(k)$ via algebraic cycles.

13.4. 1st construction: framed motives. We assume the B-S vanshing conjectures hold. In [7], a general approach to constructing the Tannaka Hopf algebra in categories like $\mathrm{MT}(k)$ via "framed extensions" is given.

Definition 13.4. A *framed mixed Tate motive* is an $M \in \mathrm{MT}(k)$ together with $f : \mathrm{gr}^W_n M \to \mathbb{Q}$, $v \in \mathrm{gr}^W_{n'} M$, $n < n'$. $(M, f, v) \sim (M', f', v')$ if there is a morphism $a : M \to M'(\ell)$ for some ℓ with $\mathrm{gr}_{n'}(a)(v) = v'(\ell)$, $f = f'(\ell) \circ \mathrm{gr}_n(a)$. Let $\mathcal{H}_{\mathrm{gm}}(k)$ denote the set of equivalence classes. Define a grading on $\mathcal{H}(k)$ by $\deg := n - n'$.

We make $\mathcal{H}_{\mathrm{gm}}(k)$ a graded abelian monoid by $(M, f, v) + (M', f', v') := (M \oplus M', f + f', (v, v'))$ (after Tate twist of (M', f', v') if needed). Make $\mathcal{H}_{\mathrm{gm}}(k)$ a graded $\mathbb{Q}$-vector space by $r \cdot (M, f, v) := (M, rf, v) \sim (M, f, rv)$ and make $\mathcal{H}_{\mathrm{gm}}(k)$ into a $\mathbb{Q}$-algebra by using $\otimes$,

The coproduct $\delta : \mathcal{H}_{\mathrm{gm}}(k) \to \mathcal{H}_{\mathrm{gm}}(k) \otimes \mathcal{H}_{\mathrm{gm}}(k)$ is defined as follows: Let (M, f, v) be in $\mathcal{H}_{\mathrm{gm},n}(k)$. For $n < m < 0$, choose a basis $\{e_{i,m}\}$ of $\mathrm{gr}^W_m M$ with dual basis $\{e^i_m\}$. Set

$$\delta(M) := 1 \otimes M + M \otimes 1 + \sum_m \sum_i (M, f, e_{i,m}) \otimes (M, e^i_m, v).$$

The antipode is the dual: $\iota(M, f, v) := (M^\vee, v^\vee, f^\vee)$. This makes $\mathcal{H}_{\mathrm{gm}}(k)$ into a graded Hopf algebra.

Next, we show how to map $\mathrm{MT}(k)$ to $\mathcal{H}_{\mathrm{gm}}(k)$ co-modules (i.e. representations of $G := \mathrm{Spec}\,\mathcal{H}_{\mathrm{gm}}(k)$). Take $M \in \mathrm{MT}(k)$. Define a graded $\mathcal{H}_{\mathrm{gm}}(k)$ co-module $V(M)$ as follows: As a graded $\mathbb{Q}$-vector space, $V(M) := \mathrm{gr}^W_* M$. Choose a basis $\{v_{i,m}\}$ for $\mathrm{gr}^W_m M$, let $\{v^i_m\}$ be the dual basis and define

$$\nu : V(M) \to \mathcal{H}(k) \otimes V(M)$$

by

$$\nu(v_{i,m}) := \sum_{n<m,j} (M, v^j_n, v_{i,m}) \otimes v_{j,n}.$$

Proposition 13.5. *Sending M to $V(M)$ identifies* $\mathrm{MT}(k)$ *with the category of graded $\mathcal{H}_{\mathrm{gm}}(k)$ co-modules (finite dimensional as a $\mathbb{Q}$-vector space), giving an isomorphism $U_{\mathrm{mot}}(k) \cong \mathrm{Spec}\,\mathcal{H}_{\mathrm{gm}}(k)$.*

The idea is that (M, f, v) acts like a "matrix coefficient" for $U_{\mathrm{mot}}(k)$: for $u \in U_{\mathrm{mot}}(k)$, $u(M)$ acts on $\mathrm{gr}^W_* M$. Set

$$(M, f, v)(u) := f(u(M)(v)).$$

See the article [7] for details.

14. Mixed Tate motives as cycle modules

As the approach via framed extensions requires some knowledge of the category MT(k) to understand the resulting group, we consider another approach that relies on the cycle complexes we constructed in Lecture 2. The idea is to use the cycle complexes to feed into the machinery of commutative differential graded algebras (cdga's). The idea of using algebraic cycles to define a category of mixed Tate motives goes back to Bloch [9]; the material described is based on a combination of [10, 34]. For a more detailed exposition, we refer the reader to [39].

14.1. Cycle algebras. To make our cycle complexes into cdga's we need to use cubes instead of simplices.

Let $\square^n := (\mathbb{P}^1 \setminus \{1\}, \{0,\infty\})^n$, i.e. $\square^n = (\mathbb{P}^1 \setminus \{1\})^n$ with distinguished divisors $t_i = 0, \infty$, $i = 1, \ldots, n$. Define $z^q(X,n)^c \subset z^q(X \times \square^n)$ just as we did $z^q(X,n)$, replacing Δ^n and its faces with $\square^n$ and all its faces.

Let $z^q(X,n)^c_{\text{alt}} \subset z^q(X,n)^c_{\mathbb{Q}}$ be the subspace of cycles which are *alternating* with respect to all permutations of coordinates in $\square^n$, and also with respect to $t_i \mapsto t_i^{-1}$.

Definition 14.1. $z^q(X,*)^c_{\text{alt}}$ is the complex with differential

$$d_n : z^q(X,n)^c_{\text{alt}} \to z^q(X,n-1)^c_{\text{alt}}$$

the sum $\sum_{j=1}^n (-1)^j (\iota^*_{j,1} - \iota^*_{j,0})$, where $\iota_{j,\epsilon} : \square^{n-1} \to \square^n$ is the identification of $\square^{n-1}$ with the face $t_j = \epsilon$.

Set $\mathcal{N}^q(X)^n := z^q(X, 2q-n)^c_{\text{alt}}$.

Proposition 14.2. *(1) There is a natural isomorphism* $z^q(X,*)_{\mathbb{Q}} \cong z^q(X,*)^c_{\text{alt}}$ *in* $D^-(\mathbb{Q})$.

(2) $(Z,W) \mapsto \pi_{\text{alt}}(Z \times W)$ *(π_{alt} the alternating projection) makes* $\mathcal{N}(k) := \oplus_{q\ge 0} \mathcal{N}^q(k)^*$ *into a graded (in q), commutative differential graded $\mathbb{Q}$-algebra (an* Adams graded cdga*). For $a \in \mathcal{N}^q(k)^n$, we call q the* Adams degree *and n the* cohomological degree *of a, written $w(a)$ and $|a|$, respectively.*

Definition 14.3. An Adams graded differential graded $\mathcal{N}(k)$ module (M,d) is a bi-graded $\mathbb{Q}$-vector space $M = \oplus_{q,n} M^n(q)$ with differential $d = \oplus_{q,n} d^n(q)$,

$$d^n(q) : M^n(q) \to M^{n+1}(q),$$

$d^2 = 0$, together with an $\mathcal{N}(k)$ module structure

$$\mathcal{N}(k) \otimes M \to M$$

such that

(1) $a \in \mathcal{N}^q(k)^n$, $m \in M^{n'}(q') \Longrightarrow am \in M^{n+n'}(q+q')$
(2) $d(am) = da \cdot m + (-1)^{|a|} a \cdot dm$

An *Adams graded cell module* over $\mathcal{N}(k)$ is an Adams graded differential graded $\mathcal{N}(k)$ module (M,d) with a finite direct sum decomposition $M = \sum_q M\langle q\rangle$ into finitedly generated free bi-graded modules $M\langle q\rangle$ for $\mathcal{N}(k)$. Morphisms are maps respecting all the structures. Denote the category of Adams graded cell modules over $\mathcal{N}(k)$ by CM($\mathcal{N}(k)$).

In CM($\mathcal{N}(k)$) we have the usual translation $M[1]^n(q) := M^{n+1}(q)$, $d^n_{M[1]} := -d^{n+1}_M$. Note that the translation does not alter the Adams degree. We also have a cone operation: if $f:(M,d_M)\to(N,d_N)$ is a map of cell modules, the $\mathrm{Cone}(f)$ is the cell module that is $N\oplus M[1]$ as a bi-graded module over $\mathcal{N}(k)$, with differential the usual matrix

$$d_{\mathrm{Cone}(f)} := \begin{pmatrix} d_N & f[1] \\ 0 & d_{M[1]} \end{pmatrix}$$

We have the usual cone sequence

$$M \xrightarrow{f} N \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} M[1].$$

$\mathrm{CM}(\mathcal{N}(k))$ has a tensor structure by taking $\otimes_{\mathcal{N}(k)}$ on bigraded $\mathcal{N}(k)$ modules, with differential given by the usual Leibniz rule (with the signs for commutation and differentiation involving the cohomological grading only). There is also a duality involution: if M has "cell decompositon"

$$M = \oplus_i \mathcal{N}(k)\cdot e_i$$

with $d(e_i)=\sum_j a_{ij}e_j$, $a_{ij}\in\mathcal{N}(k)$, $w(e_i)=w_i$, $|e_i|=n_i$, then $M^\vee := \oplus_i\mathcal{N}(k)\cdot e^i$ with $w(e^i)=-w_i$, $|e^i|=-|e_i|$ and

$$de^i := -\sum_j a_{ji}e^j.$$

A morphism $f:M\to M'$ in $\mathrm{CM}(\mathcal{N}(k))$ is a quasi-isomorphism if f is a quasi-isomorphism of the underlying complexes of $\mathbb{Q}$-vector spaces.

Definition 14.4. Let $\mathrm{Cyc}(k)$ denote the derived category of Adams graded cell modules over $\mathcal{N}(k)$, ie., the category formed from $\mathrm{CM}(\mathcal{N}(k))$ by inverting the quasi-isomorphisms.

Remark 14.5. $\mathrm{Cyc}(k)$ is a triangulated category, where the distinguished triangles are those which are isomorphic to the image of a cone sequence. The tensor operation and duality involution on $\mathrm{CM}(\mathcal{N}(k))$ make $\mathrm{Cyc}(k)$ into a rigid tensor triangulated category.

14.2. Spitzweck's representation theorem.

Theorem 14.6 (Spitzweck). *$\mathrm{Cyc}(k)$ is equivalent to $\mathrm{DMT}(k)$ as rigid tensor triangulated category.*

Idea of proof. First define a functorial version $X\mapsto\mathcal{N}^q_{\mathrm{equi}}(X)$ of $\mathcal{N}^q(X)^*$ by using cycles on $X\times\square^m\times\mathbb{A}^q$ which are equi-dimensional over $X\times\square^m$ (alternating in the $\square^m$ variables and symmetric in the $\mathbb{A}^q$ variables).

$X\mapsto\mathcal{N}_{\mathrm{equi}}(X)$ is thus a cdga-object $\mathcal{N}_{\mathrm{mot}}$ of $DM^{\mathrm{eff}}_-(k)_{\mathbb{Q}}$, in fact a cdga-object of $\mathrm{DMT}^{\mathrm{eff}}(k)$ since $X\mapsto\mathcal{N}^q_{\mathrm{equi}}(X)$ is isomorphic to $\mathbb{Q}(q)$. The pull-back map $\mathcal{N}_{\mathrm{equi}}(k)\to\mathcal{N}_{\mathrm{equi}}(X)$ makes $\mathcal{N}_{\mathrm{mot}}$ an $\mathcal{N}_{\mathrm{equi}}(k)$-algebra. $\mathcal{N}_{\mathrm{mot}}$ will act as a "tilting module" to relate $\mathrm{DMT}(k)$ and $\mathrm{Cyc}(k)$.

Let $\mathcal{N}^{\mathrm{equi},q}(X)\subset\mathcal{N}^q(X)$ be the complex of cycles equi dimensional over X (but not in general over $\square^n$). This inclusion is a quasi-isomorphism for affine X. External product makes $\oplus_q\mathcal{N}^{\mathrm{equi},q}(\mathbb{A}^q)$ an Adams graded cdga and a sub-cdga of $\mathcal{N}_{\mathrm{equi}}(k)$.

The maps $\mathcal{N}(k)\xrightarrow{\pi^*}\oplus_q\mathcal{N}^{\mathrm{equi},q}(\mathbb{A}^q)\leftarrow\mathcal{N}_{\mathrm{equi}}(k)$ are quasi-isomorphisms of cdga's hence induce equivalences of the categories of cell modules

Let M be an Adams graded cell module over $\mathcal{N}_{\text{equi}}(k)$. We have the Adams graded object $M \otimes_{\mathcal{N}_{\text{equi}}(k)} \mathcal{N} = \oplus_q [M \otimes_{\mathcal{N}_{\text{equi}}(k)} \mathcal{N}](q)$, which is a dg module over $\mathcal{N}_{\text{mot}}$. The multiplication gives us the maps

$$[M \otimes_{\mathcal{N}_{\text{equi}}(k)} \mathcal{N}](q) \otimes \mathcal{N}_{\text{mot}}(1) \to [M \otimes_{\mathcal{N}_{\text{equi}}(k)} \mathcal{N}](q+1),$$

giving by adjunction the map

$$[M \otimes_{\mathcal{N}_{\text{equi}}(k)} \mathcal{N}](q) \to \mathcal{H}om(\mathcal{N}(1), [M \otimes_{\mathcal{N}_{\text{equi}}(k)} \mathcal{N}](q+1)).$$

This yields the sequence of maps (N some fixed large integer).

$$\begin{aligned} \ldots \to \mathcal{H}om(\mathcal{N}(q), [M \otimes_{\mathcal{N}_{\text{equi}}(k)} \mathcal{N}](q+N)) \\ \to \mathcal{H}om(\mathcal{N}(q+1), [M \otimes_{\mathcal{N}_{\text{equi}}(k)} \mathcal{N}](q+N+1)) \to \ldots. \end{aligned}$$

Let

$$m_N(M) := [\operatorname{colim}_q \mathcal{H}om(\mathcal{N}(q), [M \otimes_{\mathcal{N}_{\text{equi}}(k)} \mathcal{N}](q+N))].$$

A priori, $m_N(M)$ is only in $DM_-^{\text{eff}}(k)$, but one makes an explicit computation that shows that $m_N(\mathbb{Q}(a)) \cong \mathbb{Q}(a+N)$. By an induction on the weight filtration, it follows that $m_N(M)$ is in $\text{DMT}^{\text{eff}}(k) \subset DM_{\text{gm}}^{\text{eff}}(k)$ as long as N is sufficiently large. We then define the geometric motive of M, $m(M)$ by

$$m(M) := m_N(M)(-N);$$

one shows that $m(M)$ is independent of N for all sufficiently large N. Sending M to $m(M)$ defines the desired equivalence $\text{Cyc}(k) \to \text{DMT}(k)$. $\square$

14.3. Weight filtration. The weight filtration on $\text{DMT}(k)$ is easy to describe in $\text{Cyc}(k)$. In fact, let $M = \oplus \mathcal{N}(k) e_i$ be a cell module. Write the differential as

$$d(e_i) = \sum_j a_{ij} e_j$$

Since d has Adams degree 0, we have

$$w(e_i) = w(d(e_i)) = w(a_{ij}) + w(e_j)$$

for all i, j. Noting that $\mathcal{N}(k)$ is concentrated in Adams degree ≥ 0, this forces $w(e_i) \geq w(e_j)$ if $a_{ij} \neq 0$. Thus for each n, the submodule

$$W_n M := \bigoplus_{w(e_i) \leq n} \mathcal{N}(k) e_i \subset M$$

is a sub-cell module of M. Since all the morphisms in $\text{CM}(\mathcal{N}(k))$ preserve Adams degree, the operation $M \mapsto W_n M$ defines a functorial filtration in $\text{CM}(\mathcal{N}(k))$. Similarly, W_n preserves quasi-isomorphisms, hence defines a functorial filtration in $\text{Cyc}(k)$; it is not hard to see that this goes over to our weight filtration in $\text{DMT}(k)$ via the equivalence $\text{Cyc}(k) \to \text{DMT}(k)$.

14.4. The cell-module heart. Now suppose the B-S vanishing conjectures hold. This is equivalent to saying that $\mathcal{N}(k)$ is *cohomologically connnected*, i.e.

$$H^n(\mathcal{N}(k)) = 0 \text{ for } n < 0$$
$$H^0(\mathcal{N}(k)) = H^0(\mathcal{N}_0(k)) = \mathbb{Q}.$$

Definition 14.7. Let $\text{Cyc}(k)_{\leq 0} := \{(M, d) \mid H^n(M) = 0 \text{ for } n > 0\}$, let $\text{Cyc}(k)_{\geq 0} := \{M \mid H^n(M) = 0 \text{ for } n < 0\}$.

Proposition 14.8. *Suppose the B-S vanishing conjectures hold for k. Then*
(1) $(\mathrm{Cyc}(k)_{\le 0}, \mathrm{Cyc}(k)_{\ge 0})$ *is a t-structure on* $\mathrm{Cyc}(k)$*. Let* $\mathcal{A}_{\mathrm{cyc}}(k)$ *denote the heart.*

(2) This t-structure goes over to the "motivic" t-structure on $\mathrm{DMT}(k)$ *via the equivalence* $\mathrm{Cyc}(k) \sim \mathrm{DMT}(k)$*, hence* $\mathcal{A}_{\mathrm{cyc}}(k)$ *is equivalent to* $\mathrm{MT}(k)$*.*

The proof of the proposition is essentially formal; for details, see e.g. [39].

14.5. The cycle Hopf algebra. To get a cycle-theoretic construction of the Hopf algebra $\mathcal{H}_{\mathrm{gm}}(k)$, we use the bar construction on $\mathcal{N}(k)$.

We first quickly recall the bar construction in the context of cdga's over $\mathbb{Q}$. Let $\mathcal{C} \xrightarrow{\epsilon} \mathbb{Q}$ be an augmented cdga over $\mathbb{Q}$. Form the double complex (with $\mathbb{Q}$ in degree $(0,0)$)

$$\ldots \to \mathcal{C}^{\otimes n} \to \mathcal{C}^{\otimes n-1} \to \ldots \to \mathcal{C}^{\otimes 2} \to \mathcal{C} \xrightarrow{0} \mathbb{Q}$$

by

$$\begin{aligned}[a_1|\ldots|a_n] &:= a_1 \otimes \ldots \otimes a_n \\ &\mapsto \epsilon(a_1)[a_2|\ldots|a_n] + \sum_{i=1}^{n-1}(-1)^i[a_1|\ldots|a_ia_{i+1}|\ldots|a_n] \\ &\qquad + (-1)^n[a_1|\ldots|a_{n-1}]\epsilon(a_n).\end{aligned}$$

Definition 14.9. The *reduced bar complex* of $\mathcal{C}$ is $\mathcal{B}(\mathcal{C}) := \mathrm{Tot}(\mathcal{C}^{\otimes *})$.

$\mathcal{B}(\mathcal{C})$ is a dg Hopf algebra:

The algebra structure is the *shuffle product*:

$$[a_1|\ldots|a_n] \cdot [a_{n+1}|\ldots|a_{n+m}] := \sum_{(n,m)\text{ shuffles }\sigma} \mathrm{sgn}(\sigma)[a_{\sigma(1)}|\ldots|a_{\sigma(n+m)}].$$

The coalgebra structure is

$$[a_1|\ldots|a_n] \mapsto \sum_{i=1}^{n}[a_1|\ldots|a_i] \otimes [a_{i+1}|\ldots|a_n]$$

The antipode is

$$[a_1|\ldots|a_n] \mapsto -[a_n|\ldots|a_1].$$

Definition 14.10. The cycle Hopf algebra is $\mathcal{H}_{\mathrm{cyc}}(k) := H^0(\mathcal{B}(\mathcal{N}(k)))$.

The Adams grading on $\mathcal{N}(k)$ induces a grading on $\mathcal{H}_{\mathrm{cyc}}(k)$. Give $H^0(\mathcal{B}(\mathcal{N}(k)))$ an ind-structure by taking the subcomplexes

$$\mathrm{Tot}(\mathcal{C}^{\otimes *}, * \le n) \subset sB(\mathcal{C})$$

and writing

$$H^0(\mathcal{B}(\mathcal{N}(k))) - \cup_n H^0(\mathrm{Tot}(\mathcal{C}^{\otimes *}, * \le n)).$$

This makes $\mathcal{H}_{\mathrm{cyc}}(k)$ an ind-Hopf algebra. A continuous co-module M for $\mathcal{H}_{\mathrm{cyc}}(k)$ is one whose co-action

$$M \to \mathcal{H}_{\mathrm{cyc}}(k) \otimes M$$

has image in $H^0(\mathrm{Tot}(\mathcal{C}^{\otimes *}, * \le n)) \otimes M$ for some n.

The following theorem, together with proposition 14.8, gives us our desired description of the motivic Galois group in terms of algebraic cycles.

Theorem 14.11. *Suppose the B-S vanishing conjectures hold for k. Then there is a canonical equivalence of $\mathcal{A}_{\mathrm{cyc}}(k)$ with the category of graded continuous co-modules over $\mathcal{H}_{\mathrm{cyc}}(k)$ which are finite dimensional as $\mathbb{Q}$-vector spaces.*

This equivalence, together with the equivalence $\mathcal{A}_{\mathrm{cyc}}(k) \sim \mathrm{MT}(k)$, induces an isomorphism $\mathcal{H}_{\mathrm{cyc}}(k) \cong \mathcal{H}_{\mathrm{gm}}(k)$.

14.6. Lie algebras and the $K(\pi, 1)$ conjecture. Since the essential part of our motivic Galois group is a unipotent group, we can describe everything in terms of representations of Lie algebras. This also has a direct description in terms of cycles, using our cycle cdga $\mathcal{N}(k)$ and the theory of 1-minimal models.

Let $\mathcal{C}$ be a cdga over $\mathbb{Q}$, $\mathcal{H} := H^0(\mathcal{B}(\mathcal{C}))$ the corresponding Hopf algebra. There are two ways to construct a co-Lie algebra:

1. *The 1-minimal model of $\mathcal{C}$*
2. *The cotangent space $\mathfrak{m}_{\mathcal{H}}/\mathfrak{m}_{\mathcal{H}}^2$*

These are isomorphic, by a theorem of Quillen [52]. The $K(\pi, 1)$ *conjecture* is

Conjecture 14.12. *$\mathcal{N}(k)$ is 1-minimal, i.e: Let $i_1 : \mathcal{N}(k)(1) \to \mathcal{N}(k)$ be the 1-minimal model of $\mathcal{N}(k)$. Then i_1 is a quasi-isomorphism.*

For us, the $K(\pi, 1)$ conjecture is interesting because:

Proposition 14.13. *Suppose the B-S vanishing conjectures hold. Then* $\mathrm{MT}(k) \to \mathrm{DMT}(k)$ *induces an equivalence* $D^b(\mathrm{MT}(k)) \to \mathrm{DMT}(k)$ *if and only if $\mathcal{N}(k)$ is 1-minimal.*

In practical terms an equivalence $D^b(\mathrm{MT}(k)) \to \mathrm{DMT}(k)$ means that for $A, B \in \mathrm{MT}(k)$, one has $\mathrm{Hom}_{\mathrm{DMT}(k)}(A, B[n]) = \mathrm{Ext}^n_{\mathrm{MT}(k)}(A, B)$.

Example 14.14. The $K(\pi, 1)$ conjecture holds for finite fields, number fields, function fields $\mathbb{F}_q(C)$ of curves over a finite field and the 1-variable rational function fields $\overline{\mathbb{Q}}(t)$, and $\overline{\mathbb{F}_q(C)}(t)$. We do not know if the $K(\pi, 1)$ conjecture holds for $k(t)$ with k a number field or $k = \mathbb{F}_q(C)$.

14.7. The 1-minimal model.

Definition 14.15. Call a cdga $\mathcal{C}$ over $\mathbb{Q}$ *generalized nilpotent* if there is a tower of graded subspaces $0 = \mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots \subset \mathcal{C}^{*>0}$ such that

1. as an algebra $\mathcal{C} = \cup_{n=1}^{\infty} \Lambda^* \mathcal{C}_n$
2. $d(\mathcal{C}_n) \subset \mathcal{C}_{n-1}$.

Definition 14.16. An *n-minimal model* of a cohomologically connected cdga $\mathcal{C}$ is a cdga $\mathcal{C}(n)$ together with a map $i_n : \mathcal{C}(n) \to \mathcal{C}$ of cdga's such that

1. $\mathcal{C}(n)$ is a generalized nilpotent cdga with the generating subspaces in degrees $\leq n$.
2. i_n is an isomorphism on H^i for $i \leq n$.
3. i_n is injective on H^{n+1}.

The "follow your nose" procedure constructs an n-minimal model of a cohomologically connected cdga $\mathcal{C}$; two n-minimal models $p_1 : \mathcal{C}_1 \to \mathcal{C}$, $p_2 : \mathcal{C}_2 \to \mathcal{C}$ are isomorphic with isomorphism commuting with the maps p_1, p_2 up to homotopy.

14.8. The 1-minimal model. We describe the construction explicitly:
Step 1. Take $\mathcal{C}_1 \subset \mathcal{C}^1$ a subspace representing $H^1(\mathcal{C})$. Set $d \equiv 0$ on $\mathcal{C}_1$.

Step 2. Take $\mathcal{C}_2 := \mathcal{C}_1 \oplus \ker\big(\mathcal{C}_1 \wedge \mathcal{C}_1 \to H^2(\mathcal{C})\big)$ with d on the kernel the inclusion $\ker \to \mathcal{C}_1 \wedge \mathcal{C}_1$. Then $ker(H^2(\Lambda^*\mathcal{C}_1) \to H^2(\mathcal{C})$ maps to zero in $H^2(\Lambda^*\mathcal{C}_2)$ and $H^1(\Lambda^*\mathcal{C}_1) \xrightarrow{\sim} H^1(\Lambda^*\mathcal{C}_2)$.

Repeat, construction the sequence of vector spaces

$$\mathcal{C}_1 \to \mathcal{C}_2 \to \ldots \to \mathcal{C}_n \to$$

with compatible maps $\mathcal{C}_n \to \mathcal{C}^1$, so that $ker(H^2((\Lambda^*\mathcal{C}_n) \to H^2(\mathcal{C})$ goes to zero $H^2((\Lambda^*\mathcal{C}_{n+1})$ and $H^1(\Lambda^*\mathcal{C}_n) \xrightarrow{\sim} H^1(\mathcal{C})$. Then take $\mathcal{C}(1) := \mathrm{colim}_n \Lambda^*\mathcal{C}_n$.

14.9. The co-Lie algebra. Let $\mathcal{C}(1) \to \mathcal{C}$ be the 1-minimal model, let $\mathcal{M}(\mathcal{C}) := \mathcal{C}(1)^1$. $\mathcal{C}(1) = \Lambda^*\mathcal{M}(\mathcal{C})$, so the differential in degree 1 looks like

$$d : \mathcal{M}(\mathcal{C}) \to \Lambda^2\mathcal{M}(\mathcal{C}).$$

The identity $d^2 = 0$ translates into the Jacobi identity on $\mathcal{M}^\vee$, so $\mathcal{M}$ is a co-Lie algebra with co-bracket d.

14.10. The cotangent space. Let $\mathcal{H} = H^0(\mathcal{B}(\mathcal{C}))$ for an augmented cdga $\mathcal{C}$. Then $G := \mathrm{Spec}\,\mathcal{H}$ is a pro-algebraic group, so the tangent space to the identity T_eG is a pro-Lie algebra. Since $\mathcal{H} \to k(e)$ is the augmentation ϵ, the maximal ideal $\mathfrak{m}_e$ is $\mathfrak{m}_\mathcal{H} := \ker\epsilon$ and the cotangent space $T_e^\vee$ is $\mathfrak{m}_\mathcal{H}/\mathfrak{m}_\mathcal{H}^2$.

In case $\mathcal{C}$ has an Adams grading, then the category of $\mathcal{H}$ co-modules is equivalent to the category of modules over the Lie algebra $(\mathfrak{m}_\mathcal{H}/\mathfrak{m}_\mathcal{H}^2)^*$ (with the (co)modules being finite dimensional as $\mathbb{Q}$-vector spaces).

We collect our analysis in the following theorem:

Theorem 14.17 (Summary)**.** *For an arbitrary field k, we have* $\mathrm{Cyc}(k) \sim \mathrm{DMT}(k)$ *as rigid tensor triangulated categories together with their weight filtrations. Suppose the B-S vanishing conjectures hold for k. Then*

(1) $\mathrm{MT}(k)$ *is equivalent to the category of finite dimensional co-modules over the ind-Hopf algebra of framed mixed Tate motives $\mathcal{H}_{\mathrm{gm}}(k)$.*

(2) $\mathrm{MT}(k)$ *is equivalent to the heart $\mathcal{A}_{\mathrm{cyc}}(k)$ of* $\mathrm{Cyc}(k)$.

(3) Let $\mathcal{H}_{\mathrm{cyc}} := H^0(\mathcal{B}(\mathcal{N}(k))$. Then $\mathcal{H}_{\mathrm{cyc}}(k) \cong \mathcal{H}_{\mathrm{gm}}(k)$.

(4) The co-Lie algebra $\mathfrak{m}_{\mathcal{H}_{\mathrm{gm}}}/\mathfrak{m}^2_{\mathcal{H}_{\mathrm{gm}}}$ is isomorphic to the co-Lie algebra $\mathcal{M}(k) := \mathcal{M}(\mathcal{N}(k))$ and the category of graded co-representations of this co-Lie algebra is $\mathrm{MT}(k)$.

(5) $\mathcal{N}(k)$ is 1-minimal if and only if $DTM(k) \sim D^b(\mathrm{MT}(k))$.

15. The action of $\mathrm{Gal}(\mathbb{Q})$ on $\pi_1(\mathbb{P}^1 \setminus S)$

Let X be a k-scheme, k a field, with structure morphism $p : X \to \mathrm{Spec}\,k$. Fix a geometric point $\bar{x}$ of X, which gives us the geometric point $p(\bar{x})$ of $\mathrm{Spec}\,k$, i.e., a choice of an algebraic closure $k(\bar{x})$ of k. We have the fundamental exact sequence on the algebraic fundamental groups

$$1 \to \pi_1^{alg}(\bar{X}, \bar{x}) \to \pi_1^{alg}(X, \bar{x}) \xrightarrow{p_*} \pi_1^{alg}(\operatorname{Spec} k, p(\bar{x})) = \operatorname{Gal}(k(\bar{x})/k) \to 1,$$

where $\bar{X} := X \times_k \bar{k}$. This in turn gives a map of $\operatorname{Gal}(k(\bar{x})/k)$ to the outer automorphism group $\operatorname{Out}(\pi_1^{alg}(\bar{X}, x))$. If $\bar{x}$ arises from a k-rational point x of X, then x defines a splitting of the above sequence and thereby an action of $\operatorname{Gal}(k(\bar{x})/k)$ on $\pi_1^{alg}(\bar{X}, \bar{x})$ lifting the outer action.

In this section, we show how to form a version of this picture for the groups arising from the categories of mixed Tate motives.

15.1. Mixed Tate motives over a number ring. Let k be a number field. Then $H^p(k, \mathbb{Q}(q)) = 0$ unless $p = 1$, $q > 0$ or $p = q = 0$. Let r_1 be the number of real embeddings of k, and $2r_2$ the number of complex embeddings.

$$H^1(k, \mathbb{Q}(q)) = \begin{cases} \mathbb{Q}^{r_2} & \text{for } q \geq 2 \text{ even} \\ \mathbb{Q}^{r_1+r_2} & \text{for } q \geq 3 \text{ odd} \\ k^\times_{\mathbb{Q}} & \text{for } q = 1. \end{cases}$$

Thus $\mathcal{L}_{\mathrm{mot}}(k) := \mathcal{M}(k)^*$ is the free graded Lie algebra on $\oplus_{q\geq 1} H^1(k, \mathbb{Q}(q))^*$ with $H^1(k, \mathbb{Q}(q))$ in degree $-q$.

For $k = \mathbb{Q}$, this gives:

$$\mathcal{L}_{\mathrm{mot}}(\mathbb{Q}) = \operatorname{Lie}[\oplus_{p \text{ prime}} \mathbb{Q}, s_3, s_5, \ldots].$$

The quotient Lie algebra $\mathcal{L}_{\mathrm{mot}}(\mathbb{Q})/[\oplus_{p \text{ prime}} \mathbb{Q}, \mathcal{L}_{\mathrm{mot}}(\mathbb{Q})]$ is the *unramified over* $\mathbb{Z}$ Lie algebra $\mathcal{L}_{\mathrm{mot}}(\mathbb{Z})$.

The finite dimensional graded representations of $\mathcal{L}_{\mathrm{mot}}(\mathbb{Z})$ is the category $\mathrm{MT}(\mathbb{Z}) \subset \mathrm{MT}(\mathbb{Q})$ of *mixed Tate motives over* $\mathbb{Z}$.

One makes a similar construction for $\mathcal{L}_{\mathrm{mot}}(\mathcal{O})$, where $\mathcal{O}$ is the ring of S-integers in a number field K, and its category of representations $\mathrm{MT}(\mathcal{O}) \subset \mathrm{MT}(K)$.

15.2. Mixed Tate motives over $\mathbb{P}^1_k \setminus S$. Let $k = k_0[t][1/g]$, k_0 a number field. Write $g = \prod p_i$, p_i irreducible. We suppose each p_i is linear. Each p_i gives the class $[p_i] \in H^1(k, \mathbb{Q}(1))$. Then we have the exact sequence

$$\begin{aligned} 0 \to H^1(k_0, \mathbb{Q}(q)) &\to H^1(k, \mathbb{Q}(q)) \\ &\to \oplus_i [p_i] \cdot H^1(k_0, \mathbb{Q}(q-1)) \to H^2(k, \mathbb{Q}(q)) \to 0. \end{aligned}$$

For $q = 1$, $H^2(k, \mathbb{Q}(q)) = 0$, and for $q > 1$, the first and last maps are isomorphisms. Thus $[p_k] \cup [p_j] = 0 \mod \oplus_i [p_i] \cdot H^1(k, \mathbb{Q}(1))$.

These computations imply that $\mathcal{L}_{\mathrm{mot}}(k)$ is no longer free, but is described by an exact sequence

$$0 \to \mathcal{L}_{\mathrm{geom}}(k) \to \mathcal{L}_{\mathrm{mot}}(k) \to \mathcal{L}_{\mathrm{mot}}(k_0) \to 0$$

with $\mathcal{L}_{\mathrm{geom}}(k)$ the free Lie algebra on the duals $[p_i]^\vee$ of the classes $[p_i]$ (all in degree -1).

Assuming one has the exact sequence

$$\begin{aligned} 0 \to H^1(\mathcal{O}_{k_0,S}, \mathbb{Q}(q)) &\to H^1(\mathcal{O}_{k_0,S}[t, 1/g], \mathbb{Q}(q)) \\ &\to \oplus_i [p_i] \cdot H^1(\mathcal{O}_{k_0,S}, \mathbb{Q}(q-1)) \to H^2(\mathcal{O}_{k_0,S}[t, 1/g], \mathbb{Q}(q)) \to 0. \end{aligned}$$

with the similar properties as above, we have the exact sequence

$$0 \to \mathcal{L}_{\mathrm{geom}}(k) \to \mathcal{L}_{\mathrm{mot}}(\mathcal{O}_{k_0,S}[t, 1/g]) \to \mathcal{L}_{\mathrm{mot}}(\mathcal{O}_{k_0,S}) \to 0$$

A choice of a base-point in $\mathrm{Spec}\, k$ (or a *tangential base-point*) gives a splitting to this sequence, hence an action of $\mathcal{L}_{\mathrm{mot}}(\mathcal{O}_{k_0,S})$ on $\mathcal{L}_{\mathrm{geom}}(k)$.

15.3. Mixed Tate motives over $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Take $k_0 = \mathbb{Q}$, $g = t(1-t)$, and the tangential base-point $\partial/\partial t_{|t=1}$. This gives the exact sequence

$$0 \to \mathrm{Lie}[X, Y] \to \mathcal{L}_{\mathrm{mot}}(\mathbb{P}^1_{\mathbb{Z}} \setminus \{0, 1, \infty\}) \to \mathcal{L}_{\mathrm{mot}}(\mathbb{Z}) \to 0,$$

with $X = [t]^\vee$, $Y = [1-t]^\vee$, and the action of $\mathcal{L}_{\mathrm{mot}}(\mathbb{Z})$ on $\mathrm{Lie}[X, Y]$.

This whole picture is reminiscent of the situation for π_1^{alg}:

$$1 \to \pi_1^{\mathrm{geom}}(C) \to \pi_1^{\mathrm{alg}}(C) \to \pi_1^{\mathrm{alg}}(k_0) = \mathrm{Gal}(k_0) \to 1$$

where $C = \mathrm{Spec}\, k_0[t, 1/g]$ (or the same with k_0 replaced by $\mathcal{O}_{k_0,S}$).

15.4. Realization functors. The category $DM_{\mathrm{gm}}(k)$ is universal for homology theories on $\mathbf{Sm}/k$ with a suitable form of transfer map. Classically, cohomology theories were often prefered to homology, but the point of view of Bloch-Ogus allows one to go from cohomology to Borel-Moore homology.

Using compactifications and hypercoverings, as Deligne does in constructing mixed Hodge structures on singular, non-proper schemes, one can go back and forth between homology and Borel-Moore homology.

Either using cycle classes in a Bloch-Ogus theory, or other more direct forms of transfers, one is often able to transform a given cohomology theory to a homology theory with transfers, giving rise to a functor on $DM_{\mathrm{gm}}(k)$ by its universal property.

In short, a reasonable "cohomology" theory $\mathbb{H}$ on $\mathbf{Sm}/k$ often gives rise to an exact tensor *realization functor*

$$\Re_{\mathbb{H}} : DM_{\mathrm{gm}}(k) \to D(\mathrm{Sh}^{\mathcal{A}}_\tau(\mathbf{Sm}/k))$$

for a suitable abelian category $\mathcal{A}$ and Grothendieck topology τ, extending the assignment $X \mapsto \mathbb{H}_X$, where $\mathbb{H}_X$ is a functorial complex representing the $\mathbb{H}$-cohomology theory of X.

One can also compose $\Re_{\mathbb{H}}$ with $R\pi_* : D(\mathrm{Sh}^{\mathcal{A}}_\tau(\mathbf{Sm}/k)) \to D(\mathrm{Sh}^{\mathcal{A}}_\tau(k))$ or $R\Gamma : D(\mathrm{Sh}^{\mathcal{A}}_\tau(\mathbf{Sm}/k)) \to D(\mathcal{A})$ to get related triangulated functors, or with H^0 to get a cohomological functor.

Sometimes one needs variations on this basic version, for example using Hodge complexes instead of complexes of mixed Hodge structures, or using categories of inverse systems of ℓ-adic sheaves instead of étale sheaves.

In any case, such realization functors are available in some form or other for de Rham cohomology, with its Hodge filtration, singular cohomology with its weight filtration or Hodge structure, as well as version "over a base" i.e. variations of mixed Hodge structures, and similar forms of étale cohomology.

15.5. Polylog. We use the polylog mixed Tate motive to compute the action of $\mathcal{L}_{\mathrm{mot}}(\mathbb{Z})$ on a quotient of $\mathrm{Lie}[X, Y]$.

Let $C = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, let $(-1)^{n(n-1)}\rho_n \in z^{\mathrm{equi},n}(C, 2n-1)$ be the closure of the locus parametrized by

$$(t, x_1, x_2, \ldots, x_{n-1}, 1 - x_1, 1 - \frac{x_2}{x_1}, \ldots, 1 - \frac{x_{n-1}}{x_{n-2}}, 1 - \frac{t}{x_{n-1}}),$$

followed by the alternating projection.

$\rho_n(t)$:= the fiber over $t \in C$. Also $\rho(0), \rho_n(1)$ are well-defined elements of $z^n(\mathbb{Q}, 2n-1)$ for $n > 1$; $\rho(0) = 0$.

One computes:

$$d\rho_n = [t] \cdot \rho_{n-1} \tag{15.5.1}$$

The elements $[t], [1-t], \rho_2, \ldots, \rho_n$ are in the co-Lie algebra $\mathcal{M}(C) := \mathcal{N}^{\mathrm{equi}}(C)(1)^1$: Start with the elements $[t], [1-t]$ of $\mathcal{N}^1(C)^1$. Then kill $[t] \wedge [1-t]$ by $d\rho_2$, kill $[t] \wedge \rho_2$ by ρ_3, etc., using (15.5.1). This gives the subspace $\mathrm{Li}^* \subset \mathcal{M}^{\mathrm{geom}}(C)$ spanned by $\rho_1, \rho_2, \ldots$ ($\rho_1 = [1-t]$), with co-bracket

$$d : \mathrm{Li}^* \to \mathcal{M}(C) \wedge \mathrm{Li}^*;\ d\rho_n = [t] \wedge \rho_{n-1}$$

Dualizing gives the *polylog quotient* Li of $\mathcal{L}^{\mathrm{geom}}(C)$. From (1), Li is the quotient of $\mathcal{L}^{\mathrm{geom}}(C) = \mathrm{Lie}[X, Y]$ by the subspace spanned by all brackets containing at least 2 Y's.

Let $\mathrm{Li}^*_n \subset \mathrm{Li}^*$ be the subspace spanned by $\rho_1, \ldots, \rho_n$, giving the quotient Li_n of Li. The co-bracket for Li^* restricts to

$$d : \mathrm{Li}^*_n \to \mathcal{M}(C) \wedge \mathrm{Li}^*_n$$

making Li_n a mixed Tate motive over C. Since $M_{\mathrm{gm}}(C) \in DM_{\mathrm{gm}}(\mathbb{Q})$ is itself a sum of pure Tate motives over $\mathbb{Q}$, "push-forward" makes Li_n a mixed Tate motive over $\mathbb{Q}$, in fact over $\mathbb{Z}$. This is one construction of the nth polylogarithm mixed Tate motive.

15.6. The action. The action $\mathcal{L}_{\mathrm{mot}}(\mathbb{Z}) \otimes \mathrm{Lie}[X, Y] \to \mathrm{Lie}[X, Y]$ is dual to the co-action $\mathcal{M}^{\mathrm{geom}}(C) \to \mathcal{M}(\mathbb{Z}) \otimes \mathcal{M}^{\mathrm{geom}}(C)$. This latter is induced by the composition

$$\mathcal{M}^{\mathrm{geom}}(C) \xrightarrow{d} \mathcal{M}(C) \otimes \mathcal{M}^{\mathrm{geom}}(C) \xrightarrow{sp_{t\to 1} \otimes \mathrm{id}} \mathcal{M}(\mathbb{Q}) \otimes \mathcal{M}^{\mathrm{geom}}(C).$$

Thus the co-action restricts to

$$\mathrm{Li}^* \xrightarrow{d} \mathrm{Li}^* \otimes \mathcal{M}^{\mathrm{geom}}(C) \xrightarrow{sp_{t\to 1} \otimes \mathrm{id}} \mathcal{M}(\mathbb{Z}) \otimes \mathcal{M}^{\mathrm{geom}}(C)$$

(i.e., $[1-t] \mapsto 0$, $\rho_n \mapsto \rho_n(1)$) so

$$\rho_n \mapsto \rho_{n-1}(1) \otimes [t].$$

Dualizing gives the action of $\mathcal{L}_{\mathrm{mot}}(\mathbb{Z}) = \mathrm{Lie}[s_3, s_5, \ldots]$ on Li: $s_m(Y) = 0$ and

$$s_m(X) = \sum_n s_m(\rho_{n-1}(1)) \cdot [X, [X, [\ldots [X, Y] \ldots] \ (n-1\ X\text{'s}).$$

Using the Hodge realization, one computes the regulator of $\rho_{n-1}(1)$ as the polylogarithm $Li_{n-1}(1) = \zeta_{\mathbb{Q}}(n-1)$. Thus $\rho_m(1) \in H^1(\mathbb{Q}, \mathbb{Q}(m))$ is $\neq 0$ for $m > 1$ odd.

Since s_m is a generator of the dual space $H^1(\mathbb{Q}, \mathbb{Q}(m))^*$, $s_m(\rho_m(1)) \neq 0$ for $m > 1$ odd. The other terms vanish for degree reasons.

This yields:

$$s_{2k+1}(X) = c_k[X, [X, [\ldots [X, Y] \ldots] \ (2k+1\ X\text{'s}, c_k \in \mathbb{Q}^*).$$

15.7. The conjecture of Deligne-Ihara. Consider the Galois version of this story via the exact sequence

$$1 \to \pi_1^{\mathrm{geom}}(C) \to \pi_1^{\mathrm{alg}}(C) \to \mathrm{Gal}(\mathbb{Q}) \to 1$$

with splitting given by the tangential basepoint $\partial/\partial_{|t=1}$. Note that $\pi_1^{\mathrm{geom}}(C)$ the free profinite group on two generators, with Lie algebra $\mathcal{L}_\ell^{\mathrm{geom}}$. This gives a (negative) grading to the pro-ℓ Lie algebra $\mathrm{End}_\ell(\mathcal{L}_\ell^{\mathrm{geom}})$ by using the word length. The *ad* representation of $\mathcal{L}_\ell^{\mathrm{geom}}$ on itself:

$$ad(x)(y) := [x, y]$$

induces a graded action of $\mathcal{L}_\ell^{\mathrm{geom}}$ on $\mathrm{End}_\ell(\mathcal{L}_\ell^{\mathrm{geom}})$; denote the quotient by $\mathrm{out}_\ell(\mathcal{L}_\ell^{\mathrm{geom}})$.

Ihara notes that the resulting action of $\mathrm{Gal}(\mathbb{Q})$ on pro-ℓ Lie algebra of $\pi_1^{\mathrm{geom}}(C)$ factors through the pro-ℓ unipotent completion of $\mathrm{Gal}(\mathbb{Q})$, so we have an action of the associated $\mathbb{Q}_\ell$ Lie algebra on $\mathcal{L}_\ell^{\mathrm{geom}}$. Ihara states the following conjecture, attributing it to Deligne:

Conjecture 15.1 (Deligne)**.** *The action homomorphism $\mathcal{L}_{\acute{e}t}(\mathbb{Z})_\ell \to \mathrm{out}(\mathcal{L}_\ell^{\mathrm{geom}})$ has image a free $\mathbb{Q}_\ell$ Lie algebra on generators in degrees $-3, -5, \ldots$.*

15.8. The theorem of Hain-Matsumoto.

Theorem 15.2 (Hain-Matsumoto [26])**.** *The image of the action homomorphism is generated by elements in degrees $-3, -5, \ldots$.*

The proof goes by using the étale realization functor to show that the action homomorphism $\mathcal{L}_{\text{ét}}(\mathbb{Z})_\ell \to \mathrm{End}_\ell(\mathcal{L}_\ell^{\mathrm{geom}})$ factors through the $\mathbb{Q}_\ell$-extension of the motivic action
$\mathcal{L}_{\mathrm{mot}}(\mathbb{Z}) \to \mathrm{End}(\mathrm{Lie}(X, Y))$ via the realization map $\mathcal{L}_{\text{ét}}(\mathbb{Z})_\ell \to \mathcal{L}_{\mathrm{mot}}(\mathbb{Z}) \otimes \mathbb{Q}_\ell$. Since $\mathcal{L}_{\mathrm{mot}}(\mathbb{Z}) = \mathrm{Lie}[s_3, s_5, \ldots]$ the Hain-Matsumoto result follows.

The full Deligne-Ihara conjecture would follow if the motivic action homomorphism were known to be injective. Our computation of the action on the polylog quotient shows that the generators $\{s_{2n+1}\}$ remain independent.

16. Multiple zeta values and periods of mixed Tate motives

In this last section of our lecture on mixed Tate motives, we describe work of Terasoma, Goncharov-Manin and others showing how to realized mulitple zeta values as periods of mixed Tate motives over $\mathbb{Z}$.

16.1. Periods of mixed Tate motives. Classically, a period is an integral of a form of a certain type over some integral or rational homology class. Obviously, a single period in this setting is meaningless, what is interesting is the matrix of all periods of forms of a certain type (usually Hodge type) over a natural subgroup of homology.

However, in the case of mixed Tate motives over a "small" field k_0, the de Rham realization lands in filtered (by the Hodge filtration) k_0 vector spaces, and the Betti realization lands in filtered (by the weight filtration) $\mathbb{Q}$-vector spaces, so periods of one Hodge/weight type will give complex numbers, well-defined modulo $k_0^\times$ and similar periods of lower weight.

One can also take the abstract view, considering periods of mixed Tate motives as the functions (matrix coefficients) on the pro-scheme $\mathrm{Isom}(\mathrm{gr}^W_*, \mathrm{gr}^*_F)$ where gr^W_* is essentially the Betti realization, and gr^*_F is the essentially the de Rham realization, both viewed as fiber functors.

16.2. Periods of framed mixed Tate motives. To avoid some of the ambiguity, one can take the period of a *framed* mixed Tate motive (M, f, v). We can assume $v \in \mathrm{gr}^W_n M = \mathbb{Q}(-n)$, and that $M = W_n M$. We can similarly assume that $\mathrm{gr}^W_0 M = W_0 M = \mathbb{Q}(0)$, and $f \in \mathrm{gr}^W_0 M^*$.

We take the Hodge realization, giving the framed mixed Hodge structure $(\mathrm{Hdg}M, v, f)$ (this doubles the index for W). Since $\mathrm{gr}^W_n M = \mathbb{Q}(-n)$ and $M = W_n M$, $F^n\mathrm{Hdg}M \to \mathrm{gr}^W_{2n}\mathrm{Hdg}M_{\mathbb{C}}$ is an isomorphism. So lift v uniquely to $\tilde{v}_{\mathbb{C}} \in F^n\mathrm{Hdg}M$.

Since $\mathrm{gr}^W_0 M = W_0 M$, there is a splitting $\pi : \mathrm{Hdg}M_{\mathbb{Q}} \to \mathrm{gr}^W_0\mathrm{Hdg}M$ to the inclusion. Define the period of (M, f, v) to be $(f \circ \pi)_{\mathbb{C}}(\tilde{v})$. The only ambiguity is the choice of π, so the period is well-defined modulo periods of weight $< n$.

16.3. Multiple zeta values. We have seen that the zeta-values $\zeta_{\mathbb{Q}}(2n+1)$ are periods of the polylog mixed Tate motive.

Definition 16.1. For natural numbers $k_1, \ldots, k_r$, $k_r \geq 2$, the *multiple zeta value* is

$$\zeta(k_1, \ldots, k_r) := \sum_{1 \leq n_1 < \ldots < n_r} \frac{1}{n_1^{k_1} \cdot \ldots \cdot n_r^{k_r}}$$

The *weight* of the MZV is $\sum_j k_j$.

Conjecture 16.2. *$\{$periods of framed mixed Tate motives over $\mathbb{Z}\}$ is the $\mathbb{Q}$-subspace of $\mathbb{C}$ spanned by the multiple zeta values and powers of $2\pi i$. $\{$periods of weight $\leq n\}$ is the subspace of weight $\leq n$.*

16.4. $\zeta(k_1, \ldots, k_r)$ as a period. Goncharov-Manin [22] have used the compactified moduli space $\bar{\mathcal{M}}_{0,n}$ to show that $\zeta(k_1, \ldots, k_r)$ is a period of a MTM. This result was first proved by Terasoma [58] by a different method. The starting point for Goncharov-Manin is:

$$\zeta(k_1, \ldots, k_r) = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} \omega_{k_1, \ldots, k_r}$$

where $n = \sum_j k_j$ and

$$\omega_{k_1, \ldots, k_r} := \frac{dt_1}{1 - t_1} \cdot \prod_{j=2}^{k_1} \frac{dt_j}{t_j} \cdot \ldots \cdot \frac{dt_{n-k_r+1}}{1 - t_{n-k_r+1}} \cdot \prod_{j=n-k_r+2}^{n} \frac{dt_j}{t_j}.$$

The basic facts about $\bar{\mathcal{M}}_{0,n}$:

$\bar{\mathcal{M}}_{0,n}$ is the moduli space of *stable n-pointed genus zero curves*: each point

$$[C, x_1, \ldots, x_n] \in \bar{\mathcal{M}}_{0,n}(\mathbb{C})$$

corresponds to a union C of smooth $\mathbb{P}^1$'s, joined together so that C has only ordinary double points as singularities, and no cycles, together with an n-tuple $(x_1, \ldots, x_n)$ of smooth points of C, modulo isomorphism.

The union of the C's give the *universal curve* $\bar{\mathcal{C}}_{0,n} \to \bar{\mathcal{M}}_{0,n}$; the points $x_1, \ldots, x_n$ give n sections $\sigma_i : \bar{\mathcal{M}}_{0,n} \to \bar{\mathcal{C}}_{0,n}$.

$\bar{\mathcal{M}}_{0,n}$ is a smooth projective variety. $\bar{\mathcal{M}}_{0,n+1} = \bar{\mathcal{C}}_{0,n}$ with $\bar{\mathcal{C}}_{0,n+1}$ a blow-up of $\bar{\mathcal{C}}_{0,n} \times_{\bar{\mathcal{M}}_{0,n}} \bar{\mathcal{C}}_{0,n}$. The new section σ_{n+1} is the proper transform of the diagonal.

The *open moduli space* $\mathcal{M}_{0,n}$ is exactly the scheme of n distinct points on $\mathbb{P}^1$, modulo isomorphisms of $\mathbb{P}^1$. Sending (x_1, x_2, x_n) to $(0, 1, \infty)$ gives

$$\mathcal{M}_{0,n} = (\mathbb{P}^1)^{n-3} \setminus ([\cup_{i<j} t_i = t_j] \cup [\cup_i t_i = 0] \cup [\cup_i t_i = 1] \cup [\cup_i t_i = \infty]).$$

The universal curve over $\mathcal{M}_{0,n}$ is just $\mathcal{M}_{0,n} \times \mathbb{P}^1$, with the evident sections.

The inclusions $\mathcal{M}_{0,n} \to \bar{\mathcal{M}}_{0,n}$, $\mathcal{M}_{0,n} \to (\mathbb{P}^1)^{n-3}$ extend to a birational morphism $\bar{\mathcal{M}}_{0,n} \to (\mathbb{P}^1)^{n-3}$, which can be described as an sequence of blow-ups with smooth center.

16.5. Divisors at infinity and Stasheff's associahedron. The complement $D := \bar{\mathcal{M}}_{0,n+3} \setminus \mathcal{M}_{0,n+3}$ is a divisor with normal crossing. Writing a point of $\mathcal{M}_{0,n+3}$ as $(0, t_1, \ldots t_n, 1, \infty)$ the real points $\mathcal{M}_{0,n}(\mathbb{R})$ has connected components corresponding to a choice of order on the values $0, t_1, \ldots, t_n, 1$ (with $0 < 1$) .

The standard order yields the component $\mathcal{M}_{0,n+3}(\mathbb{R})_0$: $0 < t_1 < \ldots < t_n < 1$, whose closure Δ is exactly the region over which we integrate to compute $\zeta(k_1, \ldots, k_r)$. Let B be the components of D which meet Δ.

Δ is a polyhedron, whose faces are in 1-1 correspondence with "faces" of B, by taking Zariski closure. Thus we have

$$[\Delta] \in \mathrm{gr}_0^W H^n(\bar{\mathcal{M}}_{0,n+3}, B)^*.$$

The various faces of Δ have an amusing combinatorial description: The codimension d faces are in 1-1 correspondence with the ways of inserting d "associators" $(-)$ into the word $0t_1 \ldots t_n 1$ so that the result makes sense as a partial association of the product. Inclusions of faces correspond to removing associators $(-)$. Thus Δ is Stasheff's "associahedron" of dimension n.

In terms of the blow-up $\bar{\mathcal{M}}_{0,n+3} \to (\mathbb{P}^1)^n$, the associators describe the limiting behavior of the general point $(0, t_1, \ldots t_n, 1, \infty)$ as the values come together. All the t_i associated with 0 have limit zero, and the relative order of vanishing of the differences $t_{i+1} - t_i$ or $t_i - 0$ gets larger as one goes deeper into the nest of parentheses:

$$(0(t_1 t_2))1 \Longrightarrow \nu_0(t_2 - t_1) > \nu_0(t_1) > 0.$$

Clearly $\omega_{k_1,\ldots,k_r}(t_1, \ldots, t_n)$ is a regular differential n form on $\mathcal{M}_{0,n+3}$, $n = \sum_j k_j$. In fact a direct calculation shows:

Lemma 16.3. *$\omega_{k_1,\ldots,k_r}$ has at worst log poles along D and is regular at each generic point of B.*

Thus there is a divisor $A := A_{k_1,\ldots,k_r} \subset D$ such that

1. A and B have no components in common
2. $\omega_{k_1,\ldots,k_r}$ defines an element in $\mathrm{gr}_{2n}^W H^n(\bar{\mathcal{M}}_{0,n+3} \setminus A)$.

Also:

Lemma 16.4. *The maps*

$$\mathrm{gr}_{2n}^W H^n(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A) \to \mathrm{gr}_{2n}^W H^n(\bar{\mathcal{M}}_{0,n+3} \setminus A)$$
$$\mathrm{gr}_0^W H^n(\bar{\mathcal{M}}_{0,n+3}, B) \to \mathrm{gr}_0^W H^n(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A)$$

are isomorphisms.

We can put this altogether, yielding:

Theorem 16.5 (Goncharov-Manin)**.** *Let $k_1, \ldots, k_r$ be natural numbers with $k_r \geq 2$. Let $n = \sum_j k_j$.*

(1) There is a MTM $H^n_{\mathrm{mot}}(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A) \in \mathrm{MT}(\mathbb{Z})$ with Hodge realization

$H^n(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A)$.

(2) The period of $(H^n_{\mathrm{mot}}(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A), [\Delta], [\omega])$ *is* $\zeta(k_1, \ldots, k_r)$.

Proof. Suppose we have (1). By the last lemma, $(H^n_{\mathrm{mot}}(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A), [\Delta], [\omega])$ is a well-defined framed MTM. The period is just $\int_\Delta \omega = \zeta(k_1, \ldots, k_r)$, showing (2).

For (1): Keel [33] shows that all the strata of $D_n \subset \bar{\mathcal{M}}_{0,n}$, and all the locii blown up to form $\bar{\mathcal{M}}_{0,n} \to (\mathbb{P}^1)^{n-3}$, are products of $\bar{\mathcal{M}}_{0,n'}$'s for $3 \le n' < n$. Since $\bar{\mathcal{M}}_{0,3} = \mathbb{P}^1$, this shows $M_{\mathrm{gm}}(\bar{\mathcal{M}}_{0,n})$ and $M_{\mathrm{gm}}(E)$ are in $\mathrm{DMT}(\mathbb{Q})$ for all n and for all subdivisors $E \subset D_n$.

By the Gysin sequence, the open motives $M_{\mathrm{gm}}(\bar{\mathcal{M}}_{0,n+3} \setminus A)$, $M_{\mathrm{gm}}(B \setminus A)$ are in $\mathrm{DMT}(\mathbb{Q})$ for all n and all $k_1, \ldots, k_r$. The relative motive $M_{\mathrm{gm}}(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A)$ is defined by the distinguished triangle

$$M_{\mathrm{gm}}(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A) \to M_{\mathrm{gm}}(\bar{\mathcal{M}}_{0,n+3} \setminus A) \to M_{\mathrm{gm}}(B \setminus A) \to$$

hence is also in $\mathrm{DMT}(\mathbb{Q})$.

$\mathbb{Q}$ satisfies the $K(\pi, 1)$ conjecture, so $\mathrm{DMT}(\mathbb{Q}) = D^b(\mathrm{MT}(\mathbb{Q}))$, and we have

$$H^n_{\mathrm{mot}}(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A) := H^n(M_{\mathrm{gm}}(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A)) \in \mathrm{MT}(\mathbb{Q})$$

with the correct Hodge realization.

$\bar{\mathcal{M}}_{0,n+3}$ is the extension to $\operatorname{Spec}\mathbb{Q}$ of a smooth proper moduli scheme $\bar{\mathcal{M}}_{0,n+3,\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$; the divisor at infinity D extends to a relative strict normal crossing divisor $D_{\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$. Thus, the same argument shows that the ℓ-adic étale realization of $H^n_{\mathrm{mot}}(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A)$ is unramified at all primes $p \neq \ell$; this shows $H^n_{\mathrm{mot}}(\bar{\mathcal{M}}_{0,n+3} \setminus A, B \setminus A)$ is in $\mathrm{MT}(\mathbb{Z})$. $\square$

Remarks 16.6. (1) MZV's of weight $\le n$ arise from periods of MTM's over $\mathbb{Z}$ of weight-length $\le n$, which are in turn representations of $\mathrm{Lie}[s_3, s_5, \ldots]/(s_{2m+1}, 2m+1 > n)$. This gives the bound (proved by Terasoma)

$$\dim_{\mathbb{Q}}\{\text{MZV's of weight } \le n\} \le d_n$$

where $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and $d_{i+3} = d_{i+1} + d_i$. Equality is *Zagier's conjecture* [70].

(2) The framed MTM's over $\mathbb{Z}$ constructed as above for different choices of ω, B and n fit together to form a Hopf algebra over $\mathbb{Q}$; F. Brown [13] has described this Hopf algebra combinatorially using triangulated plane polygons, and has shown that all the periods that arise this way are in the MZV subalgebra of $\mathbb{C}$.

Lecture 4. Moving Lemmas

In our fourth lecture, we consider five different moving lemmas which play a central role in the construction of the motivic categories. The moving lemmas are where the algebraic geometry plays its most important role; there will be essentially no categorical input in these constructions. This will finish up the last bit of material that we put off from Lecture 2.

The Chow-type moving lemmas are the most well-known. These go back to the projecting cone technique devised by Chow [16] to show that cycle-intersection gives a well-defined ring structure on the Chow groups of a smooth projective variety (see [56] for a modern treatment); this was extended to smooth quasi-projective varieties by Chevalley [15]

We use essentially the same geometric argument to generalize this result to Bloch's cycle complexes. The Friedlander-Lawson moving lemma is a beautiful extension of this technique, which allows for a simultaneous moving in bounded families of cycles. These moving lemmas form the basis for the functoriality of the higher Chow groups and the duality theorems for bivariant cycle cohomology, respectively.

Voevodsky's moving lemma is another descendent of the projecting cone argument, refined and modernized through Quillen's proof of Gersten's conjecture [53] and Gabber's presentation lemma. I view this result as a moving lemma because it says that the inclusion $\mathrm{Spec}\,(\mathcal{O}_{X,x}) \to X$ for x a smooth point on a variety X, can be moved (within $\mathbb{A}^1$ homotopy equivalence) by a finite correspondence to avoid any given divisor on X.

Suslin's moving lemma is quite different in flavor. His construction moves a cycle on $X \times \Delta^n$ to be equi-dimensional over Δ^n (for affine X) by a family of translations in Δ^n, varying over X. This leads to a proof that Bloch higher cycle complex is quasi-isomorphic to the subcomplex formed by the Suslin complex of equi-dimensional cycles, i.e., that the higher Chow groups compute motivic cohomology, at least for affine X.

Our last moving lemma is yet another departure from the classical method. Bloch's method of moving by blowing up faces in Δ^n is the central construction for proving the fundamental localization sequence for the higher Chow groups. I have found this technique quite useful in other settings, such as studying Voevodsky's slice filtration in $\mathcal{SH}(k)$ (see [38]).

17. Chow-type moving lemma

17.1. Projections. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n and let $L \subset \mathbb{P}^N$ be a $\mathbb{P}^{N-n-1}$, with $X \cap L = \emptyset$. The $\mathbb{P}^{N-n}$'s containing L form a $\mathbb{P}^n$, so we have the morphism

$$\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^n.$$

Restricting to X gives the finite morphism

$$\pi := \pi_{L,X} : X \to \mathbb{P}^n.$$

Let Z be an irreducible subvariety of X. Define the cycle

$$\tilde{L}(Z) := \pi^*(\pi_*(1 \cdot Z)) - Z.$$

$\tilde{L}(Z)$ is an effective cycle, well-defined since π is finite and $\mathbb{P}^n$ is smooth. The operation $\tilde{L}$ extends by linearity to cycles, mapping effective cycles to effective cycles.

For A a closed subset of X of pure dimension a with irreducible components $A_1, \ldots, A_s$ set

$$\tilde{L}(A) = \mathrm{supp}\,(\sum_i \tilde{L}(A_i)).$$

For a k-scheme T of finite type, we let $\dim T$ denote the maximum of the dimensions of the irreducible components of T; we set $\dim \emptyset = -1$.

17.2. Chow's lemma. Let A, B be closed subsets of X of pure dimension a, b.

Definition 17.1. The *excess* of A, B is

$$e(A, B; X) := \dim(A \cap B \cap X_{\mathrm{reg}}) - \max(a + b - n, -1)$$

(or 0 if $A \cap B \cap X_{\mathrm{reg}} = \emptyset$).

An irreducible component Z of $A \cap B$ with $\dim Z > a + b - n$ is a *component of excess intersection.*

Lemma 17.2. *For A, B as above with excess e, the set of L such that $e(\tilde{L}(A), B) \leq \max(e-1, 0)$ contains a non-empty open subscheme of Grass$(N-n-1, N)$.*

Proof. The "bad" L are those which intersect either the the union of the secant lines from $\overline{A \cap X_{\mathrm{reg}}}$ to $\overline{B \cap X_{\mathrm{reg}}}$ in dimension $> \max(a+b-n, -1)$ or (if $A \cap B \cap X_{\mathrm{reg}} \neq \emptyset$) the closure of the union of the tangent $\mathbb{P}^n$'s to X_{reg} at some $x \in A \cap B \cap X_{\mathrm{reg}}$ in dimension $> \dim A \cap B \cap X_{\mathrm{reg}} - 1$.

A dimension count shows that the secant set has dimension $\leq a + b + 1$, and the tangent set has dimension $\leq n + \dim A \cap B \cap X_{\mathrm{reg}}$. Thus a general L intersects these in the allowed dimension. $\square$

Remark 17.3. For general L, all the excess intersection components of $\tilde{L}(A) \cap B$ are contained in $A \cap B \cap R_L$, where R_L is the ramification locus of $\pi_L : X \to \mathbb{P}^n$. This follows from the dimension count for the secant variety and the fact that if x is in $\tilde{L}(A) \setminus R_L \cap A$ then there is a $y \in \overline{A \cap X_{\mathrm{reg}}}$, $y \neq x$, with $\pi_L(y) = \pi_L(x)$.

Indeed, we may suppose that $A = \overline{A \cap X_{\mathrm{reg}}}$. Let $I = [X \times_{\mathbb{P}^n} X]_{\mathrm{red}} \subset X \times X$, let $\Delta \subset X \times X$ be the diagonal, and let I^0 be the closure of $I \setminus \Delta$. Then

$$R_L = p_1(I^0 \cap \Delta) = p_2(I^0 \cap \Delta).$$

and for general L,

$$\tilde{L}(A) = p_2(A \times X \cap I^0).$$

Thus

$$\tilde{L}(A) \cap A = p_2(A \times A \cap I^0).$$

So, if x is in $\tilde{L}(A) \cap A$, and the only $y \in A$ with $\pi_L(y) = \pi_L(x)$ is $y = x$, then we must have $(x, x) \in A \times A \cap I^0$, hence x is in R_L.

17.3. Chow's moving lemma.

Theorem 17.4 (Chow's moving lemma)**.** *Let Z, W be cycles on X. Then Z is rationally equivalent to a cycle Z' such that $e(Z, W) = 0$, i.e., each component of excess intersection is contained in $X_{\rm sing}$.*

Proof. Suppose $e(Z, W) > 0$. By lemma 17.2, $e(L(Z), W) < e(Z, W)$ for general L.

For a general $g \in \mathrm{GL}_{n+1}$, $\pi^*(g \cdot \pi(Z))$ has 0 excess with W. GL_{n+1} is an open subscheme of $M_{n+1,n+1} = \mathbb{A}^{(n+1)^2}$, so $g \sim_r \mathrm{id}$, hence

$$\pi^*(g \cdot \pi_*(Z)) \sim_r \pi^*(\pi_* Z)$$

Thus

$$Z = \pi^*(\pi_* Z) - \tilde{L}(Z) \sim_r \pi^*(g \cdot \pi_*(Z)) - \tilde{L}(Z)$$

This lowers e by 1. Repeating the process gives the desired Z'. □

17.4. Variants on Chow's moving lemma. For affine X, one can replace $\mathbb{P}^N$ with $\mathbb{A}^N$, replacing π_L its restriction $\pi_L : \mathbb{A}^N \to \mathbb{A}^n$. By Noether normalization, for general L, the restriction to $X \subset \mathbb{A}^N$ is a finite morphism $\pi : X \to \mathbb{A}^n$. The same proof as above gives the analogous result.

One can also refine the rational equivalence to a full $\mathbb{A}^1$ family of group elements $g(t) \in \mathrm{GL}_{n+1}(k[t])$, corresponding to a linear family of translations in $\mathbb{A}^n \subset \mathbb{P}^n$.

In fact, as we are only considering the excess of the intersection taking place in the smooth locus of X, we can take U to be any smooth quasi-projective variety, X the closure of U in some $\mathbb{P}^n$ and then the moving lemma for X yields a moving lemma for U.

Somewhat more tricky is to show that, if we have cycles A, A', B on U such that A and A' intersect B properly, and A and A' are rationally equivalent on U, then $A \cdot B$ and $A' \cdot B$ are rationally equivalent (here U is smooth and quasi-projective). This is shown in [15] using the following trick: Let $C = A - A'$. Then there is a cycle $\bar{C}$ on the projective closure X of U such that $\bar{C} \cap U = C$ and $\bar{C} \sim_{\rm rat} 0$ on X. Indeed, for any closure $\bar{C}'$ of C, we have $\bar{C}'\mathrm{rat}D$ for some D supported on $X \setminus U$, and we can simply take $\bar{C} = \bar{C}' - D$.

Armed with this lifting, Chevalley shows that the projecting cone applied to a given $\mathbb{P}^1$ family of cycles $\bar{\mathcal{C}}$ with $\bar{\mathcal{C}}(0) = 0$, $\bar{\mathcal{C}}(\infty) = \bar{C}$, yields a rational equivalence of $\pi_L^*\pi_{L*}(\bar{C})$ to 0. Thus, one can move the rational equivalence in a series of steps so that each member of the resulting $\mathbb{P}^1$ family of cycles intersects B properly on U, giving the desired rational equivalence of $A \cdot B$ with $A' \cdot B$.

In somewhat more detail, Chevalley proves the following result:

Lemma 17.5. *Let X be a projective variety over an infinite field k, $U \subset X$ an open subscheme of X, contained in the smooth locus $X_{\rm reg}$, and $\mathcal{C}$ a a finite collection of closed subsets of U. Let $\mathcal{Z} \subset X \times \mathbb{A}^1$ be a cycle , flat over $\mathbb{A}^1$, with $\mathcal{Z}(0) = 0$, where $\mathcal{Z}(t) := \mathcal{Z} \cdot X \times t$. Then there is an open subscheme V of $\mathbb{A}^1$, containing $\{0, 1\}$ and positive cycles $\mathcal{Y}^+, \mathcal{Y}^-$ on $\mathbb{A}^1 \times \mathbb{A}^1 \times X$, and for each point $t \in \mathbb{A}^1$, positive cycles $\mathcal{A}_t^+, \mathcal{A}_t^-$ on $\mathbb{A}^1 \times t \times X$ such that*

1. *$\mathcal{Y}^+, \mathcal{Y}^-$ are flat over V, and $\mathcal{A}_t^+, \mathcal{A}_t^-$ are flat over $V \cap \mathbb{A}^1 \times t$ for each t.*
2. *$\mathcal{Z} = res_{0\times\mathbb{A}^1}\mathbb{Y}^+ - \mathcal{Y}^-$*
3. *$res_{\mathbb{A}^1\times 0}\mathcal{Y}^+ = res_{\mathbb{A}^1\times 0}\mathcal{Y}^-$, $\mathcal{A}_0^+ = \mathcal{A}_0^-$.*
4. *For $s \neq 0$, $(s, t) \in V \times \mathbb{A}^1$, $\mathcal{Y}^\pm(s, t)$ and $\mathcal{A}_t^\pm$ have excess 0 with respect to $\mathcal{C}$.*

(5) *For all* $t \in \mathbb{A}^1$, $\mathcal{A}_t^{\pm}(0)$ *have excess* $\leq e_{\mathcal{C}}(\mathcal{Z}(t))$.

In (3), the cycles $res_{\mathbb{A}^1 \times 0}\mathcal{Y}^{\pm}$ *are the respective closures in* $\mathbb{A}^1 \times 0 \times X$ *of the restriction of* $\mathcal{Y}^{\pm}$ *to* $V \times 0 \times X$.

Proof. Suppose X is in $\mathbb{P}^N$, and let $n = \dim X$.

Choose a general rational curve $C \subset \mathrm{GL}_{n+1}$ passing through id. Choose a general projecting linear space $L \cong \mathbb{P}^{N-n-1}$. For $s \in C$, we have the operation on cycles on X

$$W \mapsto \pi_L^*(s \cdot \pi_{L*}(W))$$

which we denote by $\tau(s)(W)$. More generally, for W a cycle on $Y \times X$, we define the cycle $\tau(s)(W)$ on $Y \times W$ by a similar formula

$$\tau(s)(W) := (\mathrm{id} \times \pi_L)^*(\mathrm{id} \times s \cdot (\mathrm{id} \times \pi_{L*}(W))$$

For W a cycle on $Y \times X$, we let $\tau(W) \subset \mathbb{A}^1 \times Y \times X$ be the closure of the cycle $\tau(W)^0$ on $C \times Y \times X$ with $\tau(W)^0 \cap s \times Y \times X = \tau(s)(W)$ for all $s \in C$.

We have already defined the cycle $L(W) := \pi_L^*(\pi_{L*}(W)) - W$; for W a cycle on $Y \times X$ we have the cycle on $Y \times X$,

$$L(W) := (\mathrm{id} \times \pi_L)^*(\mathrm{id} \times \pi_{L*}(W)).$$

One constructs $\mathcal{Y}^{\pm}$, $\mathcal{A}_t^{\pm}$ inductively as follows: For each t, write $\mathcal{Z}(t)$ as a difference of positive cycles

$$\mathcal{Z}(t) = \mathcal{Z}_t^+ - \mathcal{Z}_t^-$$

with $\mathcal{A}_t^{0+}$ and $\mathcal{A}_t^{0-}$ having no common components. Similarly, write $\mathcal{Z}$ as a difference of positive cycles

$$\mathcal{Z} = \mathcal{Z}^+ - \mathcal{Z}^-$$

also without common components. We take

$$\mathcal{Y}^{0\pm} := \mathbb{A}^1 \times \mathcal{Z}^{\pm}; \quad \mathcal{A}_t^{0\pm} := \mathbb{A}^1 \times \mathcal{Z}_t^{\pm}.$$

Suppose we have constructed families $\mathcal{Y}^{i\pm}$, $\mathcal{A}_t^{i\pm}$ for $i = 0, \dots, m-1$. Define

$$\begin{aligned}
\tilde{\mathcal{Y}}^{m+} &:= \tau(\mathcal{Y}^{m-1+}) + \mathbb{A}^1 \times L(\mathcal{Y}^{m-1-}) \\
\tilde{\mathcal{Y}}^{m-} &:= \tau(\mathcal{Y}^{m-1-}) + \mathbb{A}^1 \times L(\mathcal{Y}^{m-1+}) \\
\tilde{\mathcal{A}}_t^{m+} &:= \tau(\mathcal{A}_t^{m-1+}) + \mathbb{A}^1 \times L(\mathcal{A}_t^{m-1-}) \\
\tilde{\mathcal{A}}_t^{m-} &:= \tau(\mathcal{A}_t^{m-1+}) + \mathbb{A}^1 \times L(\mathcal{A}_t^{m-1-}),
\end{aligned}$$

where we allow ourselves to use a new curve C and a new linear space L each time.

Choose a general $D \subset \mathbb{A}^1 \times \mathbb{A}^1$, $D \cong \mathbb{A}^1$, connecting $(0,0)$ and $(1,1)$. The cycle $\tilde{\mathcal{Y}}^{m+}$ is a cycle on $\mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \times X$; we restrict to $D \times \mathbb{A}^1 \times X$ to form $\mathcal{Y}^{m+}$. The cycles $\mathcal{Y}^{m-}$, $\mathcal{A}^{m\pm}$ are defined similarly (all using the same D).

It follows from lemma 17.2 that after at most n steps, we have achieved the desired construction. □

Now, given a cycle $\bar{C}$ on X and a rational equivalence $\mathcal{Z} \subset \mathbb{A}^1 \times X$, $\mathcal{Z}(1) = \bar{C}$, $\mathcal{Z}(0) = 0$, we apply the above lemma to give the families $\mathcal{Y}^{\pm}$, $\mathcal{A}_t^{\pm}$. Assuming that $\bar{C}$ is already in good position with respect to B, it follows that the families $\mathcal{A}_1^{\pm}$ are in good position with respect to B, and the difference $\mathcal{A}_1^+ - \mathcal{A}_1^-$ defines a rational equivalence of $\bar{C} \cdot B$ with $(\mathcal{A}_1^+(1) - \mathcal{A}_1^-(1)) \cdot B$. But

$$\mathcal{A}_1^+(1) - \mathcal{A}_1^-(1) = \mathcal{Y}^+(1,1) - \mathcal{Y}^-(1,1)$$

and $\mathcal{Y}^{\pm}(1,1)$ are in good position with respect to B. Also, the families $res_{1\times\mathbb{A}^1\times X}\mathcal{Y}^{\pm}$ are in good position with respect to B and the difference of the two defines a rational equivalence of $(\mathcal{Y}^{+}(1,1)-\mathcal{Y}^{-}(1,1))\cdot B$ with $(\mathcal{Y}^{+}(1,0)-\mathcal{Y}^{-}(1,0))\cdot B$. But $\mathcal{Y}^{+}(1,0)-\mathcal{Y}^{-}(1,0)=0$, hence $(A-A')\cdot B\sim_r 0$.

17.5. Chow's moving lemma for $z^q(X,*)$.

Definition 17.6. Let X be an equi-dimensional k-scheme. Let $\mathcal{C}$ be a finite set of irreducible locally closed subsets of X_{reg}. $e:\mathcal{C}\to\mathbb{Z}_+$ a function. Then

$$z^q(X,*)_{\mathcal{C},e}\subset z^q(X,*)$$

is the subcomplex generated by irreducible $W\in z^q(X,n)$ such that for each $C\in\mathcal{C}$ and each face F of Δ^n

$$\mathrm{codim}_{C\times F}(W\cap C\times F)\geq q-e(C).$$

Theorem 17.7. *Suppose X is smooth and is either affine or projective. The inclusion $z^q(X,*)_{\mathcal{C},e}\to z^q(X,*)$ is a quasi-isomorphism.*

Proof. We prove the projective case with k infinite. The proof of this moving lemma is in two parts:

Part 1: $X=\mathbb{P}^n$. An $\mathbb{A}^1$-family $g(t)\in\mathrm{GL}_{n+1}(k[t])$ gives a "partially defined" homotopy $h:z^q(X,*)\to z^q(X,*+1)$ between id and translation by $g(1)$. The homotopy h is defined by viewing $g(t)$ as a map

$$g:X\times\mathbb{A}^1\to X$$

Pulling back cycles on $X\times\Delta^n$

$$g\times\mathrm{id}_{\Delta^n}:X\times\mathbb{A}^1\times\Delta^n\to X\times\Delta^n$$

and triangulating $\mathbb{A}^1\times\Delta^n=\Delta^1\times\Delta^n$ by the standard sum of Δ^{n+1}'s gives the partially defined map $h_n:z^q(X,n)\to z^q(X,n+1)$; the map is only partially defined because we have intorduced new faces when we formed the triangulation. More generally, if $g(t)$ is defined over a larger field $K\supset k$, we have partially defined map $h_n:z^q(X,n)\to z^q(X_K,n+1)$.

An $\mathbb{A}^1$ family of translations in $\mathbb{P}^n$ can defined by a vector $v\in\mathbb{A}^{n+1}$ and a linear form H; the translation $g(t)$ is then

$$g(t)([x]):=[x+tH(x)v].$$

The generic $\mathbb{A}^1$ family of translations $g(t)\in\mathrm{GL}_{n+1}(K[t])$ is thus defined over a pure transcendental extension $K=k(t_1,\ldots,t_s)$ of k. Since $g(t)$ is generic, the maps

$$h_n:z^q(X,n)\to z^q(X_K,n+1)$$

are defined on all of $z^q(X,n)$, giving a well-defined homotopy h. Since H and v are generic, $T^*_{g(1)}$ maps $z^q(X,*)$ to $z^q(X_K,*)_{\mathcal{C}}$ and h gives a homotopy of $T^*_{g(1)}$ with the base-extension $z^q(X,*)\to z^q(X_K,*)$.

Thus base-extension

$$\frac{z^q(X,*)}{z^q(X,*)_{\mathcal{C},e}}\to\frac{z^q(X_K,*)}{z^q(X_K,*)_{\mathcal{C},e}}$$

is homotopic to zero.

However, base-extension is injective on homotopy groups: if $x \mapsto 0$ over K, x goes to zero in $X \times U$ for some open $U \subset \operatorname{Spec} k[t_1, \dots, t_s]$. Since k is infinite, we can restrict to a "general enough" point $u \in U(k)$ to show that $x = 0$.

Thus $z^q(X, *)/z^q(X, *)_{\mathcal{C},e}$ is acyclic.

Part 2: the general case.

Since $z^q(X_K, *)_{\mathcal{C},n} = z^q(X_K, *)$, we need to show that

$$z^q(X_K, *)_{\mathcal{C},e} \to z^q(X_K, *)_{\mathcal{C},e-1}$$

is a quasi-isomorphism.

Use the generic L to define $\pi : X \to \mathbb{P}^n_K$ for a pure trancendental extension K of k and let $\mathcal{D} = \{\pi(C), C \in \mathcal{C}\}$. We have

$$\frac{z^q(X, *)_{\mathcal{C},e}}{z^q(X, *)_{\mathcal{C},e-1}} \xrightarrow{\pi_*} \frac{z^q(\mathbb{P}^n_K, *)_{\mathcal{D},e}}{z^q(\mathbb{P}^n_K, *)_{\mathcal{D},e-1}} \xrightarrow{\pi^*} \frac{z^q(X_K, *)_{\mathcal{C},e}}{z^q(X_K, *)_{\mathcal{C},e-1}}$$

and

$$\tilde{L} : \frac{z^q(X, *)_{\mathcal{C},e}}{z^q(X, *)_{\mathcal{C},e-1}} \to \frac{z^q(X_K, *)_{\mathcal{C},e}}{z^q(X_K, *)_{\mathcal{C},e-1}}$$

Then $\pi^*\pi_* = \text{base-extension} + \tilde{L}$. But $\tilde{L} = 0$ so $\pi^*\pi_* = \text{base-extension}$.

From part 1, $z^q(\mathbb{P}^n_K, *)_{\mathcal{D},e}/z^q(\mathbb{P}^n_K, *)_{\mathcal{D},e-1}$ is acyclic, so base-extension induces 0 on homology. As in part 1, base-extension is injective on homology, so the complex $z^q(X, *)_{\mathcal{C},e}/z^q(X, *)_{\mathcal{C},e-1}$ is acyclic. □

17.6. Functoriality I. Let $f : Y \to X$ be a morphism with $X \in \mathbf{Sm}/k$. Let $z^q(X, *)_f$ be the subcomplex of $z^q(X, *)$ generated by irreducible $W \in z^q(X, n)$ such that $(f \times \mathrm{id}_{\Delta^n})^*(W)$ is defined and is in $z^q(Y, n)$. Clearly the maps $(f \times \mathrm{id}_{\Delta^n})^*$ define the map of complexes

$$f^* : z^q(X, *)_f \to z^q(Y, *).$$

Theorem 17.8. *Let $f : Y \to X$ be a morphism in $\mathbf{Sm}/k$, with X either affine or projective. Then the inclusion $z^q(X, *)_f \to z^q(X, *)$ is a quasi-isomorphism.*

Proof. There are locally closed subsets $C_i \subset X$, $i = 1, \dots, s$ and non-negative integers $e(C_i)$ such that

$$z^q(X, *)_f = z^q(X, *)_{\mathcal{C},e}.$$

By the moving lemma $z^q(X, *)_{\mathcal{C},e} \to z^q(X, *)$ is a quasi-isomorphism. □

Remark 17.9. Full functoriality requires localization or Mayer-Vietoris. These facts require Bloch's technique of "moving by blow-ups".

18. Friedlander-Lawson: moving in families

Friedlander-Lawson improve Chow's moving lemma to move *families* of effective cycles. They alter the two parts of the moving lemma as follows:

The projection: Instead of using the linear projection $\pi_L : X \to \mathbb{P}^n$, they use a map of high degree

$$\pi_f(x) := (f_0(x) : \dots : f_n(x))$$

where the f_i are homogeneous polynomials of degree $d >> 0$. $\pi_f : X \to \mathbb{P}^n$ is still finite for general f. This is just a linear projection after re-embedding $X \subset \mathbb{P}^N$ via a Veronese embedding of degree d, $\mathbb{P}^N \to \mathbb{P}^M$.

For $Z \subset X$ set $\tilde{f}(Z) = \pi_f^*(\pi_{f*}(Z)) - Z$ as before. The basic result is:

Lemma 18.1. *Let Z, W be cycles of dimension r, s on X. Suppose $r + s \geq n$.Let $c(Z, W, d)$ denote the codimension in $H^0(\mathbb{P}^N, \mathcal{O}(d))^{n+1}$ of the set of f such that $e(\tilde{f}(Z), W) \geq e(Z, W)$. Then $c(Z, W, d)$ goes to infinity as $d \to \infty$.*

Proposition 18.2. *Let $\mathcal{Z} \subset T \times X$ be an effective cycle, equi-dimensional over T of relative dimension r. Let W be a dimension s cycle on X. Suppose $r+s \geq n$. Then for $d >> 0$, a general f of degree d has $e(\tilde{f}(\mathcal{Z}(t)), W) \leq \max(e(\mathcal{Z}(t), W) - 1, 0)$ for all $t \in T$.*

Indeed, just take $d >> 0$ so that, for each $t \in T$ the bad locus for $\mathcal{Z}(t)$ has codimension $> \dim T$, so the union of all the bad locii still has positive codimension in $H^0(\mathbb{P}^N, \mathcal{O}(d))^{n+1}$.

The translation: Translating by a $g \in \mathrm{GL}_{n+1}$ can be described in projective geometry as: Fix $L = \mathbb{P}^{N-n-1}$ and a complementary $\mathbb{P}^n$ in $\mathbb{P}^N$. Let L' be another $\mathbb{P}^{N-n-1}$ with $L' \cap \mathbb{P}^n = \emptyset$, and let $\mathbb{P}'$ be a second $\mathbb{P}^n$ with $L \cap \mathbb{P}' = L' \cap \mathbb{P}' = \emptyset$.

Take $x \in \mathbb{P}^n$, giving the $\mathbb{P}^{N-n}$ $L\#x$ spanned by L and x. Let $y = (L\#x) \cap \mathbb{P}'$ and let $g(x) = (L'\#y) \cap \mathbb{P}^n$. Then $x \mapsto g(x)$ is given by translation by $g \in \mathrm{GL}_{n+1}$, and each translation arises this way.

This construction generalizes by replacing $\mathbb{P}'$ with a complete intersection $\mathbf{D} := D_1 \cap \ldots \cap D_{N-n}$, D_i a hypersurface of degree d_i. For a cycle Z on $\mathbb{P}^n$, let

$$\phi_{\mathbf{D},L'}(Z) := \pi_{L'*}(L\#Z \cdot \mathbf{D}).$$

Given a cycle W in $\mathbb{P}^n$ the codimension of the "bad" $(\mathbf{D}, L')$ goes to infinity as $d_1, \ldots, d_{N-n}$ gets large:

Proposition 18.3. *Given families $\mathcal{Z} \subset T \times \mathbb{P}^n$, $\mathcal{W} \subset S \times \mathbb{P}^n$ of cycles of dimenison r, s with $r + s \geq n$, a general choice of $L_1, \ldots, L_n$, $\mathbf{D}_1, \ldots, \mathbf{D}_n$ for all sufficiently large multi-degrees satisfies:*

$\phi_{\mathbf{D}_n,L_n} \circ \ldots \circ \phi_{\mathbf{D}_1,L_1}(\mathcal{Z}(t))$ *intesects $\mathcal{Y}(s)$ properly for all $t \in T$, $s \in S$.*

Remark 18.4. $\phi_{\mathbf{D},L'}(Z)$ is NOT rationally equivalent to Z, but to $\mathbf{d} \cdot Z$, with $\mathbf{d} = \prod_i d_i$. To correct this, one takes a linear combination $a \cdot \phi_{\mathbf{D}_1,L_1} - b \cdot \phi_{\mathbf{D}_2,L_2}$ with $a\mathbf{d}_1 - b\mathbf{d}_2 = 1$.

The rational equivalence $\phi_{\mathbf{D},L'}(Z) \sim_r \mathbf{d} \cdot Z$ is constructed by degenerating each D_i to $d_i \cdot \mathbb{P}^{N-n}$. The argument used to prove the proposition shows that the rational equivalence can be chosen to give a family of maps $\phi_{\mathbf{D}_j(u),L_j}$ for $u \in U$, some open neighborhood of $0 \in \mathbb{A}^1$, so that

a. $\phi_{\mathbf{D}_j(1),L_j}(Z) = \mathbf{d}_i \cdot Z$.
b. The good intersection property of the proposition holds for all $u \in U$.

This gives the desired "rational equivalence in families".

One applies these results with $\mathcal{Z} =$ the complete family of cycles of dimension r and degree a, and $\mathcal{W} =$ the complete family of cycles of dimension s and degree b, that is T and S are the respective Chow varieties $Chow(r, a, N)$, $Chow(s, b, N)$.

18.1. The duality theorem. Our main application of the Friedlander-Lawson moving lemma is the proof of the duality quasi-isomorphism used in Lecture 2.

Recall the Nisnevich sheaf with transfers $z_r^{\mathrm{equi}}(X)$ ($X \in \mathbf{Sch}_k$), $z_r^{\mathrm{equi}}(X)(V)$ the free abelian group on subvarieties $W \subset X \times V$ which are equi-dimension of dimension r over V.

For $X \in \mathbf{Sch}_k$, $U \in \mathbf{Sm}/k$, recall as well the Nisnevich sheaf with transfers $z_r^{\mathrm{equi}}(U, X)$: $V \mapsto z_r^{\mathrm{equi}}(X)(U \times V)$.

Finally, for a presheaf with transfers $\mathcal{F}$, we have the Suslin complex $C_*(\mathcal{F})$ with homology presheaves $h_i^{\mathrm{Nis}}(\mathcal{F})$ homotopy invariant presheaves with transfers.

The duality theorem we will prove is:

Theorem 18.5 (Duality)**.** *Suppose k admits resolution of singularities. Take $X \in \mathbf{Sch}_k$, $U \in \mathbf{Sm}/k$ quasi-projective of dimension n. Then the inclusion $\mathcal{D} : z_r^{\mathrm{equi}}(U, X) \to z_{r+n}^{\mathrm{equi}}(X \times U)$ induces a quasi-isomorphism of complexes on $\mathbf{Sm}/k_{\mathrm{Zar}}$:*

$$C_*(z_r^{\mathrm{equi}}(U, X))_{\mathrm{Zar}} \to C_*(z_{r+n}^{\mathrm{equi}}(X \times U))_{\mathrm{Zar}}$$

18.2. Effective cycles. The moving lemma used in the proof requires the sheaf of monoids $z_r^{\mathrm{equi}}(X)^{\mathrm{eff}}$, with $z_r^{\mathrm{equi}}(X)^{\mathrm{eff}}(V)$ the free abelian *monoid* on subvarieties $W \subset X \times V$ which are equi-dimension of dimension r over V.

For $X \subset \mathbb{P}^N$ projective, let $z_r^{\mathrm{equi}}(X, \leq d)^{\mathrm{eff}} \subset z_r^{\mathrm{equi}}(X)^{\mathrm{eff}}$ be the subsheaf (of sets) of cycles of degree $\leq d$, i.e. $W \in z_r^{\mathrm{equi}}(X)^{\mathrm{eff}}(U)$ is in $z_r^{\mathrm{equi}}(X, \leq d)^{\mathrm{eff}}(U)$ if each fiber $W(x)$ has degree $\leq d$. Let $z_r^{\mathrm{equi}}(X, \leq d)$ be the presheaf of abelian groups generated by $z_r^{\mathrm{equi}}(X, \leq d)^{\mathrm{eff}}$

Let $z_r^{\mathrm{equi}}(Y, X, \leq d)(U) \subset z_r^{\mathrm{equi}}(Y, X)(U)$ be the subgroup generated by the effective $Z \in z_r^{\mathrm{equi}}(Y, X)(U)$ with $\mathcal{D}(Z) \in z_{r+\dim Y}^{\mathrm{equi}}(X \times Y, \leq d)(U)$.

18.3. The effective moving lemma.

Theorem 18.6. *$X \subset \mathbb{P}^N$ smooth and projective of dimension n over k, $a, b, r, s \geq 0$ integers with $r + s \geq n$. Then there are homomorphisms of abelian monoids*

$$H_U^+, H_U^- : z_r^{\mathrm{equi}}(X)^{\mathrm{eff}}(U) \to z_r^{\mathrm{equi}}(X)^{\mathrm{eff}}(U \times \mathbb{A}^1)$$

natural in $U \in \mathbf{Sm}/k$, satisfying: Let W be an effective cycle of dimension s and degree $\leq e$, and take $Z \in z_r^{\mathrm{equi}}(X, \leq d)^{\mathrm{eff}}(U)$. Then for all $u \in U$

(1a) all excess components of $H_U^{\pm}(Z)(u, 0) \cap W$ are contained in $Z(u) \cap W$
(1b) $H_U^+(Z)(u, 0) - H_U^-(Z)(u, 0) = Z(u)$.
2. For all $t \neq 0$, $H_U^{\pm}(Z)(u, t)$ intersects W properly.

Idea of proof. This follows from Friedlander-Lawson moving, by iterating the operations $Z \mapsto \tilde{f}(Z)$ and $\phi_{\mathbf{D},L}$, then separating the + and − parts. Even though the dimension of U is arbitrarily large, one need only construct $H^{\pm}$ for the universal families over $Chow(r, a, X)$ and $Chow(s, b, X)$.

The rational equivalence given by Friedlander-Lawson is only defined on an open neighborhood V of $0 \in \mathbb{A}^1$, but this is converted to a family over $\mathbb{A}^1$ by taking transfers with respect to $\Gamma \in \mathrm{Cor}(\mathbb{A}^1, V)$ with fiber $1 \cdot 0$ over $0 \in V$. $\square$

18.4. Duality for smooth projective varieties. We can now use the Friedlander-Lawson moving lemma to prove our first duality result.

Theorem 18.7. *Let X, Y be smooth projective varieties over k. Then the inclusion*

$$\mathcal{D} : z_r^{\mathrm{equi}}(Y, X) \to z_{r+\dim Y}^{\mathrm{equi}}(X \times Y)$$

induces a a quasi-isomorphism

$$C_*(z_r^{\mathrm{equi}}(Y, X)) \to C_*(z_{r+\dim Y}^{\mathrm{equi}}(X \times Y)).$$

Proof. Let $e = \deg Y$, $s = \dim Y$. We apply the effective moving lemma to family of cycles $x \times Y \subset X \times Y$: For an effective cycle W on $X \times Y \times U$ equi-dimensional of dimension $r + \dim Y$ over U, W is in $z_r^{\mathrm{equi}}(Y, X)(U)$ if and only if $W \cap x \times Y \times u$ has 0 excess for all (x, u).

The maps $H^{\pm}_{U \times \Delta^*}$ of the effective moving lemma give a homotopy of the natural map

$$C_*(\frac{z_{r+\dim Y}^{\mathrm{equi}}(X \times Y, \leq d)}{z_r^{\mathrm{equi}}(Y, X, \leq d)})(U) \to C_*(\frac{z_{r+\dim Y}^{\mathrm{equi}}(X \times Y)}{z_r^{\mathrm{equi}}(Y, X)})(U)$$

with the zero map. Passing to the limit over d, this implies that the quotient complex $C_*(\frac{z_{r+\dim Y}^{\mathrm{equi}}(X \times Y)}{z_r^{\mathrm{equi}}(Y,X)})(U)$ is acyclic. $\square$

18.5. Proof of the duality theorem. We now handle duality for general X. To shorten the proof, we just do the case of smooth and quasi-projective X.

Take smooth projective compactifications $Y \subset \bar{Y}$, $X \subset \bar{X}$, let $n = \dim Y$. We have the commutative diagram

$$\begin{tikzcd}
\Phi \arrow[r] \arrow[d] & z_r^{\mathrm{equi}}(Y, X) \arrow[d, "\mathcal{D}"] \\
z_{r+n}^{\mathrm{equi}}(\bar{Y} \times \bar{X}) \arrow[r, "\alpha"'] & z_{r+n}^{\mathrm{equi}}(Y \times X)
\end{tikzcd}$$

α is the map $Z \in z_{r+n}^{\mathrm{equi}}(\bar{Y} \times \bar{X})(V) \mapsto Z \cap (Y \times X \times V)$.

Φ is defined to make the diagram cartesian: $\Phi(V)$ is the group of cycles Z on $\bar{Y} \times \bar{X} \times V$, equi-dimensional over V, such that $\alpha(Z)$ is equi-dimensional over $Y \times V$.

The same proof as for the duality theorem for smooth projective varieties shows:

$$C_*(\Phi) \to C_*(z_{r+n}^{\mathrm{equi}}(\bar{Y} \times \bar{X}))$$

is a quasi-isomorphism of presheaves.

We have the commutative diagram of presheaves

$$\begin{tikzcd}
0 \arrow[r] & ker_1 \arrow[r] \arrow[d, "a"'] & \Phi \arrow[r] \arrow[d, "b"'] & z_r^{\mathrm{equi}}(Y, X) \arrow[r] \arrow[d, "\mathcal{D}"'] & coker_1 \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & ker(\alpha) \arrow[r] & z_{r+n}^{\mathrm{equi}}(\bar{Y} \times \bar{X}) \arrow[r, "\alpha"'] & z_{r+n}^{\mathrm{equi}}(Y \times X) \arrow[r] & coker_2 \arrow[r] & 0
\end{tikzcd}$$

a is an isomorphism and all vertical maps are monomophisms. $C_*(b)$ is a quasi-isomorphism.

$(coker_2)_{\mathrm{cdh}} = 0$ by using "flattening by blow-ups". Thus $(coker_1)_{\mathrm{cdh}} = 0$, and both $C_*(coker_1)_{\mathrm{Zar}}$ and $C_*(coker_2)_{\mathrm{Zar}}$ are acyclic, by the cdh-acyclicity theorem. Therefore $\mathcal{D} : C_*(z_r^{\mathrm{equi}}(Y, X))_{\mathrm{Zar}} \to C_*(z_{r+n}^{\mathrm{equi}}(Y \times X))$ is a quasi-isomorphism.

19. Voevodsky's moving lemma

The theorem we will prove in this section is

Theorem 19.1. *Let X be in $\mathbf{Sm}/k$, $j : U \to X$ an open subscheme, S a finite set of points of X. There there is an open neighborhood $i : V \to X$ of S in X and a $\lambda \in \mathrm{Cor}(V, U)$ such that for all homotopy invariant PST's F, the diagram*

$$\begin{tikzcd} F(X) \arrow[r, "j^*"] \arrow[d, "i^*"'] & F(U) \arrow[dl, "F(\lambda)"] \\ F(V) & \end{tikzcd}$$

commutes.

Corollary 19.2. *The map $j^* : F_{\mathrm{Zar}}(X) \to F_{\mathrm{Zar}}(U)$ is injective. If X is semi-local then $F(X) = F_{\mathrm{Zar}}(X)$.*

The proof may be viewed as a variation of Quillen's proof of Gersten's conjecture for higher K-theory [53], with the technique strengthened by the use of Gabber's presentation lemma.

19.1. Gabber's presentation lemma.

Lemma 19.3. *Let X be an irreducible normal affine k-scheme of finite type and dimension n over k, S a finite set of smooth points of X, $i : D \to X$ a divisor. Then there is a commutative diagram*

$$\begin{tikzcd} D \arrow[r, "i"] \arrow[d, "q_0"'] & X \arrow[r, "j"] \arrow[d, "q"'] & \bar{X} \arrow[d, "\bar{q}"] & \bar{X}_\infty \arrow[l, "i_\infty"'] \arrow[d, "\bar{q}_\infty"] \\ \mathbb{A}^{n-1} \arrow[r, shift right, "0"'] & \mathbb{A}^{n-1} \times \mathbb{A}^1 \arrow[l, shift right, "\pi"'] \arrow[r, "j'"'] & \mathbb{P} \arrow[ll, bend left, "\bar{\pi}"] & \mathbb{P}_\infty \arrow[l, "i'"] \end{tikzcd}$$

such that

1. j and j' are open immersions, q, q_0, $\bar{q}$ and $\bar{q}_\infty$ are finite.

2. In a neighborhood of S, q is étale and $\pi \circ q$ is smooth.

2. $\bar{X} \setminus X = \bar{X}_\infty = \bar{q}_\infty^{-1}(\mathbb{P}_\infty)$, $\mathbb{P} \setminus \mathbb{P}_\infty = \mathbb{A}^{n-1} \times \mathbb{A}^1$.

3. $\bar{\pi} : \mathbb{P} \to \mathbb{A}^{n-1}$ is a $\mathbb{P}^1$-bundle with disjoint sections 0 and ∞.

4. $\bar{X}$ is normal and there is an affine open $U \supset D \cup \bar{X}_\infty$.

Proof. (Assume k infinite). Take $X' \supset X$ the projective closure of $X \subset \mathbb{A}^N \subset \mathbb{P}^N$, with $D' \supset D$ the projective closure of D. Extend a general linear projection $\pi' : D \to \mathbb{A}^{n-1}$ to $\bar{\pi}' : D' \to H_0 := \mathbb{P}^{n-1}$. Add one more linear form L to give a "general" projection

$$(\pi' : L) : X' \to \mathbb{P}^n$$

which restricts to a finite map $X \to \mathbb{A}^n$. We may assume that π' is smooth in a neighborhood of S, so $(\pi' : L)$ is étale in a neighborhood of S for general L.

Blow up a general point of $H_\infty := \mathbb{P}^n \setminus \mathbb{A}^n$ in $\mathbb{P}^n$ and remove the proper transform of H_∞ to get the $\mathbb{P}^1$-bundle $\bar{\pi} : \mathbb{P} \to \mathbb{A}^{n-1}$. Take $\bar{X}$ to be the normalization of

$\mathbb{P} \times_{\mathbb{P}^n} X'$. $\mathbb{P}_\infty$ is the restriction of the exceptional divisor of the blow-up to $\mathbb{P}$; $\mathbb{P}_\infty$ defines the section ∞.

Let $H_0 \subset \mathbb{P}^n$ be a second general hyperplane; the pull-back of H_0 to $\mathbb{P}$ gives the section 0. If H_0, H_∞ are defined by linear forms L_0, L_∞, then using the linear function $\ell := L_0/L_\infty$ we have the isomorphism

$$(\bar{\pi}, \ell) : \mathbb{P} \setminus \mathbb{P}_\infty \to \mathbb{A}^{n-1} \times \mathbb{A}^1.$$

Let H be a general hyperplane containing $H_0 \cap H_\infty$, let $V := \mathbb{P}^n \setminus H$. Then take $U := \bar{q}^{-1}(V)$. $\square$

19.2. The proof of Voevodsky's moving lemma. We are given an open immersion in $\mathbf{Sm}/k$, $j : U \to X$, and finite set of points S of X, and we need to find a neighborhood V of S and a finite correspondence $\lambda \in \mathrm{Cor}(V, U)$ with certain properties.

We can replace U with any open subset $U' \subset U$ and use the pushforward $\mathrm{Cor}(V, U') \to \mathrm{Cor}(V, U)$, so we may replace X with an affine neighborhood of S and U with $X \setminus D$, D a divisor on X.

We can apply Gabber's presentation lemma:

$$\begin{tikzcd}
D \arrow[r, "i"] \arrow[d, "q_0"'] & X \arrow[r, "j"] \arrow[d, "q"'] & \bar{X} \arrow[d, "\bar{q}"] & \bar{X}_\infty \arrow[l, "i_\infty"'] \arrow[d, "\bar{q}_\infty"] \\
\mathbb{A}^{n-1} \arrow[r, shift right, "0"'] & \mathbb{A}^{n-1} \times \mathbb{A}^1 \arrow[l, shift right, "\pi"'] \arrow[r, "j'"'] & \mathbb{P} \arrow[ll, bend left, "\bar{\pi}"] & \mathbb{P}_\infty \arrow[l, "i'"]
\end{tikzcd}$$

For V an open neighborhood of S, form the pull-back of this diagram via $\pi : V \to \mathbb{A}^{n-1}$. Let $\Delta \subset V \times_{\mathbb{A}^{n-1}} X$ be the graph of the inclusion $V \to X$.

Since $X \to \mathbb{A}^{n-1}$ is a smooth curve in the neighborhood of S, we may assume that $V \to \mathbb{A}^{n-1}$ is smooth, so $X_V := V \times_{\mathbb{A}^{n-1}} X$ is also in $\mathbf{Sm}/k$, and thus Δ is a Cartier divisor on X_V. $X_V \to \bar{X}_V$ is an open immersion and $\Delta \cap \bar{X}_{V\infty} = \emptyset$. Taking V affine, there is an affine open neighborhood W of $D_V \cup \bar{X}_{V\infty}$ in $\bar{X}_V$.

Since $D_V \to V$ is finite, we may shrink V so that $\mathcal{O}(\Delta)$ is trivial in a neighborhood of D_V. Thus there is a regular function s on W with $s \equiv 1$ on $\bar{X}_{V\infty}$ and $\mathrm{Div} s = \Delta + \Delta'$, with $\Delta' \cap (D_V \cup \bar{X}_{V\infty}) = \emptyset$.

Extend s to a rational function f on $\bar{X}_V$. Then $\mathrm{Div} f = \Delta + \Delta''$, with $\Delta'' \cap (D_V \cup \bar{X}_{V\infty}) = \emptyset$, so $\Delta'' \subset V \times U$ is in $\mathrm{Cor}(V, U)$. Also, f is regular in a neighborhood of $D_V \cup \bar{X}_{V\infty}$ and $f \equiv 1$ on $\bar{X}_{V\infty}$.

Let g be the rational function on $\bar{X}_V \times \mathbb{A}^1$, $g = t + (1-t)f$. Then g is regular in the neighborhood $W \times \mathbb{A}^1$ of $\bar{X}_{V\infty} \times \mathbb{A}^1$ and $g \equiv 1$ on $\bar{X}_{V\infty} \times \mathbb{A}^1$.

Thus $\Delta \times \mathbb{A}^1 - \mathrm{Div} g$ is a cycle on $X_V \times \mathbb{A}^1 \subset X \times V \times \mathbb{A}^1$, finite over $V \times \mathbb{A}^1$, with fiber $-\Delta''$ over $t = 0$ and Δ over $t = 1$.

Let $\lambda = -\Delta'' \in \mathrm{Cor}(V, U)$, $i : V \to X$, $j : U \to X$ the inclusions. Then for F a PST,

$$i^* = F(\Delta) : F(X) \to F(V)$$
$$F(j_*(\lambda)) = F(\lambda) \circ j^* : F(X) \to F(V),$$

and for F a homotopy invariant PST, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "j^*"] \arrow[d, "i^*"'] & F(U) \arrow[dl, "F(\lambda)"] \\ F(V) & \end{tikzcd}$$

commutes.

20. Suslin's moving lemma

Suslin's moving lemma allows us to compare the complex of equi-dimensional cycles with the larger cycle complex of Bloch. The main result we prove in this section is:

Theorem 20.1 (Suslin [63, chapter 6]). *Suppose k admits resolution of singularities. Let X be a quasi-projective over k. Then the inclusion*

$$C_*(z_r^{\mathrm{equi}}(X))(\operatorname{Spec} k) \to z_r(X,*)$$

is a quasi-isomorphism for all $r \leq \dim X$.

20.1. Reduction to affine X: Mayer-Vietoris.

Proposition 20.2. *Suppose k admits resolution of singularities, and let $X = U \cup V$ be a Zariski open cover. Then:*

(1) The sequence

$$C_*(z_r^{\mathrm{equi}}(X))_{\mathrm{Zar}} \to C_*(z_r^{\mathrm{equi}}(U))_{\mathrm{Zar}} \oplus C_*(z_r^{\mathrm{equi}}(V))_{\mathrm{Zar}} \to C_*(z_r^{\mathrm{equi}}(U \cap V))_{\mathrm{Zar}}$$

extends to a distinguished triangle in $D^-(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}(k)))$.

(2) The sequence

$$z_r(X,*) \to z_r(U,*) \oplus z_r(V,*) \to z_r(U \cap V,*)$$

extends to a distinguished triangle in $D^-(\mathbf{Ab})$.

Proof. For Bloch's cycle complex, this follows from the localization theorem (to be proved later in this lecture).

For $z_r^{\mathrm{equi}}(-)$: The sheaf sequence

$$0 \to z_r^{\mathrm{equi}}(X) \to z_r^{\mathrm{equi}}(U) \oplus z_r^{\mathrm{equi}}(V) \to z_r^{\mathrm{equi}}(U \cap V) \to coker \to 0$$

is exact and $coker_{\mathrm{cdh}} = 0$. By the cdh-acyclicity theorem (really corollary 9.4) $C_*(coker)_{\mathrm{Zar}}$ is acyclic. Thus $C_*(coker)(\operatorname{Spec} k) = C_*(coker)_{\mathrm{Zar}}(\operatorname{Spec} k)$ is acyclic. □

20.2. The main idea. Recall $z_r^{\mathrm{equi}}(X,*) := C_*(z_r^{\mathrm{equi}}(X))(\operatorname{Spec} k)$. Clearly we have $z_r^{\mathrm{equi}}(X,0) = z_r(X,0)$. Take a finite subset $W = \{W_i\} \subset z_r(X,1)$, write $X = \operatorname{Spec} A$

Identify $(\Delta_1,(0,1),(1,0))$ with $(\mathbb{A}^1,0,1)$. Let $h : X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ be a morphism over X: $h(x,t) = (x, f(x,t))$ for some $f : X \times \mathbb{A}^1 \to \mathbb{A}^1$. Assume $f(x,0) \equiv 0$, $f(x,1) \equiv 1$, so $f \in A[T]$ is just an element of $T + (T(1-T))A[T]$. We take $f = T + T(1-T)g$, $g \in A$.

Lemma 20.3. *For all g of sufficiently high degree, $h^*(W_i)$ is in $z_r^{\mathrm{equi}}(X,1)$ for all i.*

Proof. By embedding $X \subset \mathbb{A}^m$, can replace X with $\mathbb{A}^m$, so "degree" makes sense for g. Let $(q_1(X,T),\ldots,q_r(X,T))$ be the ideal of W_i.

$W_i \cap \mathbb{A}^m \times \{0,1\}$ has dimension r, so $W_i \to \mathbb{A}^1$ is equi-dimensional over $\{0,1\}$. The fiber of $h^*(W_i)$ over $t \in \mathbb{A}^1$ has dimension $> r$ iff $h(\mathbb{A}^m \times t)$ contains a dense subset of W_i iff

$$q_j(X, t + t(1-t)g(X)) \equiv 0$$

for all j. Writing $q_j(X,T) = \sum_{i=0}^N q_{j,i}(X)T^i$, with $q_{j,i}(X)$ all of degree $\leq M$, the above identities are impossible if g has degree $> M$ and $t \neq 0, 1$. $\square$

In addition, from the explicit nature of the construction, h^* is $\mathbb{A}^1$-homotopic to the identity: Given some $h = (\mathrm{id}, f)$, $f = t + t(1-t)g$ as above, we have the $\mathbb{A}^1$ family

$$h(s) := (\mathrm{id}, t + st(1-t)g).$$

For sufficiently high degree g as above, $h(s)$ satisfies the condition of the lemma for all $s \neq 0$.

20.3. The inductive construction. We continue the construction inductively. Start with a collection of finitely generated subgroups $W(i) \subset z_r(X,i)$, $i = 0,\ldots,n$ closed under d. Suppose we have for each i an $\mathbb{A}^1$ family of maps over X,

$$h_i(s) : X \times \Delta^i \to X \times \Delta^i; \quad i = 0,\ldots,n-1,$$

such that

1. $h_i(0) = \mathrm{id}$, $h_0 = \mathrm{id}$
2. $h_i(s)^*(W(i)) \subset z_r^{\mathrm{equi}}(X,i)$ for all $s \neq 0$, $i = 1,\ldots,n-1$
3. For $1 \leq i \leq n-1$, and for each face map $\delta_j : \Delta^{i-1} \to \Delta^i$,

$$(\mathrm{id} \times \delta_j) \circ h_{i-1}(s) = h_i(s) \circ (\mathrm{id} \times \delta_j).$$

Let $\partial\Delta^n := \cup_{j=0}^{n+1} \delta_j(\Delta^{n-1})$. By (3) there is a unique $\mathbb{A}^1$ family of maps over X:

$$\partial h_n(s) : X \times \partial\Delta^n \to X \times \partial\Delta^n$$

with $(\mathrm{id} \times \delta_j) \circ h_{n-1}(s) = \partial h_n(s) \circ (\mathrm{id} \times \delta_j)$ for all j.

Using an elaboration of the degree considerations used for the case of Δ^1, Suslin proves:

Lemma 20.4. *For all "sufficiently general" $\mathbb{A}^1$ families $h_n(s) : X \times \Delta^n \to X \times \Delta^n$ over X extending $\partial h_n(s) \cup \mathrm{id}_{s=0}$, we have*

$$h_n(s)^*(W(n)) \subset z_r^{\mathrm{equi}}(X,n)$$

for all $s \neq 0$.

We consider each h_n as a map

$$h_n : X \times \Delta^n \times \Delta^1 \to X \times \Delta^n.$$

Triangulate $\Delta^n \times \Delta^1$ in the usual way via maps $t_{n,i} : \Delta^{n+1} \to \Delta^n \times \Delta^1$ and let

$$H_n := \sum_i \pm(t_{n,i}\times)^* \circ h_n^* : z_{r+n}(\Delta^n \times X) \to z_{r+n+1}(\Delta^{n+1} \times X).$$

The H_n define a homotopy of the identity with 0 on finitely generated subcomplexes of

$$\frac{z_r(X,*)}{z_r^{\mathrm{equi}}(X,*)},$$

which shows that $z_r^{\mathrm{equi}}(X,*) \to z_r(X,*)$ is a quasi-isomorphism.

21. Bloch's moving lemma

Bloch proves the localization theorem for $z_r(-,*)$ by a process of "moving by blow-ups". In this section, we outline this technique and sketch the proof of:

Theorem 21.1 (Bloch [11]). *Let $j : U \to X$ be an open immersion in $\mathbf{Sch}_k$. Let $z_r(U_X,*) \subset z_r(U,*)$ be the image of*

$$j^* : z_r(X,*) \to z_r(U,*)$$

Then $z_r(U_X,*) \to z_r(U,*)$ is a quasi-isomorphism.

This has as nearly immediate consequence the localization theorem for the higher Chow groups.

Corollary 21.2 (localization). *Let $j : U \to X$ be an open immersion in $\mathbf{Sch}_k$ with closed complement $i : W \to X$. Then the sequence*

$$z_r(W,*) \xrightarrow{i_*} z_r(X,*) \xrightarrow{j^*} z_r(U,*)$$

extends canonically to a distinguished triangle in $D^-(\mathbf{Ab})$.

Indeed, the sequence

$$0 \to z_r(W,*) \xrightarrow{i_*} z_r(X,*) \xrightarrow{j^*} z_r(U_X,*) \to 0$$

is term-wise exact.

Localization in turn yields the Mayer-Vietoris property.

Corollary 21.3 (Mayer-Vietoris). *Let $X = U \cup V$ be a Zariski open cover of $X \in \mathbf{Sch}_k$. Then the sequence*

$$z_r(X,*) \to z_r(U,*) \oplus z_r(V,*) \to z_r(U \cap V,*)$$

extends canonically to a distinguished triangle in $D^-(\mathbf{Ab})$.

Proof. Let $W := X \setminus U = V \setminus U \cap V$. By localization, the cones of the restriction maps

$$z_r(X,*) \to z_r(U,*)$$
$$z_r(V,*) \to z_r(U \cap V,*)$$

are both quasi-isomorphic to $z_r(W,*)[1]$. The Mayer-Vietoris sequence follows from this and the octahedral axiom. $\square$

21.1. Blowing up faces: the global picture. Let $\partial S = \sum_i \partial_i S$ be a strict normal crossing divisor on some $S \in \mathbf{Sm}/k$. A *face* of ∂S is an irreducible component of some intersection $\cap_j \partial_{i_j} S$. A *vertex* is a face of dimension 0.

Let F be a face of ∂S, and let $\mu : S' \to S$ be the blow-up of S along F. Then $\mu^{-1}(\partial S)_{\mathrm{red}}$ is again a SNC divisor with irreducible components $\mu^{-1}[\partial_i S]$ and $E := \mu^{-1}(F)$.

Thus we can iterate, forming a *sequence of blow-ups of faces*

$$S_N \to S_{N-1} \to \ldots \to S_1 \to S_0 = S$$

and each S_j has its "boundary divisor" ∂S_j.

Proposition 21.4 (Bloch)**.** *Let $j : U \to X$ be an open subscheme of some $X \in \mathbf{Sch}_k$. Let $W \subset U \times S$ be a pure dimension r closed subset, intersecting $U \times F$ properly for each face F of ∂S. Then*

(1) For each sequence of blow-ups of faces

$$S_N \to S_{N-1} \to \ldots \to S_1 \to S_0 = S;\ \mu : S_N \to S,$$

$(\mathrm{id} \times \mu)^{-1}(W) \subset U \times S_N$ *intersects $U \times F$ properly for each face F of ∂S_N.*

(2) There exists a sequence of blow-ups of faces as in (1) such that $\overline{(\mathrm{id} \times \mu)^{-1}(W)} \subset X \times S_N$ intersects $X \times F$ properly for each face F of ∂S_N.

We will say a word about the proof after describing the local theory of blow-ups of faces.

21.2. Blowing up faces: the local picture. Let $\partial \mathbb{A}^n := \sum_{i=1}^n (x_i = 0) \subset \mathbb{A}^n$, so a face is a linear subspace

$$x_{i_1} = \ldots = x_{is} = 0.$$

There is a single vertex $0 \in \mathbb{A}^n$.

Blow up a face $F :\ x_1 = \ldots = x_i = 0$, $\mu : (S, \partial S) \to (\mathbb{A}^n, \partial \mathbb{A}^n)$. Then

$$\partial S := \mu^{-1}(\partial \mathbb{A}^n)_{\mathrm{red}} = \sum_{j=1}^{i} \mu^{-1}[x_j = 0] + \sum_{j=i+1}^{n} \mu^{-1}(x_j = 0) + F \times \mathbb{P}^{i-1}.$$

$\mu^{-1}(F) = F \times \mathbb{P}^{i-1}$ and ∂S has i new vertices $v_1, \ldots, v_i$ lying over $0 \in \mathbb{A}^n$, with v_j the unique vertex in the complement of $\mu^{-1}[x_j = 0]$.

Let $S_j := S \setminus \mu^{-1}[x_j = 0]$, with coordinates $t_1, \ldots, t_n$:

$$t_a := \begin{cases} x_a & \text{for } a = j, i+1, \ldots, n \\ x_a/x_j & \text{for } a = 1, \ldots, i, a \neq j. \end{cases}$$

These coordinates yield an isomorphism $(S_j, \partial S_j, v_j) \cong (\mathbb{A}^n, \partial \mathbb{A}^n, 0)$, so we can blow up again.

21.3. Hironaka's game. Take an effective divisor $D \subset \mathbb{A}^n$.

Hironaka's game has two players A and B. The game starts by giving the divisor D; the game is only interesting if $0 \in D$. Player A choses a face F of $\partial \mathbb{A}^n$ with $F \subset D$, and blows up $\mathbb{A}^n$ along F. Player B then choses a vertex in the blow-up that is in $\mu^{-1}[D]$, if he can! $(D, \mathbb{A}^n, 0)$ gets replaced with $(\mu^{-1}[D] \cap S_v, S_v, v)$ and another round begins.

A wins when B can't move, i.e., the proper transform of D contains no new vertex.

By a series of clever reductions, Bloch reduces the blow-up proposition to the theorem of Spivikovsky

Theorem 21.5 (Spivikovsky). *Let $D \subset \mathbb{A}^n$ be an effective divisor. Then Player A has a winning strategy for Hironaka's game.*

Idea of proof. The behavior of the proper transform of D under local blow-ups can be described by the linear geometry of polyhedra in $\mathbb{R}^n_+$, by associating to D the Newton polyhedron N_D of the defining equation g of D in $\mathbb{A}^n$: the positive convex hull of the points in $\mathbb{Z}^n_+$ corresponding to the exponents of the monomials appearing in g.

The whole game can be rephrased in terms of explicit linear transformations on the polyhedron N_D, so one can avoid all mention of the divisor D. Spivikovsky gives an explicit algorithm for A's moves, which end in transforming N_D (regardless of how clever B is) to the maximal polyhedron $\mathbb{R}^n_+$. This means that the corresponding D has a non-zero constant term and A wins. $\square$

This finishes the blow-up part of the construction. But there is a problem: If one starts with a cycle W on $U \times \Delta^n$, then after a blow-up, one has a cycle $\mu^{-1}(W)$ on $U \times S$ that extends to a good cycle on $X \times S$. But we wanted W to extend to a good cycle on $X \times \Delta^n$!

21.4. Little cubes. The trick is in the cubical nature of the blow-up. The local blow-up picture shows that the choice of a vertex v in the blow-up gives a coordinate system adapted to the divisors passing through v.

Because of this cubical structure, it is better to start with $(\square^n, \partial\square^n)$ rather than $(\Delta^n, \partial\Delta^n)$, i.e., use the cubical complex $z_r(-,*)^{cube}$ that appeared in our discussion of the motivic cdga. Since we don't require a commutative multiplication, we do not require the alternating projection, so the construction works integrally.

We start with canonical coordinates $(x_1^v, \dots, x_n^v)$ for each vertex $v \in \partial\square^n$ by $x_i = t_i$ or $x_i = t_i^{-1}$ (recall that $\square^n := (\mathbb{P}^1 \setminus \{1\})^n$). Choose a general "center" $c \in \square^n \setminus \partial\square^n$. For each vertex v, form the little cube with coordinates $x^{v,c}$ by

$$x_i^{v,c} := x_i^v / x_i^v(c).$$

Let $\partial^{v,c}\square^n$ be the divisor $\sum_i (x_i^{v,c} = 0) + \sum_i (x_i^{v,c} = 1)$. This divides the big cube $(\square^n, \partial\square^n)$ into 2^n pointed little cubes $(\square^n, \partial^{v,c}\square^n, v)$ each with canonical coordinates $x^{v,c}$. v and c are in "opposite corners" of $\partial^{v,c}\square^n$.

For each blow-up $(S, \partial S) \to (\square^n, \partial\square^n)$ and each vertex w of ∂S, we have the image vertex $v \in \partial\square^n$. The coordinate system $x^{v,c}$ around v induces the coordinate system $x^{w,c}$ around w: For 1 blow-up, this is the local coordinate system as already described. In general, iterate.

In slightly more detail:

Lemma 21.6. *Let $\mu : (S, \partial S) \to (\square^n, \partial\square^n)$ be a sequence of blow-ups of faces, let w be a vertex of ∂S and $v := \mu(w)$.*

(1) The coordinate system $x^{w,c}$ determines a morphism

$$\lambda^{w,c} : (U^{w,c}, \partial_0 U^{w,c}, 0) \to (S, \partial S, w)$$

where $U^{w,c} \subset \square^n$ is an open neighborhood of all the vertices in $\partial\square^n$ and $\partial_0 U^{w,c} := \sum_i (x_i = 0)$.

(2) $\mu \circ \lambda^{w,c}$ extends to a morphism

$$\Lambda^{w,c} : \square^n \to \square^n$$

with $\Lambda^{w,c}(\partial_0\square^n) \subset \partial_v\square^n$, $\partial_v\square^n :=$ the components in $\partial\square^n$ containing v.

To define a map of complexes, one takes a signed sum

$$\phi(S) := \sum_w \mathrm{sgn}(w)\Lambda^{w,c}$$

In this sum, the contributions of the divisor $\partial_1\square^n := \sum_i (x_i = 1)$ cancel out.

This leads to a map of complexes

$$\Phi(S)^* : z_r(U,*)^{cube}_{\mathrm{fin}} \to z_r(U,*)^{cube}_{\mathrm{fin}}$$

Here $z_r(U,*)^{cube}_{\mathrm{fin}}$ is a homotopy colimit over subcomplexes of $z_r(U,*)^{cube}$ generated by finitely many cycles W on $U \times \square^n$:

For each added cycle, one moves the center c into more general position.

Proposition 21.7. *For $\mu : S \to \square^n$, $\Phi(S)^* : z_r(U,*)^{cube}_{\mathrm{fin}} \to z_r(U,*)^{cube}_{\mathrm{fin}}$ has the following properties:*

1. $\Phi(S)^*$ restricts to a map $\Phi(S)^* : z_r(U_X,*)^{cube}_{\mathrm{fin}} \to z_r(U_X,*)^{cube}_{\mathrm{fin}}$

2. The map on the quotients

$$\bar\Phi(S)^* : \frac{z_r(U,*)^{cube}_{\mathrm{fin}}}{z_r(U_X,*)^{cube}_{\mathrm{fin}}} \to \frac{z_r(U,*)^{cube}_{\mathrm{fin}}}{z_r(U_X,*)^{cube}_{\mathrm{fin}}}$$

is homotopic to the identity.

3. Let $\mathcal{W} \subset z_r(U,*)^{cube}$ be a subcomplex generated by cycles $W \subset U \times \square^n$ such that the closure

$$\overline{(\mathrm{id}_U \times \mu)^{-1}(W)} \subset X \times S$$

intersects all faces on $X \times \partial S$ properly. Then $\bar\Phi(S)^*(\mathcal{W}) = 0$.

This plus the blow-up proposition proves the main theorem.

Remarks 21.8. (1) This all works over an arbitrary field k. No resolution of singularities assumption is needed.

(2) Spivikovsky's theorem on Hironaka's polyhedral game generalizes to codimension two subsets of $\mathbb{A}^n_{\mathcal{O}}$, $\mathcal{O}$ a DVR, even in mixed characteristic (see [36]). This extends Bloch's blow-up proposition to mixed characteristic X. The localization theorem extends as well (see [37] for details).

21.5. Functoriality II. Localization extends the functoriality for Bloch's cycle complexes from the affine case to the quasi-projective case.

Theorem 21.9. *Let $X \in \mathbf{Sch}_k$ be quasi-projective, $\mathcal{C}$ a finite set of locally closed subsets of X_{reg}, $e : \mathcal{C} \to \mathbb{Z}_+$ a function. Then the inclusion $z_r(X,*)_{\mathcal{C},e} \to z_r(X,*)$ is a quasi-isomorphism.*

Proof. Let $X \subset \bar{X}$ be a projective compactification, $W := \bar{X} \setminus X$. The same argument as for the localization theorem gives the map of distinguished triangles

$$\begin{tikzcd}
z_r(W,*) \arrow[r, "i_*"] \arrow[d, equal] & z_r(\bar{X},*)_{\mathcal{C},e} \arrow[r, "j^*"] \arrow[d, "\alpha"'] & z_r(X,*)_{\mathcal{C},e} \arrow[r] \arrow[d, "\beta"'] & z_r(W,*)[1] \arrow[d, equal] \\
z_r(W,*) \arrow[r, "i_*"'] & z_r(\bar{X},*) \arrow[r, "j^*"'] & z_r(X,*) \arrow[r] & z_r(W,*)[1]
\end{tikzcd}$$

Since $\mathcal{C}$ is contained in the smooth locus of $\bar{X}$, the arrow α is a quasi-isomorphism by the moving lemma for the projective variety $\bar{X}$, so β is also a quasi-isomorphism. □

Corollary 21.10. *Let $f : Y \to X$ be a morphism in $\mathbf{Sm}/k$. Then there is a pull-back morphism $f^* : z^q(X,*) \to z^q(Y,*)$ in $D^-(\mathbf{Ab})$, with $(fg)^* = g^*f^*$.*

This follows from the theorem, as in the projective/affine case.

Lecture 5. An introduction to motivic homotopy theory

The last two lectures in our series deal with a refinement of the triangulated category of motives, the *motivic stable homotopy category.* This theory, initiated by Morel and Voevodsky [47], is formally parallel to the stable homotopy theory of topological spaces, but has algebraic varieties for its most basic building blocks. More precisely, the motivic stable homotopy category over a field k, $\mathcal{SH}(k)$, contains objects corresponding to smooth schemes over k, as well as objects corresponding to classical spaces (simplicial sets) or, more generally, spectra. These are put together on a more or less equal footing, so that one can even apply standard topological constructions, such as quotient spaces or smash products, to algebraic varieties without any technical difficulties.

The category $\mathcal{SH}(k)$ is the natural place for *bi-graded cohomology theories* on $\mathbf{Sm}/k$ to live. The well-known examples of such: algebraic K-theory and motivic cohomology, are both represented in $\mathcal{SH}(k)$, but these are only the two most well-known of infinitely many examples. The systematic study of motivic homotopy theory is just beginning; we hope this brief introduction and overview will give enough of the flavor of the subject to encourage the reader to look further. We suggest the following sources for a detailed description: [31, 44, 45, 46, 47].

22. A bird's-eye view of classical homotopy theory

We begin with a very quick review of the foundations of classical homotopy theory. Rather than the category of topologicial spaces, we use as starting point the somewhat more combinatorial category of simplicial sets; to keep the topological flavor, these are called *spaces.* We discuss the homotopy theory of this category, which gives us unstable homotopy theory. Stable homotopy theory is constructed by inverting the suspension operator on spaces, giving the category of spectra and the associated homotopy category, the stable homotopy category. This latter category is the home of *generalized cohomology theories* by the Brown representability theorem; extending this picture to the setting of presheaves on the category of smooth varieties gives a well-defined notion of generalized cohomology of algebraic varieties.

22.1. Simplicial sets and the unstable theory. A combinatorial framework for homotopy theory is given by the category of simplicial sets.

Let **Ord** be the category with objects the finite ordered sets $[n] := \{0, 1, \ldots, n\}$, $n = 0, 1, \ldots$ and with morphisms the order-preserving maps.

Definition 22.1. The category **Spc** of *spaces* is the category of functors S : $\mathbf{Ord}^{\mathrm{op}} \to \mathbf{Sets}$, i.e. presheaves of sets on **Ord**, i.e., simplicial sets. Functors to pointed sets gives the category of pointed spaces $\mathbf{Spc}_*$.

As a functor category, **Spc** inherits the basic operations in **Sets** by performing them objectwise, most essentially: limits and colimits.

Simplices and internal Hom. The representable presheaves are

$$\Delta[n] := \mathrm{Hom}(-, [n])$$

There is an *internal Hom*:

$$\mathcal{H}om(A, B)([n]) := \mathrm{Hom}(A \times \Delta[n], B),$$

with a natural isomorphism

$$\mathrm{Hom}(A\times B, C)\cong \mathrm{Hom}(A, \mathcal{H}om(B, C)).$$

The *interval* $(I, 0, 1)$ is $\Delta[1]$ with the two points $0, 1$, given by the maps

$$i_0, i_1 : * = \Delta[0]\to\Delta[1]$$

coming from the two maps $[0]\to[1]$ in **Ord**.

Pointed versions. For (A, a), (B, b) pointed spaces, the coproduct is $A\vee B := A\amalg B/a\sim b$. We have also the *smash product*

$$A\wedge B := A\times B/A\vee B.$$

The pointed internal Hom is

$$\mathcal{H}om_*(A, B)([n]) := \mathrm{Hom}(A\wedge\Delta[n]_+, B)$$

with natural isomorphism

$$\mathrm{Hom}_*(A\wedge B, C)\cong\mathrm{Hom}_*(A, \mathcal{H}om_*(B, C)).$$

The *n-spheres* are

$$S^0 := \{*, 1\}, S^1 := I/\{0, 1\};\ S^n := S^1\wedge\ldots\wedge S^1(n \text{ factors}).$$

Suspension and loops. The two fundamental operations on $\mathbf{Spc}_*$ are *suspension*

$$X\mapsto\Sigma X := X\wedge S^1$$

and the *loop space*

$$X\mapsto\Omega X := \mathcal{H}om_*(S^1, X)$$

with the adjunction isomorphism

$$\mathrm{Hom}_*(\Sigma X, Y)\cong\mathrm{Hom}_*(X, \Omega Y).$$

Geometric realization. Let

$$\Delta_n := \{(t_0, \ldots, t_n)\in\mathbb{R}^{n+1}\mid\sum_i t_i = 1, t_i\geq 0\},$$

with vertices v_i^n: $t_i = 1$, $t_j = 0$ for $j\neq i$.

Sending n to Δ_n extends to a functor (i.e a cosimplicial topological space)

$$\Delta : \mathbf{Ord}\to\mathbf{Top} :$$

for $g : [n]\to[m]$, $\Delta(g) : \Delta_n\to\Delta_m$ is the convex linear extension of the map on vertices $v_i^n\mapsto v_{g(i)}^m$.

Definition 22.2. For $A\in\mathbf{Spc}$, the *geometric realization* $|A|$ is the CW complex

$$|A| := \amalg_n A([n])\times\Delta_n/\sim$$

with $(x, \Delta(g)(t))\sim(A(g)(x), t)$ for $g : [n]\to[m]$ in **Ord**, $x\in A([m])$, $t\in\Delta_n$.

For example, $|\Delta[n]| = \Delta_n$.

Homotopy and homotopy equivalence.

Definition 22.3. (1) Two maps $f, g : A \to B$ are *simply homotopic* if there is a map $h : A \times I \to B$ with $f = h \circ i_0$, $g = h \circ i_1$. Maps f and g are *homotopic* if they are related by a chain of simple homotopies, written $f \sim_h g$.

(2) A map $f : A \to B$ is a *homotopy equivalence* if there is a map $g : B \to A$ with $gf \sim_h \mathrm{id}_A$ and $fg \sim_h \mathrm{id}_B$.

There are pointed versions of (1) and (2) as well.

(3) The *homotopy groups* of a simplicial set X are defined as the homotopy groups of the geometric realization: $\pi_n(X) := \pi_n(|X|)$.

(4) A map $f : A \to B$ is a *weak homotopy equivalence* if f induces an isomorphism on π_n for all n and all choices of base-point.

Warning! A homotopy equivalence is a weak homotopy equivalence but not always the other way around: The map $[I \amalg I/0 \sim 0, 1 \sim 1] \to I/\{0,1\}$ collapsing the second I to $*$ is a weak equivalence but has no homotopy inverse. Similarly, one can define simplicial homotopy groups as homotopy classes of maps $S^n \to X$, but this does not in general agree with the homotopy groups of the geometric realization.

However ... Let $\Lambda_{n,i} \subset \Delta[n]$ be the sub-simplicial set of maps $f : [m] \to [n]$ such that $[n] \setminus \{i\} \not\subset f([m])$. $|\Lambda_{n,i}| \subset \Delta_n$ is the union of faces $t_j = 0$, $j \neq i$.

Definition 22.4. A map of spaces $f : X \to Y$ is a *Kan fibration* if there exists a lifting $\dashrightarrow$ for every commutative diagram

$$\begin{tikzcd} \Lambda_{n,i} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

If f is also a weak equivalence, call f a *trivial fibration.* X is *fibrant* if $X \to *$ is a Kan fibration.

A map of spaces $i : A \to B$ is a *cofibration* if $i_n : A([n]) \to B([n])$ is a monomorphism for all n, a *trivial cofibration* if i is also a weak equivalence.

Theorem 22.5 (Kan). *(1) Let $f : X \to Y$ be a map of fibrant spaces. Then f is a weak equivalence if and only if f is a homotopy equivalence.*

(2) Let X be fibrant. Then maps $f, g : A \to X$ are simply homotopic if and only if they are homotopic.

(3) If X is fibrant then the simplicial homotopy groups agree with the homotopy groups of $|X|$, where the simplicial homotopy groups of X with base-point $x_0 \in X([0])$ are defined as the pointed maps $S^n \to (X, x_0)$ modulo pointed homotopy. Similarly, the set of simplicial path components of X is the same as the set of path components of $|X|$.

(4) Consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

(RLP) f is a fibration iff a lifting exists for all trivial cofibrations i.
(LLP) i is a cofibration iff a lifting exists for all trivial fibrations f.

22.2. A word on model categories. The categories **Spc** and $\mathbf{Spc}_*$ are examples of *simplicial model categories*. Without going into details, in a simplicial model category $\mathcal{M}$ one has:

- Arbitrary small limits and colimits, hence an initial object $\emptyset$ and a final object $*$.
- Three distinguished classes of morphisms: cofibrations, fibrations and weak equivalences
- An extension of the usual Hom-sets to Hom-simplicial sets, $\mathcal{H}om$.
- Operations $X \mapsto X^K$ and $X \mapsto X \times K$ for $X \in \mathcal{M}$, $K \in \mathbf{Spc}$.

These of course satisfy a number of axioms, which we won't specify here. For details, see the book by Hovey [30].

The homotopy category.

Definition 22.6. Let $\mathcal{M}$ be a simplicial model category with weak equivalences $w\mathcal{M}$. The *homotopy category* $\mathcal{HM}$ is the category $\mathcal{M}[w\mathcal{M}^{-1}]$.

Remark 22.7. As in theorem 1.7(ii), the functor $\mathcal{HM} \to \mathcal{M}[w\mathcal{M}^{-1}]$ is defined as the universal functor inverting the morphisms in $w\mathcal{M}$. As model categories are almost never small categories, a construction of $\mathcal{M}[w\mathcal{M}^{-1}]$ via left or right fractions, or even its existence, is not completely obvious. One reason Quillen invented model categories is to show that homotopy categories $\mathcal{M}[w\mathcal{M}^{-1}]$ exist and to give a (hopefully) manageable description of the morphisms.

Fibrant and cofibrant replacements.

Definition 22.8. An object A in a a model category $\mathcal{M}$ is *cofibrant* if the canonical map $\emptyset \to A$ is a cofibration. $Y \in \mathcal{M}$ is *fibrant* if $Y \to *$ is a fibration.

It follows from the axioms that for each $X \in \mathcal{M}$, there is a weak equivalence $f : X_c \to X$ with X_c cofibrant and f a fibration. There is also a natural weak equivalence $i : X \to X_f$ with i a cofibration and X_f fibrant. $X_c \to X$ is the *cofibrant replacement* of X and $X \to X_f$ is the *fibrant replacement* of X.

Formally, $X_c \to X$ is like a projective resolution, and $X \to X_f$ is like an injective resolution.

The main theorem of homotopical algebra. The simplicial structure on $\mathcal{H}om(A, B)$ gives a notion of homotopy of maps and homotopy equivalence of objects: the set of maps $A \to B$ in the underlying category is just the set of vertices $\mathcal{H}om(A, B)_0$ and the set of homotopy classes in $\mathrm{Hom}(A, B)$ is $\pi_0(\mathcal{H}om(A, B))$.

Theorem 22.9 (Quillen [51])**.** *Let $\mathcal{M}$ be a simplicial model category, $A \in \mathcal{M}$ a cofibrant object, $X \in \mathcal{M}$ a fibrant object. Then*

$$\mathrm{Hom}_{\mathcal{HM}}(A, X) = \mathrm{Hom}_{\mathcal{M}}(A, X)/\sim_h = \pi_0(\mathcal{H}om_{\mathcal{M}}(A, B)).$$

In particular, $\mathcal{HM}$ is equivalent to the category with objects the fibrant and cofibrant objects of $\mathcal{M}$, and maps the homotopy classes of maps in $\mathcal{M}$.

This should remind us of the fact that, for $\mathcal{A}$ an abelian category with enough injectives, $D^+(\mathcal{A})$ is equivalent to the homotopy category of complexes of injectives.

This theorem gives one reason that the model structure is useful.

Quillen pairs. Model categories also allow the construction of "derived functors" on respective homotopy categories.

Definition 22.10. Let $F : \mathcal{M} \to \mathcal{N}$ be a functor of the underlying categories of model categories $\mathcal{M}, \mathcal{N}$, having a right adjoint $G : \mathcal{N} \to \mathcal{M}$. Call (F, G) a *Quillen pair* if F preserves cofibrations and trivial cofibrations (equivalently, G preserves fibrations and trivial fibrations).

A Quillen pair (F, G) is a *Quillen equivalence* if for $M \in \mathcal{M}$ cofibrant and $N \in \mathcal{N}$ fibrant, a morphism $FM \to N$ is a weak equivalence in $\mathcal{N}$ if and only if the adjoint map $M \to GN$ is a weak equivalence in $\mathcal{M}$.

Derived functors. Let $(F : \mathcal{M} \to \mathcal{N}, G : \mathcal{N} \to \mathcal{M})$ be a Quillen pair. We have the *left derived functor* $LF : \mathcal{HM} \to \mathcal{HN}$ and the *right derived functor* $RG : \mathcal{HN} \to \mathcal{HM}$ defined by

$$LF(X) := F(X_c)$$
$$RG(Y) := G(Y_f)$$

where $X_c \to X$ is a cofibrant replacement and $Y \to Y_f$ is a fibrant replacement. This should remind you of defining the left derived functor by taking a projective resolution (cofibrant replacement) and the right derived functor by taking an injective resolution (fibrant replacement). The utility of this construction is given by:

Theorem 22.11. *Let $(F : \mathcal{M} \to \mathcal{N}, G : \mathcal{N} \to \mathcal{M})$ be a Quillen pair. Then the left derived functor $LF : \mathcal{HM} \to \mathcal{HN}$ is left adjoint to the right derived functor $RG : \mathcal{HN} \to \mathcal{HM}$. If (F, G) is a Quillen equivalence, then (LF, RG) define inverse equivalences on the homotopy categories.*

22.3. Spaces and topological spaces. Now that we have a purely combinatorial homotopy theory of "spaces", what about the "real" version?

The category **Top** of compactly generated Hausdorff topological spaces also has a model structure: weak equivalences are maps inducing isomorphisms on all homotopy groups, fibrations are Serre fibrations, and cofibrations are those with LLP for all trivial fibrations.

We have the geometric realization functor $| - | :$ **Spc** $\to$ **Top**; to go the other way, use the *singular simplices* construction::

$\mathrm{Sing}X$ is the simplicial set of singular simplices of X:

$$\mathrm{Sing}X([n]) := \{f : \Delta_n \to X\}$$

with the simplicial structure induced by the cosimplicial structure of Δ_*. These functors form a Quillen equivalence and thus give an equivalence of the respective homotopy categories.

22.4. Spectra and stable homotopy theory.

Definition 22.12. A *spectrum* X is a sequence of pointed spaces $X_0, X_1, \ldots$ together with *bonding maps* $\epsilon_n : \Sigma X_n \to X_{n+1}$. Morphisms are sequences of pointed maps respecting the bonding maps. This defines the category of spectra **Spt**.

For $X \in \mathbf{Spt}$ define the *stable homotopy group*

$$\pi_n^s(X) := \lim_{N\to\infty} \pi_{n+N}(X_N).$$

A morphism $f : X \to Y$ in **Spt** is a *stable weak equivalence* if $\pi_n^s(f)$ is an isomorphism for all n.

Note that $\pi_n^s(X)$ is defined (and is an abelian group) for all $n \in \mathbb{Z}$.

Remark 22.13. **Spt** is a model category with weak equivalences the stable weak equivalences.

The cofibrations can be defined explicitly: $f : X \to Y$ is a cofibration if $f_0 : X_0 \to Y_0$ is a cofibration in $\mathbf{Spc}_*$ and if $f_n \cup \epsilon_{n-1} : X_n \cup_{\Sigma X_{n-1}} \Sigma Y_{n-1} \to Y_n$ is a cofibration in **Spt** for all $n > 0$.

The fibrations are determined by having the RLP for all trivial cofibrations.

Definition 22.14. The homotopy category $\mathcal{H}\mathbf{Spt}$ is the *stable homotopy category*, denoted $\mathcal{SH}$.

The suspension and loop space functors on $\mathbf{Spc}_*$ give two essential functors between $\mathbf{Spc}_*$ and **Spt**:

1. The *suspension spectrum functor* $\Sigma^\infty : \mathbf{Spc}_* \to \mathbf{Spt}$: $\Sigma^\infty A := (A, \Sigma A, \Sigma^2 A, \ldots)$ with the identity bonding maps.
2. The *0-space functor* $\Omega^\infty : \mathbf{Spt} \to \mathbf{Spc}_*$: $\Omega^\infty(X_0, X_1, \ldots) := X_0$.

These are adjoint, in fact a Quillen adjoint pair:

$$\mathrm{Hom}_{\mathbf{Spc}_*}(A, \Omega^\infty X) = \mathrm{Hom}_{\mathbf{Spt}}(\Sigma^\infty A, X).$$

Thus this whole structure passes to the homotopy categories, giving adjoint functors

$$\mathcal{H}_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{SH}$$

Suspension and loops in $\mathcal{SH}$. The suspension and loop space functors on $\mathbf{Spc}_*$ also extend (up to stable weak equivalence) to give two essential functors on **Spt** (also a Quillen pair):

1. $\Sigma : \mathbf{Spt} \to \mathbf{Spt}$, $\Sigma(X_0, X_1, \ldots) := (X_1, X_2, \ldots)$.
2. $\Omega : \mathbf{Spt} \to \mathbf{Spt}$, $\Omega(X_0, X_1, \ldots) := (\Omega X_0, X_0, X_1, \ldots)$.

These are in fact a Quillen equivalence, so induce inverse equivalences on $\mathcal{SH}$. Thus: $\Sigma^\infty : \mathcal{H}_* \to \mathcal{SH}$ inverts the suspension functor Σ.

Triangulated structure. In $\mathbf{Spc}_*$, we have the *mapping cone*: For $f : X \to Y$,

$$M(f) := \frac{X \times I \amalg Y}{x \times 0 \cup * \times I \sim *, x \times 1 \sim f(x)}.$$

This gives the *cone sequence* (also called the homotopy cofiber sequence)

$$X \xrightarrow{f} Y \xrightarrow{i} M(f) \xrightarrow{p} \Sigma X.$$

The mapping cone construction passes to $\mathbf{Spt}$ term-wise.

Theorem 22.15. *$\mathcal{SH}$ is a triangulated category, with translation $X[1] := \Sigma X$. The distinguished triangles are those isomorphic to a cone sequence.*

Additive structure. In particular, $\mathcal{SH}$ is a *additive category*: the Hom-sets are naturally abelian groups. This follows from

$$\mathrm{Hom}_{\mathcal{SH}}(A, B) = \mathrm{Hom}_{\mathcal{SH}}(A \wedge S^1, B \wedge S^1);$$

the co-group structure on S^1,

$$S^1 \to S^1 \vee S^1,$$

gives a group structure to $\mathrm{Hom}_{\mathcal{SH}}(A \wedge S^1, -)$. Suspending again gives the commutativity of the group operation.

Ω-spectra.

Definition 22.16. An Ω-spectrum X is a spectrum $(X_0, X_1, \ldots)$ such that the X_n are all fibrant in $\mathbf{Spc}_*$ and the adjoint $X_n \to \Omega X_{n+1}$ of the bonding map $\Sigma X_n \to X_{n+1}$ is a weak equivalence in $\mathbf{Spc}_*$ for all n.

Remarks 22.17. (1) A fibrant spectrum is an Ω-spectrum.

(2) For $X = (X_0, X_1, \ldots)$, one has the weakly equivalent Ω-spectrum $X \to \tilde{X} := ([\mathrm{colim}_n \Omega^n X_n]^{\mathrm{fib}}, [\mathrm{colim}_n \Omega^n X_{n+1}]^{\mathrm{fib}}, \ldots)$, where $(-)^{\mathrm{fib}}$ is a fibrant model.

(3) From (2), we see that the derived 0-space functor applied to an arbitrary spectrum $X = (X_0, X_1, \ldots)$ is weakly equivalent to $\mathrm{colim}_n \Omega^n X_n$.

An "abstract" description of $\mathcal{SH}$ using Ω-spectra is:

1. First define a *projective weak equivalence* $f : E \to F$ in $\mathbf{Spt}$ by: $f_n : E_n \to F_n$ is a weak equivalence in $\mathbf{Spc}_*$ for all n. Fibrations are $f : E \to F$ such that $f_n : E_n \to F_n$ is a fibration in $\mathbf{Spc}_*$ for all n. Cofibrations are given by the LLP for each fibration which is a projective weak equivalence. This yields the *projective model structure* $\mathbf{Spt}_p$ and the projective homotopy category $\mathcal{H}\mathbf{Spt}_p$.

The 0-space functor $\Omega^\infty : \mathbf{Spt}_p \to \mathbf{Spc}_*$ has right derived functor $\Omega^\infty : \mathcal{H}\mathbf{Spt}_p \to \mathcal{H}_*$. Let $\Sigma : \mathbf{Spt}_p \to \mathbf{Spt}_p$ be the shift functor

$$\Sigma(E_0, E_1, \ldots) := (E_1, E_2, \ldots)$$

giving $\Sigma : \mathcal{H}\mathbf{Spt}_p \to \mathcal{H}\mathbf{Spt}_p$.

2. Call $f : E \to F$ in $\mathcal{H}\mathbf{Spt}_p$ a *stable weak equivalence* if f induces an isomorphism $\Omega^\infty\Sigma^n E \to \Omega^\infty\Sigma^n F$ in $\mathcal{H}_*$ for all $n \geq 0$.

Let $\mathbf{Spt}_{st} = \mathbf{Spt}_p$ as a category. Cofibrations are the same in both categories. A fibration in $\mathbf{Spt}_{st}$ is a map having the RLP for each cofibration that is a stable weak equivalence.

Let $\mathbf{Spt}^\Omega \subset \mathbf{Spt}_{st}$ be the full subcategory of Ω-spectra and $\mathcal{H}\mathbf{Spt}^\Omega$ the localization of $\mathbf{Spt}^\Omega$ by the stable weak equivalences.

Proposition 22.18. *(1)* $\mathbf{Spt}_{st} = \mathbf{Spt}$ *as model categories, so* $\mathcal{H}\mathbf{Spt}_{st} = \mathcal{SH}$*.*
(2) The map $\mathcal{H}\mathbf{Spt}^\Omega \to \mathcal{SH}$ *is a weak equivalence.*

Brown representability. Each $E \in \mathbf{Spt}$ gives rise to a contravariant functor on $\mathbf{Spc}$ to graded abelian groups by

$$E^n(X) := \mathrm{Hom}_{\mathcal{SH}}(\Sigma^\infty X_+, \Sigma^n E)$$

The functor $E^* : \mathbf{Spc}_*^{\mathrm{op}} \to \mathrm{Gr}\mathbf{Ab}$ satisfies the axioms of a *generalized cohomology theory*:

Compatibility with pointed union: $E^*(\vee_\alpha X_\alpha) \cong \prod_\alpha E^*(X_\alpha)$.
Homotopy invariance: $E^*(X \times I) = E^*(X)$.
Suspension: $E^*(\Sigma X_+) = E^{*-1}(X)$
Cofiber sequence: Let $f : X \to Y$ be a map in $\mathbf{Spc}_*$. Then the mapping cone sequence for f induces a long exact sequence

$$\ldots \to E^{n-1}(X) \to E^n(M(f)) \to E^n(Y) \to E^n(X) \to \ldots$$

These properties follow from the structures built into the triangulated category $\mathcal{SH}$, plus that fact that $E^n(X) = \mathrm{Hom}_{\mathcal{SH}}(\Sigma^\infty X, \Sigma^n E)$.

The Brown representability theorem says

Theorem 22.19. *Each functor* $E^* : \mathbf{Spc}_*^{\mathrm{op}} \to \mathrm{Gr}\mathbf{Ab}$ *satisfying the axioms of a generalized cohomology theory is represented by a spectrum* $E \in \mathcal{SH}$*.*

If we consider functors $\mathbf{Spt} \to \mathrm{Gr}\mathbf{Ab}$, we have an equivalence of categories between $\mathcal{SH}$ and generalized cohomology theories on $\mathbf{Spt}$.

Examples 22.20. 1. For A an abelian group, let $K(A, n)$ be a simplicial set with $A = \pi_n$ and all other homotopy groups trivial. One can choose the $K(A, n)$ to be fibrant, functorial in A and with natural weak equivalences

$$K(A, n) \to \Omega K(A, n+1)$$

Let $EM(A)$ be the Ω-spectrum $(K(A, 0), K(A, 1), \ldots)$, the *Eilenberg-Maclane spectrum* of A. $EM(A)$ represents singular cohomology with A-coefficients: $EM(A)^* := H^*(-, A)$.

2. Let BU be a model for the classifying space of the infinite unitary group U. One model for BU is the doubly infinite complex Grassmann variety:

$$BU_n := \mathrm{Gr}_{\mathbb{C}}(n, \infty) := \mathrm{colim}_N \mathrm{Gr}_{\mathbb{C}}(n, N)$$
$$BU := \mathrm{Gr}_{\mathbb{C}} := \mathrm{colim}_n \mathrm{Gr}_{\mathbb{C}}(n, \infty).$$

Bott periodicity gives a weak equivalence

$$BU \to \Omega^2 BU.$$

This yields the *topological K-theory spectrum*

$$K_{top} := (BU \times \mathbb{Z}, \Sigma BU \times \mathbb{Z}, BU \times \mathbb{Z}, \ldots).$$

3. Over $BU_n := \mathrm{Gr}(n, \infty)$ we have the universal complex n-plane bundle

$$U_n \to BU_n.$$

Adding a trivial line bundle e defines $i_n : BU_n \to BU_{n+1}$ with

$$i_n^*(U_{n+1}) = U_n \oplus e.$$

Let MU_{2n} be the Thom space $Th(U_n) := \mathbb{P}(U_n \oplus e)/\mathbb{P}(U_n)$. Since $Th(V \oplus e) = Th(V) \wedge S^2$, the i_n induce maps $MU_{2n} \wedge S^2 \to MU_{2n+2}$.

This gives the *Thom spectrum*

$$MU := (MU_0, \Sigma MU_0, MU_2, \Sigma MU_2, \ldots).$$

representing *complex cobordism* MU^*.

5. The sphere spectrum: $S := \Sigma^\infty S^0 := (S^0, S^1, \ldots)$. The stable homotopy groups $\pi_n^s S$ are just the stable homotopy groups of spheres. S plays the role of $\mathbb{Z}$ in homological algebra, as every spectrum is an "S-module".

$\mathcal{SH}$ and $D(\mathbf{Ab})$. Sending an abelian group to the Eilenberg-Maclane spectrum extends to a (non-full!) embedding on the derived category

$$D(\mathbf{Ab}) \to \mathcal{SH}.$$

This allows one to think of stable homotopy theory as an extension of homological algebra.

Roughly speaking the *Dold-Kan correspondence* identifies the category of simplicial abelian groups with homological complexes supported in degrees ≥ 0, $C_{\geq 0}(\mathbf{Ab})$, with weak equivalence corresponding to quasi-isomorphism.

The process of making spectra out of simplicial abelian groups translates into passing from $C_{\geq 0}(\mathbf{Ab})$ to $C(\mathbf{Ab})$.

Summary:

$$\begin{tikzcd}
\mathbf{Spc} \arrow[r, "+"] \arrow[d] & \mathbf{Spc}_* \arrow[loop above, "\Sigma"] \arrow[loop below, "\Omega"] \arrow[r, shift left, "\Sigma^\infty"] \arrow[d, "\text{invert weak}"', "\text{equivalences}"] & \mathbf{Spt} \arrow[l, shift left, "\Omega^\infty"] \arrow[loop above, "\Sigma"] \arrow[loop below, "\Omega"] \arrow[d] \\
\mathcal{H} \arrow[r, "+"] & \mathcal{H}_* \arrow[loop above, "\Sigma"] \arrow[loop below, "\Omega"] \arrow[r, shift left, "\Sigma^\infty"] & \mathcal{SH} \arrow[l, shift left, "\Omega^\infty"] \arrow[loop above, "\Sigma"] \arrow[loop below, "\Omega"]
\end{tikzcd}$$

$\Omega = \Sigma^{-1}$ on $\mathcal{SH}$. $\mathcal{SH}$ is a triangulated category with distinguished triangles the homotopy (co)fiber sequences.

23. Motivic homotopy theory: a quick overview

The scheme of construction of classical homotopy theory can be carried out, with the necessary changes, to give a good stable homotopy theory for scheme. Here is a dictionary from the world of classical homotopy theory to the motivic one.

1. Replace spaces **Spc** with *presheaves of spaces on* $\mathbf{Sm}/k$:

$$\mathbf{Spc}_{ps}(k) := \{\text{Functors } A : \mathbf{Sm}/k^{\mathrm{op}} \to \mathbf{Spc}\}.$$

Examples: $X \in \mathbf{Sm}/k$ gives $X \in \mathbf{Spc}_{ps}(k)$, $X(Y) := \mathrm{Hom}(Y, X)$. $A \in \mathbf{Spc}$ gives $A \in \mathbf{Spc}_{ps}(k)$, $A(Y) := A$.

2. Replace suspension $\Sigma A := A \wedge S^1$ with T-suspension $\Sigma_T A := A \wedge \mathbb{P}^1$.

3. Replace spectra

$$\mathbf{Spt} = \{E = (E_0, E_1, \ldots), E_n \in \mathbf{Spc}_* \ + \epsilon_n : \Sigma E_n \to E_{n+1}\}$$

with T-spectra:

$$\mathbf{Spt}_T(k) := \{E = (E_0, E_1, \ldots), E_n \in \mathbf{Spc}_{ps*}(k) \ + \epsilon_n : \Sigma_T E_n \to E_{n+1}\}.$$

4. Use Bousfield localization to impose Mayer-Vietoris and homotopy invariance in the resulting homotopy categories.

Now for the details.

24. The unstable motivic homotopy category

The construction of the unstable motivic category is roughly parallel to that of the classical unstable category. Our "motivic spaces" are presheaves of spaces on the category $\mathbf{Sm}/k$ of smooth k schemes. We use the machinery of model categories to impose homotopy invariance and Nisnevich excision in the resulting homotopy category.

24.1. Motivic spaces.

Definition 24.1. Let $\mathbf{Spc}_{ps}(k)$ be the category of presheaves of spaces on $\mathbf{Sm}/k$.

Make $\mathbf{Spc}_{ps}(k)$ a model category by defining cofibrations and weak equivalences objectwise, for instance $f : \mathcal{X} \to \mathcal{Y}$ is a cofibration in $\mathbf{Spc}_{ps}(k)$ if and only if $f(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$ is a cofibration in **Spc** for each $U \in \mathbf{Sm}/k$. Fibrations are determined by having the RLP for trivial cofibrations. Write $\mathcal{H}_{ps}(k)$ for the homotopy category $\mathcal{H}\mathbf{Spc}_{ps}(k)$.

Sending $X \in \mathbf{Sch}_k$ to the (discrete) representable presheaf gives a functor $\mathbf{Sch}_k \to \mathbf{Spc}_{ps}(k)$ which is a full embedding on $\mathbf{Sm}/k$.

Sending $A \in \mathbf{Spc}$ to the constant presheaf $A \in \mathbf{Spc}_{ps}(k)$ gives a full embedding $\mathbf{Spc} \to \mathbf{Spc}_{ps}(k)$.

Advantage: $\mathbf{Spc}_{ps}(k)$ has all small limits and colimits, in fact "all" constructions in **Spc** can be carried out in $\mathbf{Spc}_{ps}(k)$. For example, we have objects ΣX for $X \in \mathbf{Sch}_k$; $X \wedge Y$ for pointed $X, Y \in \mathbf{Sch}_k$, X/Y for $Y \subset X$, $X, Y \in \mathbf{Sch}_k$.

Disadvantage: The Nisnevich topology is nowhere in sight, so there is for example no Mayer-Vietoris property for Zariski open covers. $\mathbb{A}^1$ is not contractible. We correct this by *Bousfield localization.*

24.2. Bousfield localization. Start with a simplicial model category $\mathcal{M}$. Let $\mathcal{S}$ be a collection of morphisms between cofibrant objects in $\mathcal{M}$. Call a fibrant $Z \in \mathcal{M}$ *$\mathcal{S}$-local* if for all $f : A \to B$ in $\mathcal{S}$, the map on simplicial Hom sets

$$f^* : \mathrm{Hom}_{\mathcal{M}}(B, Z) \to \mathrm{Hom}_{\mathcal{M}}(A, Z)$$

is a weak equivalence in **Spc**.

Call $f : A \to B$ in $\mathcal{M}$ an *$\mathcal{S}$-weak equivalence* if

$$f^* : \mathrm{Hom}_{\mathcal{M}}(B, Z) \to \mathrm{Hom}_{\mathcal{M}}(A, Z)$$

is a weak equivalence for all $\mathcal{S}$-local Z.

Definition 24.2. An $\mathcal{S}$-cofibration is the same as a cofibration in $\mathcal{M}$. An $\mathcal{S}$-fibration is a map in $\mathcal{M}$ that has the RLP for all $\mathcal{S}$-cofibrations that are $\mathcal{S}$-weak equivalences.

Theorem 24.3. *Under a certain "smallness" condition on the cofibrations in $\mathcal{M}$ and the maps in $\mathcal{S}$, the classes of $\mathcal{S}$-cofibrations, $\mathcal{S}$-fibrations and $\mathcal{S}$-weak equivalences define a simplicial model structure on $\mathcal{M}$.*

See Hirschhorn [28] for a precise statement.

24.3. The motivic unstable homotopy category.

Definition 24.4. A map $f : A \to B$ in $\mathbf{Spc}_{ps}(k)$ is a *Nisnevich local weak equivalence* if for each point $x \in X \in \mathbf{Sm}/k$, the map on Nisnevich stalks $f_x : A_x \to B_x$ is a weak equivalence.

Let $\mathbf{Spc}_{\mathrm{Nis}}(k)$ be the category $\mathbf{Spc}_{ps}(k)$ with the Nisnevich local model structure: cofibrations are the same as in $\mathbf{Spc}_{ps}(k)$, weak equivalences the Nisnevich local ones, and fibrations determined by having the RLP for trivial cofibrations.

The homotopy category $\mathcal{H}_{\mathrm{Nis}}(k) := \mathcal{H}\mathbf{Spc}_{\mathrm{Nis}}(k)$ is the *Nisnevich local unstable homotopy category over k.*

Remarks 24.5. (1) $\mathbf{Spc}_{\mathrm{Nis}}(k)$ can be defined as a Bousfield localization of $\mathbf{Spc}_{ps}(k)$, with $\mathcal{S}$ either the collection of Nisnevich local weak equivalences, or (a much more manageable collection) the maps of the form

$$U \amalg_{U\times_X V} V \to X,$$

where

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ U \arrow[r, "j"'] & X \end{tikzcd}$$

is an *elementary Nisnevich square*: j is an open immersion, f is étale and $f : V \setminus U \times_X V \to X \setminus U$ is an isomorphism. Here $U \amalg_{U\times_X V} V$ denotes the push-out in the diagram

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \\ U; & \end{tikzcd}$$

note that $U \times_X V \to V$ is a cofibration.

(2) Let $\mathbf{Spc}_{\mathrm{Nis},s}(k)$ be the category of simplicial Nisnevich sheaves of sets. Make $\mathbf{Spc}_{\mathrm{Nis},s}(k)$ a model category by defining cofibrations and weak equivalences stalkwise. Fibrations have the RLP for trivial cofibrations.

The inclusion $i : \mathbf{Spc}_{\mathrm{Nis},s}(k) \to \mathbf{Spc}_{\mathrm{Nis}}(k)$ has as left adjoint the *sheafification functor* $a_{\mathrm{Nis}} : \mathbf{Spc}_{\mathrm{Nis}}(k) \to \mathbf{Spc}_{\mathrm{Nis},s}(k)$.

The pair (a_{Nis}, i) is a Quillen equivalence, so we could have defined $\mathcal{H}_{\mathrm{Nis}}(k)$ as the homotopy category $\mathcal{H}\mathbf{Spc}_{\mathrm{Nis},s}(k)$. This is the approach used by Morel-Voevodsky.

Definition 24.6. $\mathbf{Spc}(k)$ is $\mathbf{Spc}_{\mathrm{Nis}}(k)$ with the model structure given by the Bousfield localization with respect to the collection of maps $X \times \mathbb{A}^1 \to X$, $X \in \mathbf{Sm}/k$. $\mathbf{Spc}(k)$ is the model category of *motivic spaces over* k.

We thus have the homotopy category $\mathcal{H}(k) := \mathcal{H}\mathbf{Spc}(k)$, the *unstable motivic homotopy category over* k. Pointed versions are defined similarly.

Remark 24.7. Roughly speaking, we have formed $\mathcal{H}(k)$ by localizing $\mathcal{H}\mathbf{Spc}_{ps}(k)$ with respect to Nisnevich Mayer-Vietoris, and $\mathbb{A}^1$-homotopy, similar to our construction of $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ out of $K^b(\mathrm{Cor}(k))$.

24.4. Structures in spaces over k. The basic operations in $\mathbf{Spc}_*$ extend objectwise to $\mathbf{Spc}_{ps*}(k)$: We have $\vee$ and $\wedge$, internal Hom $\mathcal{H}om$, suspension Σ and loops Ω, which are adjoint. For instance, for $\mathcal{X}$, $\mathcal{Y}$ in $\mathbf{Spc}_*(k)$, we have the smash product $\mathcal{X} \wedge \mathcal{Y}$ defined by

$$\mathcal{X} \wedge \mathcal{Y}(U) := \mathcal{X}(U) \wedge \mathcal{Y}(U)$$

for each $U \in \mathbf{Sm}/k$.

These all induce operations with expected properties on the homotopy categories $\mathcal{H}_{ps*}(k)$, $\mathcal{H}_{\mathrm{Nis}*}(k)$ and $\mathcal{H}(k)$.

Elementary properties. In $\mathcal{H}(k)$ we have:

- $X \cong U \amalg_{U \cap V} V$ for $X = U \cup V$ an open cover in $\mathbf{Sm}/k$.
- $\mathbb{P}^1 \cong \Sigma\mathbb{G}_m$: $\mathbb{P}^1 \cong \mathbb{A}^1 \amalg_{\mathbb{G}_m} \mathbb{A}^1 \cong * \amalg^h_{\mathbb{G}_m} * \cong \Sigma\mathbb{G}_m$. Here $\amalg^h$ means the homotopy colimit over the diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}$$

in the pointed category $\mathbf{Spc}_{ps*}(k)$.
- For $V \to X$ an affine-space bundle, $V \cong X$: this is true for $V = X \times \mathbb{A}^n$ by the homotopy property, then use Mayer-Vietoris.

Definition 24.8. Let $E \to X$ be a vector bundle over $X \in \mathbf{Sm}/k$ with zero-section s. Define the *Thom space*

$$Th(E) := E/(E \setminus s(X)).$$

A central result in the theory is the Morel-Voevodsky purity theorem:

Theorem 24.9 (Purity [47, theorem 3.2.23])**.** *Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$, with normal bundle $N \to W$. There is a canonical isomorphism in $\mathcal{H}_*(k)$:*

$$Th(N) \cong X/(X \setminus W).$$

Idea of proof. Use the deformation to the normal bundle as in Lecture 4: Let $Def(i)$ be the blow-up of $X \times \mathbb{A}^1$ along $W \times 0$, with the proper transform of $X \times 0$ removed.

$$[W \times \mathbb{A}^1 \to Def(i)] \to \mathbb{A}^1$$

is smooth over $\mathbb{A}^1$ with fiber $W \xrightarrow{i} X$ over 1 and fiber $W \xrightarrow{s} N$ over 0.

It suffices to show:

$$X/(X \setminus W) \to Def(i)/(Def(i) \setminus W \times \mathbb{A}^1)$$

and

$$Th(N) \to Def(i)/(Def(i) \setminus W \times \mathbb{A}^1)$$

are isomorphisms in $\mathcal{H}_*(k)$.

For this, work locally on X in the Nisnevich topology. This reduces to the case $X = W \times \mathbb{A}^n$, $i =$ 0-section. Then $Def(i) = W \times \mathbb{A}^1 \times \mathbb{A}^n$ with $s, i : W \times \mathbb{A}^1 \to Def(i)$ the 0-section, resp., 1-section, so the result is trivially true. ☐

24.5. Algebraic K-theory. Let $\mathrm{Gr}(n, m)$ be the Grassmann variety (over k) of n-planes in m-space. We have $\mathrm{Gr}(n, m) \to \mathrm{Gr}(n, m+1)$ by adding a new coordinate in the ambient space, and $\mathrm{Gr}(n, m) \to \mathrm{Gr}(n+1, m+1)$ by adding the same coordinate to both spaces.

Let

$$\mathrm{Gr} := \lim_{n,m\to\infty} \mathrm{Gr}(n, m)$$

(as a presheaf).

Theorem 24.10 ([47, theorem 4.3.13])**.** $\mathrm{Gr} \times \mathbb{Z}$ *represents higher algebraic K-theory as a presheaf on $\mathbf{Sm}/k$:*

$$\mathrm{Hom}_{\mathcal{H}_*(k)}(\Sigma^n X_+, \mathrm{Gr} \times \mathbb{Z}) \cong K_n(X)$$

for $X \in \mathbf{Sm}/k$.

Morel-Voevodsky prove this, roughly speaking, by first showing that the classifying space $B\mathrm{GL}_n$ represents the functor of rank n-vector bundles in $\mathcal{H}_{\mathrm{Nis}}(k)$, and then that the topological group-completion of the monoid $\amalg_n B\mathrm{GL}_n$ represents Quillen K-theory in $\mathcal{H}_{\mathrm{Nis}*}(k)$. As K-theory is $\mathbb{A}^1$-homotopy invariant, this shows that this group-completion represents K-theory in $\mathcal{H}_*(k)$. Via explicit $\mathbb{A}^1$ homotopies on Grassmannians, one shows that monoid $\amalg_n B\mathrm{GL}_n$ in $\mathcal{H}(k)$ is commutative and has $\pi_0^{\mathbb{A}^1} = \mathbb{N}$. Using the universal property of the plus construction, it follows that the canonical map of $B\mathrm{GL}_\infty \times \mathbb{Z}$ to the group completion of $\amalg_n B\mathrm{GL}_n$ is therefore an $\mathbb{A}^1$ weak equivalence.

Riou [55] has shown how one can use this result to construct the λ-ring structure on $K_*(-)$ via self-maps of $\mathrm{Gr} \times \mathbb{Z}$ in $\mathcal{H}_*(k)$.

24.6. Endomorphisms of $\mathbb{P}^{1\wedge n}$. Let k be a field of characteristic $\neq 2$. The *Grothendieck-Witt group of* k, $GW(k)$, is formed from the monoid of isomorphism classes of quadratic forms over k (under orthogonal direct sum) by group completion.

For $u \in k^\times$, let $<u>$ be the quadratic form ux^2. Since every quadratic form can be diagonalized, the elements $<u> \in GW(k)$ generate. Let

$$\times u : \mathbb{P}^1 \to \mathbb{P}^1$$

be the map $(x_0 : x_1) \mapsto (x_0 : ux_1)$. Morel has proven a fundamental result showing that $GW(k)$ appears naturally in motivic homotopy theory.

Theorem 24.11 (Morel [45])**.** *Let k be a perfect field of characteristic $\neq 2$. For each $n \geq 2$, there is an isomorphism*

$$GW(k) \cong \mathrm{Hom}_{\mathcal{H}_*(k)}(\mathbb{P}^{1\wedge n}, \mathbb{P}^{1\wedge n}).$$

The isomorphism sends $<u> \in GW(k)$ to the map

$$\times u \wedge \mathrm{id} : \mathbb{P}^1 \wedge (\mathbb{P}^1)^{\wedge n-1} \to \mathbb{P}^1 \wedge (\mathbb{P}^1)^{\wedge n-1}$$

25. T-spectra and the motivic stable homotopy category

We follow the construction of **Spt** from $\mathbf{Spc}_*$ in the presheaf category, replacing Σ with $\Sigma_T := (-) \wedge \mathbb{P}^1$, then localize with respect to T-stable weak equivalences.

The Nisnevich local structure and the $\mathbb{A}^1$-local structure are inherited from $\mathcal{H}_*(k)$.

25.1. T-spectra.

Definition 25.1. $\mathbf{Spt}_{T,p}(k)$ is the category of *presheaves of T-spectra on* $\mathbf{Sm}/k$: objects are sequences $X = (X_0, X_1, \ldots)$ in $\mathbf{Spc}_*(k)$ plus bonding maps $\epsilon_n : \Sigma_T X_n \to X_{n+1}$, where

$$\Sigma_T A := A \wedge (\mathbb{P}^1, \infty).$$

Remark 25.2. For technical reasons, we really should be using $T = \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$ rather than $\mathbb{P}^1$. However, these two objects are isomorphic in $\mathcal{H}_*(k)$, so we are not too far off from the truth when we use the more geometric model. The careful reader with replace $\mathbb{P}^1$ with $\mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$, and in the spots where $\mathbb{P}^1$ is used in an essential way, replace identities with the Morel-Voevodsky purity isomorphism.

For $A \in \mathbf{Spc}_*(k)$, let $\Omega_T A := \mathcal{H}om_{\mathbf{Spc}_*(k)}(\mathbb{P}^1, A)$.

Definition 25.3. A map $f : E \to F$ is a *levelwise weak equivalence* if $f_n : E_n \to F_n$ is an isomorphism in $\mathcal{H}_*(k)$ for all n. f is a *levelwise fibration* if $f_n : E_n \to F_n$ is a fibration in $\mathbf{Spc}_*(k)$ for all n.

$E \in \mathbf{Spt}_{T,p}(k)$ is an *Ω_T-spectrum* if each E_n is fibrant in $\mathbf{Spc}_*(k)$ and the map $E_n \to \Omega_T E_{n+1}$ is a weak equivalence in $\mathcal{H}_*(k)$ for all n.

We give $\mathbf{Spt}_{T,p}(k)$ the *projective model structure*:

Fibrations and weak equivalences are the levelwise ones. Cofibrations are the maps having the LLP for each trivial fibration. Explicitly, $f : E \to F$ is a cofibration if

$$E_n(U) \amalg_{\Sigma_T E_{n-1}(U)} \Sigma_T F_{n-1}(U) \to F_n(U)$$

is a cofibration in $\mathbf{Spc}_*$ for each $U \in \mathbf{Sm}/k$.

Operations and structures. The basic structures in $\mathbf{Spt}$ extend objectwise to $\mathbf{Spt}_{T,p}(k)$: We have Quillen adjoint pair of suspension and loops functors (Σ_T, Ω_T):

$$\Sigma_T(E_0, E_1, \ldots) := (E_1, E_2, \ldots)$$
$$\Omega_T(E_0, E_1, \ldots) := (\Omega_T E_0, E_0, E_1, \ldots),$$

and the adjoint pair of functors

$$\Sigma_T^\infty : \mathcal{H}_*(k) \to \mathcal{H}\mathbf{Spt}_{T,p}(k); \qquad \Sigma_T^\infty A := (A, \Sigma_T A, \Sigma_T^2 A, \ldots)$$
$$\Omega_T^\infty : \mathcal{H}\mathbf{Spt}_{T,p}(k) \to \mathcal{H}_*(k); \qquad \Omega_T^\infty(E_0, E_1, \ldots) := \mathrm{colim}_n \Omega_T^n E_n^{\mathrm{fib}}.$$

Definition 25.4. Call $f : E \to F$ a *T-stable weak equivalence* if

$$\Omega_T^\infty \Sigma_T^n f : \Omega_T^\infty \Sigma_T^n E \to \Omega_T^\infty \Sigma_T^n F$$

is a weak equivalence in $\mathcal{H}_*(k)$ for all $n \geq 0$.

25.2. Stable model structure.

Definition 25.5. The model category of *T-spectra over k*, $\mathbf{Spt}_T(k)$, is the Bousfield localization of $\mathbf{Spt}_{T,p}(k)$ with respect to the T-stable weak equivalences.

The homotopy category $\mathcal{SH}(k) := \mathcal{H}\mathbf{Spt}_T(k)$ is the *motivic stable homotopy category of T-spectra over k*.

Proposition 25.6. *Let $\mathbf{Spt}_T^\Omega(k)$ be the full subcategory of $\mathbf{Spt}_T(k)$ with objects the Ω_T-spectra. Let $\mathcal{SH}_T^\Omega(k)$ be the category formed by inverting naive weak equivalences in $\mathbf{Spt}_T^\Omega(k)$. Then the evident map $\mathcal{SH}_T^\Omega(k) \to \mathcal{SH}(k)$ is an equivalence.*

(Σ_T, Ω_T) induce inverse equivalences Σ_t and Ω_t on $\mathcal{SH}(k)$. on $\mathcal{SH}(k)$. Similarly, we have the Quillen pair of adjoint functor $(\Sigma_T^\infty, \Omega_T^\infty)$ which induce the adjoint pair of derived functors

$$\Sigma_t^\infty : \mathcal{H}_*(k) \to \mathcal{SH}(k)$$
$$\Omega_t^\infty : \mathcal{SH}(k) \to \mathcal{H}_*(k)$$

Since $\mathbb{P}^1 \cong \Sigma\mathbb{G}_m$ in $\mathcal{H}_*(k)$, the "usual" suspension and loops functors $\Sigma_{S^1} := (-) \wedge S^1$ and $\Omega_{S^1} := \mathcal{H}om(S^1, -)$ also induce inverse weak equivalences Σ_s, Ω_s on $\mathcal{SH}(k)$.

Remark 25.7. There are at least three different operations on T-spectra that, up to isomorphism, induce the derived suspension operation Σ_t in $\mathcal{SH}(k)$:

(1) The shift operator we have already define:
$$\Sigma_T^1(E_0, E_1, \ldots) = (E_1, E_2, \ldots)$$
with bonding maps the "same" as for $(E_0, E_1, \ldots)$.

(2) The operator $- \wedge T$:
$$\Sigma_T^2(E_0, E_1, \ldots) = (E_0 \wedge T, E_1 \wedge T, \ldots)$$
with bonding maps $\epsilon_n \wedge \mathrm{id} : E_n \wedge T \to E_{n+1} \wedge T$.

(3) The operator $T \wedge -$
$$\Sigma_T^3(E_0, E_1, \ldots) = (T \wedge E_0, T \wedge E_1, \ldots)$$
with bonding maps $\mathrm{id}_T \wedge \epsilon_n$.

We can map $\Sigma_T^2 E$ to $\Sigma_T^1 E$ by the sequence of maps

$$\epsilon_n : E_n \wedge T \to E_{n+1};$$

it is easy to check that this gives a well-defined map of spectra, which is a stable $\mathbb{A}^1$ weak equivalence. However, if we try this for Σ_T^3, we run into trouble, we do not in general have commutativity in the diagram

$$\begin{tikzcd}
T \wedge E_n \wedge T \arrow[r, "\mathrm{id}\wedge\epsilon_n"] \arrow[d, "\tau\wedge\mathrm{id}"'] & T \wedge E_{n+1} \arrow[d, "\tau"] \\
E_n \wedge T \wedge T \arrow[r, "\epsilon_n\wedge\mathrm{id}"'] & E_{n+1} \wedge T.
\end{tikzcd}$$

Here τ denotes the appropriate commutativity constraint.

Nonetheless, the fact that the cyclic permutation of $T \wedge T \wedge T$ is the identity implies that $\Sigma_T^3 \cong \Sigma_T^1$ on $\mathcal{SH}(k)$ (see [31, §2.3 and corollary 3.21]). We have actually used this result in secret, in our claim that the T-suspension operators on $\mathcal{H}_\bullet(k)$ and $\mathcal{SH}(k)$ are intertwined by Σ_t^∞: this is not the case in $\mathbf{Spc}_\bullet(k)$ and $\mathbf{Spt}_T(k)$ and relies on the fact that the cyclic permutation of $T^{\wedge 3}$ is the identity for this to be the case in $\mathcal{H}_\bullet(k)$ and $\mathcal{SH}(k)$. We hope the reader will forgive us this "simplification".

The same issue arises for classical spectra $\mathcal{SH}$ and for S^1-spectra $\mathcal{SH}_{S^1}(k)$, and is resolved the same way.

$\mathcal{SH}(k)$ is a triangulated category with the "usual" translation Σ_s and distinguished triangles given by homotopy cofiber sequences. The translation Σ_t is not involved in the triangulated structure, but shows up later as giving a "weight filtration" on $\mathcal{SH}(k)$.

The additive structure comes again from the fact that S^1 (as a presheaf) is a co-group and S^2 is a co-commutative co-group.

Summary. The following diagram summarizes the constructions:

$$\begin{tikzcd}
\mathbf{Spc}(k) \arrow[r, "+"] \arrow[d] & \mathbf{Spc}_*(k) \arrow[loop above, "{\Sigma_T,\Sigma_{S^1}}"] \arrow[loop below, "{\Omega_T,\Omega_{S^1}}"'] \arrow[r, shift left, "\Sigma_T^\infty"] \arrow[d, "\text{invert } \mathbb{A}^1\text{-weak}"', "\text{equivalences}"] & \mathbf{Spt}_T \arrow[loop above, "{\Sigma_T,\Sigma_{S^1}}"] \arrow[loop below, "{\Omega_T,\Omega_{S^1}}"'] \arrow[l, shift left, "\Omega_T^\infty"] \arrow[d] \\
\mathcal{H}(k) \arrow[r, "+"] & \mathcal{H}_*(k) \arrow[loop above, "{\Sigma_t,\Sigma_s}"] \arrow[loop below, "{\Omega_t,\Omega_s}"'] \arrow[r, shift left, "\Sigma_t^\infty"] & \mathcal{SH}(k) \arrow[loop above, "{\Sigma_t,\Sigma_s}"] \arrow[loop below, "{\Omega_s,\Omega_s}"'] \arrow[l, shift left, "\Omega_t^\infty"]
\end{tikzcd}$$

$\Omega_t = \Sigma_t^{-1}$ and $\Omega_s = \Sigma_s^{-1}$ on $\mathcal{SH}(k)$. $\mathcal{SH}(k)$ is a triangulated category with translation $\Sigma_s := (-) \wedge S^1$.

25.3. Bi-graded cohomology. A striking feature of $\mathcal{SH}(k)$ is that we have *two* suspension operators: the simplicial suspension Σ_s, giving the translation in the triangulated structure, and the "Tate" suspension Σ_t. This allows us to make a doubly-indexed cohomology theory out of $E \in \mathbf{Spt}_T(k)$.

Definition 25.8. Take $E \in \mathbf{Spt}_T(k)$, $X \in \mathbf{Sm}/k$, and integers m, n. Define

$$E^{n,m}(X) := \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_T^\infty X_+, \Sigma_t^m \Sigma_s^{n-2m} E).$$

Remark 25.9. The simplicial suspension $- \wedge S^1$ is "constant" i.e. does the same thing independent of the choice of k. The Tate suspension $- \wedge \mathbb{P}^1$ "varies" in k. If we can map $k \to \mathbb{C}$, then $\mathbb{P}^1 \mapsto \mathbb{P}^1(\mathbb{C}) = S^2$, while if we can map $k \to \mathbb{R}$, then $\mathbb{P}^1 \mapsto \mathbb{P}^1(\mathbb{R}) = \mathbb{RP}^1 = S^1$.

25.4. Why T-spectra? There is another spectrum category which we will define later: the category of S^1*-spectra over* k, $\mathbf{Spt}_{S^1}(k)$.

$\mathbf{Spt}_{S^1}(k)$ is the category of presheaves of "usual" spectra: $E := (E_0, E_1, \ldots)$, $E_n \in \mathbf{Spc}_*(k)$, plus bonding maps $\Sigma_s E_n \to E_{n+1}$, with the model structure similar to $\mathbf{Spt}_T(k)$.

You might think $\mathbf{Spt}_{S^1}(k)$ is the more obvious analog of $\mathbf{Spt}$ in the motivic world, so what purpose does it serve to invert $(-) \wedge \mathbb{P}^1$ rather than $(-) \wedge S^1$?

There are many reasons, but one consequence of the Σ_T-invertibility in $\mathcal{SH}(k)$ is the existence of "wrong-way" or *transfer maps* for $E^{*,*}$ in certain situations.

Take a closed immersion $i : Z \to X$ in $\mathbf{Sm}/k$. For $E \in \mathbf{Spt}_T(k)$, applying $\mathrm{Hom}_{\mathcal{SH}(k)}(-, E)$ to the homotopy cofibration sequence

$$X \setminus Z \xrightarrow{j} X \to X/(X \setminus Z) \to \Sigma_s(X \setminus Z)_+$$

gives the long exact sequence of "cohomology with supports"

$$\to \ldots E^{n-1,m}(X \setminus Z) \xrightarrow{\partial} E_Z^{n,m}(X) \xrightarrow{i_*} E^{n,m}(X) \xrightarrow{j^*} E^{n,m}(X \setminus Z) \to \ldots$$

where

$$E_Z^{n,m}(X) := \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_t^\infty X/(X \setminus Z), \Sigma_t^m \Sigma_s^{n-2m} E).$$

We have the purity isomorphism (in $\mathcal{H}_*(k)$)

$$X/(X \setminus Z) \cong Th(N_{Z/X}).$$

If we shrink Z so that the normal bundle is trivial, a choice of trivialization gives

$$X/(X \setminus Z) \cong Th(\mathbb{A}^d \times Z \to Z) = (\mathbb{A}^d/(\mathbb{A}^d \setminus 0)) \wedge Z_+ \cong \Sigma_T^d Z_+,$$

where $d = \mathrm{codim}_X Z$. Putting in the isomorphism $\Sigma_T^d Z_+ \cong X/(X \setminus Z)$ and using the invertibility of Σ_t gives

$$\begin{aligned} E_Z^{n,m}(X) &= \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_t^\infty \Sigma_T^d Z_+, \Sigma_t^m \Sigma_s^{n-2m} E) \\ &= \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_t^d \Sigma_t^\infty Z_+, \Sigma_t^m \Sigma_s^{n-2m} E) \\ &= \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_t^\infty Z_+, \Sigma_t^{m-d} \Sigma_s^{n-2d-2(m-d)} E) \\ &= E^{n-2d,m-d}(Z). \end{aligned}$$

Fitting this into the cohomology with supports sequence gives the long exact *Gysin sequence*

$$\ldots \to E^{n-1,m}(X \setminus Z) \xrightarrow{\partial} E^{n-2d,m-d}(Z) \xrightarrow{i_*} E^{n,m}(X) \xrightarrow{j^*} E^{n,m}(X \setminus Z) \to \ldots$$

This would not have been possible if we had only inverted $(-)\wedge S^1$, although in the classical world of topology (of complex manifolds), this does suffice:

$$Th_{top}(\mathbb{C}^d \times Z \to Z) = S^{2d} \wedge Z_+ = \Sigma_s^{2d} Z_+.$$

25.5. Brown representability for $\mathcal{SH}(k)$. Each T-spectrum $E \in \mathbf{Spt}_T(k)$ gives rise to a contravariant functor on $\mathbf{Spc}(k)$ to bi-graded abelian groups: $X \mapsto E^{*,*}(X)$ satisfying the axioms of a *generalized Bloch-Ogus cohomology theory*:

Compatibility with disjoint union: $\prod_\alpha E^{*,*}(X_\alpha) \cong E^{*,*}(\amalg_\alpha X_\alpha)$.
Homotopy invariance: $E^{*,*}(X \times \mathbb{A}^1) = E^{*,*}(X)$.
s,t-suspension: $E^{*,*}(\Sigma_T X_+) = E^{*-2,*-1}(X)$;

$$E^{*,*}(\Sigma X_+) = E^{*-1,*}(X).$$

Cofiber sequence: Let $f : X \to Y$ be a map in $\mathbf{Spc}(k)$. Then the mapping cone sequence for f induces a long exact sequence

$$\ldots \to E^{n-1,*}(X) \to E^{n,*}(M(f)) \to E^{n,*}(Y) \to E^{n,*}(X) \to \ldots$$

Nisnevich excision: Let $U \to X$ be an open immersion, $f : V \to X$ an étale map with $V \setminus V \times_X U \cong X \setminus U$. Then

$$E^{*,*}(X) \to E^{*,*}(U \amalg_X V)$$

is an isomorphism.

The motivc Brown representability theorem says

Theorem 25.10. *Each functor $E^{*,*} : \mathbf{Spc}_*(k)^{\mathrm{op}} \to \mathrm{Gr}^2\mathbf{Ab}$ satisfying the axioms of a generalized Bloch-Ogus cohomology theory, plus some additional technical conditions, is represented by a T-spectrum $E \in \mathcal{SH}_T(k)$.*

Remarks 25.11. Brown representability in its modern form has been developed in works of Neeman and others (see e.g. [49]). To explain the statement of the theorem, first of all, what are the "additional technical conditions"? This contains at least the condition that $E^{*,*}$ extends to the subcategory of compact objects $\mathcal{SH}_T(k)^c$ in $\mathcal{SH}_T(k)$. In the case of spaces and specta (with a graded functor E^*), this is a mild condition, but in the motivic setting, this seems to be more complicated and is at present not well understood.

Secondly, one needs a countability condition to extend from $\mathcal{SH}_T(k)^c$ in $\mathcal{SH}_T(k)$. This condition is supplied by a result stated by Voevodsky [62, proposition 5.5], and proven in detail by Naumann and Spitzweck [48], but requires that the base-scheme S is a union of affine open subschemes $U_\alpha = \operatorname{Spec} R_\alpha$ with R_α a *countable* ring. Thus, in the arithmetic setting, we are in good shape, but over e.g., $\mathbb{C}$ the situation is more complicated.

Finally, there is the question of uniqueness, and the related question of phantom maps in $\mathcal{SH}_T(k)$. This complication exists already in the classical case (see e.g. [17])

Examples 25.12. *1. Motivic Eilenberg-Maclane spectra.* Let $i : Z \to X$ be a closed immersion in $\mathbf{Sch}_k$ (for simplicity, we assume that X is quasi-projective). Define the space over k:

$$\mathrm{Sym}^n(X/Z) := \mathrm{Sym}^n X / \cup\, im(X \times \ldots \times Z \times \ldots X).$$

In somewhat more detail: $\mathrm{Sym}^n X$ is the quotient of X^n by the action of the symmetric group $\mathfrak{S}_n$ permuting the factors. For X affine, $X = \mathrm{Spec}\, R$, $\mathrm{Sym}^n X$ is $\mathrm{Spec}\,(R^{\otimes_k n})^{\mathfrak{S}_n}$. Since X is quasi-projective, X^n is covered by affine open subschemes of the form U^n, for $U \subset X$ affine, giving the affine cover of $\mathrm{Sym}^n X$ by the $\mathrm{Sym}^n U$; in particular, $\mathrm{Sym}^n X$ exists in $\mathbf{Sch}_k$. The space $\mathrm{Sym}^n(X/Z)$ is by definition the cofiber of the map in $\mathbf{Spc}(k)$,

$$\amalg_{i=1}^n X \times \ldots \times Z \times \ldots \times X \to \mathrm{Sym}^n X,$$

where in the ith component, Z occurs in the ith factor.

For (Y_i, y_i) pointed schemes, this defines $\mathrm{Sym}^n Y_1 \wedge \ldots \wedge Y_m$. We have the map $\mathrm{Sym}^n Y_1 \wedge \ldots \wedge Y_m \to \mathrm{Sym}^{n+1} Y_1 \wedge \ldots \wedge Y_m$ by "adding $*$ to the sum". Thus we have

$$\mathrm{Sym}^\infty Y_1 \wedge \ldots \wedge Y_m := \mathrm{colim}_n \mathrm{Sym}^n Y_1 \wedge \ldots \wedge Y_m \in \mathbf{Spc}_*(k)$$

Let $M\mathbb{Z} := (S^0_k, \mathrm{Sym}^\infty \mathbb{P}^1, \mathrm{Sym}^\infty(\mathbb{P}^{1\wedge 2}), \ldots)$. The bonding maps are induced by

$$\mathrm{Sym}^n(\mathbb{P}^{1\wedge m}) \wedge \mathbb{P}^1 \to \mathrm{Sym}^n(\mathbb{P}^{1\wedge m+1})$$

by $(\sum_{i=1}^m x_i, x_{m+1}) \mapsto \sum_{i=1}^m (x_i, x)$.

If we form the topological version, with $\mathbb{P}^1 \rightsquigarrow S^2$, the fact that the resulting spectrum is the Eilenberg-Maclane spectrum $EM(\mathbb{Z})$ follows from a theorem of Dold-Thom.

One can think of a map $Z \to \mathrm{Sym}^n Y$ as an effective cycle on $Z \times Y$, finite of degree n over Z. This allows one to show that (at least if $\mathrm{char} k = 0$) $M\mathbb{Z}$ represents motivic cohomology:

$$M\mathbb{Z}^{n,m}(X) \cong H^n(X, \mathbb{Z}(m)).$$

Using cycles throughout rather than symmetric products represents motivic cohomology in arbitrary characteristic. Using cycles with A-coefficients rather than $\mathbb{Z}$-coefficients defines the motivic Eilenberg-Maclane spectrum MA.

2. Algebraic K-theory We have $i : \mathbb{P}^1 \to BGL$ classifying the virtual tautological bundle $\mathcal{O}(1) - \mathcal{O}$. Let $BGL \wedge \mathbb{P}^1 \to BGL$ be the composition

$$BGL \wedge \mathbb{P}^1 \xrightarrow{\mathrm{id}\wedge i} BGL \wedge BGL \xrightarrow{\otimes} BGL.$$

Define the *algebraic K-theory spectrum*

$$\mathcal{K} := (BGL \times \mathbb{Z}, BGL \times \mathbb{Z}, \ldots)$$

with the bonding maps as above.

We already know that $BGL \times \mathbb{Z}$ represents algebraic K-theory in $\mathcal{H}(k)$. The map $BGL \times \mathbb{Z} \to \Omega_T BGL \times \mathbb{Z}$ adjoint to our bonding map, evaluated on $\Sigma^n X_+$ for $X \in \mathbf{Sm}/k$, gives a map

$$\begin{aligned} K_n(X) &:= \mathrm{Hom}_{\mathcal{H}_*(k)}(\Sigma^n X_+, BGL) \\ &\to \mathrm{Hom}_{\mathcal{H}_*(k)}(\Sigma^n X_+, \Omega_T BGL) = \mathrm{Hom}_{\mathcal{H}_*(k)}(\Sigma^n X_+ \wedge \mathbb{P}^1, BGL) \\ &= \ker[K_n(X \times \mathbb{P}^1) \xrightarrow{s_\infty^*} K_n(X)], \end{aligned}$$

where $s_\infty : X \to X \times \mathbb{P}^1$ is the infinity section.

The projective bundle formula for K-theory says that $\ker s_\infty^*$ is isomorphic to $K_n(X)$ via

$$K_n(X) \xrightarrow{([\mathcal{O}(1)]-[\mathcal{O}])\cup p_1^*} K_n(X \times \mathbb{P}^1),$$

Thus $\mathcal{K}$ is an Ω-spectrum, and the bonding maps translate into Bott periodicity if we replace $\mathrm{Gr}(\infty,\infty)$ with the complex points $\mathrm{Gr}_{\mathbb{C}}(\infty,\infty) \sim BU$.

Hornbostel [29] has shown that *Hermitian K-theory* and Balmer's *Witt theory* are representable in $\mathcal{SH}(k)$.

3. Algebraic cobordism Let $U_n \to BGL_n := \mathrm{Gr}(n,\infty)$ be the universal bundle, $i_n : \mathrm{Gr}(n,\infty) \to \mathrm{Gr}(n+1,\infty)$ the map classifying $U_n \oplus e$.

As for MU, i_n induces a map $Th(U_n) \wedge \mathbb{P}^1 \to Th(U_{n+1})$, giving the *motivic Thom spectrum*

$$MGL := (MGL_0, MGL_1, MGL_2, \ldots)$$

with $MGL_n := Th(U_n)$. The cohomology theory $MGL^{*,*}$ is called *algebraic cobordism*

Some basic facts about MGL-theory are known: Hopkins and Morel have constructed a spectral sequence ($X \in \mathbf{Sm}/k$)

$$E_2^{p,q} := \oplus_{a\geq 0} H^{p+2a}(X, \mathbb{Z}(q+a)) \otimes_{\mathbb{Z}} MU^{-a}(pt) \Longrightarrow MGL^{p,q}(X).$$

which degenerates after $\otimes\mathbb{Q}$.

Levine and Morel [40] have defined a "geometric theory" Ω^* on $\mathbf{Sm}/k$, with a natural map

$$\theta_X^n : \Omega^n(X) \to MGL^{2n,n}(X).$$

The spectral sequence shows that θ_X^n is surjective with torsion kernel, and an isomorphism for $X = \operatorname{Spec} k$. We conjecture that θ_X^n is always an isomorphism.[1]

4. The motivic sphere spectrum. As in $\mathcal{SH}$, the sphere spectrum is of fundamental importance. The definition in $\mathcal{SH}(k)$ is exactly analogous:

$$S_k := \Sigma_T^\infty S_k^0 = (S_k^0, \mathbb{P}^1, \mathbb{P}^1 \wedge \mathbb{P}^1, \ldots).$$

Morel's theorem computing $\mathrm{End}_{\mathcal{H}_*(k)}(\mathbb{P}^{1\wedge n})$ immediately gives the T-stable result:

Theorem 25.13 (Morel [45, theorem 6.2.1])**.** *Let k be a perfect field of characteristic different from two. The map sending $u \in k^\times$ to the map*

$$\begin{aligned} \times u : \mathbb{P}^1 &\to \mathbb{P}^1 \\ (x_0 : x_1) &\mapsto (x_0 : ux_1). \end{aligned}$$

descends via the map

$$\begin{aligned} k^\times &\to \mathrm{GW}(k) \\ u &\mapsto <u> \end{aligned}$$

to a natural isomorphism of rings

$$\mathrm{GW}(k) \cong \pi_{0,0}(S_k) := \mathrm{Hom}_{\mathcal{SH}(k)}(S_k, S_k)$$

[1]This has been verified in [41], using the Hopkins-Morel spectral sequence as starting point.

Lecture 6. The Postnikov tower in motivic stable homotopy theory

In this last lecture, we take a look at one aspect of classical stable homotopy theory, the Postnikov tower. Voevodsky [65, 66] has described an interesting version of this construction in motivic stable homotopy theory, which involves a truncation with respect to "weights" rather than the classical notion of connectedness. As we will see, this motivic version of the Postnikov tower is closely related to Grothendieck's *coniveau filtration* on the cohomology of an algebraic variety.

26. Classical Postnikov towers

26.1. The $n-1$-connected cover. Let E be a spectrum, n an integer. The $n-1$-*connected cover* $E{<}n{>} \to E$ of E is a map of spectra such that

1. $\pi^s_m(E{<}n{>}) = 0$ for $m \leq n-1$ and
2. $\pi^s_m(E{<}n{>}) \to \pi^s_m(E)$ is an isomorphism for $m \geq n$.

One can construct $E{<}n{>} \to E$ by killing all the homotopy groups of E in degrees $\geq n$, forming $\pi_n : E \to E(n)$ by successively coning off each element and taking the homotopy colimit, and then take the homotopy fiber of π_n.

There is a structural approach as well: Let $\Sigma^n\mathcal{SH}^{\mathrm{eff}} \xrightarrow{i_n} \mathcal{SH}$ be the full subcategory of $n-1$ connected spectra. A modification of the proof of Brown's representability theorem shows that the functor on $\Sigma^n\mathcal{SH}^{\mathrm{eff}}$

$$A \mapsto \mathrm{Hom}_{\mathcal{SH}}(i_n(A), E)$$

is representable in $\Sigma^n\mathcal{SH}^{\mathrm{eff}}$; the representing object is the $n-1$-connected cover $E{<}n{>} \to E$.

26.2. The Postnikov tower. In fact, we have the tower of subcategories

$$\ldots \subset \Sigma^{n+1}\mathcal{SH}^{\mathrm{eff}} \subset \Sigma^n\mathcal{SH}^{\mathrm{eff}} \subset \ldots \subset \mathcal{SH}$$

so we have for each E the corresponding *Postnikov tower* of connected covers

$$\ldots E{<}n+1{>} \to E{<}n{>} \to \ldots \to E$$

natural in E.

The layers. Form the cofiber $E{<}n+1{>} \to E{<}n{>} \to E{<}n/n+1{>}$. Clearly

$$\pi^s_m(E{<}n/n+1{>}) = \begin{cases} 0 & \text{for } m \neq n \\ \pi^s_n(E) & \text{for } m = n. \end{cases}$$

Obstruction theory gives an isomorphism of $E{<}n/n+1{>}$ with the Eilenberg-Maclane spectrum $\Sigma^n(EM(\pi^s_n(E))) = EM(\pi^s_n(E)[n])$.

The Atiyah-Hirzebruch spectral sequence. Roughly speaking, the Postnikov tower shows how a spectrum is built out of Eilenberg-Maclane spectra. From the point of view of the cohomology theory represented by E, the Postnikov tower yields the *Atiyah-Hirzebruch spectral sequence*

$$E_2^{p,q} := H^p(X, \pi^s_{-q}(E)) \Longrightarrow E^{p+q}(X)$$

(there are convergence problems in general).

This is constructed just like the the spectral sequence for a filtered complex, by linking all the long exact sequences coming from applying $\mathrm{Hom}_{\mathcal{SH}}(\Sigma^\infty X_+, -)$ to the cofiber sequence $E<n+1> \to E<n> \to E<n/n+1>$.

Examples 26.1. 1. The layers of $EM(A)$ are: $EM(A)<0/1> = EM(A)$, $EM(A)<n/n+1> = 0$ for $n \neq 0$.

2. The layers of K_{top} are: $K_{top}<n/n+1> = EM(\mathbb{Z})$ for n even, 0 for n odd (Bott periodicity).

3. The layers of S are $S<0/1> = EM(\mathbb{Z})$, $S<n/n+1> = EM(\text{torsion})$ for $n > 0$, 0 for $n < 0$.

27. The motivic Postnikov tower

27.1. The effective subcategory. From now on, k is a perfect field. Voevodsky has defined the "Tate" analog of the Postnikov tower:

Let $\mathcal{SH}^{\mathrm{eff}}(k)$ be the smallest full triangulated subcategory of $\mathcal{SH}(k)$ containing all the T-suspension spectra $\Sigma^\infty_T A$, $A \in \mathbf{Spc}_*(k)$, and closed under arbitrary direct sums.

Taking T-suspensions gives the tower of full triangulated localizing subcategories
$$\ldots \subset \Sigma_t^{n+1}\mathcal{SH}^{\mathrm{eff}}(k) \subset \Sigma_t^n\mathcal{SH}^{\mathrm{eff}}(k) \subset \ldots \subset \mathcal{SH}(k)$$
$n \in \mathbb{Z}$.

Lemma 27.1. *The inclusion functor $i_n : \Sigma_t^n\mathcal{SH}^{\mathrm{eff}}(k) \to \mathcal{SH}(k)$ admits an exact right adjoint $r_n : \mathcal{SH}(k) \to \Sigma_t^n\mathcal{SH}^{\mathrm{eff}}(k)$.*

This follows by a Brown representability result [49] applied to the functor on $\Sigma_t^n\mathcal{SH}^{\mathrm{eff}}(k)$
$$F \mapsto \mathrm{Hom}_{\mathcal{SH}(k)}(i_n(F), E)$$
for each $E \in \mathcal{SH}(k)$.

Define $f_n := i_n r_n : \mathcal{SH}(k) \to \mathcal{SH}(k)$, giving the natural tower
$$\ldots \to f_{n+1}E \to f_nE \to \ldots \to E$$
which we call the *motivic Postnikov tower*. The layers
$$s_nE := \mathrm{cofib}(f_{n+1}E \to f_nE)$$
are Voevodsky's *slices*.

27.2. Some results. We list the some computations for the motivic Postnikov tower:

1. For the motivic sphere spectrum over k, S_k, $s_0(S_k) = M\mathbb{Z}$ (a theorem of Voevodsky [67]).

2. $s_n(M\mathbb{Z}) = 0$ for $n \neq 0$, $s_0(M\mathbb{Z}) = M\mathbb{Z}$.

3. $s_n(\mathcal{K}) = \Sigma_t^n(M\mathbb{Z})$. (see [38]) The spectral sequence for the motivic Postnikov tower yields the Atiyah-Hirzebruch spectral sequence for K-theory:
$$E_2^{p,q} := H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X)$$

This is the same one as constructed by Bloch-Lichtenbaum [12] (for fields) and extended to arbitrary X by Friedlander-Suslin [20].

To prove these results, it is useful to give a version of the motivic Postnikov tower in *S^1-spectra over k.*

28. S^1-SPECTRA

28.1. The construction. Rather than inverting Σ_T on $\mathbf{Spc}_*(k)$, one can just invert Σ_{S^1}.

Definition 28.1. $\mathbf{Spt}_{S^1}(k)$ is the category of *presheaves of spectra on* $\mathbf{Sm}/k$: objects are sequences $X = (X_0, X_1, \ldots)$ in $\mathbf{Spc}_{p*}(k)$ plus bonding maps $\epsilon_n : \Sigma X_n \to X_{n+1}$.

A map $f : E \to F$ is a *levelwise weak equivalence* if $f_n : E_n \to F_n$ is a weak equivalence in $\mathbf{Spc}_*(k)$ for all n. This extends to a the *projective model structure* just as for T-spectra, so we have a well-defined *0-space functor*

$$\Omega^\infty : \mathcal{H}_{\mathrm{proj}}\mathbf{Spt}_{S^1}(k) \to \mathcal{H}_*(k)$$

and shift functor

$$\Sigma_s : \mathcal{H}_{\mathrm{proj}}\mathbf{Spt}_{S^1}(k) \to \mathcal{H}_{\mathrm{proj}}\mathbf{Spt}_{S^1}(k)$$

induced by $(E_0, E_1, \ldots) \mapsto (E_1, E_2, \ldots)$

Definition 28.2. $f : E \to F$ in $\mathbf{Spt}_{S^1,p}(k)$ is a *stable* $\mathbb{A}^1$ *weak equivalence* if

$$\Omega_s^\infty \Sigma_s^n(f) : \Omega_s^\infty \Sigma_s^n E \to \Omega_s^\infty \Sigma_s^n F$$

is an isomorphism in $\mathcal{H}_*(k)$ for all n.

We give $\mathbf{Spt}_{S^1}(k)$ the Bousfield localization model structure with respect to stable $\mathbb{A}^1$ weak equivalence, just as for T-spectra, giving us the homotopy category $\mathcal{SH}_s(k)$: the stable homotopy category of *motivic S^1-spectra.*

Remark 28.3. Recall that the model structure in $\mathbf{Spc}_*(k)$ already incorporates both Nisnevich-local and $\mathbb{A}^1$-local structures. This builds Nisnevich excision and $\mathbb{A}^1$-homotopy invariance into the model structure in $\mathbf{Spt}_{S^1}(k)$.

28.2. Triangulated structure. The shift Σ_s and shift in the other direction Ω_s form a Quillen pair on $\mathbf{Spt}_{S^1}(k)$, and the derived functors

$$\Sigma_s : \mathcal{SH}_s(k) \to \mathcal{SH}_s(k)$$
$$\Omega_s : \mathcal{SH}_s(k) \to \mathcal{SH}_s(k)$$

are inverse equivalences.

With Σ_s as translation and with the distinguished triangles given by mapping cone sequences, $\mathcal{SH}_s(k)$ becomes a triangulated category.

28.3. From S^1-spectra to T-spectra. We have constructed T-spectra $\mathbf{Spt}_T(k)$ out of spaces $\mathbf{Spc}_*(k)$, inverting the operation Σ_T on the homotopy category by replacing a space with a sequence of spaces plus bonding maps for Σ_T.

We can make exactly the same construction, starting with $\mathbf{Spt}_s(k)$ rather than $\mathbf{Spc}_*(k)$. Since Σ_s is already invertible in $\mathcal{SH}(k)$ (because $\mathbb{P}^1 \cong S^1 \wedge \mathbb{G}_m$) the category of "$T$-spectra of S^1-spectra" has homotopy category equivalent to $\mathcal{SH}(k)$ (see [31, §§3.2, 3.3] for details).

We will go back and forth between these two constructions of $\mathcal{SH}(k)$ as needed. In the next few sections, we carry out some of the details of the construction.

28.4. ***s, t*-spectra.** The T-suspension functor Σ_T on $\mathbf{Spc}_*(k)$ induces the functor Σ_T on $\mathbf{Spt}_{S^1}(k)$ by

$$\Sigma_T(E_0, E_1, \ldots) := (\Sigma_T E_0, \Sigma_T E_1, \ldots)$$

with right adjoint the similarly defined T-loops functor Ω_T.

Definition 28.4. An *s, t-spectrum* E is a sequence $(E_0.E_1, \ldots)$, $E_n \in \mathbf{Spt}_{S^1}(k)$ plus bonding maps $\epsilon_n : \Sigma_T E_n \to E_{n+1}$.

This defines the category of s, t-spectra over k, $\mathbf{Spt}_{s,t}(k)$.

Model structure. We use the same two-step procedure as we did for T-spectra and S^1-spectra to define a model structure on $\mathbf{Spt}_{s,t}(k)$:

1st define the projective model structure relative to $\mathbf{Spt}_{S^1}(k)$.

Next, use the shift functor Σ_t and the T-0-space functor

$$\Omega_t^\infty : \mathbf{Spt}_{s,t}(k) \to \mathbf{Spt}_{S^1}(k)$$

to define $\mathbb{A}^1$-stable weak equivalences.

Finally, define the $\mathbb{A}^1$-stable model structure on $\mathbf{Spt}_{s,t}(k)$ as the Bousfield localization of the projective model structure by the $\mathbb{A}^1$-stable weak equivalences. We write $\mathcal{SH}_{s,t}(k)$ for the homotopy category of $\mathbf{Spt}_{s,t}(k)$.

28.5. $\mathbf{Spt}_T(k)$ and $\mathbf{Spt}_{s,t}(k)$. Given an $E = (E_0, E_1, \ldots) \in \mathbf{Spt}_{s,t}(k)$ we can form its termwise s-0-space spectrum $((E_0)_0, (E_1)_0, \ldots)$, giving an object $\Omega^\infty_{S^1} E$ of $\mathbf{Spt}_T(k)$.

Similarly, given $E = (E_0, E_1, \ldots)$ in $\mathbf{Spt}_T(k)$, we can form the sequence of S^1-suspension spectra $(\Sigma_s^\infty E_0, \Sigma_s^\infty E_1, \ldots)$, giving an object $\Sigma^\infty_{S^1} E$ in $\mathbf{Spt}_{s,t}(k)$.

Proposition 28.5. *The functors*

$$\Sigma^\infty_{S^1} : \mathbf{Spt}_T(k) \to \mathbf{Spt}_{s,t}(k)$$
$$\Omega^\infty_{S^1} : \mathbf{Spt}_{s,t}(k) \to \mathbf{Spt}_T(k)$$

define a Quillen equivalence; in particular, the derived functors $(\Sigma_s^\infty, \Omega_s^\infty)$ *define inverse equivalences of the homotopy categories* $\mathcal{SH}_{s,t}(k)$, $\mathcal{SH}(k)$.

Summary. The diagram summarizes all these constructions:

$$\begin{tikzcd}
 & & \mathcal{SH}(k) \arrow[dl, shift left, "\Omega_t^\infty"] \arrow[dr, shift left, "\Sigma_s^\infty"] & \\
\mathcal{H}(k) \arrow[r, "+"] & \mathcal{H}_*(k) \arrow[ur, shift left, "\Sigma_t^\infty"] \arrow[rr, shift left, "\Sigma_{s,t}^\infty"] \arrow[dr, shift left, "\Sigma_s^\infty"] & & \mathcal{SH}_{s,t}(k) \arrow[ul, shift left, "\Omega_s^\infty"] \arrow[ll, shift left, "\Omega_{s,t}^\infty"] \arrow[dl, shift left, "\Omega_t^\infty"] \\
 & & \mathcal{SH}_s(k) \arrow[ul, shift left, "\Omega_s^\infty"] \arrow[ur, shift left, "\Sigma_t^\infty"] &
\end{tikzcd}$$

$\Omega_s = \Sigma_s^{-1}$ on $\mathcal{SH}_s(k)$. $\mathcal{SH}_s(k)$ is a triangulated category with distinguished triangles the homotopy (co)fiber sequences. Σ_t operates on $\mathcal{SH}_s(k)$, but is *not* an equivalence. Finally, $\mathcal{SH}(k)$ and $\mathcal{SH}_{s,t}(k)$ are equivalent via Ω_s^∞, Σ_s^∞.

Remark 28.6. We have the same commutativity problems here as with T-suspension in $\mathcal{SH}(k)$, but these are solved in a similar way and will be ignored.

28.6. The Postnikov tower in $\mathcal{SH}_s(k)$. Just as one can form an unstable Postnikov tower in $\mathcal{H}_*$, we have the "semi-stable" motivic Postnikov tower in $\mathcal{SH}_s(k)$.

Take the tower of full triangulated subcategories

$$\ldots \subset \Sigma_t^{n+1}\mathcal{SH}_s(k) \subset \Sigma_t^n\mathcal{SH}_s(k) \subset \ldots \subset \Sigma_t\mathcal{SH}_s(k) \subset \mathcal{SH}_s(k)$$

The inclusions $i_{n,s} : \Sigma_t^n\mathcal{SH}_s(k) \to \mathcal{SH}_s(k)$ have a right adjoint $r_{n,s} : \mathcal{SH}_s(k) \to \Sigma_t^n\mathcal{SH}_s(k)$, giving us the truncation functors

$$f_{n,s} : \mathcal{SH}_s(k) \to \mathcal{SH}_s(k),$$

and for $E \in \mathcal{SH}_s(k)$, the *S^1-motivic Postnikov tower*

$$\ldots \to f_{n+1,s}E \to f_{n,s}E \to \ldots \to f_{1,s}E \to E.$$

Let $s_{n,s}E$ be the cofiber of $f_{n+1,s}E \to f_{n,s}E$.

The T-suspension functor

$$\Sigma_t^\infty : \mathcal{SH}_s(k) \to \mathcal{SH}(k)$$

maps $\Sigma_t^n\mathcal{SH}_s(k)$ to $\Sigma_t^n\mathcal{SH}(k)$, giving the natural map

$$\Sigma_t^\infty f_{n,s}E \to f_n\Sigma_t^\infty E.$$

28.7. The connectedness conjecture. The behavior of the 0-space functor

$$\Omega_t^\infty : \mathcal{SH}(k) \to \mathcal{SH}_s(k)$$

is more subtle. Voevodsky conjectured the following analogs of the classical *Freudenthal suspension theorem*:

Conjecture 28.7. *If E is in $\Sigma_t^n\mathcal{SH}_s(k)$, then so is $\Omega_t\Sigma_tE$.*

Conjecture 28.8. *If E is in $\Sigma_t^n\mathcal{SH}_s(k)$ then so is $\Omega_t^\infty\Sigma_t^\infty E$.*

Since $\Omega_t^\infty\Sigma_t^\infty E$ is the colimit of $(\Omega_t\Sigma_t)^nE$, the second conjecture follows from the first.

The Freudenthal suspension theorem is the statement that if a pointed space A is n-connected, then ΣA is $n+1$-connected. Since $\pi_m(\Omega A) = \pi_{m-1}(A)$, $\Omega\Sigma A$ is as connected as A.

These conjectures are interesting for us, since the 2nd conjecture implies

Conjecture 28.9 (Compatibility of truncations)**.** *For $\mathcal{E} \in \mathcal{SH}(k)$, there is a natural isomorphism*

$$f_{n,s}(\Omega_t^\infty\mathcal{E}) \cong \Omega_t^\infty(f_n\mathcal{E})$$

for all $n \geq 0$.

Indeed, since $\Sigma_t^n\mathcal{SH}^{\mathrm{eff}}(k)$ is generated by $\Sigma_t^\infty(\Sigma_t^n\mathcal{SH}_s(k))$, the 2nd conjecture implies that

$$\Omega_t^\infty(\Sigma_t^n\mathcal{SH}^{\mathrm{eff}}(k) \subset \Sigma_t^n\mathcal{SH}_s(k).$$

Therefore $\Omega_t^\infty(f_n\mathcal{E})$ is in $\Sigma_t^n\mathcal{SH}_s(k)$.

Defining K by the exactness of

$$\Omega_t^\infty(f_n\mathcal{E}) \to \Omega_t^\infty(\mathcal{E}) \to K$$

it follows that $\mathrm{Hom}(\Sigma_t^n\mathcal{SH}_s(k), K) = 0$ since Ω_t^∞ is a right adjoint. Thus

$$\Omega_t^\infty(f_n\mathcal{E}) \to \Omega_t^\infty(\mathcal{E})$$

satisfies the universal property of $f_{n,s}\Omega_t^\infty(\mathcal{E})$.

The 1st conjecture is a consequence of

Theorem 28.10 (Levine [38])**.** *There is a natural isomorphism of functors on $\mathcal{SH}_s(k)$:*

$$f_{n,s} \circ \Omega_t \cong \Omega_t \circ f_{n+1,s}.$$

Indeed, F is in $\Sigma_t^n\mathcal{SH}_s(k)$ if and only if the canonical map $f_{n,s}F \to F$ is an isomorphism. For $E \in \Sigma_t^n\mathcal{SH}_s(k)$, $\Sigma_t E$ is in $\Sigma_t^{n+1}\mathcal{SH}_s(k)$ so

$$\begin{aligned} f_{n,s} \circ \Omega_t\Sigma_t E &\cong \Omega_t f_{n+1,s}\Sigma_t E \\ &\cong \Omega_t\Sigma_t E \end{aligned}$$

28.8. An application: the slices of K-theory. Voevodsky shows how these conjectures, together with the computation/theorem:

$$s_0(S_k) \cong M\mathbb{Z}$$

easily leads to a computation of all the slices of $\mathcal{K}$.

1. $s_n(\mathcal{K}) = \Sigma_t^n(s_0\mathcal{K})$. This follows from Bott periodicity for $\mathcal{K} := \Sigma_t\mathcal{K} \cong \mathcal{K}$.

2. To compute $s_0\mathcal{K}$: The unit $S_k \to \mathcal{K}$ gives the map on the 0th slice

$$M\mathbb{Z} = s_0(S_k) \xrightarrow{\epsilon} s_0(\mathcal{K}).$$

using Voevodsky's theorem that $M\mathbb{Z} = s_0(S_k)$.

3. To show ϵ is an isomorphism: It suffices to show $\Omega_t^\infty(\epsilon)$ is an isomorphism in $\mathcal{SH}_s(k)$. We have

$$\Omega_t^\infty(M\mathbb{Z}) = EM_s(\mathbb{Z}): \text{ the “naive” Eilenberg-Maclane spectrum}$$

and $\Omega_t^\infty(\mathcal{K})$ is the Ω_s-spectrum

$$(\mathrm{Gr} \times \mathbb{Z}, \Omega_{\mathbb{G}_m}\mathrm{Gr} \times \mathbb{Z}, \ldots, \Omega_{\mathbb{G}_m}^n\mathrm{Gr} \times \mathbb{Z}, \ldots).$$

We have the canonical map

$$\Sigma_s^\infty(\mathrm{Gr} \times \mathbb{Z}) \to \Omega_t^\infty(\mathcal{K});$$

a technical result of Voevodsky's [66, proposition 4.4] reduces us to showing that $s_0(\Sigma_s^\infty\mathrm{Gr}) = 0$.

This is easy: $\mathrm{Gr}(m,n)$ contains an open cell $U_{m,n} \cong \mathbb{A}^{m(n-m)}$. By purity $\Sigma_s^\infty\mathrm{Gr}(m,n) \cong \Sigma_s^\infty(\mathrm{Gr}(m,n)/U_{m,n})$ is in $\Sigma_t^1\mathcal{SH}_s(k)$.

We give a proof of theorem 28.10 in the next section, using the *homotopy coniveau tower.*

29. The homotopy coniveau tower

Here is an outline of the proof of theorem 28.10:

• The construction of the spectral sequence from motivic cohomology to K-theory (Bloch-Lichtenbaum, Friedlander-Suslin) generalizes to the *homotopy coniveau tower* for an arbitrary $E \in \mathbf{Spt}(k)$, $X \in \mathbf{Sm}/k$:

$$\ldots \to E^{(n+1)}(X) \to E^{(n)}(X) \to \ldots \to E^{(0)}(X) = E^{(-1)}(X) = \ldots.$$

• $X \mapsto E^{(n)}(X)$ can be made functorial in X.

• For $i : W \to X$ a codimension d closed immersion with open complement U, Bloch's moving lemma gives a localization distinguished triangle in $\mathcal{SH}$:

$$E^{(n-d)}(W) \xrightarrow{i_*} E^{(n)}(X) \xrightarrow{j^*} E^{(n)}(U) \to E^{(n-d)}(W)[1]$$

• There is a natural isomorphism $E^{(n)} \cong f_{n,s}E$. The localization distinguished triangle applied to $X \times 0 \to X \times \mathbb{P}^1$ proves the theorem

$$f_{n,s} \circ \Omega_t \cong \Omega_t \circ f_{n+1,s}.$$

• For $\mathcal{E} = (E_0, E_1, \ldots, E_m, \ldots) \in \mathcal{SH}(k)$, the sequence

$$(E_0^{(n)}, E_1^{(n+1)}, \ldots, E_m^{(n+m)}, \ldots)$$

gives a model for $f_n\mathcal{E}$. Thus: the motivic Postnikov tower is just a homotopy invariant version of the coniveau filtration.

29.1. The construction of $E^{(n)}(X)$. $\Delta^n := \operatorname{Spec} k[t_0, \ldots, t_n]/\sum_i t_i - 1$
A *face* F of Δ^n is a closed subscheme defined by
$t_{i_1} = \ldots = t_{i_r} = 0$.

$n \mapsto \Delta^n$ extends to the cosimplicial scheme

$$\Delta^* : \mathbf{Ord} \to \mathbf{Sm}/k.$$

For $E \in \mathbf{Spt}(k)$, $X \in \mathbf{Sm}/k$, $W \subset X$ closed, set

$$E^W(X) := \mathrm{fib}(E(X) \to E(X \setminus W)).$$

For $X \in \mathbf{Sm}/k$, set

$$\mathcal{S}_X^{(p)}(n) := \{W \subset X \times \Delta^n, \text{closed}, \operatorname{codim}_{X\times F} W \cap (X \times F) \geq p\}.$$

For $E \in \mathbf{Spt}(\mathbf{Sm}/k)$ set

$$E^{(p)}(X, n) := \operatorname*{hocolim}_{W \in \mathcal{S}_X^{(p)}(n)} E^W(X \times \Delta^n).$$

This gives the simplicial spectrum $E^{(p)}(X)$: $n \mapsto E^{(p)}(X, n)$, and the *homotopy coniveau tower*

$$\ldots \to E^{(p+1)}(X) \to E^{(p)}(X) \to \ldots \to E^{(0)}(X) = E^{(-1)}(X) = \ldots$$

Remark 29.1. $X \mapsto E^{(p)}(X)$ is functorial in X for *flat* maps.

We denote the homotopy cofiber of $E^{(p+1)}(X) \to E^{(p)}(X)$ by $E^{(p/p+1)}(X)$, giving the distinguished triangle in $\mathcal{SH}$

$$E^{(p+1)}(X) \to E^{(p)}(X) \to E^{(p/p+1)}(X) \to E^{(p+1)}(X)[1]$$

29.2. The S^1-stable theory. Fix an $E \in \mathbf{Spt}(k)$. We will assume 2 basic properties hold for E:

(1) *homotopy invariance:* For all $X \in \mathbf{Sm}/k$, $E(X) \to E(X \times \mathbb{A}^1)$ is a stable weak equivalence.
(2) *Nisnevich excision:* Let $f : Y \to X$ be an étale map in $\mathbf{Sm}/k$. Suppose $W \subset X$ is a closed subset such that $f : f^{-1}(W) \to W$ is an isomorphism. Then $f^* : E^W(X) \to E^{f^{-1}(W)}(Y)$ is a stable weak equivalence.

We also assume that k is an infinite field.

The two properties imply that the fibrant replacement $E \to E_{\text{fib}}$ is a objectwise stable weak equivalence, so we can use E instead of E_{fib}.

Theorem 29.2. *Let E be in $\mathbf{Spt}(k)$ satisfying properties 1 and 2. Then*
(1) $X \mapsto E^{(p)}(X)$ extends (up to weak equivalence) to a functor $E^{(p)} : \mathbf{Sm}/k^{\text{op}} \to$ $\mathbf{Spt}$ satisfying properties 1 and 2.

(2) Localization. *Let $i : W \to X$ be a closed codimension d closed embedding in $\mathbf{Sm}/k$, with trivialized normal bundle, and open complement $j : U \to X$. There is a natural homotopy fiber sequence in $\mathcal{SH}$*

$$(\Omega_t^d E)^{(p-d)}(W) \to E^{(p)}(X) \xrightarrow{j^*} E^{(p)}(U)$$

(3) Delooping. *There is a natural weak equivalence*

$$(\Omega_t^m E)^{(n)} \xrightarrow{\sim} \Omega_t^m(E^{(n+m)})$$

Idea of proof. (1) Functoriality: this is proven using Chow's moving lemma, just as for Bloch's cycle complexes.

(2) Localization: this is proven using Bloch's moving lemma (blowing up) just as for Bloch's cycle complexes.

Note. The techniques for (1) and (2) are essentially homological, so one needs to use a version of the Hurewicz theorem: if the stable homology of a spectrum is 0 in all degrees, then the stable homotopy groups are also all 0.

(3) Delooping follows from the localization sequence:

$$(\Omega_t E)^{(n)}(X \times 0) \to E^{(n+1)}(X \times \mathbb{P}^1) \to E^{(n+1)}(X \times \mathbb{A}^1)$$

and the natural weak equivalence

$$\text{fib}(F(X \times \mathbb{P}^1) \to F(X \times \mathbb{A}^1)) \cong (\Omega_t F)(X).$$

$\square$

Corollary 29.3 (Birationality)**.** *Take $E \in \mathbf{Spt}(k)$, $X \in \mathbf{Sm}/k$, $Z \subset X$ closed,* $\text{codim}_X Z \geq d$. *Then*

$$(E^{(d/d+1)})^Z(X) \cong \bigoplus_{z \in Z \cap X^{(d)}} (\Omega_T^d E)^{(0/1)}(k(z)).$$

In particular, for $d = 0$, $Z = X$ irreducible, this gives

$$E^{(0/1)}(X) \cong E^{(0/1)}(k(X)).$$

Proof. To restrict from X to $X \setminus Z$, stratify, so can assume Z is smooth, irreducible codimension $q \geq d$, with trivial normal bundle in X. By localization

$$(E^{(d/d+1)})^Z(X) \cong (\Omega^q E)^{(d-q/d-q+1)}(Z).$$

If $\mathrm{codim}_X Z > d$, then

$$(\Omega^q E)^{(d-q)}(Z) = (\Omega^q E)^{(0)}(Z) = (\Omega^q E)^{(d-q+1)}(Z)$$

so the cofiber $(\Omega^q E)^{(d-q/d-q+1)}(Z)$ is 0. This reduces everything to the generic points of Z which are codimension d on X.

For $z \in X$ closed codimension d, localization gives as above

$$(E^{(d/d+1)})^Z(X) \cong (\Omega^q E)^{(0/1)}(z).$$

□

29.3. The idempotence theorem. We iterate the operation $E \mapsto E^{(p)}$.

Theorem 29.4. *Suppose E satisfies properties 1, 2. Then*

$$(E^{(q)})^{(p)} \cong E^{(\max\{p,q\})}$$

in $\mathcal{SH}_s(k)$.

Proof. The proof uses the *semi-local* Δ^*: For a field F

$$\Delta^n_{F,0} = \mathrm{Spec}\,(\mathcal{O}_{\Delta^n,v^n})$$

where v^n is the set of vertices of Δ^n. The $\Delta^n_{F,0}$ form a cosimplicial subscheme of Δ^*.

By definition, we have, for $X \in \mathbf{Sm}/k$ irreducible

$$E^{(0/1)}(X) = E(\Delta^*_{k(X),0}).$$

The essential case is to show that

$$(E^{(q)})^{(0/1)} = 0$$

for $q > 0$, i.e., $(E^{(q)})^{(1)} \cong E^{(q)}$.

For any $Y \in \mathbf{Sm}/k$ and "good" $F \in \mathbf{Spt}_{S^1}(k)$,

$$F^{(0/1)}(Y) = F(\Delta^*_{k(Y),0})$$

Thus

$$(E^{(q)})^{(0/1)}(X) = E^{(q)}(\Delta^*_{k(X),0}).$$

$E^{(q)}(\Delta^*_{k(X),0})$ is the total spectrum of the bi-simplicial spectrum

$$(n,m) \mapsto E^{(q)}(n,m) := \mathrm{colim}_{W \subset' \Delta^n_{k(X),0} \times \Delta^m} E^W(\Delta^n_{k(X),0} \times \Delta^m)$$

where the limit is over W of codimension $\geq q$, intersecting all products of faces $F_0' \times F$ properly.

Let $E^{(q)}(-,m)$ be the total spectrum of $n \mapsto E^{(q)}(n,m)$. These satisfy a homotopy invariance property: the restriction to a face

$$\delta_i^* : E^{(q)}(-,m+1) \to E^{(q)}(-,m)$$

is a weak equivalence.

Thus the inclusion $E^{(q)}(-,0) \to E^{(q)}(-,-)$ is a weak equivalence. But if $W \subset \Delta^n_{k(X),0}$ has codimension $\geq q > 0$ and intersects all faces properly, then W contains no vertex of $\Delta^n_{k(X),0}$. Thus $W = \emptyset$, and $E^{(q)}(-,0) = 0$. □

29.4. The comparison theorem. We first prove 2 lemmas

Lemma 29.5. *For $p \geq q$, the map*

$$(f_{q,s}E)^{(p)} \to E^{(p)}$$

is an isomorphism in $\mathcal{SH}_s(k)$.

Idea of proof. Take $W \subset Y$, and $E \in \mathbf{Spt}_{S^1}(k)$ fibrant. Then

$$E^W(Y) = \mathcal{H}om(\Sigma_s^\infty Y/(Y \setminus W), E).$$

If $\mathrm{codim}_Y W \geq p$, then the Thom isomorphism + a stratification argument $\Longrightarrow$

$$\Sigma_s^\infty Y/(Y \setminus W) \in \Sigma_t^p \mathcal{SH}_s(k)$$

But $f_{q,s}E \to E$ is universal for maps $F \to E$, $F \in \Sigma_t^q \mathcal{SH}_s(k) \supset \Sigma_t^p \mathcal{SH}_s(k)$, so

$$(f_{q,s}E)^W(Y) \to E^W(Y)$$

is an isomorphism in $\mathcal{SH}$.

Apply this to $W \subset X \times \Delta^n$ and take colim over all "good" codim $\geq p$ W:

$$(f_{q,s}E)^{(p)}(X,n) \to E^{(p)}(X,n)$$

is a weak equivalence for all $n \Longrightarrow (f_{q,s}E)^{(p)}(X) \to E^{(p)}(X)$ is a weak equivalence for all X. □

Lemma 29.6. *For $E \in \Sigma_t^p \mathcal{SH}_s(k)$ the natural map*

$$\phi_{E,q} : E^{(q)} \to E$$

is an isomorphism in $\mathcal{SH}_s(k)$ for $p \geq q$.

We will in fact show that

(*) $$E \in \Sigma_t^p \mathcal{SH}_s(k) \Longrightarrow E^{(n/n+1)} = 0 \text{ for } 0 \leq n < p.$$

Since all the layers in

$$E^{(q)} \to E$$

are of this form, (*) implies the result.

Proof. Write $E = \Sigma_t^p F$. We first construct a splitting $E \to E^{(p)}$ to $E^{(p)} \to E$.

By the delooping isomorphism, we have:

$$\Omega_t^p(E^{(p)}) \cong \Omega_t^p E = \Omega_t^p \Sigma_t^p F.$$

The natural map

$$F \to \Omega_t^p \Sigma_t^p F \cong \Omega_t^p(E^{(p)})$$

thus gives by adjunction the map

$$E = \Sigma_t^p F \to E^{(p)}.$$

Apply $(-)^{(n/n+1)}$ to the diagram

$$\begin{tikzcd} E \arrow[rr, "\mathrm{id}"] \arrow[dr] & & E \\ & E^{(p)} \arrow[ur] & \end{tikzcd}$$

By the idempotence theorem

$$(E^{(p)})^{(n/n+1)} = 0$$

for $0 \le n < p$.

Thus $\mathrm{id} : E \to E$ induces the 0 map on $E^{(n/n+1)}$ for $0 \le n < p$, hence

$$E^{(n/n+1)} = 0.$$

□

Corollary 29.7. *For $E \in \mathcal{SH}_s(k)$:*

(1) The natural map

$$\phi_{f_{p,s}E,q} : (f_{p,s}E)^{(q)} \to f_{p,s}E$$

is an isomorphism for $p \ge q$.

(2) $E^{(p)}$ is in $\Sigma^p_t \mathcal{SH}_s(k)$ for all $p \ge 0$.

Proof. (1) follows from the 2nd lemma, since $f_{p,s}E$ is in $\Sigma^p_t\mathcal{SH}_s(k)$.

(2) follows from the the two lemmas: $f_{p,s}E$ is in $\Sigma^p_t\mathcal{SH}_s(k)$, so by the 2nd lemma $(f_{p,s}E)^{(p)}$ is in $\Sigma^p_t\mathcal{SH}_s(k)$. By the 1st lemma

$$(f_{p,s}E)^{(p)} \cong E^{(p)}$$

so $E^{(p)}$ is in $\Sigma^p_t\mathcal{SH}_s(k)$.

The map $\phi_E : E^{(p)} \to E$ thus induces the canonical map κ fitting into the commutative diagram

$$\begin{tikzcd} & E^{(p)} \arrow[dl, "\kappa"'] \arrow[d, "\phi_E"] \\ f_{p,s}E \arrow[r, "\tau"'] & E \end{tikzcd}$$

Theorem 29.8. *κ is an isomorphism.*

This is easy: Apply $(-)^{(p)}$ to the canonical map $\tau : f_{p,s}E \to E$, giving the commutative diagram

$$\begin{tikzcd} (f_{p,s}E)^{(p)} \arrow[r, "\tau^{(p)}"] \arrow[d, "\phi_{f_{p,s}E}"'] & E^{(p)} \arrow[d, "\phi_E"] \\ f_{p,s}E \arrow[r, "\tau"'] & E \end{tikzcd}$$

We have already shown that the maps $\tau^{(p)}$ and $\phi_{f_{p,s}E}$ are isomorphisms, so we can fill in the diagram with an isomorphism $\kappa' : E^{(p)} \to f_{p,s}E$:

$$\begin{tikzcd} (f_{p,s}E)^{(p)} \arrow[r, "\tau^{(p)}"] \arrow[d, "\phi_{f_{p,s}E}"'] & E^{(p)} \arrow[d, "\phi_E"] \arrow[dl, dashed, "\kappa'"'] \\ f_{p,s}E \arrow[r, "\tau"'] & E \end{tikzcd}$$

Since κ is unique, $\kappa = \kappa'$. □

29.5. Generalized cycle complexes. The idempotence theorem gives a description of $E^{(p/p+1)}$ as a "generalized cycle complex".

Let $X^{(p)}(n)$ be the set of codimension p points $w \in X \times \Delta^n$ with closure $\bar{x}$ intersecting $X \times F$ in codimension $\geq p$ for all faces F of Δ^n.

Theorem 29.9. *Take $E \in \mathbf{Spt}_{S^1}(k)$ satisfying properties 1 and 2. Then there is a simplicial spectrum $E^{(p)}_{s.l.}(X)$, with*

$$E^{(p)}_{s.l.}(X)(n) \cong \bigoplus_{w \in X^{(p)}(n)} (\Omega^p_t E)^{(0/1)}(w)$$

and with $E^{(p/p+1)}(X)$ is isomorphic in $\mathcal{SH}$ to $E^{(p)}_{s.l.}(X)$.

That is: $E^{(p/p+1)}(X)$ is the "generalized cycle complex" with coefficients in $(\Omega^p_t E)^{(0/1)}$. We use the birational natural of $(\Omega^p_t E)^{(0/1)}$ to get well-defined boundary maps.

Proof. To prove the theorem, use the idemotence theorem:

$$E^{(p/p+1)}(X) \cong (E^{(p/p+1)})^{(p)}(X)$$

But for $W \subset X \times \Delta^n$ of codimension p, the birationality theorem gives

$$(E^{(p/p+1)})^W(X \times \Delta^n) = \sum_{w \in W_{\text{gen}}} (\Omega^p_t E)^{(0/1)}(w)$$

as desired. $\square$

Remarks 29.10. (1) One can calculate $K^{(p/p+1)}(X)$ directly using this result. It is not hard to see that

$$(\Omega^p_t K)^{(0/1)}(w) = K^{(0/1)}(w) = EM(K_0(k(w)), 0) = EM(\mathbb{Z}, 0),$$

so we get $K^{(p)}_{s.l.}(X) = z^p(X, *)$.

(2) The "coefficient spectrum" $E^{(p)}_{s.l.}(X)$ has been computed explicitly for some other E, for example $E(X) = K(X; \mathcal{A})$ for $\mathcal{A}$ a central simple algebra over k (with Bruno Kahn [32]). We get

$$(\Omega^p_t K_{\mathcal{A}})^{(0/1)}(w) = K^{(0/1)}_{\mathcal{A}}(w) = EM(K_0(k(w) \otimes_k \mathcal{A}), 0).$$

30. The T-stable theory

The description of the S^1-stable Postnikov tower via the homotopy coniveau tower can be extended to give a similar description of the T-stable Postnikov tower; this relies on the localization properties of the homotopy coniveau tower to give a canonical T-delooping. We conclude with a discussion of the 0th slice of the motivic sphere spectrum, giving a "cycle-theoretic" approach to the theorem of Voevodsky identifying $s_0(S_k)$ with motivic cohomology.

30.1. The T-stable homotopy coniveau tower. Let

$$\mathcal{E} := (E_0, E_1, \ldots, E_n, \ldots)$$
$$\epsilon_n : E_n \to \Omega_T E_{n+1}$$

be an s,t-spectrum over k. We assume that the ϵ_n are weak equivalences.

For each n, m we have the weak equivalence $\epsilon_n^{<m>}$:

$$E_n^{(n+m)} \xrightarrow{(\epsilon_n)^{(n+m)}} (\Omega_T E_{n+1})^{(n+m)} \xrightarrow{\text{deloop}} \Omega_T(E_{n+1}^{(n+m+1)})$$

Set:

$$\mathcal{E}{<}m{>} := (E_0^{(m)}, E_1^{(m+1)}, \ldots, E_n^{(m+n)}, \ldots)$$

with bonding maps $\epsilon_n^{<m>}$.

The homotopy coniveau towers

$$\ldots \to E_n^{(m+n+1)} \to E_n^{(m+n)} \to \ldots$$

fit together to form the *T-stable homotopy coniveau tower*

$$\ldots \to \mathcal{E}{<}m+1{>} \to \mathcal{E}{<}m{>} \to \ldots \to \mathcal{E}{<}0{>} \to \mathcal{E}{<}{-1}{>} \to \ldots \to \mathcal{E}.$$

in $\mathcal{SH}(k)$.

30.2. The comparison theorem.

Proposition 30.1. *For $\mathcal{E} \in \mathcal{SH}(k)$, $\mathcal{E}{<}n{>}$ is in $\Sigma_t^n \mathcal{SH}^{\text{eff}}(k)$.*

The universal property of $f_n\mathcal{E} \to \mathcal{E}$, applied to $\mathcal{E}{<}m{>} \to \mathcal{E}$, gives the canonical map $h : \mathcal{E}{<}m{>} \to f_n\mathcal{E}$.

Theorem 30.2. *For each $\mathcal{E} \in \mathcal{SH}(k)$, the map $h : \mathcal{E}{<}m{>} \to f_n\mathcal{E}$ is an isomorphism.*

These results follow easily from the S^1 results.

30.3. The 0th slice of S_k.

Theorem 30.3 (0th-slice)**.** $s_0(S_k) \cong M\mathbb{Z}$.

Idea of proof. By definition

$$s_0(S_k) \cong S_k{<}0/1{>} := ((\Sigma_t^\infty \operatorname{Spec} k_+)^{(0/1)}, (\Sigma_t^\infty \mathbb{P}^1)^{(1/2)}, \ldots, \Sigma_t^\infty (\mathbb{P}^{1\wedge p})^{(p/p+1)}, \ldots)$$

The individual term $\Sigma_t^\infty (\mathbb{P}^{1\wedge p})^{(p/p+1)}$ can be interpreted as a "cycle complex":

$$\Sigma_t^\infty (\mathbb{P}^{1\wedge p})^{(p/p+1)} = \text{codim } p \text{ cycle complex with coefficients}$$
$$(\Omega_t^\infty \Sigma_t^\infty \operatorname{Spec} k_+)^{(0/1)}.$$

This description allows one to construction a "reverse cycle map"

$$\text{rev} : M\mathbb{Z} \to S_k{<}0/1{>}$$

inverse to the canonical map $S_k{<}0/1{>} \to M\mathbb{Z}{<}0/1{>} = M\mathbb{Z}$. $\square$

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