Mixed Motives

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Summary. This is a first version of the chapter on mixed motives for the *K-Theory Handbook*

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1 Introduction

1.1 Mixed motives

During the early and mid-eighties, Beilinson[2] and Deligne [24] independently described a conjectural abelian tensor category of *mixed motives* over a given base field k, $\mathcal{M}\mathcal{M}_k$, which, in analogy to the category of mixed Hodge structures, should contain Grothendieck's category of pure (homological) motives as the full subcategory of semi-simple objects, but should have a rich enough structure of extensions to allow one to recover the weight-graded pieces of algebraic K-theory. More specifically, one should have, for each smooth scheme X of finite type over a given field k, an object $h(X)$ in the derived category $D^b(\mathcal{M}\mathcal{M}_k)$, as well as Tate twists $h(X)(n)$, and natural isomorphisms

$$\mathrm{Hom}_{D^b(\mathcal{M}\mathcal{M}_k)}(1, h(X)(n)[m]) \otimes \mathbb{Q} \cong K_{2n-m}(X)^{(n)},$$

where $K_p(X)^{(n)}$ is the weight n eigenspace for the Adams operations. The abelian groups

$$H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) := \mathrm{Hom}_{D^b(\mathcal{M}\mathcal{M}_k)}(1, h(X)(q)[p])$$

should form the universal Bloch-Ogus cohomology theory on smooth k-schemes of finite type; as this theory should arise from mixed motives, it is called *motivic* cohomology.

This category $\mathcal{M}\mathcal{M}_k$ should on the one hand give a natural framework for Beilinson's unified conjectures on the relation of algebraic K-theory to values of L-functions, and on the other hand, give a direct relation of singular cohomology and the Chow ring. For this, conjectures of Beilinson, Bloch and Murre [74] suggest a decomposition (with $\mathbb{Q}$-coefficients)

$$h(X)_{\mathbb{Q}} = \oplus_{i=0}^{2d} h^i(X)[-i]$$

for X a smooth projective variety of dimension d over k, with the $h^i(X)$ semi-simple objects in $\mathcal{MM}_k \otimes \mathbb{Q}$. This yields a decomposition

$$H^{2n}(X, \mathbb{Q}(n)) = \oplus_{i=0}^{2d} \mathrm{Ext}^{2n-i}_{\mathcal{MM}_k \otimes \mathbb{Q}}(1, h^i(X)(n));$$

since one expects $H^{2n}(X, \mathbb{Q}(n)) = K_0(X)^{(n)} = \mathrm{CH}^n(X)_{\mathbb{Q}}$, this would give an interesting decomposition of the Chow group $\mathrm{CH}^n(X)_{\mathbb{Q}}$. For instance, the expected properties of the $h^i(X)$ would lead to a proof of *Bloch's conjecture*:

Conjecture 1.1. Let X be a smooth projective surface over $\mathbb{C}$ with $H^2(X, \mathcal{O}_X) = 0$. Let $A^2(X)$ be the kernel of the degree map $\mathrm{CH}^2(X) \to \mathbb{Z}$. Then the Albanses map $\alpha_X : A^2(X) \to \mathrm{Alb}(X)(\mathbb{C})$ is an isomorphism.

The relation of the conjectural category of mixed motives to various generalizations of Bloch's conjecture and other fascinating conjectures of a geometric nature, as well as to values of L-functions, has been widely discussed in the literature and we will not discuss these topics in any detail in this article. For some more details on the conjectured properties of $\mathcal{MM}_k$ and applications, we refer the reader to [21, 52, 53, 75, 80, 81, 82, 103], as well as additional articles in [104] and the article of Goncharov [35] in this volume.

The category $\mathcal{MM}_k$ has yet to be constructed. However, in the nineties, progress was made toward the construction of the derived category $D^b(\mathcal{MM}_k)$, that is, the construction of a triangulated tensor category $DM(k)$ that has many of the structural properties expected of $D^b(\mathcal{MM}_k)$. In particular, we now have a very good candidate for motivic cohomology $H^p_{\mathcal{M}}(X, \mathbb{Z}(q))$, which, roughly speaking, satisfies all the expected properties which can be deduced from the existence of a triangulated tensor category of mixed motives, without assuming there is an underlying abelian category whose derived category is $DM(k)$, or even that $DM(k)$ has a reasonable t-structure.

In addition to the triangulated candidates for $D^b(\mathcal{MM}_k)$, there are also constructions of candidates for $\mathcal{MM}_k$; these however are not known to have all the desired properties, e.g., the correct relation to K-theory.

In this article, we will outline the constructions and basic properties of various versions of categories of mixed motives which are now available. We will also cover in some detail the known theory of the subcategory of *mixed Tate motives*, that is, the subcategory (either triangulated or abelian) generated by the rational Tate objects $\mathbb{Q}(n)$.

We will make some mention of the relevance of these construction for the mod n-theory, the Beilinson-Lichtenbaum conjectures and the Bloch-Kato conjectures, but as these themes have been amply explained elsewhere (see e.g. [32], [55]), we will not make more than passing reference to this topic.

The discussion of mixed Tate motivic categories in §5 is based in large part on a seminar on this topic that ran during the fall of 2002 at the University of Essen while I was visiting there. I would like to thank the participants of that seminar, and especially Sviataslav Archava, Najmuddin Fakhruddin, Marco Schlichting, Stefan Müller-Stach and Helena Verrill, and for their lectures and discussions; a more detailed discussion of mixed Tate motivic categories arising from this seminar is now in the process of being written. I would also like to thank the Mathematics Department at the University of Essen and especially my hosts, Hélène Esnault and Eckart Viehweg, for their hospitality and support, which helped so much in the writing of this article.

1.2 Notations and conventions

If $A(-)$ is a simplicial abelian group $n \mapsto A(n)$, we have the associated (homological) complex $A(*)$, with $A(*)_n = A(n)$ and $d_n : A(n) \to A(n-1)$ the alternating sum

$$d_n := \sum_{j=0}^{n} (-1)^j A(\delta_j)$$

where the δ_j are the standard co-face maps.

We let $C^{\pm}(\mathbf{Ab})$ denote the category of cohomological complexes, bounded below $(+)$ or bounded above $(-)$. We let $C_{\pm}(\mathbf{Ab})$ denote the category of homological complexes, bounded below $(+)$ or bounded above $(-)$. In both categories, we have the suspension operation $C \mapsto C[1]$, and cone sequences

$$A \xrightarrow{f} B \to \mathrm{Cone}(f) \to A[1].$$

Thus, in the cohomological category $(A[1])^n = A^{n+1}$ and in the homological category $(A[1])_n = A_{n-1}$. We extend these notations to the respective derived categories.

For a scheme S, we let $\mathbf{Sch}_S$ denote the category of schemes of finite type over S, $\mathbf{Sm}_S$ the full sub-category of smooth quasi-projective S-schemes. If $S = \operatorname{Spec} A$ for some ring A, we write $\mathbf{Sch}_A$ and $\mathbf{Sm}_A$ for $\mathbf{Sch}_{\operatorname{Spec} A}$ and $\mathbf{Sm}_{\operatorname{Spec} A}$.

For a noetherian commutative ring R, we let R-mod denote the category of finitely generated R-modules; for a field F, we let F-Vec be the category of vector spaces over F (not necessarily of finite dimension). If G is a profinite group, we let $\mathbb{Q}_p[G]$- mod denote the category of finitely dimensional $\mathbb{Q}_p$-vector spaces with a continuous G-action.

2 Motivic complexes

In this first section, we begin with a discussion of Bloch's seminal work in the weight-two case. We then give an overview of the conjectures of Beilinson and Lichtenbaum on absolute cohomology, as a prelude to our discussion of mixed motives and motivic cohomology. After this, we describe two constructions of theories of absolute cohomology: Bloch's construction of the higher Chow groups, and the Friedlander-Suslin construction of motivic complexes. For later use, we also give some details on associated cubical versions of these complexes.

The relation of the Zariski cohomology of $\mathbb{G}_m$ to K_0 and K_1 was well-known from the very beginning: the Picard group $H^1(X_{\mathrm{Zar}}, \mathbb{G}_m)$ appears as a quotient of the reduced K_0 and the group of global units $H^0(X_{\mathrm{Zar}}, \mathbb{G}_m)$ is likewise a quotient of $K_1(X)$, both via a determinant mapping. Hilbert's theorem 90 says that $H^i(X_{\mathrm{Zar}}, \mathbb{G}_m) \to H^i(X_{\text{ét}}, \mathbb{G}_m)$ is an isomorphism for $i = 0, 1$; the Kummer sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \to 1$$

relates the torsion and cotorsion in $H^*(X_{\text{ét}}, \mathbb{G}_m)$ to $H^*(X, \mu_n)$. Rationally, $H^1(X_{\mathrm{Zar}}, \mathbb{G}_m)$ and $H^0(X, \mathbb{G}_m)$ give the weight-one portion of $K_0(X)$ and $K_1(X)$, respectively (the weight-zero portion of K_0 is similarly given by $H^0(X_{\mathrm{Zar}}, \mathbb{Z})$).

The idea behind motivic complexes is, rather than arranging K-theory by the K-theory degree, one can also collect together the pieces of the same weight (for the Adams operations), and by doing so, one should be able to construct the universal Bloch-Ogus cohomology theory with integral coefficients. For weight one, this is given by the cohomology of the single sheaf $\mathbb{G}_m$, but for weight $n > 1$, one would need a complex of length at least $n - 1$. Later on, the complexes assumed a secondary role as explicit representatives for the total derived functors $R\mathrm{Hom}_{D^b(\mathcal{MM}_k)}(1, h^0(X)(n))$, where $\mathcal{MM}_k$ is the conjectural category of mixed motives over k, see §3.1.

Our discussion is historically out of order, in that quasi-isomorphism of Bloch's complexes with the Friedlander-Suslin construction was only constructed after Voevodsky introduced the machinery of finite correspondences [100] and showed how to adapt Quillen's proof of Gersten's conjecture to this setting in the course of his construction of a triangulated category of mixed motives. However, it is now apparant that one can deduce the Mayer-Vietoris properties of the Friedlander-Suslin complexes from Bloch's complexes, and conversely, one can acheive a more natural functoriality for Bloch's complexes from the Friedlander-Suslin version, without giving any direct relation to categories of mixed motives.

2.1 Weight-two complexes

Before a general framework emerged in the early '80's, there was a lively development of the weight-two case, starting with Bloch's Irvine notes [8], in which he related:

1. the relations defining K_2 of a field F
2. the indecomposable K_3 of F
3. the values of the dilogarithm function
4. the Borel regulator on K_3 of a number field.

These relations were made more precise by Suslin's introduction of the 5-term dilogarithm relation [88], [87], uniting Bloch's work with Dupont and Sah's study [30] of the homology of SL_2 and the scissors congruence group. Lichtenbaum [68], building on Bloch's introduction of the relative K_2 of the semi-local ring $F[t]_{t(1-t)}$, constructed a length-two complex which computed the weight-two portions of K_2 and K_3, up to inverting small primes. These constructions formed the basis for the general picture, as conjectured by Beilinson and Lichtenbaum, as well as the later constructions of Goncharov [38], [39], Bloch [13] and Voevodsky-Suslin-Friedlander [100].

Bloch's complexes

In [8], Bloch constructs 3 complexes:

(1) Let F be a field. Let $R(F) = F[t]_{t(1-t)}$, i.e., the localization of the polynomial ring $F[t]$ formed by inverting all polynomials $P(t)$ with $P(0)P(1) \neq 0$. Let $I(F) = t(1-t)R$. We have the *relative K-groups* $K_n(R;I)$, which fits into a long exact sequence

$$\ldots \to K_{n+1}(R/I) \to K_n(R;I) \to K_n(R) \to K_n(R/I) \to \ldots$$

Using the localization sequence in K-theory, we have the boundary map

$$K_2(R) \to \oplus_{\substack{x\in\mathbb{A}^1_F\setminus\{0,1\},\\ x \text{ closed}}} k(x)^*;$$

composing with $K_2(R;I) \to K_2(R)$ gives the length-one complex

$$K_2(R;I) \xrightarrow{tame} \oplus_{\substack{x\in\mathbb{A}^1_F\setminus\{0,1\},\\ x \text{ closed}}} k(x)^* \tag{1}$$

Bloch shows

Proposition 2.1. *There are canonical isomorphisms*

$$\ker\partial \cong K_3^{\mathrm{ind}}(F)$$
$$\mathrm{coker}\partial \cong K_2(F)$$

Here $K_3^{\mathrm{ind}}(F)$ is the quotient of $K_3(F)$ by the image of the cup-product map $K_1(F)^{\otimes 3} \to K_3(F)$.

To make the comparision with the other two complexes, one needs an extension of Matsumoto's presentation of K_2 of a field to the relative case: For a semi-local PID A with Jacobson radical J and quotient field L, there is an isomorphism (*cf.* [101])

$$K_2(A,J) \cong (1+J)^* \otimes_{\mathbb{Z}} L^*/\langle f\otimes(1-f) \mid f \in 1+J\rangle.$$

In particular, $K_2(R, I)$ contains the subgroup of symbols $\{1 + I, F^*\}$; taking the quotient of (1) by this subgroup yields the exact sequence (assuming F algebraically closed)

$$0 \to \frac{K_3^{\text{ind}}(F)}{\text{Tor}_1(F^*, F^*)} \to \frac{K_2(R, I)}{\{1 + I, F^*\}} \to F^* \otimes F^* \to K_2(F) \to 0.$$

(2) Let $\mathcal{A}(F)$ be the free abelian group on $F \setminus \{0, 1\}$, and form the complex

$$\mathcal{A}(F) \xrightarrow{\lambda} F^* \otimes F^*$$

by sending $x \in F \setminus \{0, 1\}$ to $\lambda := x \otimes (1 - x) \in F^* \otimes F^*$. By Matsumoto's theorem, $K_2(F) = \text{coker}(\lambda)$. Let $\mathcal{B}(F)$ be the kernel of λ, giving the exact sequence

$$0 \to \mathcal{B}(F) \to \mathcal{A}(F) \to F^* \otimes F^* \to K_2(F) \to 0$$

(3) Start with the exponential sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp 2\pi i} \mathbb{C}^* \to 1.$$

Tensor with $\mathbb{C}^*$ (over $\mathbb{Z}$), giving the complex $\mathbb{C} \otimes \mathbb{C}^* \to \mathbb{C}^* \otimes \mathbb{C}^*$ and the exact sequence

$$0 \to \text{Tor}_1(\mathbb{C}^*, \mathbb{C}^*) \to \mathbb{C}^* \to \mathbb{C} \otimes \mathbb{C}^* \to \mathbb{C}^* \otimes \mathbb{C}^* \to 1.$$

The image of $\text{Tor}_1(\mathbb{C}^*, \mathbb{C}^*)$ in $\mathbb{C}^*$ is the torsion subgroup; let $\tilde{\mathbb{C}}^*$ be the torsion-free quotient, yielding the exact sequence

$$0 \to \tilde{\mathbb{C}}^* \to \mathbb{C} \otimes \mathbb{C}^* \to \mathbb{C}^* \otimes \mathbb{C}^* \to 1.$$

To relate these three complexes, Bloch defines two maps on $\mathcal{A}(\mathbb{C})$. For $x \in \mathbb{C} \setminus \{0, 1\}$, let $\epsilon(x) \in \mathbb{C} \otimes \mathbb{C}^*$ be defined by

$$\epsilon(x) := \big[\frac{1}{2\pi i} \log(1 - x) \otimes x\big] + \big[1 \otimes \exp\big(\frac{-1}{2\pi i} \int_0^x \log(1 - t) \frac{dt}{t}\big)\big].$$

In this formula, define

$$\log(1 - t) := -\int_0^t \frac{dt}{1 - t},$$

and use the same path of integration for all the integrals. Bloch shows that $\epsilon(x)$ is then well-defined and independent of the choice of path from 0 to x. Extending ϵ to $\mathcal{A}(\mathbb{C})$ by linearity gives the commutative triangle

$$\begin{array}{ccc} & \mathcal{A}(\mathbb{C}) & \\ {}^{\epsilon}\swarrow & & \searrow^{\lambda} \\ \mathbb{C} \otimes \mathbb{C}^* & \xrightarrow[\exp 2\pi i \otimes \text{id}]{} & \mathbb{C}^* \otimes \mathbb{C}^*. \end{array}$$

The second map $\eta : \mathcal{A}(\mathbb{C}) \to K_2(R(\mathbb{C}), I(\mathbb{C}))$ is defined explicitly by

$$\eta(x) := \{1 - \frac{xt^2}{(t-1)^3 - xt^2(t-1)}, \frac{t}{t-1}\}^6 \in K_2(R(\mathbb{C}), I(\mathbb{C})).$$

This all yields the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \to & \frac{K_3(\mathbb{C})^{\text{ind}}}{\text{Tor}_1(\mathbb{C}^*,\mathbb{C}^*)} & \to & \frac{K_2(R,I)}{\{1+I,\mathbb{C}^*\}} & \xrightarrow{\textit{tame}} & \mathbb{C}^* \otimes \mathbb{C}^* & \to & K_2(\mathbb{C}) & \to & 0 \\
 & & \uparrow{\scriptstyle \Psi} & & \uparrow{\scriptstyle \eta} & & \| & & \| & & \\
0 & \to & \mathcal{B}(\mathbb{C}) & \to & \mathcal{A}(\mathbb{C}) & \xrightarrow{\lambda} & \mathbb{C}^* \otimes \mathbb{C}^* & \to & K_2(\mathbb{C}) & \to & 0 \\
 & & {\scriptstyle \theta}\downarrow & & {\scriptstyle \epsilon}\downarrow & & \| & & \| & & \\
0 & \to & \tilde{\mathbb{C}}^* & \to & \mathbb{C} \otimes \mathbb{C}^* & \xrightarrow[\exp 2\pi i \otimes \text{id}]{} & \mathbb{C}^* \otimes \mathbb{C}^* & \to & K_2(\mathbb{C}) & \to & 0
\end{array}$$

where Ψ and θ are the maps induced by ϵ and η.

The dilogarithm

Composing ϵ with

$$\mathbb{C} \otimes \mathbb{C}^* \xrightarrow{\text{real part} \otimes \text{id}} \mathbb{R} \otimes \mathbb{C}^* \xrightarrow{\text{id} \otimes \log|-|} \mathbb{R} \otimes \mathbb{R} \xrightarrow{\text{multiply}} \mathbb{R}$$

yields the map $D : \mathcal{A}(\mathbb{C}) \to \mathbb{R}$. On generators $x \in \mathbb{C} \setminus \{0, 1\}$, $D(x)$ is the *Bloch-Wigner dilogarithm*

$$D(x) = \arg(1-x)\log|x| - \text{Im}\Big(\int_0^x \log(1-t)\frac{dt}{t}\Big).$$

Bloch shows how to relate D to the Borel regulator on $K_3(\mathbb{C})$ via the map θ. If $F \subset \mathbb{C}$ is a number field and one has explicit elements in $\mathcal{B}(F)$ which form a basis for $K_3(F)^{\text{ind}}$, this gives an explict formula for the value of the Borel regulator for $K_3(F)$.

Example 2.2. Let $F = \mathbb{Q}(\zeta)$, where $\zeta = \exp(\frac{2\pi i}{\ell})$ and ℓ is an odd prime. An easy calculations shows that $\ell[\zeta^i]$ is in $\mathcal{B}(F)$ for all i; one shows that $\ell[\zeta^1], \ldots, \ell[\zeta^{\ell-1/2}]$ maps to a basis of $K_3(F)_{\mathbb{Q}}$ under θ. Using the explicit formula

$$D(\zeta^i) = \text{Im}\Big(\sum_{m=1}^{\infty} \frac{\zeta^{mi}}{m^2}\Big),$$

Bloch computes: the lattice in $\mathbb{R}^{\ell-1)/2}$ generated by the vectors

$$(D(\ell[\zeta^i)]), \ldots, D(\ell[\zeta^{ij})]), \ldots, D(\ell[\zeta^{i(\ell-1)/2})]); \quad i = 1, \ldots, \frac{\ell-1}{2},$$

has volume $2^{-(\ell-1)/2}\ell^{3(\ell-1)/4}\prod_{\chi \text{ odd}}|L(2,\chi)|$, where χ runs over the odd characters of $(\mathbb{Z}/l\mathbb{Z})^*$ and $L(s,\chi) := \sum \chi(n)n^{-s}$ is the Dirichlet L-function.

The Bloch-Suslin complex

Suslin [87] refined Bloch's construction of the complex $\mathcal{A}(F) \to F^* \otimes F^*$ by imposing the five-term relation satisfied by the dilogarithm function:

$$[x] - [y] + [y/x] - [\frac{y-1}{x-1}] + [\frac{y(x-1)}{x(y-1)}]$$

One checks that this element goes to zero in $F^* \wedge F^*$, giving the complex

$$A(F) \xrightarrow{\bar{\lambda}} \Lambda^2 F^* \tag{2}$$

with $A(F)$ being the above-mentioned quotient of $\mathcal{A}(F)$, and $\lambda(x) = x \wedge (1-x)$. Since $\{x, y\} = \{y, x\}^{-1}$ in $K_2(F)$, the cokernel of λ is still $K_2(F)$; Suslin shows

Proposition 2.3. *Let F be an infinite field. There is a natural isomorphism*

$$\ker \bar{\lambda} \cong K_3^{\text{ind}}(F)/\widetilde{\text{Tor}_1}(F^*, F^*),$$

where $\widetilde{\text{Tor}_1}(F^*, F^*)$ is an extension of $\text{Tor}_1(F^*, F^*)$ by $\mathbb{Z}/2$.

Higher weight

The construction of the Bloch-Suslin complex (2) has been generalized by Goncharov [38], [39] to give complexes $C(n)$ of the form

$$\begin{aligned} A_F(n) \to A_F(n-1) \otimes F^* \to A_F(n-2) \otimes \Lambda^2 F^* \to \ldots \\ \to A_F(2) \otimes \Lambda^{n-2} F^* \to \Lambda^n F^* \end{aligned}$$

These are homological complexes with $\Lambda^n F^*$ in degree n.

The groups $A_F(i)$ are defined inductively: Each $A_F(i)$ is a quotient of $\mathbb{Z}[F \cup \{\infty\}]$; denote the generator corresponding to $x \in F$ as $[x]_i$. For $i > 2$, the map

$$A_F(i) \otimes \Lambda^{n-i} F^* \to A_F(i-1) \otimes \Lambda^{n-i+1} F^*$$

sends $[x]_i \otimes \eta$ to $[x]_{i-1} \otimes x \wedge \eta$ for $x \neq 0, \infty$ and sends $[0]_i$, and $[\infty]_i$ to 0. $A_F(1) = F^*$, with $[x]_1$ mapping to $x \in F^*$ and $A_F(2)$ is the Bloch-Suslin construction $A(F)$ (set $[0]_i = [1]_i = [\infty]_i = 0$ for $i = 1, 2$). The map

$$A_F(2) \otimes \Lambda^{n-2} F^* \to \Lambda^n F^*$$

sends $[x]_2 \wedge \eta$ to $x \wedge (1-x) \wedge \eta$ for $x \neq 0, 1, \infty$.

To define $A_F(i)$ as a quotient of $\mathcal{A}_F(i) := \mathbb{Z}[F \cup \{\infty\}]$ for $i > 2$, Goncharov imposes "all rational relations": Let $\mathcal{B}_F(i)$ be the kernel of

$$\begin{aligned} &\mathcal{A}_F(i) \to A_F(i-1) \otimes F^*, \\ &[x] \mapsto [x]_{i-1} \otimes x. \end{aligned}$$

For $\sum_j n_j[x_j(t)]$ in $\mathcal{B}_{F(t)}(i)$, t a variable, each $x_j(t)$ defines a morphism $x_j : \mathbb{P}^1_F \to \mathbb{P}^1_F$, and so $x_j(a) \in F \cup \{\infty\}$ is well-defined for all $a \in F$. Let $\mathcal{R}_F(i) \subset \mathcal{A}_F(i)$ be the subgroup generated by $[0]$, $[\infty]$ and elements of the form

$$\sum_j n_j[x_j(1)] - \sum_j n_j[x_j(0)],$$

with $\sum_j n_j[x_j(t)] \in \mathcal{B}_{F(t)}(i)$, and set $A_F(i) := \mathcal{A}_F(i)/\mathcal{R}_F(i)$. One checks that this does indeed form a complex.

The role of these complexes and their applications to a number of conjectures is explained in detail in Goncharov's article [35]. We will only mention that the homology $H_p(C(n))$ is conjectured to be the weight n K-group $K_p(F)^{(n)}$ for $n \leq p \leq 2n-1$.

Remark 2.4. In addition to inspiring later work on the construction of motivic complexes, Bloch's introduction of the relative K_2 to study K_3^{ind} was later picked up by Merkurjev-Suslin [71] and Levine [66] in their computation of the torsion and co-torsion of K_3^{ind} of fields.

2.2 Beilinson-Lichtenbaum complexes

In the early '80's Beilinson and Lichtenbaum gave conjectures for versions of universal cohomology which would arise as hypercohomology (in the Zariski, resp. étale topology) of certain complexes of sheaves. The conjectures describe sought-after properties of these representing complexes.

Beilinson's conjectures

In [5], Beilinson gives a simultaneous generalization of a number of conjectures on values of L-functions (see Kahn's article [56] for details). A major part of this work involved generalizing the Borel regulator using Deligne cohomology and Gillet's Chern classes for higher K-theory. He also states:

"It is thought that for any schemes[*sic*] there exists a universal cohomology theory $H^j_{\mathcal{A}}(X, \mathbb{Z}(i))$ satisfying Poincaré duality and related to Quillen's K-theory in the same way as in topology the singular cohomology is related to K-theory. $H^*_{\mathcal{A}}$ must be closely related to the Milnor ring".

The reader should note that, at this point, Beilinson is speaking of a "universal" cohomology theory, but *not* "motivic" cohomology. In particular, one should expect that the rational version $H^j_{\mathcal{A}}(X, \mathbb{Q}(i))$ is weight-graded K-theory, and the integral version is related to Milnor K-theory, but there is as yet no direct connection to motives. In any case, here is a more precise formulation describing absolute cohomology:

Conjecture 2.5 (Beilinson [6]). For $X \in \mathbf{Sm}_k$ there are complexes $\Gamma_{\text{Zar}}(r)$, $r \geq 0$, in the derived category of sheaves of abelian groups on X_{Zar}, (functorial in X) with functorial graded product, and

(0) $\Gamma_{\rm Zar}(0) \cong \mathbb{Z}$, $\Gamma_{\rm Zar}(1) \cong \mathbb{G}_m[-1]$
(1) $\Gamma_{\rm Zar}(r)$ is acyclic outside $[1,r]$ for $r \geq 1$.
(2) $\Gamma_{\rm Zar}(r)\otimes^L \mathbb{Z}/n \cong \tau_{\leq r} R\alpha\mu_n^{\otimes r}$ if n is invertible on X, where $\alpha : X_{\rm \acute{e}t} \to X_{\rm Zar}$ is the change of topology morphism.
(3) ${\rm gr}^r_\gamma K_j(X) \otimes \mathbb{Q} \cong \mathbb{H}^{2r-i}(X_{\rm Zar}, \Gamma_{\rm Zar}(r))_\mathbb{Q}$ (or up to small primes)
(4) $\mathcal{H}^r(\Gamma_{\rm Zar}(r)) = \mathcal{K}^M_r$.

Here $\mathcal{K}^M_r$ is the sheaf of Milnor K-groups, where the stalk $\mathcal{K}^M_{r,x}$ for $x \in X$ is the kernel of the symbol map

$$K^M_r(k(X)) \to \oplus_{x\in X^{(1)}} K^M_{r-1}(k(x)).$$

Lichtenbaum's conjectures

Lichtenbaum's conjectures seem to be motivated more by the search for an integral cohomology theory that would explain the values of L-functions. As the ℓ-part of these values was already seen to have a close connection with ℓ-adic étale cohomology, it is natural that these complexes would be based on the étale topology.

Conjecture 2.6 (Lichtenbaum [69, 67]). For $X \in \mathbf{Sm}_k$ there are complexes $\Gamma_{\rm \acute{e}t}(r)$, $r \geq 0$, in the derived category of sheaves of abelian groups on $X_{\rm \acute{e}t}$, (functorial in X) with functorial graded product, and

(0) $\Gamma_{\rm \acute{e}t}(0) \cong \mathbb{Z}$, $\Gamma_{\rm \acute{e}t}(1) \cong \mathbb{G}_m[-1]$
(1) $\Gamma_{\rm \acute{e}t}(r)$ is acyclic outside $[1,r]$ for $r \geq 1$.
(2) $R^{r+1}\alpha\Gamma_{\rm \acute{e}t}(r) = 0$
(3) $\Gamma_{\rm \acute{e}t}(r) \otimes^L \mathbb{Z}/n \cong \mu_n^{\otimes r}$ if n is invertible on X.
(4) ${\rm gr}^r_\gamma \mathcal{K}^{\rm \acute{e}t}_j \cong \mathcal{H}^{2r-i}(\Gamma(r))$ (up to small primes), where $\mathcal{K}^{\rm \acute{e}t}_j$ and $\mathcal{H}^{2r-i}(\Gamma_{\rm \acute{e}t}(r))$ are the respective Zariski sheaves.
(5) For a field F, $H^r(\Gamma_{\rm \acute{e}t}(r)(F)) = K^M_r(F)$.

The two constructions should be related by

$$\tau_{\leq r} R\alpha_*\Gamma_{\rm \acute{e}t}(r) = \Gamma(r);\ \ \Gamma_{\rm \acute{e}t}(r) = \alpha^*\Gamma_{\rm Zar}(r).$$

The relations (2) and (4) in Beilinson's conjectures and (2), (3) and (5) in Lichtenbaum's version are generalizations of the Merkujev-Suslin theorem (the case $r = 2$); Lichtenbaum's condition (2) is a direct generalization of the classical Hilbert Theorem 90, and also the generalization for K_2 due to Merkurjev and Suslin [72]. These conjectures, somewhat reinterpreted for motivic cohomology, are now known as the Beilinson-Lichtenbaum conjectures (see [32] and also §2.4 for additional details).

2.3 Bloch's cycle complexes

In [13], Bloch gives a construction for complexes on X_{Zar} which satisfy some of the conjectured properties of Beilinson, and whose étale sheafification satisfies some of the properties conjectured by Lichtenbaum. The construction and basic properties of these complexes are discussed in [32]; we will use his notations here, but restrict ourselves mainly to the case of schemes of finite type over a field.

Cycle complexes and higher Chow groups

Fix a field k. In [13], Bloch constructs, for each k-scheme X of finite type and equi-dimensional over k, and each integer $q \geq 0$, a simplicial abelian group $n \mapsto z^q(X,n)$. The associated homological complex $z^q(X,*)$ is called Bloch's *cycle complex* and the *higher Chow groups* $\mathrm{CH}^q(X,n)$ are defined by

$$\mathrm{CH}^q(X,n) := H_n(z^q(X,*)).$$

We recall some details of this construction here for later use.

The *algebraic n-simplex* is the scheme

$$\Delta^n := \operatorname{Spec}\mathbb{Z}[t_0,\ldots,t_n]/\big(\sum_{i=0}^n t_i - 1\big).$$

The *vertex* v_i^n of Δ^n is the closed subscheme defined by $t_j = 0$, $j \neq i$. More generally, a *face* of Δ^n is a closed subscheme defined by equations of the form $t_{i_1} = \ldots = \ldots t_{i_s} = 0$. We let $v(n)$ denote the set of vertices of Δ^n; sending i to v_i^n defines a bijection $\nu_n : \underline{n} \to v(n)$. The choice of an index $i \in \underline{n}$ determines an isomorphism $\Delta^n \cong \mathbb{A}^n$ via the coordinates $t_0,\ldots,t_{i-1},t_{i+1},\ldots,t_n$. Note that each face $F \subset \Delta^n$ is isomorphic to Δ^m for some $m \leq n$, we set $\dim F := m$. Let R_n denote the coordinate ring $\mathbb{Z}[t_0,\ldots,t_n]/(\sum_{i=0}^n t_i - 1)$.

If $g : \underline{n} \to \underline{m}$ is a map of sets, let $g^* : R_m \to R_n$ be the map defined by $g^*(t_i) = \sum_{j\in g^{-1}(i)} t_j$ (so $g^*(t_i) = 0$ if i is not in the image of g). We thus have the map $\Delta(g) : \Delta^n \to \Delta^m$, and this forms the cosimplicial scheme $\Delta^* : \Delta \to \mathbf{Sch}$. More generally, if X is a k-scheme, we have the cosimplicial k-scheme $X \times \Delta^*$.

Definition 2.7. For a finite type k-scheme X and integer n, let $z_p(X,n) \subset z_{p+n}(X \times \Delta^n)$ be the subgroup generated by integral closed subscheme W of $X \times \Delta^n$ with

$$\dim_k \big(W \cap (X \times F)\big) \leq \dim F + p.$$

for each face F of Δ^n.

If X is locally equi-dimensional over k, let $z^p(X,n) \subset z^p(X \times \Delta^n)$ be the subgroup generated by integral closed subscheme W of $X \times \Delta^n$ with

$$\operatorname{codim}_{X\times F}\big(W \cap (X \times F)\big) \geq p.$$

for each face F of Δ^n.

For $g : \underline{n} \to \underline{m}$ a map in Δ, we let $g^* : z_p(X, m) \to z_p(X, n)$ be the map induced by

$$g^*(\bar{w}) := (\mathrm{id} \times g)^* : z_{p+m}(X \times \Delta^m)_{\mathrm{id} \times g} \to z_{p+n}(X \times \Delta^n).$$

If X is locally equi-dimensional over k, the map $g^* : z^p(X, m) \to z^p(X, n)$ is defined similarly.

The assignment

$$\begin{aligned} &n \mapsto z_p(X, n), \\ &(g : \underline{n} \to \underline{m}) \mapsto (g^* : z_p(X, m) \to z_p(X, n)) \end{aligned}$$

forms the simplicial abelian group $z_p(X, -)$. We let $z_p(X, *)$ denote the associated complex of abelian groups. If X is locally equi-dimensional over k, we have the simplicial abelian group $z^p(X, -)$ and the complex $z^p(X, *)$; if X has pure dimension d over k, then $z^p(X, -) = z_{d-p}(X, -)$.

Definition 2.8. Let X be a k-scheme of finite type. Set

$$\mathrm{CH}_p(X, n) := \pi_n(z_p(X, -)) = H_n(z_p(X, *)).$$

If X is locally equi-dimenisonal over k, we set

$$\mathrm{CH}^p(X, n) := \pi_n(z^p(X, -)) = H_n(z^p(X, *)).$$

Elementary functorialities

The complexes $z_p(X, *)$ and groups $\mathrm{CH}_p(X, n)$ satisfy the following functorialities:

1. Let $f : Y \to X$ be a proper map in $\mathbf{Sch}_k$. Then the maps $(f \times \mathrm{id}_{\Delta^n})_*$ give rise to the map of complexes

$$f_* : z_p(Y, *) \to z_p(X, *)$$

   yielding $f_* : \mathrm{CH}_p(X, n) \to \mathrm{CH}_p(Y, n)$. The maps f_* satisfy the functoriality $(gf)_* = g_* \circ f_*$ for composable proper maps f, g.
2. Let $f : Y \to X$ be an equi-dimensional l.c.i.map in $\mathbf{Sch}_k$ with fiber dimension d. Then the maps $(f \times \mathrm{id}_{\Delta^n})^*$ give rise to the map of complexes

$$f^* : z_p(X, -) \to z_{p+d}(Y, -),$$

   yielding $f^* : \mathrm{CH}_p(X, n) \to \mathrm{CH}_{p+d}(Y, n)$. The maps f^* satisfy the functoriality $(gf)^* = g^* \circ f^*$ for composable equi-dimensional l.c.i.maps f, g.

Classical Chow groups

The groups $\mathrm{CH}_p(X,0)$ are by definition the cokernel of the map

$$\delta^*_{0,0} - \delta^*_{0,1} : z_{p+1}(X,1) \to z_p(X,0) = z_p(X)$$

From this, one has the identity $\mathrm{CH}_p(X,0) = \mathrm{CH}_p(X)$.

Remark 2.9. All the above extends to schemes essentially of finite type over k by taking the evident direct limit over finite-type models. One can also extend the definitions to scheme ove finite type over a regular base B of Krull dimension one : for $X \to B$ finite type and locally equi-dimensional, the definition of $z^p(X,*)$ is word-for-word the same. The definition of $z_p(X,-)$ for X a finite-type B-scheme requires only a reasonable notion of dimension to replace $\dim_k$. The choice made in [62] is as follows: Suppose that B is integral with generic point η. Let $p : W \to B$ be of finite type, with W integral. If the generic fiber W_η is non-empty, set $\dim W := \dim_{k(\eta)} W_\eta + 1$; if on the other hand $p(W) = x$ is a closed point of B, set $\dim W := \dim_{k(x)} W$. In particular, one has a good definition of the higher Chow groups $\mathrm{CH}_p(X,n)$ for X of finite type over the ring of integers $\mathcal{O}_F$ in a number field F.

Fundamental properties and their consequences

We now list the fundamental properties of the complexes $z_p(X,*)$.

Theorem 2.10. *[Homotopy property [13]] Let X be in $\mathbf{Sch}_k$ and let $\pi : X \times \mathbb{A}^1 \to X$ be the projection. Then the map $\pi^* : z_p(X,*) \to z_{p+1}(X \times \mathbb{A}^1,*)$ is a quasi-isomorphism, i.e., the map $\pi^* : \mathrm{CH}_p(X,n) \to \mathrm{CH}_{p+1}(X \times \mathbb{A}^1,n)$ is an isomorphisms for $n = 0,1,\ldots$.*

Theorem 2.11. *[Localization [10]] Let X be in $\mathbf{Sch}_k$, let $i : W \to X$ be a closed subscheme and $j : U \to X$ the open complement $X \setminus W$. Then the sequence*

$$z_p(W,*) \xrightarrow{i_*} z_p(X,*) \xrightarrow{j^*} z_p(U,*)$$

induces a quasi-isomorphism

$$z_p(W,*) \to \mathrm{Cone}(z_p(X,*) \xrightarrow{j^*} z_p(U,*))[-1].$$

Definition 2.12. Let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$, with Y and X locally equi-dimensional over k. Let $z^p(X,n)_f \subset z^p(X,n)$ be the subgroup generated by irreducible $W \subset X \times \Delta^n$ with $1\cdot W \in z^p(X,n)$ and $1\cdot Z \in z^p(Y,n)$ for each irreducible component Z of $(\mathrm{id} \times f)^{-1}(W)$. This forms a subcomplex $z^p(X,*)_f$ of $z^p(X,*)$.

Theorem 2.13. *[Moving Lemma [63, Part I, Chap. II, §3.5]] Let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$ with X in $\mathbf{Sm}_k$. Suppose X is either affine or projective over k. Then the inclusion $z^p(X,*)_f \to z^p(X,*)$ is a quasi-isomorphism.*

These results have the following consequences

Mayer-Vietoris

Let X be in $\mathbf{Sch}_k$, $U, V \subset X$ open subschemes with $X = U \cup V$. Then the sequence (the maps are the evident restriction maps)

$$z_p(X, *) \to z_p(U, *) \oplus z_p(V, *) \to z_p(U \cap V, *)$$

gives a quasi-isomorphism

$$z_p(X, *) \to \mathrm{Cone}(z_p(U, *) \oplus z_p(V, *) \to z_p(U \cap V, *))[-1].$$

This yields the usual long exact Mayer-Vietoris sequence for the higher Chow groups.

Functoriality

Let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$ with $X \in \mathbf{Sm}_k$ and Y locally equi-dimensional over k. Take an affine cover $\mathcal{U} = \{U_1, \ldots, U_m\}$ of X, and let $\mathcal{V} := \{V_1, \ldots, , V_m\}$ be the cover $V_j := f^{-1}(U_j)$ of Y. For $I \subset \{1, \ldots, m\}$ let $U_I = \cap_{i \in I} U_i$, define V_I similarly, and let $f_I : U_I \to V_I$ be the morphism induced by f.

Form the Čech complex $z^p(\mathcal{U}, *)$ as the total complex of the evident double complex

$$\oplus_i z^p(U_i, *) \to \oplus_{i<j} z^p(U_i \cap U_j, *) \to \ldots \to z^p(\cap_{i=1}^m U_i, *)$$

and define $z^p(\mathcal{V}, *)$ similarly. Replacing $z^p(U_I, *)$ with $z^p(U_I, *)_{f_I}$ yields the subcomplex $z^p(\mathcal{U}, *)_f$ of $z^p(\mathcal{U}, *)$; the pull-backs f_I^* yield the map of complexes

$$f^* : z^p(\mathcal{U}, *)_f \to z^p(\mathcal{V}, *).$$

By the moving lemma (Theorem 2.13), the inclusion $z^p(\mathcal{U}, *)_f \to z^p(\mathcal{U}, *)$ is a quasi-isomorphism. We thus have the morphism $f^* : z^p(X, *) \to z^p(Y, *)$ in $D^-(\mathbf{Ab})$ defined by the zig-zag diagram

$$z^p(X, *) \to z^p(\mathcal{U}, *) \leftarrow z^p(\mathcal{U}, *)_f \xrightarrow{f^*} z^p(\mathcal{V}, *) \leftarrow z^p(Y, *).$$

One shows that this makes the assignment $X \mapsto z^p(X, *) \in D^-(\mathbf{Ab})$ into a functor $z^p(-, *) : \mathbf{Sm}_k^{\mathrm{op}} \to D^-(\mathbf{Ab})$. In particular, $X \mapsto \mathrm{CH}^p(X, n)$ becomes a functor

$$\mathrm{CH}^p(-, n) : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{Ab}.$$

Products

The cycle complexes admit natural associative and commutative external products

$$\cup_{X,Y} : z_p(X, *) \otimes z_q(Y, *) \to z_{p+q}(X \times Y, *)$$

in $D^-(\mathbf{Ab})$; for X smooth over k, one has natural cup products in $D^-(\mathbf{Ab})$

$$\cup_X := \delta^* \circ \cup_{X,X} :: z^p(X,*) \otimes z^q(X,*) \to z^{p+q}(X,*),$$

where $\delta : X \to X \times_k X$ is the diagonal. The cup products make $\oplus_p z^p(X,*)$ an associative commutative ring in the derived category, with unit the fundamental class $1 \cdot X \in z^0(X,0)$. In particular, this makes $\oplus_{p,q}\mathrm{CH}^p(X,q)$ into a bigraded ring (commutative in the p-grading, graded-commutative in the q-grading), functorial in X.

One easily verifies the projection formula

$$p_*(p^*\alpha \cup \beta) = \alpha \cup p_*\beta$$

for a proper map $p : Y \to X$ in $\mathbf{Sm}_k$.

The external products are essentially given by the usual external product of cycles. However, as the external product of cycle on $X \times \Delta^n$ and a cycle on $Y \times \Delta^m$ yields a cycle on $X \times Y \times \Delta^n \times \Delta^m$, not a cycle on $X \times Y \times \Delta^{n+m}$, the natural target of the external product is the total complex of the double complex $z_{p+q}(X \times_k Y, *, *)$, where $z^p(X \times Y, n, m)$ is the subgroup of $z^p(X \times Y \times \Delta^n \times \Delta^m)$ of cycles in good position with respect to "bi-faces" $X \times Y \times F \times F'$. One then needs to map $\mathrm{Tot}z_{p+q}(X \times_k Y, *, *)$ back to $z_{p+q}(X \times_k Y, *)$, in the derived category. There are two techniques for doing this:

1. Use the standard triangulation of $\Delta^n \times \Delta^m$ into $n+m$-simplices
2. Show that the inclusion

$$z_{p+q}(X \times_k Y, *) = z_{p+q}(X \times_k Y, *, 0) \subset \mathrm{Tot}z_{p+q}(X \times_k Y, *, *)$$

   is a quasi-isomorphism

Both these techniques work and give the same product structure, see e.g. [64] or [33] for details.

Projective bundle formula

For an invertible sheaf $\mathcal{L}$ on $X \in \mathbf{Sm}_k$, we may choose a Cartier divisor D on X with $\mathcal{O}_X(D) \cong \mathcal{L}$. Sending $\mathcal{L}$ to the class of D in $\mathrm{CH}^1(X,0) = \mathrm{CH}^1(X)$ gives a homomorphism

$$c_1 : \mathrm{Pic}(X) \to \mathrm{CH}^1(X,0).$$

If $\mathcal{E} \to X$ is a locally free sheaf of rank $n+1$, and $q : \mathbb{P}(\mathcal{E}) \to X$ the associated $\mathbb{P}^n$-bundle $\mathrm{Proj}_{\mathcal{O}_X}(\mathrm{Sym}^*\mathcal{E})$, we have the tautological invertible (quotient) sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$; let $\xi := c_1(\mathcal{O}(1))$. $\mathrm{CH}^*(\mathbb{P}(\mathcal{E}),*)$ is a $\mathrm{CH}^*(X,*)$-module via q^*; in fact, $\mathrm{CH}^*(\mathbb{P}(\mathcal{E}),*)$ is a free $\mathrm{CH}^*(X,*)$-module with basis $1, \xi, \ldots, \xi^n$.

Relation with K-theory

Once one has the projective bundle fomula, one can apply the technique of Gillet [34] to give natural Chern class maps

$$c_{p,q} : K_{2q-p}(X) \to \mathrm{CH}^q(X, 2q-p)$$

and a multiplicative Chern character

$$\mathrm{ch}_* : K_*(X)_{\mathbb{Q}} \to \oplus_{p,q}\mathrm{CH}^p(X, q)_{\mathbb{Q}}$$

We let $K_n(X)^{(p)}$ denote the weight p subspace of $K_n(X)_{\mathbb{Q}}$, i.e.

$$K_n(X)^{(p)} = \{x \in K_n(X)_{\mathbb{Q}}) \mid \psi_k(x) = k^p \cdot x \text{ for all } k \geq 2\},$$

where ψ_k is the kth Adams operation on $K_n(X)$.

Theorem 2.14 ([64], [10]). *Let X be in $\mathbf{Sm}_k$. The Chern character gives a isomorphism*

$$K_n(X)^{(p)} \to \mathrm{CH}^p(X, n)_{\mathbb{Q}}.$$

Milnor K-theory

As a special case of Theorem 2.14, we have the isomorphism

$$\mathrm{CH}^n(F, n) \cong K_n(F)^{(n)}$$

for F a field. From work of Suslin [89], we know that the canonical map of Milnor K-theory to Quillen K theory identifies $K_n^M(F)_{\mathbb{Q}}$ with $K_n(F)^{(n)}$. In fact, one has

Theorem 2.15 (Nestorenko-Suslin [76], Totaro [92]). *Let F be a field. There is a natural isomorphism*

$$K_n^M(F) \cong \mathrm{CH}^n(F, n).$$

The case $n = 1$ is a special case of the result in [13]:

Proposition 2.16. *Let X be in $\mathbf{Sm}_k$. Sending a unit $u \in H^0(X, \mathcal{O}_X^*)$ to the subscheme $(u-1)t_1 = u$ of $X \times \Delta^1$ defines an isomorphism $H^0(X, \mathcal{O}_X^*) \cong \mathrm{CH}^1(X, 1)$. For $n \neq 1$, $\mathrm{CH}^1(X, n) = 0$.*

2.4 Suslin homology and Friedlander-Suslin cohomology

We describe Suslin's construction of "abstract homology" for algebraic varieties, and various modifications. For further details on this construction, we refer the reader to the article [42] in this volume.

Finite cycles and quasi-finite cycles

Definition 2.17. Take Y in $\mathbf{Sm}_k$ and X in $\mathbf{Sch}_k$.
(1) Let $z_{\mathrm{fin}}(X)(Y)$ be the subgroup of $z_*(Y \times_k X)$ generated by integral closed subschemes $W \subset Y \times_k X$ such that $p_1 : W \to Y$ is finite and dominant over an irreducible component of Y.
(2) Let $z_{\mathrm{q.fin}}(X)(Y)$ be the subgroup of $z_*(Y \times_k X)$ generated by integral closed subschemes $W \subset Y \times_k X$ such that $p_1 : W \to Y$ is quasi-finite and dominant over an irreducible component of Y.

For a morphism $f : Y' \to Y$ in $\mathbf{Sm}_k$, the morphism $f \times \mathrm{id} : Y' \times_k X \to Y \times_k X$ is an l.c.i.-morphism; the finiteness, resp., quasi-finiteness conditions imply that cycle-pull-back gives well-defined homomorphisms

$$f^* : z_{\mathrm{fin}}(X)(Y) \to z_{\mathrm{fin}}(X)(Y'); f^* : z_{\mathrm{q.fin}}(X)(Y) \to z_{\mathrm{q.fin}}(X)(Y),$$

making $z_{\mathrm{fin}}(X)$ and $z_{\mathrm{q.fin}}(X)$ into presheaves of abelian groups on $\mathbf{Sm}_k$. It is easy to see that these are in fact sheaves for the étale topology on $\mathbf{Sm}_k$.

Let $\mathcal{F}$ be a presheaf of abelian groups on $\mathbf{Sm}_k$. For $Y \in \mathbf{Sm}_k$, we may apply $\mathcal{F}$ to the cosimplicial scheme $Y \times \Delta^*$, giving the simplicial abelian group $\mathcal{F}(Y \times \Delta^*)$.

Definition 2.18. Let $\mathcal{F}$ be a presheaf of abelian groups on $\mathbf{Sm}_k$. (1) The *Suslin complex* $C_*^{\mathrm{Sus}}(\mathcal{F})$ is the complex of presheaves

$$Y \mapsto C_*^{\mathrm{Sus}}(\mathcal{F})(Y),$$

where $C_*^{\mathrm{Sus}}(\mathcal{F})(Y)$ is the complex associated to the simplicial abelian group $\mathcal{F}(Y \times \Delta^*)$.
(2) For $Y \in \mathbf{Sm}_k$, write $\mathbb{Z}_{\mathrm{FS},Y}(q)$ for the (cohomological) complex of sheaves on Y_{Zar}

$$U \mapsto C_{2q-*}(z_{\mathrm{q.fin}}(\mathbb{A}^q))(U).$$

and $\mathbb{Z}_{\mathrm{FS}}(q)$ for the corresponding complex of sheaves on $\mathbf{Sm}_k^{\mathrm{Nis}}$.
(3) For $X \in \mathbf{Sch}_k$, and abelian group A, the *Suslin homology* of X, $H_*^{\mathrm{Sus}}(X, A)$ is defined by

$$H_n^{\mathrm{Sus}}(X, A) := H_n(C_*^{\mathrm{Sus}}(z_{\mathrm{fin}}(X)(k) \otimes A)).$$

(4) For $Y \in \mathbf{Sm}_k$, the *Friedlander-Suslin cohomology* $H^p_{\mathrm{FS}}(Y, A(q))$ is defined by

$$H^p_{\mathrm{FS}}(Y, A(q)) := \mathbb{H}^p(Y_{\mathrm{Zar}}, \mathbb{Z}_{\mathrm{FS},Y}(q) \otimes A).$$

Remark 2.19. Let $\mathcal{F}$ be a presheaf on $\mathbf{Sm}_k$. The homology presheaf on $\mathbf{Sm}_k$

$$Y \mapsto H_n(C_*^{\mathrm{Sus}}(\mathcal{F})(Y))$$

is homotopy invariant, i.e., the natural map

$$p^* : H_n(C_*^{\mathrm{Sus}}(\mathcal{F})(Y)) \to H_n(C_*^{\mathrm{Sus}}(\mathcal{F})(Y \times \mathbb{A}^1))$$

is an isomorphism. See e.g. [100, Chap. 3, Prop. 3.6] for a proof.

Comparison with the higher Chow groups

For
$$W \in z_{\mathrm{q.fin}}(\mathbb{A}^q)(Y \times \Delta^n) \subset z^q(Y \times \mathbb{A}^q \times \Delta^n),$$
and $F \subset \Delta^n$ a face, the intersection $W \cap (Y \times \mathbb{A}^q \times F)$ is quasi-finite over $Y \times F$, hence
$$\mathrm{codim}_{Y\times\mathbb{A}^q\times F} W \cap (Y \times \mathbb{A}^q \times F) \geq q.$$
Thus, we have inclusions
$$z_{\mathrm{q.fin}}(\mathbb{A}^q)(Y \times \Delta^n) \subset z^q(Y \times \mathbb{A}^q, n) \subset z^q(Y \times \mathbb{A}^q \times \Delta^n),$$
giving the inclusion of complexes
$$\alpha^q_Y : C_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))(Y) \to z^q(Y \times \mathbb{A}^q, *).$$

Let $\mathbb{Z}_{\mathrm{Bl},Y}(q)$ be the sheaf of (cohomological) complexes on Y_{Zar} associated to the presheaf
$$U \mapsto z^q(U \times \mathbb{A}^q, 2q - *).$$
The maps α^q_U thus give the map of sheaves of complexes
$$\alpha^q : \mathbb{Z}_{\mathrm{FS},Y}(q) \to \mathbb{Z}_{\mathrm{Bl},Y}(q)$$

The main result of this section is

Theorem 2.20 (Suslin [100, Chap. 6]). *The map* $\alpha^q : \mathbb{Z}_{\mathrm{FS},Y}(q) \to \mathbb{Z}_{\mathrm{Bl},Y}(q)$ *is a quasi-isomorphism for all* $Y \in \mathbf{Sm}_k$*.*

Corollary 2.21. *Let* Y *be in* $\mathbf{Sm}_k$*. Then* α^q *induces an isomophism*
$$H^p_{\mathrm{FS}}(Y, \mathbb{Z}(q)) \to \mathrm{CH}^q(Y, 2q - p).$$

Proof of the corollary. By the Mayer-Vietoris property for the complexes $z^q(U \times \mathbb{A}^q, *)$, the natural map
$$H_{2q-n}(z^q(Y \times \mathbb{A}^q, *)) \to \mathbb{H}^n(Y_{\mathrm{Zar}}, \mathbb{Z}_{\mathrm{Bl},Y}(q))$$
is an isomorphism for all n. By the homotopy property (Theorem 2.10), the pull-back map
$$p^* : z^q(\mathbb{A}^q, *) \to z^q(Y \times \mathbb{A}^q, *)$$
is a quasi-isomorphism, so we have the isomorphisms
$$\mathrm{CH}^q(Y, 2q - n) = H_{2q-n}(z^q(Y, *)) \cong H_{2q-n}(z^q(Y \times \mathbb{A}^q, *)).$$
Thus, we have isomorphisms
$$H^p_{\mathrm{FS}}(Y, \mathbb{Z}(q)) = \mathbb{H}^p(Y_{\mathrm{Zar}}, \mathbb{Z}_{\mathrm{FS},Y}(q)) \cong \mathrm{CH}^q(Y, 2q - p).$$
□

The proof of Theorem 2.20 goes in two steps: First one uses Suslin's technique [100, Chap. 6, Thm. 2.1] to show that $C_*(z_{\text{q.fin}}(\mathbb{A}^q)(\operatorname{Spec} F)) \to z^q(\mathbb{A}^q_F, *)$ is a quasi-isomorphism for F a field. One may then use any one of several versions of a result of Voevodsky, that for $\mathcal{F}$ a homotopy invariant presheaf with transfers, $\mathcal{O}$ the local ring of a smooth point on a scheme of finite type over k with quotient field F, the map

$$\mathcal{F}(\mathcal{O}) \to \mathcal{F}(F)$$

is injective. One particular nice way to do this is the version due to Ojanguran-Panin [79], which allows one to use a fairly restricted form of transfers, namely:

1. For $f : X \to Y$ a finite separable morphism in $\mathbf{Sm}_k$, there is a homomorphism $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$.
2. Let $f : X \to Y$ be as in (1), let $i_D : D \to Y$ be the inclusion of a smooth divisor, and suppose that $\bar{f} : E := X \times_Y D \to D$ is étale. Let $i_E : E \to X$ be the inclusion. Then $\bar{f}_* \circ i_E^* = i_D^* \circ f_*$.

It is not hard, using the moving lemma (Theorem 2.13) to show that the presheaf

$$Y \mapsto H_n(\operatorname{Cone}(\alpha^q_Y))$$

has the structure of a presheaf on $\mathbf{Sm}_k$ with Ojanguran-Panin transfers; as this presheaf vanishes on fields, it follows that α^q is a quasi-isomorphism.

Remark 2.22. Via Proposition 2.16 and Theorem 2.20, we have an isomorphism

$$u : \mathbb{G}_m[-1] \to \mathbb{Z}_{\mathrm{FS}}(1)$$

in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathbf{Sm}_k)$.

The mod-n theory

Suslin and Voevodsky show in [86] that, for an algebraically closed field k and n prime to the characteristic of k, and for A a regular Henselian local k algebra with residue field k, there is a natural isomorphism of complexes

$$\mathbb{Z}_{\mathrm{FS}}(q)(\operatorname{Spec} A) \otimes^L \mathbb{Z}/n \cong \mu_n^{\otimes q})(\operatorname{Spec} A)$$

(actually, this is only shown for k of characteristic zero, but using de Jong's theory of alterations, the same argument works in positive characteristics). This verifies part (3) of Lichtenbaum's conjectures 2.6.

The analogous conjecture, part (2) of Beilinson's conjectures 2.5, which essentially asserts the existence of a natural isomorphism

$$H^p(F, \mathbb{Z}/n(q)) \cong H^p_{\text{ét}}(F, \mu_n^{\otimes q})$$

for fields F finitely generated over a chosen base-field k (not necessarily algebraically closed) was shown in [90] and later in [33] to be equivalent to the *Bloch-Kato conjecture*:

Conjecture 2.23 ([17]). Let F be a field, n an integer prime to the characteristic of F. Then the Galois symbol

$$\theta : K_q^M(F)/n \to H^p_{\mathrm{Gal}}(F, \mu_n^{\otimes q})$$

is an isomorphism.

Here, the Galois symbol is the map sending a symbol $\{a_1, \ldots, a_q\} \in K_q^M(F)$ to the cup product $l(a_1) \cup \ldots \cup l(a_q)$, where $l(a) \in H^1_{\mathrm{Gal}}(F, \mu_n)$ is the image of $a \in F^*$ under the Kummer sequence

$$H^0_{\mathrm{Gal}}(F, \mathbb{G}_m) \xrightarrow{\times n} H^0_{\mathrm{Gal}}(F, \mathbb{G}_m) \to H^1_{\mathrm{Gal}}(F, \mu_n).$$

This conjecture is known as the *Bloch-Kato conjecture*. One reduces directly to the case $n = \ell^\nu$, ℓ a prime. The case $\ell = 2$ is also known as part of the *Milnor conjecture* [73] proven by Voevodsky [99], [98]. The case of odd ℓ has been recently reduced by Voevodsky [93] to results of Rost (as yet unpublished) on the construction and properties of so-called "generic splitting varieties".

2.5 Cubical versions

One can also use cubes instead of simplices to define the various versions of the cycle complexes. The major advantage is that the product structure for the cubical complexes is easier to define, and with $\mathbb{Q}$-coefficients, one can construct cycle complexes which have a strictly commutative and associative product. This approach is used by Hanamura in his construction of a category of mixed motives, as well as in the construction of categories of Tate motives by Bloch, Bloch-Kriz and Kriz-May.

Cubical complexes

Let $(\square^1, \partial\square^1)$ denote the pair $(\mathbb{A}^1, \{0, 1\})$, and $(\square^n, \partial\square^n)$ the n-fold product of $(\square^1, \partial\square^1)$. Explicitly, $\square^n = \mathbb{A}^n$, and $\partial\square^n$ is the divisor $\sum_{i=1}^n (x_i = 0) + \sum_{i=1}^n (x_i = 1)$, where $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{A}^n$. A *face* of $\square^n$ is a face of the normal crossing divisor $\partial\square^n$, i.e., a subscheme defined by equations of the form $t_{i_1} = \epsilon_1, \ldots, t_{i_s} = \epsilon_s$, with the ϵ_j in $\{0, 1\}$. If a face F has codimension m in $\square^n$, we write $\dim F = n - m$.

For $\epsilon \in \{0, 1\}$ and $j \in \{1, \ldots, n\}$ we let $\iota_{j,\epsilon} : \square^{n-1} \to \square^n$ be the closed embedding defined by inserting an ϵ in the jth coordinate. We let $\pi_j : \square^n \to \square^{n-1}$ be the projection which omits the jth factor.

Definition 2.24. Let X be in $\mathbf{Sch}_k$. Let $\hat{z}_p(X, n)^{\mathrm{cb}}$ be the subgroup of $z_{p+n}(X \times \square^n)$ generated by integral subschemes $W \subset X \times \square^n$ such that

$$\dim_k W \cap (X \times F) \le p + \dim F.$$

If X is equi-dimensional over k of dimension d, we write $\hat{z}^p(X,n)^{\mathrm{cb}}$ for $\hat{z}_{d-p}(X,n)^{\mathrm{cb}}$ and extend to locally equi-dimensional X by taking direct sums over the connected components of X.

Clearly the pull-back of cycles $\iota_{j,\epsilon}^*: \hat{z}_p(X,n)^{\mathrm{cb}} \to \hat{z}_p(X,n-1)^{\mathrm{cb}}$ and $\pi_j^*: \hat{z}_p(X,n-1)^{\mathrm{cb}} \to \hat{z}_p(X,n)^{\mathrm{cb}}$ are defined. We let $z_p(X,n)^{\mathrm{cb}}$ be the quotient

$$z_p(X,n)^{\mathrm{cb}} := \hat{z}_p(X,n)^{\mathrm{cb}} / \sum_{j=1}^n \pi_j^*(\hat{z}_p(X,n-1)^{\mathrm{cb}}).$$

One easily checks that

$$\sum_{j=1}^n (-1)^{j-1}\iota_{j,1}^* - \sum_{j=1}^n (-1)^{j-1}\iota_{j,0}^* : \hat{z}_p(X,n)^{\mathrm{cb}} \to \hat{z}_p(X,n-1)^{\mathrm{cb}}$$

descends to

$$d_n : z_p(X,n)^{\mathrm{cb}} \to z_p(X,n-1)^{\mathrm{cb}}$$

and that $d_{n-1}\circ d_n = 0$. Thus, we have the complex $z_p(X,*)^{\mathrm{cb}}$, and for X locally equi-dimensional over k the complex $z^p(X,*)^{\mathrm{cb}}$.

We let $\mathbb{Z}_{\mathrm{Bl},X}(p)^{\mathrm{cb}}$ denote the sheafification of the presheaf on X_{Zar}, $U \mapsto z^p(U\times\mathbb{A}^p,*)^{\mathrm{cb}}$.

Replacing $z^p(X,n)^{\mathrm{cb}}$ with $z_{\mathrm{q.fin}}(\mathbb{A}^p)(X\times\square^n)/\sum_{j=1}^n \pi_j^* z_{\mathrm{q.fin}}(\mathbb{A}^p)(X\times\square^{n-1})$ and using the similarly defined differential, we have the cubical version of Suslin's complex, $C_*^{\mathrm{cb}}(z_{\mathrm{q.fin}}(\mathbb{A}^p)(X))$ and the sheaf of complexes $\mathbb{Z}_{\mathrm{FS},X}(p)^{\mathrm{cb}}$ on X_{Zar}.

Cubes and simplices

The main comparison results are

Theorem 2.25. *Let X be in $\mathbf{Sch}_k$. (1) There is an isomorphism in $D^-(\mathbf{Ab})$*

$$z_p(X,*)^{\mathrm{cb}} \cong z_p(X,*),$$

natural with respect to flat pull-back and proper push-forward.

(2) There is a natural (in the same sense as above) isomorphism in $D^-(\mathbf{Ab})$

$$C_*^{\mathrm{cb}}(z_{\mathrm{q.fin}}(\mathbb{A}^p))(X) \cong C_*(z_{\mathrm{q.fin}}(\mathbb{A}^p)(X)).$$

The proof of (1) is given in, e.g., [64, Thm. 4.7]; the same argument (in fact somewhat easier) also proves (2).

This has as immediate corollary:

Corollary 2.26. *For $X \in \mathbf{Sm}_k$, there is an isomorphism in the derived category of sheaves on X_{Zar}*

$$\mathbb{Z}_{\mathrm{Bl},X}(p)^{\mathrm{cb}} \cong \mathbb{Z}_{\mathrm{FS},X}(p)^{\mathrm{cb}},$$

natural with respect to pull-back by maps in $\mathbf{Sm}_k$.

Remark 2.27. Let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$, with X and Y locally equi-dimensional over k. One can define the subcomplex $z^p(X,*)^{\rm cb}_f \subset z^p(X,*)^{\rm cb}$ as the cubical version of the subcomplex $z_p(X,*)_f \subset z_p(X,*)$. The argument of [64, Thm. 4.7] mentioned above shows in addition that the isomorphism $z_p(X,*)^{\rm cb} \cong z_p(X,*)$ induces an isomorphism $z^p(X,*)^{\rm cb}_f \cong z^p(X,*)_f$, and thus, in case X is in $\mathbf{Sm}_k$ and is affine, the inclusion $z^p(X,*)^{\rm cb}_f \to z^p(X,*)^{\rm cb}$ is a quasi-isomorphism. Thus, sending X to $z^p(X,*)^{\rm cb}$ extends to a functor

$$z^p(-,*)^{\rm cb} : \mathbf{Sm}_k^{\rm op} \to D^-(\mathbf{Ab}).$$

This explains the naturality assertion in the above corollary.

Products

As already mentioned, the cubical complexes are convenient for defining products. Indeed, the simple external product of cycles (after rearranging the terms in the product) defines the map of complexes

$$\cup_{X,Y} : z_p(X,*)^{\rm cb} \otimes z_q(Y,*)^{\rm cb} \to z_{p+q}(X \times_k Y,*)^{\rm cb}$$

Thus, we have a cup product

$$\cup_X := \delta_X^* \circ \cup_{X,X} : z^p(X,*)^{\rm cb} \otimes z^q(X,*)^{\rm cb} \to z^{p+q}(X,*)^{\rm cb}$$

in $D^-(\mathbf{Ab})$, and the isomorphism of Theorem 2.25 respects the two products.

Alternating complexes

The symmetric group Σ_n acts on $z_p(X,n)^{\rm cb}$ be permuting the factors of $\square^n$. Extending coefficients to $\mathbb{Q}$, we let $z_p(X,n)^{\rm Alt}$ be the subspace of $z_p(X,n)^{\rm cb}_{\mathbb{Q}}$ on which Σ^n acts by the sign representation, and let$\pi_n^{\rm Alt} : z_p(X,n)^{\rm cb}_{\mathbb{Q}} \to z_p(X,n)^{\rm Alt}$ be the (Σ-equivariant) projection on this summand. One checks (see [11, Lemma 1.1]) that the differential on $z_p(X,*)^{\rm cb}_{\mathbb{Q}}$ descends to give a map

$$d_n^{\rm Alt} : z_p(X,n)^{\rm Alt} \to z_p(X,n-1)^{\rm Alt}$$

forming the subcomplex $z_p(X,*)^{\rm Alt}$ of $z_p(X,*)^{\rm cb}_{\mathbb{Q}}$.

Lemma 2.28 ([64, Thm. 4.11]). *The inclusion* $z_p(X,*)^{\rm Alt} \to z_p(X,*)^{\rm cb}_{\mathbb{Q}}$ *is a quasi-isomorphism, with inverse the alternating projection* $\pi^{\rm Alt} : z_p(X,*)^{\rm cb}_{\mathbb{Q}} \to z_p(X,*)^{\rm Alt}$.

We may define an external product on the alternating complexes by

$$\cup^{\rm Alt}_{X,Y} := \pi^{\rm Alt}_{p+q} \circ \cup_{X,Y} : z^p(X,*)^{\rm Alt} \otimes z^q(Y,*)^{\rm Alt} \to z^{p+q}(X \times_k Y,*)^{\rm Alt}.$$

This agrees (up to homotopy) with the product on $z^*(-,*)^{\rm cb}_{\mathbb{Q}}$.

In particular, for $X = \operatorname{Spec} k$, we have the commutative, associative product

$$\cup^{\text{Alt}} : z^p(k,*)^{\text{Alt}} \otimes_{\mathbb{Q}} z^q(k,*)^{\text{Alt}} \to z^{p+q}(k,*)^{\text{Alt}},$$

satisfying the Leibniz rule

$$d(a \cup^{\text{Alt}} b) = da \cup^{\text{Alt}} b + (-1)^{\deg a} a \cup^{\text{Alt}} b.$$

We will see in §5.2 how the complexes $z^q(k,*)^{\text{Alt}}$ form a (graded) commutative differential graded algebra over $\mathbb{Q}$, which may be used to give a concrete description of the category of mixed Tate motives over k.

3 Abelian categories of mixed motives

We will now proceed to examine framework proposed by Beilinson and Deligne for a category of mixed motives in somewhat more detail. Before doing so, however, we will fix some ideas concerning Bloch-Ogus cohomology and Tannakian categories. Having done this, we give a formulation of some of the hoped-for properties of the abelian category of mixed motives, and then describe two very different approaches to a construction. The first, following Jannsen and Deligne, attempts to define a "mixed motive" by its singular/étale/de Rham realizations. The second, due to Nori, first considers the ring of natural endomorphisms of the singular cohomology functor on pairs of schemes, and then defines a mixed motive as a module over this ring (roughly speaking). As we mentioned in the introduction, it is not at all clear what relation K-theory has to the cohomology theory arising from these constructions.

We will not discuss the theory of "pure" motives here at all. As a reference, we refer the reader to the relevant articles in [104], as well as [56, Section 3] in this volume, where in addition some of the material in this section is handled in shorter form.

3.1 Background and conjectures

We formulate a version of Bloch-Ogus cohomology, somewhat modified from the original definition in [19] to fit our purposes. We recall some notions from the theory of Tannakian categories, and then give a version of the properties one would like in an abelian category of mixed motives.

Bloch-Ogus cohomology

Let $\Gamma(*) := \oplus_{n\geq 0}\Gamma(n)$ be a graded object in $C(\text{Sh}^{\text{Zar}}(k))$ (with $\Gamma(n)$ in graded degree $2n$), together with a graded product $\mu : \Gamma(*) \otimes^L \Gamma(*) \to \Gamma(*)$ in $D(\text{Sh}^{\text{Zar}}(k))$. For $X \in \mathbf{Sm}_k$, we set

$$H^n_\Gamma(X,m) := \mathbb{H}^n(X_{\text{Zar}}, \Gamma(m))$$

and for $W \subset X$ a closed subset, set

$$H^n_{\Gamma,W}(X,m) := \mathbb{H}^n_W(X_{\mathrm{Zar}}, \Gamma(m)).$$

We note that, if $W \subset W' \subset X$ are closed subsets of $X \in \mathbf{Sm}_k$, we have the natural map

$$H^n_{\Gamma,W}(X,m) \to H^n_{\Gamma,W'}(X,m).$$

Definition 3.1. We say that $\Gamma(*)$ defines a *Bloch-Ogus cohomology theory* if

1. The product μ is associative and commutative in $D(\mathrm{Sh}^{\mathrm{Zar}}(k))$.
2. $\Gamma(*)$ is *homotopy invariant*: $p^* : H^*_\Gamma(X,m) \to H^*_\Gamma(X \times \mathbb{A}^1, m)$ is an isomorphism for all m.
3. $\Gamma(*)$ satisfies *purity*: Let $W \subset X$ be a closed subset, with $X \in \mathbf{Sm}_k$. If $\mathrm{codim}_X W \geq q$ for some integer q, then $H^p_{\Gamma,W}(X,q) = 0$ for $p < 2q$.
4. $\Gamma(*)$ admits natural *cycle classes*: Let $W \subset X$ be an irreducible closed codimension q subset with X in $\mathbf{Sm}_k$. Then there is a *fundamental class* $[W] \in H^{2q}_{\Gamma,W}(X,q)$ satisfying:
   a) *Naturality*: Let $z := \sum_i n_i W_i$ be in $z^q(X)$, let W be the support of z, and set $\mathrm{cl}(z) = \sum_i n_i[W_i] \in H^{2q}_{\Gamma,W}(X,q)$. Let $f : Y \to X$ be a morphism in $\mathbf{Sm}_k$ such that $f^{-1}(W)$ has codimension q on Y. Then
   $$f^*(\mathrm{cl}(z)) = \mathrm{cl}(f^*(z)) \in H^{2q}_{\Gamma,f^{-1}(W)}(Y,q).$$
   b) *Gysin isomorphism*: Suppose that $W \subset X$ is a pure codimension q closed subset, with X and W in $\mathbf{Sm}_k$. Suppose that the inclusion $i : W \to X$ is split by a smooth morphism $p : X \to W$. Then the composition
   $$H^n_\Gamma(W,m) \xrightarrow{p^*} H^n_\Gamma(X,m) \xrightarrow{\cup[W]} H^{n+2q}_{\Gamma,W}(X,m+q)$$
   is an isomorphism.
   c) *Products* For $X_i \in \mathbf{Sm}_k$, $z_i \in z^{q_i}(X_i)$ with support W_i, $i = 1,2$, we have
   $$\mathrm{cl}(z_1 \times z_2) = p_1^*\mathrm{cl}(z_1) \cup p_2^*\mathrm{cl}(z_2)$$
   in $H^{2q_1+2q_2}_{\Gamma,W_1\times W_2}(X_1 \times_k X_2, q_1+q_2)$.
5. *Coefficients*: For $p : X \to \mathrm{Spec}\, k$ in $\mathbf{Sm}_k$, X irreducible, the map
   $$p^* : H^0_\Gamma(\mathrm{Spec}\, k, 0) \to H^0_\Gamma(X,0)$$
   is an isomorphism.

The functor $X \mapsto \oplus_{p,q} H^p_\Gamma(X,q)$ is called the *Bloch-Ogus theory on* $\mathbf{Sm}_k$ *represented by* $\Gamma(*)$. The ring $H^0_\Gamma(\mathrm{Spec}\, k, 0)$ is called the *coefficient ring* of the theory Γ.

Remark 3.2. This notion of a Bloch-Ogus cohomology theory is somewhat more general than that considered by Gillet in [34], in that Gillet requires

1. A structure map $\mathbb{G}_m[-1] \to \Gamma(1)$ in $D(\mathrm{Sh}^{\mathrm{Zar}}(\mathbf{Sm}_k))$.
2. The complexes $\Gamma(n)$ should be in $C^+(\mathrm{Sh}^{\mathrm{Zar}}(k))$.

The existence of the structure map (1) follows from the cycle class map discussed in §6.1; see Remark 6.4 for a precise statement. Allowing the $\Gamma(q)$ to be unbounded forces one to take a bit more care in the definition of the universal Chern classes on the simplicial ind-scheme BGL, in that one needs to use the *extended* total complex to define $\Gamma(n)(B\mathrm{GL}_N)$:

$$\Gamma(n)(B\mathrm{GL}_N)^m := \prod_{p\geq 0} \Gamma(n)^{m-p}(B\mathrm{GL}_N)_p,$$

and then take

$$\Gamma(n)(B\mathrm{GL}) := \lim_N \Gamma(n)(B\mathrm{GL}_N).$$

Having made this definition, Gillet's argument extends word-for-word to allow for $\Gamma(n) \in C(\mathrm{Sh}^{\mathrm{Zar}}(k))$, giving a good theory of Chern classes

$$c_\Gamma^{q,p} : K_{2q-p}(X) \to H^p_\Gamma(X, q)$$

for a Bloch-Ogus theory in our sense.

Examples 3.3. The standard cohomology theories: singular cohomology, ℓ-adic étale cohomology, de Rham cohomology and Deligne cohomology, are all examples which can be fit into the framework of the above. Also, motivic cohomology, represented by $\Gamma(n) := \mathbb{Z}_{\mathrm{FS}}(n)$ (see §2.4), is an example.

Tannakian formalism

We use [22, 28, 83] as references for this section.

Let F be a field. An F-linear abelian tensor category $\mathcal{A}$ is called *rigid* if there exists internal Homs in $\mathcal{A}$, i.e., for each pair of objects A, B of $\mathcal{A}$, there is an object $\mathcal{H}om(A, B)$ and a natural isomorphism of functors

$$(C \mapsto \mathrm{Hom}_{\mathcal{A}}(A \otimes C, B)) \cong (C \mapsto \mathrm{Hom}_{\mathcal{A}}(C, \mathcal{H}om(A, B))).$$

For example, the abelian tensor category of finite-dimensional F-vector spaces, F-mod, has the internal Hom $\mathcal{H}om(V, W) := V^\vee \otimes W$.

An F-linear rigid abelian tensor category $\mathcal{A}$ is a *Tannakian category* if there exists an exact faithful F-linear tensor functor to F'-mod for some field extension F' of F; such a functor is called a *fiber functor*. If a fiber functor to F-mod exists, we call $\mathcal{A}$ a *neutral* Tannakian category.

The primary example of a neutral Tannakian category is the category $\mathrm{Rep}_F(G)$ of representations of an affine group scheme G over F in finite dimensional F-vector spaces; the forgetful functor $\mathrm{Rep}_F(G) \to F$-mod is the evident fiber functor. Note that, if A is the Hopf algebra $\Gamma(G, \mathcal{O}_G)$, so that $G = \mathrm{Spec}\, A$, then $\mathrm{Rep}_F(G)$ is isomorphic to the category of co-representations of A in F-mod, $\text{co-rep}_F(A)$.

Neutral Tannakian categories are of interest because they are all given as categories of representations: If $\mathcal{A}$ is a Tannakian category over F with fiber functor $\omega : \mathcal{A} \to F$-mod, then there is an affine group scheme G over F with a canonical isomorphism

$$G(F) \cong \mathrm{Aut}(\omega)$$

and an equivalence of $\mathcal{A}$ with $\mathrm{Rep}_F(G)$ with ω going over to the forgetful functor. G is canonically determined by $\mathcal{A}$ and ω, and different choices of ω lead to isomorphic G's. G is called the *Galois group* of $\mathcal{A}$.

The category of mixed motives

In [2] and [24], a framework for a category of mixed motives over a base field k is proposed. There are many articles describing the consequences of such a theory, e.g., [21, 53, 75]. We give here a quick description of the properties one should expect in this category, derived from [2] and [24].

Conjecture 3.4. Let k be a field. There is a rigid tensor category $\mathcal{MM}_k$ containing "Tate objects" $\mathbb{Z}(n)$, $n \in \mathbb{Z}$, and a functor

$$h : \mathbf{Sm}_k^{\mathrm{op}} \to D^b(\mathcal{MM}_k)$$

such that

1. Setting $H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) := \mathrm{Ext}^p_{\mathcal{MM}_k}(1, h(X) \otimes \mathbb{Z}(q))$, the functor $X \mapsto \oplus_{p,q} H^p_{\mathcal{M}}(X, \mathbb{Z}(q))$ is the universal Bloch-Ogus cohomology theory on $\mathbf{Sm}_k$.
2. Let Γ be a Bloch-Ogus theory on $\mathbf{Sm}_k$, and $R\Gamma : \mathbf{Sm}_k^{\mathrm{op}} \to D(\mathbf{Ab})$ the functor $X \mapsto \tilde{\Gamma}(q)(X)$, where $\tilde{\Gamma}(q)(X)$ is as in §3.1 the global sections on X of a functorial flasque model for $\Gamma(q)$. Then there is a "realization functor"
$$\Re_\Gamma : \mathcal{MM}_k \to D(\mathbf{Ab})$$
such that the induced map $D^b\Re_\Gamma : D^b(\mathcal{MM}_k) \to D(\mathbf{Ab})$ yields a factorization of $R\Gamma$ as $D^b\Re_\Gamma \circ R$. Applying H^p yields the canonical natural transformation
$$H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) \to H^p_\Gamma(X, q)$$
given by (1).
3. In the $\mathbb{Q}$-extension $\mathcal{MM}_k \otimes \mathbb{Q}$, the full subcategory of semi-simple objects is equivalent to the category $\mathcal{M}_k$ of homological motives over k, and for each $X \in \mathbf{Sm}_k$, the object $h^i(X)(q) := H^i(h(X)) \otimes \mathbb{Q}(q)$ is in $\mathcal{M}_k$.
4. For X smooth and projective over k, there is a decomposition (not necessarily unique) in $D^b(\mathcal{MM}_k \otimes \mathbb{Q})$
$$h(X)_\mathbb{Q} = \oplus_{i=0}^{2\dim_k X} h^i(X)[-i].$$

5. Let $\sigma : k \to \mathbb{C}$, and let $\Re_{\mathrm{sing},\sigma}$ be the realization functor corresponding to singular cohomology $H^*_{\mathrm{sing}}(X^\sigma(\mathbb{C}), \mathbb{Z}(2\pi i)^j)$, where $X^\sigma(\mathbb{C})$ is the analytic manifold of $\mathbb{C}$-points of $X \times_k \mathbb{C}$. Then the functor
$$H^0 \circ \Re_{\mathrm{sing},\sigma} : \mathcal{MM}_k \otimes \mathbb{Q} \to \mathbb{Q}\text{-mod}$$
is a fiber functor, making $\mathcal{MM}_k \otimes \mathbb{Q}$ a neutral Tannakian category over $\mathbb{Q}$. Also, if $\Re_{\text{ét},\ell}$ is the realization functor corresponding to $X \mapsto H^*_{\text{ét}}(X \times_k \bar{k}, \mathbb{Q}_l(*))$, then
$$H^0 \circ \Re_{\text{ét},\ell} : \mathcal{MM}_k \otimes \mathbb{Q}_\ell \to \mathbb{Q}_\ell\text{-mod}$$
is a fiber functor, making $\mathcal{MM}_k \otimes \mathbb{Q}_\ell$ a neutral Tannakian category over $\mathbb{Q}_\ell$.
6. For each object M in $\mathcal{MM}_k$, there is a natural finite *weight filtration*
$$0 = W_{n-1}M \subset W_nM \subset \ldots \subset W_mM = M.$$
The graded quotients $\mathrm{gr}^*_W M$ are in $\mathcal{M}_k$ (after passing to the $\mathbb{Q}$-extension). For $M = h^i(X)$, the weight filtration is sent to the weight filtration for singular cohomology, respectively étale cohomology, under the respective realization functor.
7. There are natural isomorphisms
$$H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) \otimes \mathbb{Q} \cong K_{2q-p}(X)^{(q)}.$$
These should arise from a natural spectral sequence of Atiyah-Hirzebruch type
$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X),$$
which degenerates at E_2 after tensoring with $\mathbb{Q}$.

Remark 3.5. Rather than limiting oneself to motives over a field k, Beilinson suggests in [2] that one should look for a theory of “mixed motivic sheaves” $\mathcal{MM}/S$ over a base-scheme S, analogous to the category of say sheaves of abelian groups or perverse sheaves or constructible étale sheaves, or mixed Hodge modules. In any case, one would want to have the Grothendieck-Verdier formalism of four functors f_*, f^*, $f_!$ and $f^!$, as well as a relation with K-theory and the realization properties analogous to $D^b(\mathcal{MM}_k)$. However, as suggested by Deligne [21], one might ask rather for a triangulated tensor category $\mathcal{D}(S)$ with a t-structure whose heart is the abelian category $\mathcal{MM}/S$, but without necesarily requiring that $D^b(\mathcal{MM}/S) = \mathcal{D}(S)$. Voevodsky [95] has axiomatized the situation in his theory of “crossed functors”, and has announced a construction of a category of mixed motives over S which satisfies the necessary conditions. As the theory is still in its beginning stages, we will not discuss these result further.

The motivic Galois groups

Suppose k admits an embedding $\sigma : k \to \mathbb{C}$, giving us the fiber functor $F_\sigma := H^0 \circ \Re_{\mathrm{sing}}$ over $\mathbb{Q}$ corresponding to singular cohomology. Let MotGal_k be the Galois group of the Tannakian category $\mathcal{MM}_k \otimes \mathbb{Q}$, and let $\mathrm{MotGal}_k^{\mathrm{ss}}$ be the Galois group for the semi-simple subcategory $\mathcal{M}_k$. Taking the associated graded for the weight filtration defines a functor $\mathcal{MM}_k \to \mathcal{M}_k$, and hence a homomorphism $\mathrm{MotGal}_k^{\mathrm{ss}} \to \mathrm{MotGal}_k$ splitting the map induced by the restriction functor $\mathrm{MotGal}_k \to \mathrm{MotGal}_k^{\mathrm{ss}}$. The map $\mathrm{MotGal}_k^{\mathrm{ss}} \to \mathrm{MotGal}_k$ is thought of as an analog of the map on the algebraic π_1:

$$\pi_1(\bar{X}, *) \to \pi_1(X, *)$$

corresponding to the projection $\bar{X} := X \times_k \bar{k} \to X$ for a scheme X over k. The split surjection $\mathrm{MotGal}_k \to \mathrm{MotGal}_k^{\mathrm{ss}}$ yields the exact sequence

$$1 \to \widehat{\mathcal{U}}_k \to \mathrm{MotGal}_k \to \mathrm{MotGal}_k^{\mathrm{ss}} \to 1;$$

$\widehat{\mathcal{U}}_k$ is a connected pro-unipotent algebraic group scheme over $\mathbb{Q}$, encoding the extension information in $\mathcal{MM}_k$.

One can restrict to the category of *mixed Tate motives* TM_k, i.e., the full abelian subcategory (of $\mathcal{MM}_k \otimes \mathbb{Q}$) closed under extensions and generated by the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$. The abelian subcategory $\overline{\mathrm{TM}}_k$ of TM_k generated by the $\mathbb{Q}(n)$'s is equivalent to the category of graded finite dimensional $\mathbb{Q}$-vector spaces, i.e., the category of representations of $\mathbb{G}_m$ in $\mathbb{Q}$-mod. As taking the associated graded for the weight filtration defines an exact tensor functor $\mathrm{TM}_k \to \overline{\mathrm{TM}}_k$ splitting the inclusion, we have the split surjection

$$\mathrm{GalTM}_k \to \mathbb{G}_m \to 1$$

with kernel $\mathcal{U}_k$ a pro-unipotent algebraic group with $\mathbb{G}_m$-action. Since the action of $\mathbb{G}_m$ just gives the information of a grading, we thus have an equivalence of TM_k with the category of graded representations of $\mathcal{U}_k$ on finite dimensional $\mathbb{Q}$-vector spaces. More about this in section 5 on Tate motives.

3.2 Motives by compatible realizations

Building on Deligne's theory of absolute Hodge cycles [28], Jannsen [54] constructs an abelian category of "simultaneous realizations", as an attempt to capture the idea of a mixed motive by looking at structures modeled on singular, de Rham and étale cohomology, together with comparison isomorphisms between these structures. The known comparisons between singular, de Rham and étale cohomology of a scheme X yields objects $H^n(X)$ in this category, and a reasonable approximation to a good catgory of motives is the sub-abelian category generated by these and their duals. Deligne has also given a construction from this point of view in [23], adding a crystalline component to the collection of realizations. The viewpoint of compatible realizations has also been used in the setting of triangulated categories by Huber [48], see §4.2 for some details of this construction.

The category of realizations

Let k be a field finitely generated over $\mathbb{Q}$, $\bar{k}$ the algebraic closure of k. Let $G_k = \mathrm{Gal}(\bar{k}.k)$. Form the category of *mixed realizations* MR_k, with objects tuples of the form $H := (H_{\mathrm{DR}}, H_\ell, H_\sigma, I_{\infty,\sigma}, I_{\ell,\bar{\sigma}})$, with ℓ running over prime numbers, σ over embeddings $k \to \mathbb{C}$ and $\bar{\sigma}$ over embeddings $\bar{k} \to \mathbb{C}$, where

(a) H_{DR} is a finite dimensional k-vector space with an exhaustive decreasing filtration $F^n H_{\mathrm{DR}}$, and an exhaustive increasing filtration $W_n H_{\mathrm{DR}}$.
(b) H_ℓ is a finite-dimensional $\mathbb{Q}_\ell$-vector space with a continuous G_k-action, and an exhaustive increasing G_k-stable filtration $W_n H_\ell$.
(c) H_σ is a $\mathbb{Q}$-mixed Hodge structure: H_σ is a finite dimensional $\mathbb{Q}$-vector space with an exhaustive decreasing filtration F^n on $H_\sigma \otimes \mathbb{C}$, and an exhaustive increasing filtration W_n on H_σ inducing a pure $\mathbb{Q}$-Hodge structure of weight m on $\mathrm{gr}_n^W H_\sigma$, i.e., there is a direct sum decomposition

$$(\mathrm{gr}_n^W H_\sigma) \otimes \mathbb{C} = \oplus_{p+q=m} H^{p,q}$$

with $H^{q,p} = \overline{H^{p,q}}$ and with

$$\mathrm{gr}_n^W F^a H_\sigma \otimes \mathbb{C} = \oplus_{p \geq a} H^{p,q}.$$

(d) $I_{\infty,\sigma} : H_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \to H_{\mathrm{DR}} \otimes_k \mathbb{C}$ is an isomorphism, identifying the F- and W-filtrations.
(e) $I_{\ell,\bar{\sigma}} : H_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H_\ell$ is an isomorphism identifying the W-filtrations. In addition, for each $\rho \in G_k$, the diagram

$$\begin{array}{ccc} & & H_\ell \\ & \overset{I_{\ell,\bar{\sigma}}}{\nearrow} & \\ H_\sigma \otimes \mathbb{Q}_\ell & & \uparrow \rho \\ & \underset{I_{\ell,\bar{\sigma}\rho}}{\searrow} & \\ & & H_\ell \end{array}$$

commutes.

The various W-filtrations (resp. F-filtrations) are called *weight* filtrations (resp. *Hodge* filtrations) and the isomorphisms I are called *comparison isomorphisms.*

Morphisms $H \to H'$ in MR_k are $(k, \mathbb{Q}_\ell, \mathbb{Q})$-linear maps

$$(H_{\mathrm{DR}}, H_\ell, H_\sigma) \to (H'_{\mathrm{DR}}, H'_\ell, H'_\sigma)$$

respecting the various filtrations and comparison isomorphisms. Defining the operations componentwise, one has a tensor product, dual and internal Hom. In addition, for $H = (H_{\mathrm{DR}}, H_\ell, H_\sigma)$, the weight-filtrations on $H_{\mathrm{DR}}, H_\ell, H_\sigma$ are all compatible via the comparison isomorphisms, so we have the functor

$$W_n : \mathrm{MR}_k \to \mathrm{MR}_k$$
$$W_n((H_{\mathrm{DR}}, H_\ell, H_\sigma)) := (W_n H_{\mathrm{DR}}, W_n H_\ell, W_n H_\sigma).$$

From the exactness of W_n in the category of $\mathbb{Q}$-mixed Hodge structures, it follows that W_n is exact.

Tannakian structure

The main structural result for MR_k is

Theorem 3.6 ([54, Theorem 2.13]). MR_k *is a neutral Tannakian category over* $\mathbb{Q}$.

In fact, the functor $H \mapsto H_\sigma$ for a single choice of σ gives the fiber functor.

The category of mixed motives

Let (X, Y) be a pair consisting of a finite type k-scheme X and a closed subscheme Y. For an embedding $\sigma : k \to \mathbb{C}$, let (X_σ, Y_σ) be the pair of topological spaces given by the $\mathbb{C}$-points of $(X \times_k \mathbb{C}, Y \times_k \mathbb{C})$, with the $\mathbb{C}$-topology. Let $H^n(X_\sigma, Y_\sigma; \mathbb{Q})$ be the singular cohomology of the pair (X_σ, Y_σ). Let $H^n_{\text{ét}}(X, Y; \mathbb{Q}_\ell)$ be the $\mathbb{Q}_\ell$-étale cohomology of the pair and let $H_{\mathrm{DR}}(X, Y)$ be the deRham cohomology.

Let X be a smooth quasi-projective k-scheme and take $Y = \emptyset$. Give $H^n(X_\sigma, ; \mathbb{Q})$ the mixed Hodge structure of Deligne [26]. Give $H^n_{\mathrm{DR}}(X)$ the analogous weight and Hodge filtration: Take a smooth projective variety $\bar{X}$ containing X as a dense open subscheme with normal crossing divisor $D := \bar{X} \setminus X$ at infinity. One then has

$$H^n_{\mathrm{DR}}(X) = \mathbb{H}^n(\bar{X}, \Omega^*_{\bar{X}/k}(\log D)).$$

The stupid filtration on the deRham complex $\Omega^*_{\bar{X}/k}(\log D)$ gives the Hodge filtration on $H^n_{\mathrm{DR}}(X)$ and the weight-filtration on $\Omega^*_{\bar{X}/k}(\log D)$ by number of components in the polar locus of a form induces (after shift by n) the weight-filtration on $H^n_{\mathrm{DR}}(X)$. Similarly, we identify the dual of the relative cohomology $H^{2\dim X - n}_{\text{ét}}(\bar{X}_{\bar{k}}, D_{\bar{k}}, \mathbb{Q}_\ell)$ with $H^n_{\text{ét}}(X_{\bar{k}}; \mathbb{Q}_\ell)$; the skeletal filtration on D induces the weight-filtration on $H^{2\dim X - n}_{\text{ét}}(\bar{X}_{\bar{k}}, D_{\bar{k}}, \mathbb{Q}_\ell)$ and thus on $H^n_{\text{ét}}(X_{\bar{k}}; \mathbb{Q}_\ell)$.

The classical deRham theorem gives comparison isomorphisms

$$I_{\infty,\sigma} : H^n(X_\sigma, \mathbb{Q}) \otimes \mathbb{C} \to H^n_{\mathrm{DR}}(X) \otimes_k \mathbb{C}$$

and Artin's comparison isomorphism yields

$$I_{\ell,\bar{\sigma}} :: H^n(X_\sigma, \mathbb{Q}) \otimes \mathbb{Q}_\ell \to H^n_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell).$$

Jannsen shows that setting

$$H_\sigma := H^n(X_\sigma, Q)$$
$$H_{\mathrm{DR}} := H^n_{\mathrm{DR}}(X)$$
$$H_\ell := H^n_{\text{ét}}(X_{\bar{k}}; \mathbb{Q}_\ell)$$

with the above filtrations and comparison isomorphisms defines an object $H^n(X)$ in MR_k, functorial in X, giving the functor

$$H^n : \mathbf{Sm}_k^{\mathrm{op}} \to \mathrm{MR}_k$$

Definition 3.7. Jannsen's category of *mixed motives by realizations over* k, JMM_k, is the smallest full Tannakian subcategory of MR_k containing all the objects $H^n(X)$ for X smooth and quasi-projective over k. The objects of JMM_k are called *mixed motives*. The smallest Tannakian subcategory M_k of JMM_k containing all objects $H^n(X)$ for X smooth and projective over k, and closed under taking direct summands is called the subcategory of *pure* motives.

Remark 3.8. As mentioned above, Deligne [23] has also described a category of motives over $\mathbb{Q}$ by compatible realizations, adding a crystalline component to the list of possible realizations. This yields a category analogous to the category $\mathrm{MR}_{\mathbb{Q}}$. However, Deligne gives no precise definition of the subcategory analogous to $\mathrm{JMM}_{\mathbb{Q}}$, saying that the objects should be those systems of compatible realizations of *geometric origin* but explicitly leaving the definition of this term open.

Remarks 3.9. (1) Jannsen shows that the objects of JMM_k are exactly the subquotients of $H^n(U) \otimes H^m(V)^\vee$ for smooth, quasi-projective U and V over k. In addition, JMM_k is stable under the functors W_n and gr^W_n.

(2) For each $M \in \mathrm{JMM}_k$, the weight-filtration W_*M is finite and exhaustive, and the graded pieces $\mathrm{gr}^W_n M$ are all pure motives. Thus each mixed motive is a successive extension of pure motives. The category of pure motives is semi-simple.

(3) The method of cubical hyperresolutions of Guillen and Navarro-Aznar [41] extends H^n to a functor on arbitrary pairs of finite type k-schemes, sending (X, Y) to the deRham/étale/singular cohomology

$$H^n(X, Y) := (H^n_{\mathrm{DR}}(X, Y), H^n_{\text{ét}}(X, Y, \mathbb{Q}_\ell, H^n_\sigma(X_\sigma, Y_\sigma, \mathbb{Q}))$$

with the canonical mixed Hodge over $\mathbb{Q}$/weight filtration/$\mathbb{Q}$-mixed Hodge structure gives an object in M_k. Also, for a triple (X, Y, Z), the connecting morphism $H^n(X, Y) \to H^{n+1}(Y, Z)$ is a morphism in M_k.

3.3 Motives by Tannakian formalism

Let k be a subfield of $\mathbb{C}$. For a pair consisting of a finite type k-scheme X and a closed subscheme Y, one has the singular homology $H_*(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z})$, which we denote by $H_*(X,Y)$. Nori constructs an abelian category of mixed motives by considering the ring of all natural endomorphisms α of the functor $(X,Y) \mapsto H_*(X,Y)$, with the additional requirement that α should commute with all boundary maps $\partial_i : H_i(X,Y) \to H_{i-1}(Y,Z)$ for all triples $X \supset Y \supset Z$. The external products in homology make this ring into a bi-algebra; dualizing and inverting the resulting character corresponding to the Tate object $H^2(\mathbb{P}^1,\mathbb{Z})$ yields a Hopf algebra χ_{mot}. The category of co-modules of χ_{mot} in finitely generated abelian groups is then Nori's abelian category of mixed motives. In this section, we give some details regarding this construction. Some of these results involve relations with the triangulated categories of motives constructed by Voevodsky; for the notations involved, we refer the reader to §4.5.

Remark 3.10. Unfortunately, there are at present no public manuscripts detailing Nori's construction. We have relied mainly on [31], with some additional detail coming from [77]. Hopefully, one of these will soon be available to the public.

A universal construction

A *small diagram* D consists of a set of objects $O(D)$ and for each pair of objects (p,q) a set of morphisms $M(p,q)$ (but no composition law). If $\mathcal{C}$ is a category, a *representation* of D in $\mathcal{C}$, $F : D \to \mathcal{C}$ is given by assigning an object Fp of $\mathcal{C}$ for each $p \in O(D)$, and a morphism $Fm : Fp \to Fq$ in $\mathcal{C}$ for each $m \in M(p,q)$. For a noetherian commutative ring R, we let R-mod denote the abelian category of finite R-modules.

Example 3.11. Let $H_*\mathbf{Sch}_k$ be the diagram with objects the triples (X,Y,i), where X is a k-scheme of finite type, Y a closed subscheme of X and i an integer. There are two types of morphisms: for $f : X \to X'$ a morphism of k-schemes which restricts to a morphism of closed subschemes $Y \to Y'$ (i.e. a morphism of pairs $f : (X,Y) \to (X',Y')$), we have the morphism $f_* : (X,Y,i) \to (X',Y',i)$. For a triple (X,Y,Z) of closed subschemes $X \supset Y \supset Z$, we have the morphism $d : (X,Y,i) \to (Y,Z,i-1)$.

Sending (X,Y,i) to $H_i(X,Y) := H_i(X(\mathbb{C}),Y(\mathbb{C});\mathbb{Z})$, $f_* : (X,Y,i) \to (X',Y',i)$ to $f_* : H_i(X,Y) \to H_i(X',Y')$ and $d : (X,Y,i) \to (Y,Z,i-1)$ to the boundary map $\partial_i : H_i(X,Y) \to H_{i-1}(Y,Z)$ in the long exact homology sequence of the triple $(X(\mathbb{C}),Y(\mathbb{C}),Z(\mathbb{C}))$ defines a represention

$$H_* : H_*\mathbf{Sch}_k \to \mathbf{Ab}.$$

Reversing the arrow f_* to $f^* : (X',Y',i) \to (X,Y,i)$ and changing d to $d : (X,Y,i) \to (Y,Z,i+1)$ gives the cohomological version $H^*\mathbf{Sch}_k$ and the representation

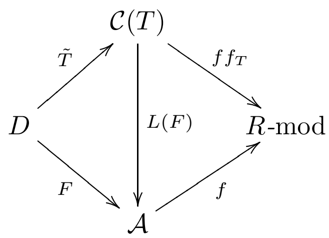
$$H^* : H^*\mathbf{Sch}_k \to \mathbf{Ab},$$

$H^*((X,Y,i)) = H^i(X,Y) := H^i(X(\mathbb{C}),Y(\mathbb{C});\mathbb{Z})$.

The main theorem regarding representations of diagrams is

Theorem 3.12. *Let $T : D \to R$-mod be a representation of a small diagram D. Then there is an R-linear abelian category $\mathcal{C}(T)$, a faithful exact R-linear functor $ff_T : \mathcal{C}(T) \to R$-mod and a representation $\tilde{T} : D \to \mathcal{C}(T)$ such that*

1. $ff_T \circ \tilde{T} = T$
2. *$\tilde{T}$ is universal: if $\mathcal{A}$ is an R-linear abelian category with a faithful exact R-linear functor $f : \mathcal{A} \to R$-mod and $F : D \to \mathcal{A}$ is a representation such that $f \circ F = T$, then there is a unique R-linear functor $L(F) : \mathcal{C}(T) \to \mathcal{A}$ such that the diagram*



commutes.

The construction of $\tilde{T} : D \to \mathcal{C}(T)$ follows the Tannakian pattern: Suppose first that D is a finite set. Let $\mathrm{End}(T)$ be ring of (left) endomorphisms of T, that is, the subset of $\prod_{p \in O(D)} \mathrm{End}_R(Tp)$ consisting of all tuples $e = \prod_p e_p$ such that, for all $m \in M(p, q)$, the diagram

$$\begin{array}{ccc} Tp & \xrightarrow{Tm} & Tq \\ {\scriptstyle e_p}\downarrow & & \downarrow{\scriptstyle e_q} \\ Tp & \xrightarrow[Tm]{} & Tq \end{array}$$

commutes. It is clear that $\mathrm{End}(T)$ is a sub-R-algebra of the product algebra $\prod_{p \in O(D)} \mathrm{End}_R(Tp)$; since each Tp is a finite R-module and D is finite, $\mathrm{End}(T)$ is an R-algebra, finite as an R-module. We let $\mathcal{C}(T)$ be the category of finitely generated $\mathrm{End}(T)$-modules, and $ff_T : \mathcal{C}(T) \to R$-mod the forgetful functor. By construction, each Tp is a left $\mathrm{End}(T)$-module by the projection $\mathrm{End}(T) \to \mathrm{End}_R(Tp)$, and each map $Tm : Tp \to Tq$ is $\mathrm{End}(T)$-linear. This yields the lifting $\tilde{T} : D \to \mathcal{C}(T)$.

In general, we apply the above construction to all finite subsets $O(F)$ of $O(D)$, i.e., to all "finite, full" subdiagrams F of D (where we use the same sets of morphisms $M(p, q)$ for all F). If $F \subset F' \subset D$ are two such finite full subdiagrams, the projection

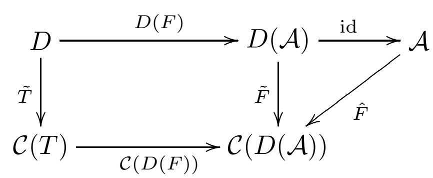
$$\prod_{p \in O(F')} \mathrm{End}_R(Tp) \to \prod_{p \in O(F)} \mathrm{End}_R(Tp)$$

yields a homomorphism $\mathrm{End}(T_{|F'}) \to \mathrm{End}(T_{|F})$, and hence an exact faithful functor $\mathcal{C}(T_{|F}) \to \mathcal{C}(T_{|F'})$. Define

$$\mathcal{C}(T) := \varinjlim_{\text{finite } F\subset D} \mathcal{C}(T_{|F});$$

the forgetful functors $\mathcal{C}(T_{|F}) \to R$-mod and the liftings $\tilde{T}_{|F}$ fit together to give $ff_T : \mathcal{C}(T) \to R$-mod and $\tilde{T} : D \to \mathcal{C}(T)$.

To prove the universality, it suffices to consider the case of a small abelian category $\mathcal{A}$ with a faithful exact functor $f : \mathcal{A} \to R$-mod. Let $D(\mathcal{A})$ be the diagram associated to $\mathcal{A}$, i.e., the objects and morphisms are the same as $\mathcal{A}$, just forget the composition law. The above construction is obviously natural in D, so we have the commutative diagram



with $\hat{F}$ an exact R-linear functor. Nori shows that $\hat{F}$ is an equivalence; an inverse to $\hat{F}$ yields the desired functor $\mathcal{C}(T) \to \mathcal{A}$.

Abelian categories of effective motives

We apply the universal construction to the representations H_* and H^*.

Definition 3.13. Let k be a subfield of $\mathbb{C}$. Let $\mathrm{EHM}(k) = \mathcal{C}(H_*)$ and $\mathrm{ECM}(k) = \mathcal{C}(H^*)$

Nori shows that these categories are independent of the choice of embedding $k \subset \mathbb{C}$. The universal property of the $\mathcal{C}$-construction yields faithful exact functors

$$\begin{aligned} &\mathrm{ECM}(k) \to \mathrm{MHS} \qquad (3)\\ &\mathrm{ECM}(k) \to \mathrm{Gal}(k)\text{-Rep}\\ &\mathrm{ECM}(k) \to \mathrm{Period}(k) \end{aligned}$$

Here MHS is the category of mixed Hodge structures, $\mathrm{Gal}(k)$-Rep is the category of representations of $\mathrm{Gal}(\bar{k}/k)$ on finitely generated abelian groups, and $\mathrm{Period}(k)$ is the category of tuples (L, V, ϕ, ∇), where L is a finitely generated abelian group, V a finite-dimensional k-vector space, $\phi : L \otimes \mathbb{C} \to V \otimes_k \mathbb{C}$ an isomorphism of $\mathbb{C}$-vector spaces and $\nabla : V \to \Omega^1_k \otimes V$ the Gauß-Manin connection, i.e. a $\mathbb{Q}$-linear connection with $\nabla^2 = 0$ and with regular singular points at infinity. Similarly, using Remark 3.9(3), the universal property yields an exact faithful functor

$$\mathrm{ECM}(k) \to \mathrm{JMM}_k$$

The basic lemma and applications

We have the functors H_i from pairs of finite-type k-schemes to $\mathrm{EHM}(k)$; in order to define the total derived functor

$$m : \mathbf{Sch}_k \to D_b(\mathrm{EHM}(k)),$$

Nori shows that affine finite type k-schemes have a type of "cellular decomposition" which, from the point of cohomology, looks like the usual cellular decomposition of a CW-complex. Specifically, the basic result is

Theorem 3.14 ([78]). *Let X be a finite type affine k-scheme of dimension n over $k \subset \mathbb{C}$. Let $Z \subset X$ be a closed subset with $\dim Z \leq n-1$. Then there exists a closed subset Y of X containing Z such that*

1. $\dim Y \leq n-1$
2. $H_i(X, Y) = 0$ *for* $i \neq n$
3. $H_n(X, Y)$ *is a finitely generated abelian group.*

Remark 3.15. Nori has mentioned to me that at the time of his proof of Theorem 3.14, he was unaware that Beilinson had already proven this result (actually, a stronger result, as Beilinson proves the above in characteristic $p > 0$ as well) in [3, Lemma 3.3], by a different argument. He has also remarked that the same method was used by Kari Vilonen in his Harvard University Masters' thesis to prove Artin's comparison theorem.

To construct m, let X be an affine k-scheme of finite type. Applying Theorem 3.14 repeatedly, there is a filtration X_* of X by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_{n-1} \subset X_n = X$$

such that $H_i(X_j, X_{j-1}) = 0$ for $i \neq j$ and $H_j(X_j, X_{j-1})$ is a finitely generated abelian group for all j. Call such a filtration a *good* filtration of X. Form the complex $C_*(X_*)$ with $C_j = H_j(X_j, X_{j-1})$ and with differential the boundary map $H_j(X_j, X_{j-1}) \to H_{j-1}(X_{j-1}, X_{j-2})$. This is clearly a complex in $\mathrm{EHM}(k)$, and is natural in $\mathrm{EHM}(k)$ with respect to morphisms $f : X \to X'$ which are compatible with the chosen filtrations X_* and X'_*.

Let $\lim_\to \mathrm{EHM}(k)$ be the category of ind-systems in $\mathrm{EHM}(k)$, and let $Ch(\lim_\to \mathrm{EHM}(k))$ be the category of chain complexes in $\lim_\to \mathrm{EHM}(k)$ which have bounded homology in $\mathrm{EHM}(k)$. Taking the system of good filtrations X_* of X (or equivalently, all filtrations) yields the functor

$$C_* : \mathbf{Aff}(k) \to Ch(\lim_{\to} \mathrm{EHM}(k)).$$

Passing to the derived categories

$$D(Ch(\lim_{\to} \mathrm{EHM}(k))) \sim D_b(\mathrm{EHM}(k))$$

and using a Čech construction yields the functor

$$m : \mathbf{Sch}_k \to D_b(\mathrm{EHM}(k)).$$

As a second application, replace the diagram category $H_*\mathbf{Sch}_k$ with the diagram category $H_*\mathbf{Sch}'_k$ of "good triples" (X, Y, i), i.e., those having $H_j(X, Y) = 0$ for $j \neq i$, and let $\mathrm{EHM}(k)' = \mathcal{C}(H'_*)$, where $H'_* : H_*\mathbf{Sch}'_k \to \mathbf{Ab}$ is the restriction of H_*. He shows

Proposition 3.16. *The natural map* $\mathrm{EHM}(k)' \to \mathrm{EHM}(k)$ *is an equivalence of abelian categories*

As applications of this result, Nori defines a tensor structure on $\mathrm{EHM}(k)$ by considering the map of diagrams:

$$\times : H_*\mathbf{Sch}'_k \times H_*\mathbf{Sch}'_k \to H_*\mathbf{Sch}'_k$$
$$(X, Y) \times (X', Y') = (X \times_k X', X \times_k Y' \cup Y \times_k X').$$

and the representation $H_* \times H_* : H_*\mathbf{Sch}'_k \times H_*\mathbf{Sch}'_k \to \mathbf{Ab} \times \mathbf{Ab}$. This gives the commutative diagram

$$\begin{array}{ccc} H_*\mathbf{Sch}'_k \times H_*\mathbf{Sch}'_k & \xrightarrow{\times} & H_*\mathbf{Sch}'_k \\ \downarrow{\scriptstyle H_* \times H_*} & & \downarrow{\scriptstyle H_*} \\ \mathbf{Ab} \times \mathbf{Ab} & \xrightarrow[\otimes]{} & \mathbf{Ab} \end{array}$$

Noting that $\mathcal{C}(H_* \times H_*) = \mathrm{EHM}(k)' \times \mathrm{EHM}(k)'$, the universal property of $\mathcal{C}$ yields the exact functor

$$\otimes : \mathrm{EHM}(k)' \times \mathrm{EHM}(k)' \to \mathrm{EHM}(k)';$$

via Proposition 3.16, this gives the tensor product operation

$$\otimes : \mathrm{EHM}(k) \times \mathrm{EHM}(k) \to \mathrm{EHM}(k).$$

Nori constructs a duality functor

$$\vee : \mathrm{EHM}(k)' \to \mathrm{ECM}(k)^{\mathrm{op}}$$

respecting the representations H_* and H^* via the usual duality

$$\mathrm{Hom}(-, \mathbb{Z}) : \mathbf{Ab} \to \mathbf{Ab},$$

by sending a good pair (X, Y, n) to (X, Y, n), noting that

$$H^n(X, Y) = H_n(X, Y)^\vee$$

for a good pair (X, Y). This induces an equivalence on the derived categories

$$\vee : D_b(\mathrm{EHM}(k)) \to D^b(\mathrm{ECM}(k))^{\mathrm{op}}.$$

Finally, using Theorem 3.14, Nori shows that the restriction of C_* to $\mathbf{Sm}_k$ factors through the embedding $\Gamma : \mathbf{Sm}_k \to \mathrm{Cor}(k)$ (see §4.5 for the notation):

Proposition 3.17. *Let $W \subset X \times_k Y$ be an effective finite correspondence, X, Y in $\mathbf{Sm}_k$ and affine. Then there is a map $W_* : C_*(X) \to C_*(Y)$, satisfying*

1. *For a morphism $f : X \to Y$ with graph Γ, $f_* = \Gamma_*$.*
2. *$(W \circ W')_* = W_* \circ W'_*$.*

Using this result, Nori shows that the restriction of m to $\mathbf{Sm}_k$ extends to a functor

$$\Pi : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to D^b(\mathrm{EHM}(k)).$$

making

$$\begin{array}{ccc} & & DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \\ & \nearrow & \\ \mathrm{Cor}(k) & & \downarrow \Pi \\ & \searrow & \\ & & D^b(\mathrm{EHM}(k)) \end{array}$$

commute.

Motives

For a finite subdiagram F of $H_*\mathbf{Sch}_k$, let $A(F)$ be the dual of $\mathrm{End}(H_{*|F})$:

$$A(F) := \mathrm{Hom}(\mathrm{End}(H_{*|F}), \mathbb{Z}).$$

Let A be the limit

$$A := \varinjlim_{F} A(F).$$

The ring structure on $\mathrm{End}(H_{*|F})$ makes $A(F)$ and A into a co-algebra (over $\mathbb{Z}$). Nori shows

Lemma 3.18. $\mathrm{EHM}(k)'$ *is equivalent to the category of left comodules M over A, which are finitely generated as abelian groups.*

The tensor product on $\mathrm{EHM}(k)'$ induces a comultiplication

$$\mathrm{End}(H_{*|F}) \otimes \mathrm{End}(H_{*|F'}) \to \mathrm{End}(H_{*|F \cdot F'}),$$

where $O(F \cdot F')$ is the set of triples of the form $(X, Y, i) \times (X', Y', i')$ for $(X, Y, i) \in F$, $(X', Y', i') \in F'$. This yields a commutative, associative multiplication $A \otimes A \to A$, making A into a bi-algebra over $\mathbb{Z}$.

Let $\mathbb{Z}(1) = H_1(\mathbb{G}_m)$, as an object of $\mathrm{EHM}(k)'$. As a $\mathbb{Z}$-module, $\mathbb{Z}(1) = \mathbb{Z}$. If F is a finite diagram containing $(\mathbb{G}_m, \emptyset, 1)$, then $\mathbb{Z}(1)$ is an $\mathrm{End}(H_{*|F})$-module; sending $a \in \mathrm{End}(H_{*|F})$ to $a \cdot 1 \in \mathbb{Z}(1) = \mathbb{Z}$ determines an element

$$\chi_F \in A(F) = \mathrm{Hom}(\mathrm{End}(H_{*|F}), \mathbb{Z}).$$

The image of χ_F in A is independent of the choice of F, giving the element $\chi \in A$. Let A_χ be the localization of A by inverting χ.

Theorem 3.19. *A_χ is a Hopf algebra.*

Let $G_{\rm mot}(k)$ be the corresponding affine group-scheme Spec A_χ.

Definition 3.20. Nori's category of mixed motives over k, NMM(k), is the category of representations of $G_{\rm mot}(k)$ in finitely generated $\mathbb{Z}$-modules, i.e., the category of co-modules over A_χ which are finitely generated as an abelian group.

Since $\otimes\mathbb{Z}(1) : \mathrm{NMM}(k) \to \mathrm{NMM}(k)$ is invertible, the functor Π : $DM^{\rm eff}_{\rm gm}(k) \to D^b(\mathrm{EHM}(k))$ extends to

$$\Pi : DM_{\rm gm}(k) \to D^b(\mathrm{NMM}(k)). \tag{4}$$

Similarly, the functors (3) and the functor $\mathrm{ECM}(k) \to \mathrm{JMM}_k$ extend to functors on NMM_k.

There are a number of classical conjectures one can restate or generalize using this formalism. For example, Beilinson's conjectures on the existence of an abelian category of mixed motives over k with the desired properties can be restated as

Conjecture 3.21 (restatement of Beilinson's conjecture). The functor $\Pi_{\mathbb{Q}} : DM^{\rm eff}_{\rm gm}(k)_{\mathbb{Q}} \to D^b(\mathrm{EHM}(k)_{\mathbb{Q}})$ is fully faithful.

The Hodge conjecture can be generalized as

Conjecture 3.22. The functor $hs : \mathrm{ECM}(k) \to MHS$ induces a fully faithful functor $\mathrm{NMM}(k)_{\mathbb{Q}} \to \mathrm{MHS}_{\mathbb{Q}}$. Equivalently, the map from the Mumford-Tate group MT to $G_{\rm mot}(k)$ corresponding to hs gives a surjective map MT $\to G_{\rm mot}(k)_{\mathbb{Q}}$. Equivalently, for all V in NMM(k), the map

$$\mathrm{Hom}_{\mathrm{NMM}(k)}(1, V)_{\mathbb{Q}} \to \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{Q}}}(1, hs(V))$$

is surjective.

Suppose that k is finitely generated over $\mathbb{Q}$. Using the universal property of NMM(k) with respect to p-adic étale cohomology, one has an exact functor

$$\mathrm{NMM}(k) \to \mathbb{Q}_p[\mathrm{Gal}(\bar{k}/k)]\text{-mod},$$

equivalently, a homomorphism of $\mathrm{Gal}(\bar{k}/k)$ to the $\mathbb{Q}_p$-points of $G_{\rm mot}(k)$. The Tate conjecture generalizes to

Conjecture 3.23. The image of $\mathrm{Gal}(\bar{k}/k) \to G_{\rm mot}(k)(\mathbb{Q}_p)$ is Zariski dense in $G_{\rm mot}(k)_{\mathbb{Q}_p}$. Equivalently, let $\mathrm{NMM}(k)_{\mathbb{Q}_p}$ be the $\mathbb{Q}_p$-extension of NMM(k). Then the functor

$$\mathrm{NMM}(k)_{\mathbb{Q}_p} \to \mathbb{Q}_p[\mathrm{Gal}(\bar{k}/k)]\text{-mod}$$

induced by $\mathrm{NMM}(k) \to \mathbb{Q}_p[\mathrm{Gal}(\bar{k}/k)]$-mod is fully faithful.

4 Triangulated categories of motives

One can attempt a construction of a *triangulated* category of motives, which should ideally have the properties expected by the derived category of Beilinson's conjectural abelian category of mixed motives. In this direction there are two essentially different approaches: One, due to Huber, is via simultaneous realizations, and the second (the approach used by Hanamura, Levine and Voevodsky) builds a category out of some form of algebraic cycles or correspondences.

The main problem in the second approach is that the composition of arbitrary correspondences is not defined, unless one passes to a suitable equivalence relation. If however, one imposes the equivalence relation first (as in Grothendieck's construction) one would lose most of the extension data that the category of motives is supposed to capture. Thus, one is forced to modify the notion of correspondence in some way so that all compositions are defined. Hanamura, Levine and Voevodsky all use different approaches to solving this problem. The constructions of Levine and Voevodsky both lead to equivalent categories; while it is not at present clear that Hanamura's construction is also equivalent to the other two (at least with $\mathbb{Q}$-coefficients) the resulting $\mathbb{Q}$-motivic cohomology is the same, and so one expects that this category is equivalent as well.

4.1 The structure of motivic categories

All the constructions of triangulated categories of motives enjoy some basic structural properties, which we formulate in this section. We give both a cohomological as well as a homological formulation.

Let A be a subring of $\mathbb{R}$ which is a Dedekind domain. By a *cohomological triangulated category of motives* over a field k with A-coefficients, we mean an A-linear triangulated tensor category $\mathcal{D}$, equipped with a functor

$$h : \mathbf{Sm}_k^{\mathrm{op}} \to \mathcal{D}$$

and *Tate objects* $A(n)$, $n \in \mathbb{Z}$, with the following properties (we write f^* for $h(f)$):

1. *Additivity.* $h(X \amalg Y) = h(X) \oplus h(Y)$.
2. *Homotopy.* The map $p^* : h(X) \to h(X \times \mathbb{A}^1)$ is an isomorphism.
3. *Mayer-Vietoris.* Let $U \cup V$ be a Zariski open cover of $X \in \mathbf{Sm}_k$, $i_U : U \cap V \to U$, $i_V : U \cap V \to V$, $j_U : U \to X$ and $j_V : V \to X$ the inclusions. Then the sequence

$$h(X) \xrightarrow{(j_U^*, j_V^*)} h(U) \oplus h(V) \xrightarrow{(i_U^*, -i_V^*)} h(U \cap V)$$

extends canonically and functorially to a distinguished triangle.

4. *Künneth isomorphism.* For $\alpha \in \mathcal{D}$, write $\alpha(n)$ for $\alpha \otimes A(n)$. There are associative, commutative external products

$$\cup_{X,Y} : h(X)(n) \otimes h(Y)(m) \to h(X \times_k Y)(n+m)$$

which are isomorphisms. $A(0)$ is the unit for the tensor structure. We let

$$\cup_X : h(X) \otimes h(X) \to h(X)$$

be the composition $\delta_X^* \circ \cup_{X,X}$, where $\delta_X : X \to X \times_k X$ is the diagonal.

Remark 4.1. For the definition of an A-linear triangulated tensor category, we refer the reader to [70, Chapter 8A]

5. *Gysin distinguished triangle.* For each closed codimension q embedding in $\mathbf{Sm}_k$, $i : W \to X$, there is a distinguished triangle

$$h(W)(-d)[-2d] \xrightarrow{i_*} h(X) \xrightarrow{j^*} h(X \setminus W) \to h(W)(-d)[1-2d]$$

which is natural in the pair (W, X). Here $j : X \setminus W \to X$ is the inclusion, and "natural" means with respect to both to morphisms of pairs $f : (W', X') \to (W, X)$ such that W' is the pull-back of W, as well as the functoriality $(i_1 \circ i_2)_* = i_{1*} \circ i_{2*}$ for a composition of closed embeddings in $\mathbf{Sm}_k$. Also, if $i : W \to X$ is an open component of X, then i_* is the inclusion of the summand $h(W)$ of $h(X) = h(W) \oplus h(X \setminus W)$.

6. *Cycle classes.* For $X \in \mathbf{Sm}_k$, there are homomorphisms

$$\mathrm{cl}^q : \mathrm{CH}^q(X) \to \mathrm{Hom}_{\mathcal{D}}(A(0), h(X)(q)[2q]).$$

The maps cl^q are compatible with external products, and pull-back morphisms. If $i : W \to X$ is a codimension d closed embedding in $\mathbf{Sm}_k$, and Z is in $\mathrm{CH}^{q-d}(W)$, then $\mathrm{cl}^q(i_*(Z)) = i_* \circ \mathrm{cl}^{q-d}(Z)$.

7. *Unit.* The map $\mathrm{cl}^0([\mathrm{Spec}\, k]) : A(0) \to h(\mathrm{Spec}\, k)$ is an isomorphism.

8. *Motivic cohomology.* For $X \in \mathbf{Sm}_k$, set

$$H^p(X, A(q)) := \mathrm{Hom}_{\mathcal{D}}(A(0), h(X)(q)[p]).$$

As a consequence of the above axioms, the bi-graded group $\oplus_{p,q} H^p(X, A(q))$ becomes a bi-graded commutative ring (with product $\cup_X$), with $H^p(X, A(q))$ in bi-degree $(p, 2q)$. The element $\mathrm{cl}^0(1 \cdot X)$ is the unit.

9. *Projective bundle formula.* Let $\mathcal{E}$ be a rank $n+1$ locally free sheaf on $X \in \mathbf{Sm}_k$ with associated $\mathbb{P}^n$-bundle $\mathbb{P}(\mathcal{E}) \to X$ and invertible quotient tautological sheaf $\mathcal{O}(1)$. Let $c_1(\mathcal{O}(1)) \in \mathrm{CH}^1(\mathbb{P}(\mathcal{E}))$ be the 1st Chern class of $\mathcal{O}(1)$, and set

$$\xi := \mathrm{cl}^1(c_1(\mathcal{O}(1)) \in H^2(\mathbb{P}(\mathcal{E}), A(1)).$$

Letting $\alpha_i : h(X)(-i)[-2i] \to h(\mathbb{P}(\mathcal{E}))$ be the map $(- \cup_{\mathbb{P}(\mathcal{E})} \xi^i) \circ q^*$, the sum

$$\sum_{i=0}^{n} \alpha_i : \oplus_{i=0}^n h(X)(-i)[-2i] \to h(\mathbb{P}(\mathcal{E}))$$

is an isomorphism.

Remarks 4.2. It follows from (4) and (7) that $A(n) \otimes A(m)$ is canonically isomorphic to $A(n+m)$, and thus we have isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(\alpha, \beta) \cong \mathrm{Hom}_{\mathcal{D}}(\alpha(n), \beta(n))$$

for all α, β in $\mathcal{D}$ and all $n \in \mathbb{Z}$.

All the properties of $h(X)$ induce related properties for $H^*(X, A(*))$ by taking long exact sequences associated to $\mathrm{Hom}_{\mathcal{D}}(A(0), -)$.

Using (5) and (9), one can define a push-forward map

$$f_* : h(Y)(-d)[-2d] \to h(X)$$

for a projective morphism $f : Y \to X$ of relative dimension d. For this, one factors f as $q \circ i$, with $i : Y \to \mathbb{P}^n \times X$ a closed embedding and $q : \mathbb{P}^n \times X \to X$ the projection. We use (5) to define i_* and let

$$q_* : h(\mathbb{P}^n \times X) \to h(X)(-n)[-2n]$$

be the inverse of the isomorphism $\sum_{i=0}^n \alpha_i$ of (9) (with $\mathcal{E} = \mathcal{O}_X^{n+1}$) followed by the projection onto $h(X)(-n)[-2n]$. One sets $f_* := q_* \circ i_*$, shows that f_* is independent of the choices made and that $(fg)_* = f_* g_*$. For details on this construction, see, e.g., [63, Part 1, Chap. III, §2].

Remark 4.3. To define a *homological* triangulated category $\mathcal{D}$ of motives over a field k with A-coefficients, one replaces the functor h with an additive functor

$$m : \mathbf{Sm}_k \to \mathcal{D}$$

and denote $m(f)$ by f_*. The properties (1)-(4) remain the same, reversing the arrows in (2) and (3). The Gysin map i_* in (5) becomes $i^* : m(X) \to m(W)(d)[2d]$.

We define $H^p(X, A(q)) := \mathrm{Hom}_{\mathcal{D}}(m(X)(-q)[-p], A(0))$. The cycle classes in (6) become maps $\mathrm{cl}^q : \mathrm{CH}^q(X) \to H^{2q}(X, A(q))$, with the same functoriality and properties as in (6) and (8). The projective bundle formula (9) becomes the isomorphism

$$\sum_{i=0}^n \alpha_i : \oplus_{i=0}^n m(X)(i)[2i] \to m(\mathbb{P}(\mathcal{E})),$$

One uses the projective bundle formula to define a pull-back map

$$q^* : m(X)(n)[2n] \to m(\mathbb{P}(\mathcal{E}))$$

by setting $q^* := \alpha_n$. This allows one to define a functorial Gysin map $f^* : m(X)(d)[2d] \to m(Y)$, for $f : Y \to X$ projective of relative dimension d, as we defined f_* in the cohomological setting.

In short, the opposite of a cohomological category of motives is a homological category of motives, after changing the signs in the Tate objects.

Definition 4.4. Let $\mathcal{D}$ be a cohomological triangulated category of motives over k with A-coefficients. A *duality* on $\mathcal{D}$ is an exact pseudo-tensor functor $\vee : \mathcal{D} \to \mathcal{D}^{\mathrm{op}}$, together with maps $\delta_\alpha : A(0) \to \alpha^\vee \otimes \alpha$, $\epsilon_\alpha : \alpha \otimes \alpha^\vee \to A(0)$ for each α in $\mathcal{D}$, such that

1. For each α in $\mathcal{D}$, $(\epsilon_\alpha \otimes \mathrm{id}_\alpha) \circ (\mathrm{id}_\alpha \otimes \delta_\alpha) = \mathrm{id}_\alpha$ and $(\mathrm{id}_\alpha \otimes \epsilon_\alpha) \circ (\delta_\alpha \otimes \mathrm{id}_\alpha) = \mathrm{id}_{\alpha^\vee}$.
2. For each smooth projective X of dimension d over k, $h(X)^\vee = h(X)(d)[2d]$ and $\delta_{h(X)}$ and $\epsilon_{h(X)}$ are the compositions of pull-back and push-forward for
$$\mathrm{Spec}\,(k) \leftarrow X \to \Delta_X X \times_k X$$

In the homological case, we just change (2) to

$$m(X)^\vee = m(X)(-d)[-2d]$$

Remark 4.5. A duality on $\mathcal{D}$ is a duality in the usual sense of tensor categories, that is, for each α, β in $\mathcal{D}$, $\alpha^\vee \otimes \beta$ is an internal Hom object in $\mathcal{D}$. In fact, the map
$$\mathrm{Hom}_{\mathcal{D}}(\alpha \otimes \gamma, \beta) \to \mathrm{Hom}_{\mathcal{D}}(\gamma, \alpha^\vee \otimes \beta)$$
induced by sending $f : \alpha \otimes \gamma \to \beta$ to the composition

$$\gamma \cong A(0) \otimes \gamma \xrightarrow{\delta \otimes \mathrm{id}} \alpha^\vee \otimes \alpha \otimes \gamma \xrightarrow{\mathrm{id} \otimes f} \alpha^\vee \otimes \beta$$

is an isomorphism for all α, β and γ, with inverse similarly constructed using ϵ_α instead of δ_α. For details, see [63, Part 1, Chap. IV, §1]

Definition 4.6. Let $\mathcal{D}$ be a (co)homological triangulated category of motives over k, with coefficients in A. We say that $\mathcal{D}$ is a *fine* category of motives if, for each $X \in \mathbf{Sm}_k$ there are homomorphisms

$$\mathrm{cl}^{p,q} : \mathrm{CH}^q(X, 2q-p) \to H^p(X, A(q))$$

satisfying:

1. $\mathrm{cl}^{2q,q} = \mathrm{cl}^q$
2. The maps $\mathrm{cl}^{p,q}$ are functorial with respect to pull-back, products and push-forward for closed embeddings in $\mathbf{Sm}_k$.
3. The maps $\mathrm{cl}^{*,q}$ commute with the boundary maps in the Mayer-Vietoris sequences for $H^*(-, A(q))$ and $\mathrm{CH}^q(-, 2q-*)$.
4. The A-linear extension of $\mathrm{cl}^{p.q}$,
$$\mathrm{cl}^{p,q}_A : \mathrm{CH}^q(X, 2q-p) \otimes A \to H^p(X, A(q))$$
is an isomorphism for all $X \in \mathbf{Sm}_k$ and all p, q.

An overview

We give below sketches of four constructions of triangulated categories of motives, due to Huber, Hanamura, Levine and Voevodsky. Huber's construction yields a cohomological triangulated category of motives over k with $\mathbb{Q}$-coefficients, with duality. Hanamura's construction, assuming k admits resolution of singularties, yields a fine cohomological triangulated category of motives over k with $\mathbb{Q}$-coefficients, with duality. Levine's construction yields a fine cohomological triangulated category of motives over k with $\mathbb{Z}$-coefficients; the category has duality if k admits resolution of singularities. Voevodsky's construction yields a fine homological triangulated category of motives over k with $\mathbb{Z}$-coefficients; the category has duality if k admits resolution of singularties. In addition, if if k admits resolution of singularties, Levine's category is equivalent to Voevodsky's category.

4.2 Huber's construction

Let k be a field finitely generated over $\mathbb{Q}$. Huber's construction of a triangulated category of mixed motives over k [48] is, roughly speaking, a combination of Jannsen's abelian category MR_k of compatible realizations, and Beilinson's category of mixed Hodge complexes [4]. In somewhat more detail, Huber considers compatible systems of bounded below complexes and comparison maps

$$
\begin{aligned}
&((C_{\mathrm{DR}}, W_*, F^*)(C_\ell, W_*), (C_\sigma, W_*), (C'_\sigma, W_*), (C'_\ell, W_*))\\
&I_\sigma : C_{\mathrm{DR}} \otimes_k \mathbb{C} \to C'_\sigma\\
&I'_\sigma : C_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \to C'_\sigma\\
&I'_{\ell,\sigma} : C_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to C'_\ell\\
&I_{\ell,\sigma} : C_\ell \to C'_\ell
\end{aligned}
$$

where

1. σ runs over embedding $k \to \mathbb{C}$ and ℓ runs over prime numbers.
2. C_{DR} is a bounded below complex of finite dimensional bi-filtered k-vector spaces, with strict differentials. W_* is an increasing filtration and F^* is a decreasing filtration
3. C_σ (resp. C'_σ) is a bounded below complex of finite dimension $\mathbb{Q}$-vector spaces (resp. $\mathbb{C}$-vector spaces), with decreasing filtration W_* and with strict differentials.
4. C'_ℓ (resp. C_ℓ) is a bounded below complex of finite dimension $\mathbb{Q}_\ell$-vector spaces (resp. with continuous G_k-action), with decreasing filtration W_* and with strict differentials.
5. I_σ, I'_σ $I'_{\ell,\sigma}$ and $I_{\ell,\sigma}$ are filtered quasi-isomorphisms of complexes (with respect to the W-filtrations).
6. For each n, the tuple of cohomologies $(H^n(C_{\mathrm{DR}}), H^n(C_\ell), H^n(C_\sigma))$ with the induced filtrations is an object in MR_k, where we give $H^n(C_\sigma) \otimes \mathbb{C}$ the Hodge filtration induced from the F-filtration on $H^n(C_{\mathrm{DR}})$.

7. The G_k-module $H^n(C_\ell)$ is *mixed* (we don't define this term here, see [48, Definition 9.1.4] for a precise definition. Roughly speaking, the G_k-action should arise from an inverse system of actions on finitely generated $\mathbb{Z}/\ell^\nu$-modules and $\mathrm{gr}^m_W H^n(C_\ell))$ should be pure of weight m for almost all Frobenius elements in G_k).

Inverting quasi-isomorphisms of tuples yields Huber's triangulated tensor category of mixed motives, $\mathcal{D}_{\mathrm{MR}}(k)$.

The category $\mathcal{D}_{\mathrm{MR}}(k)$ has the structural properties given in §4.1 for a cohomological triangulated category of motives over k with $\mathbb{Q}$-coefficients, and with duality. In addition, the functor $h : \mathbf{Sm}^o_k p \to \mathcal{D}_{\mathrm{MR}}(k)$ extends to smooth simplicial schemes over k. This extension is important in the applications given by Huber and Wildeshaus [51] to the Tamagawa number conjecture of Bloch and Kato.

4.3 Hanamura's construction

We give a sketch of Hanamura's construction of the category $\mathcal{D}(k)$ as the pseudo-abelianization of a subcategory $\mathcal{D}_{\mathrm{fin}}(k)$; in [46] $\mathcal{D}(k)$ is constructed as a subcategory of a larger category $\mathcal{D}_{inf}(k)$, which we will not describe here.

The basic object is a *higher correspondence*: Let X and Y be irreducible smooth projective varieties over k. Let

$$\mathrm{HCor}((X,n),(Y,m))^a := z^{m-n+\dim_k X}(X\times Y,-a)^{\mathrm{Alt}}.$$

For irreducible $W \in \mathrm{HCor}((X,n),(Y,m))^a$, $W' \in \mathrm{HCor}((Y,m),(Z,l))^b$ we say that $W'\circ W$ is defined if the external product $W \cup_{X\times Y, Y\times Z} W'$ is in the subcomplex

$$z^{l-n+\dim_k X+\dim_k Y}(X\times Y\times Y\times Z,*)^{\mathrm{Alt}}_{\mathrm{id}_X\times\delta_Y\times\mathrm{id}_Z} \subset z^{l-n+\dim_k X+\dim_k Y}(X\times Y\times Y\times Z,*)^{\mathrm{Alt}}.$$

In general, if $W=\sum_i n_i W_i$, $W'=\sum_i m_j W'_j$, we say $W'\circ W$ is defined if $W'_j\circ W_i$ is defined for all i,j.

If $W'\circ W$ is defined, we set

$$W'\circ W := p_{X\times Z*}((\mathrm{id}_X\times\delta_Y\times\mathrm{id}_Z)^*(W\cup_{X\times Y,Y\times Z} W'))$$

The definition of the complex HCor is extended to *formal symbols*, i.e. finite formal sums $\oplus_\alpha(X_\alpha,n_\alpha)$, by the formula

$$\mathrm{HCor}(\oplus_\alpha(X_\alpha,n_\alpha),\oplus_\beta(Y_\beta,m_\beta)) := \prod_\alpha \oplus_\beta \mathrm{HCor}((X_\alpha,n_\alpha),(Y_\beta,m_\beta)).$$

0 is the empty sum. We let 1 denote the formal symbol $(\mathrm{Spec}\, k,0)$.

If $K=\oplus_\alpha(X_\alpha,n_\alpha)$ is a formal symbol, we set $z^0(K) := \mathrm{HCor}(1,K)$. Thus, $z^0((X,n))^* = z^n(X,-*)$. We set $(X,n)^\vee := (X,\dim_k X - n)$ and extend to formal symbols by linearity. Similarly, we define a tensor product operation $K\otimes L$ as the bilinear extension of $(X,n)\otimes(Y,m) := (X\times_k Y, n+m)$.

Definition 4.7. A *diagram* $K := (K^m, f^{m,n})$ consists of formal symbols K^m, $m \in \mathbb{Z}$, together with elements $f^{m,n} \in \mathrm{HCor}(K^n, K^m)^{n-m+1}$, $n < m$ such that:

1. For all but finitely many m, $K^m = 0$.
2. For all sequences $m_1 < \ldots < m_s$ the composition $f^{m_s,m_{s-1}} \circ \ldots \circ f^{m_2,m_1}$ is defined.
3. For all $n < m$ we have the identity
$$(-1)^m \partial f^{m,n} + \sum_l f^{m,l} \circ f^{l,n} = 0.$$

Here ∂ is the differential in the complex $\mathrm{HCor}(K^n, K^m)^*$.

The yoga of duals and tensor products of usual complexes in an additive category extend to give operations $K \mapsto K^\vee$ and $(K, L) \mapsto K \otimes L$ for diagrams; we refer the reader to [63, Part 2, Chap II, §1] or [46] for detailed formulas.

The diagrams (resp. finite diagrams) are objects in a triangulated category $\mathcal{D}_{\mathrm{fin}}(k)$. In order to describe the morphisms $\mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(K, L)$, we need the notion of a *distinguished subcomplex* of $z^p(X, *)^{\mathrm{Alt}}$.

Definition 4.8. Let X be a smooth projective variety. A distinguished subcomplex of $z^p(X, *)^{\mathrm{Alt}}$ is a subcomplex of the form $z^p(X, *)^{\mathrm{Alt}}_f$ for some projective map $f : Y \to X$ in $\mathbf{Sch}_k$, with Y locally equi-dimensional over k. If $K = \oplus_\alpha (X_\alpha, n_\alpha)$ is a formal symbol, a distinguished subcomplex of $z^0(K)$ is a subcomplex of the form $\oplus_\alpha z^{n_\alpha}(X_\alpha, -*)_{f_\alpha}$, with $f_\alpha : Y_\alpha \to X_\alpha$ as above.

For $f : (X, n) \to (Y, m)$ in $\mathrm{HCor}((X, n), (Y, m))^*$, we say that f is defined on a distinguished subcomplex $z^0((X, n))' := z^n(X, -*)'$ if $f \circ \eta$ is defined for all $\eta \in z^n(X, -*)'$ (where we identify $z^n(X, -*)$ with $\mathrm{HCor}(1, (X, n))$. This notion extends in the evident manner to $f \in \mathrm{HCor}(K, K')$ for formal symbols K and K'.

Let $K = (K^m, f^{m,n})$ be a diagram. A collection of distinguished subcomplexes $m \mapsto z^0(K^m)'$ is *admissible* for K if, for each sequence $m_1 < \ldots < m_s$, the correspondence $f_*^{m_s,m_{s-1}} \circ \ldots \circ f_*^{m_2,m_1}$ is defined on $z^0(K^{m_1})'$ and maps $z^0(K^{m_1})'$ to $z^0(K^{m_s})'$. If a collection $m \mapsto z^0(K^m)'$ is admissible for K, we define the corresponding cycle complex for K, $z^0(K)'$ by
$$z^0(K)'^j := \oplus_i z^0(K^i)'^{j+i}$$
with differential $d^j : z^0(K)'^j \to z^0(K)'^{j+1}$ given by
$$d^j := \sum_i (-1)^i \partial_i + \sum_{i,i'} f_*^{i',i}.$$
Here ∂_i is the differential in $z^0(K^i)^*$.

Lemma 4.9. *For each formal symbol K, there is an admissible collection of distinguished complexes, and two different choices of such admissible collections, $m \mapsto z^0(K^m)'$ and $m \mapsto z^0(K^m)''$, result in canonically quasi-isomorphic cycle complexes $z^0(K)'$ and $z^0(K)''$.*

Thus, we may denote by $z^0(K)$ the image of a $z^0(K)'$ in $D(\mathbf{Ab})$.

Definition 4.10. Let K and L be diagrams. Set

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(K, L) := H^0(z^0(K^\vee \otimes L)).$$

Unwinding this definition, we see that the complex $z^0(K^\vee \otimes L)$ is built out of the complexes $\mathrm{HCor}(K^m, L^n)$, and so a morphism $\phi : K \to L$ is built out of higher correspondences $\phi^{n,m} : K^m \to L^n$, which satisfy some additional compatibility conditions. In particular, the composition of higher correspondences induces an associative composition

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(L, M) \otimes \mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(K, L) \to \mathrm{Hom}_{\mathcal{D}_{\mathrm{fin}}(k)}(K, M)$$

One mimics the definition of the translation operator and cone operator of complexes (this type of extension was first considered by Kapranov in the construction of the category of complexes over a DG-category, see [58], [63] or [46] for details).

Theorem 4.11 (Hanamura, [47], [46]). *The category $\mathcal{D}_{\mathrm{fin}}(k)$ with the above structures of shift, cone sequence, dual and tensor product is a rigid triangulated tensor category.*

"Rigid" means that, setting $\mathcal{H}om(K, L) := K^\vee \otimes L$, the objects $\mathcal{H}om(K, L)$ form an internal Hom object in $\mathcal{D}_{\mathrm{fin}}(k)$.

Definition 4.12. The triangulated tensor category $\mathcal{D}(k)$ is the pseudo-abelian hull of $\mathcal{D}_{\mathrm{fin}}(k)$.

By the results of [1], $\mathcal{D}(k)$ has a canonical structure of a triangulated tensor category.

For X a smooth projective k-scheme, set $\mathbb{Q}_X(n) := (X, n)[-2n]$; we write $\mathbb{Q}(n)$ for $\mathbb{Q}_{\mathrm{Spec}\, k}(n)$. More or less by construction we have

$$\mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Q}(0), \mathbb{Q}_X(n)[m]) = H_{2n-m}(z^n(X, *)^{\mathrm{Alt}}) = \mathrm{CH}^n(X, 2n-m)_{\mathbb{Q}}. \quad (5)$$

Sending X to $\mathbb{Q}_X(0) := h(X)$ defines a functor

$$h : \mathbf{SmProj}_k^{\mathrm{op}} \to \mathcal{D}(k),$$

where $\mathbf{SmProj}_k$ is the full subcategory of $\mathbf{Sch}_k$ with objects the smooth projective k-schemes.

Remark 4.13. Suppose that k admits resolution of singularities, and let X be a smooth irreducible quasi-projective k-scheme of dimension n. Let $\bar{X} \supset X$ be a smooth projective k-scheme containing X as a dense open subscheme, such that the complement $D := \bar{X} \setminus X$ is a strict normal crossing divisor.

Write $D = \sum_{i=1}^m D_i$, with the D_i irreducible. For $I \subset \{1, \ldots, m\}$ let $D_I := \cap_{i \in I} D_i$, and let $D^{(i)} = \coprod_{|I|=i} D_I$ (so $D^{(0)} = \bar{X}$).

Consider the diagram $(X, \bar{X}) :=$

$$(D^{(n)}, -n) \to (D^{(n-1)}, -n+1) \to \ldots \to (D^{(1)}, -1)) \to (D^{(0)}, 0)$$

where the correspondence $(D^{(i)}, -i) \to (D^{(i-1)}, -i+1)$ is the signed sum of inclusions $i_{I,j} : D_{I \cup \{j\}} \to D_I$, $|I| = i-1$, $j \notin I$, and the sign is $(-1)^r$ if there are exactly r elements $i \in I$ with $i < j$. Hanamura [44] shows that $(X, \bar{X})$ in $\mathcal{D}(k)$ is independent of the choice of $\bar{X}$ (up to canonical isomorphism) and that sending X to $\mathbb{Q}_X(0) := (X, \bar{X})$ extends the functor h on $\mathbf{SmProj}_k$ to

$$h : \mathbf{Sm}_k^{\mathrm{op}} \to \mathcal{D}(k).$$

The identification (5) extends to a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Q}(0), h(X)(n)[m]) \cong \mathrm{CH}^n(X, 2n-m)$$

for $X \in \mathbf{Sm}_k$.

Using the method of cubical hyperresolutions [41], Hanamura [44] extends h further to a functor

$$h : \mathbf{Sch}_k^{\mathrm{op}} \to \mathcal{D}(k).$$

In any case, assuming resolution of singularities for k, the category $\mathcal{D}(k)$ is a fine cohomological triangulated category of motives over k, with $\mathbb{Q}$ coefficients and with duality.

Remark 4.14. In [45], Hanamura shows that, assuming the standard conjectures of Grothendieck along with extensions by Murre and Soulé-Beilinson, there is t-structure on $\mathcal{D}(k)$ whose heart $\mathcal{H}$ is a good candidate for $\mathcal{MM}_k$. It is not clear what relation $\mathcal{H}$ has to say Nori's category NMM_k.

4.4 Levine's construction

Rather than using the moving lemma for the complexes $z^p(X, *)$ as above, Levine adds extra data to the category $\mathbf{Sm}_k$ so that pull-back of cycles becomes a well-defined operation.

The category $\mathcal{L}(k)$

Definition 4.15. Let $\mathcal{L}(k)$ be the category of pairs $(X, f : X' \to X)$ where

1. f is a morphism in $\mathbf{Sm}_k$.
2. f admits a smooth section $s : X \to X'$

The choice of the section s is not part of the data. For $(X, f : X' \to X)$ and $(Y, g : Y' \to Y)$ in $\mathcal{L}(k)$, $\mathrm{Hom}_{\mathcal{L}(k)}((X, f), (Y, g))$ is the subset of $\mathrm{Hom}_{\mathbf{Sm}_k}(X, Y)$ consisting of those maps $h : X \to Y$ such that there exists a *smooth* morphism $q : X' \to Y'$ making

$$\begin{array}{ccc} X' & \xrightarrow{q} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[h]{} & Y \end{array}$$

commute. Composition is induced by the composition in $\mathbf{Sm}_k$.

The condition that $f : X' \to X$ admit a smooth section is just saying that X' admits a decomposition as a disjoint union $X' = X'_0 \coprod X'_1$ where f restricted to X'_0 is an isomorphism $X'_0 \cong X$.

Definition 4.16. For $(X, f : X' \to X) \in \mathbf{Sm}_k$, let $z^q(X)_f$ be the subgroup of $z^q(X)$ generated by integral codimension q closed subschemes $W \subset X$ such that

$$\mathrm{codim}_{X'} f^{-1}(W) \geq q.$$

The basic fact that makes things work is

Lemma 4.17. *Let $h : (X, f) \to (Y, g)$ be a morphism in $\mathcal{L}(k)$. Then*

1. *h^* is defined on $z^q(X)_f$, i.e. for all $W \in z^q(X)_f$,*

$$\mathrm{codim}_Y h^{-1}(\mathrm{supp}\,(W)) \geq q.$$

2. *h^* maps $z^q(X)_f$ to $z^q(Y)_g$.*

The proof is elementary.

The category $\bar{\mathcal{A}}_{\mathbf{mot}}(k)$

We use the cycle groups $z^q(X)_f$ to construct a graded tensor category $\bar{\mathcal{A}}_{\mathrm{mot}}(k)$ in a series of steps.

(i) $\mathcal{A}_1(k)$ has objects $\mathbb{Z}_X(n)_f[m]$ for $(X, f) \in \mathcal{L}(k)$, X irreducible, and $n, m \in \mathbb{Z}$. The morphism-groups are given by

$$\mathrm{Hom}_{\mathcal{A}_1(k)}(\mathbb{Z}_X(n)_f[m], \mathbb{Z}_Y(n')_g[m']) = \begin{cases} \mathbb{Z}[\mathrm{Hom}_{\mathcal{L}(k)}((Y, g), (X, f))] & \text{if } n = n',\ m = m' \\ 0 & \text{otherwise.} \end{cases}$$

We also allow finite formal direct sums, with the Hom-groups defined for such sums in the evident manner. The composition is induced from the composition in $\mathcal{L}(k)$. We write $\mathbb{Z}_X(n)_f$ for $\mathbb{Z}_X(n)_f[0]$, $\mathbb{Z}_X(n)$ for $\mathbb{Z}_X(n)_{\mathrm{id}}$ and $\mathbb{Z}(n)$ for $\mathbb{Z}_{\mathrm{Spec}\, k}(n)$. For $p : (Y, m) \to (X, n)$ in $\mathcal{L}(k)$, we write the corresponding morphism in $\mathcal{A}_1(k)$ as $p^* : \mathbb{Z}_X(n)_f \to \mathbb{Z}_Y(m)_g$.

Setting $\mathbb{Z}_X(n)_f \otimes \mathbb{Z}_Y(m)_g := \mathbb{Z}_{X \times_k Y}(n + m)_{f \times g}$ extends to give $\mathcal{A}_1(k)$ the structure of a tensor category, graded with respect to the shift operator $\mathbb{Z}_X(n)_f[m] \mapsto \mathbb{Z}_X(n)_f[m + 1]$.

(ii) $\mathcal{A}_2(k)$ is formed from $\mathcal{A}_1(k)$ by adjoining (as a graded tensor category) an object $*$ and morphisms

$$[Z] : * \to \mathbb{Z}_X(n)_f[2n]$$

for each $Z \in z^n(X)_f$, with the relations:

1. $[aZ + bW] = a[Z] + b[W]$; $Z, W \in z^n(X)_f$, $a, b \in \mathbb{Z}$.
2. $p^* \circ [Z] = [p^*(Z)]$ for $p : (Y, g) \to (X, f)$ in $\mathcal{L}(k)$ and $Z \in z^n(X)_f$.
3. The exchange involution $\tau_{*,*} : * \otimes * \to * \otimes *$ is the identity.
4. For $Z \in z^n(X)_f$, $W \in z^m(Y)_g$, $[Z] \otimes [W] = [\mathrm{Spec}\, k] \otimes [Z \times W] = [Z \times W] \otimes [\mathrm{Spec}\, k]$ as maps $* \otimes * \to \mathbb{Z}_{X \times_k Y}(n + m)_{f \times g}[2n + 2m]$.

(iii) $\bar{\mathcal{A}}_{\mathrm{mot}}(k)$ is the full additive subcategory of $\mathcal{A}_2(k)$ with objects sums of $*^{\otimes m} \otimes \mathbb{Z}_X(n)_f$, $m \geq 0$.

The categories $D^b_{\mathrm{mot}}(k)$ and $\mathcal{DM}(k)$

Let $C^b(\bar{\mathcal{A}}_{\mathrm{mot}}(k))$ be the category of bounded complexes over $\bar{\mathcal{A}}_{\mathrm{mot}}(k)$, and $K^b_{\mathrm{mot}}(k)$ the homotopy category $K^b(\bar{\mathcal{A}}_{\mathrm{mot}}(k))$. $K^b_{\mathrm{mot}}(k)$ is a triangulated tensor category, where the shift operator and distinguished triangles are the usual ones. Note that one needs to modify the definition of morphisms in $C^b(\bar{\mathcal{A}}_{\mathrm{mot}}(k))$ slightly to allow one to identify the shift in $\bar{\mathcal{A}}_{\mathrm{mot}}(k)$ with the usual shift of complexes (see [63, Part 2, Chap. II, §1.2] for details). The tensor product in $\bar{\mathcal{A}}_{\mathrm{mot}}(k)$ makes $K^b_{\mathrm{mot}}(k)$ a triangulated tensor category.

To form $D^b_{\mathrm{mot}}(k)$, we localize $K^b_{\mathrm{mot}}(k)$; we first need to introduce some notation.

Let $(X, f : X' \to X)$ be in $\mathcal{L}(k)$, let $W \subset X$ be a closed subset, and $j : U \to X$ be the open complement. Define $\mathbb{Z}^W_X(n)_f$ by

$$\mathbb{Z}^W_X(n)_f := \mathrm{Cone}(j^* : \mathbb{Z}_X(n)_f \to \mathbb{Z}_U(n)_{f_U})[-1],$$

where $f_U : U' \to U$ is the projection $U \times_X X' \to U$. If Z is in $z^n(X)_f$ and is supported in W, then $j^*Z = 0$, so the morphism $[Z] : * \to \mathbb{Z}_X(n)_f[2n]$ lifts canonically to the morphism

$$[Z]^W : * \to \mathbb{Z}^W_X(n)_f[2n]$$

(in $C^b(\mathcal{A}_2(k))$).

If $\mathcal{C}$ is a triangulated category, and S a collection of morphisms, we let $\mathcal{C}[S^{-1}]$ be the localization of $\mathcal{C}$ with respect to the thick subcategory generated by objects $\mathrm{Cone}(f)$, $f \in S$. If $\mathcal{C}$ is a triangulated tensor category, let $\mathcal{C}[S^{-1}]_\otimes$ be the triangulated tensor category formed by localizing $\mathcal{C}$ with respect to the small thick subcategory containing the object $\mathrm{Cone}(f)$, $f \in S$, and closed under $\otimes X$ for X in $\mathcal{C}$; $\mathcal{C}[S^{-1}]_\otimes$ is a triangulated tensor category, called the triangulated tensor category formed from $\mathcal{C}$ by inverting the morphisms in S. We can extend these notions to inverting finite zig-zag diagrams by taking the cone of the direct sum of the sources mapping to the direct sum of the targets.

Definition 4.18. Let $D^b_{\mathrm{mot}}(k)$ be the triangulated tensor category formed from $K^b_{\mathrm{mot}}(k)$ by localizing as a triangulated tensor category:

1. *Homotopy.* For all X in $\mathbf{Sm}_k$, invert the map $p^* : \mathbb{Z}_X(n) \to \mathbb{Z}_{X \times \mathbb{A}^1}(n)$.

2. *Nisnevic excision.* Let $(X, f : X' \to X)$ be in $\mathcal{L}(k)$, and let $p : Y \to X$ be an étale map, $W \subset X$ a closed subset such that $p : p^{-1}(W) \to W$ is a isomorphism. Then invert the map $p^* : \mathbb{Z}_X^W(n) \to \mathbb{Z}_Y^{p^{-1}(W)}(n)$.
3. *Unit.* Invert the map $[\operatorname{Spec} k] \otimes \mathrm{id}^* : * \otimes 1 \to 1 \otimes 1 = 1$.
4. *Moving lemma.* For all (X, f) in $\mathcal{L}(k)$, invert the map $\mathrm{id}^* : \mathbb{Z}_X(n)_f \to \mathbb{Z}_X(n)$.
5. *Gysin isomorphism.* Let $q : P \to X$ be a smooth morphism in $\mathbf{Sm}_k$ with section s and let $W = s(X) \subset P$. Let $d = \dim_X P$. Invert the zig-zag diagram:

$$\mathbb{Z}_X(-d)[-2d] \xrightarrow{q^*} \mathbb{Z}_P(-d)[-2d] = 1 \otimes \mathbb{Z}_P(-d)[-2d]$$
$$\xleftarrow{[\operatorname{Spec} k]\otimes \mathrm{id}} * \otimes \mathbb{Z}_P(-d)[-2d] \xrightarrow{[W]^W \otimes \mathrm{id}} \mathbb{Z}_P^W(d)[2d] \otimes \mathbb{Z}_P(-d)[-2d]$$
$$= \mathbb{Z}_{P\times P}^{W\times P}(0) \xleftarrow{\mathrm{id}^*} \mathbb{Z}_{P\times P}^{W\times P}(0)_{\delta_P} \xrightarrow{\delta_P^*} \mathbb{Z}_P^W(0),$$

where $\delta_P : P \to P \times P$ is the diagonal.

Remark 4.19. Our description of $D^b_{\mathrm{mot}}(k)$ is slightly different than that given in [63], but yields an equivalent triangulated tensor category $D^b_{\mathrm{mot}}(k)$. The categroy $\bar{\mathcal{A}}_{\mathrm{mot}}(k)$ described here is denoted $\mathcal{A}^0_{\mathrm{mot}}(\mathcal{V})^*$ in [63].

The category $\mathcal{DM}(k)$ is now defined as the pseudo-abelianization of $D^b_{\mathrm{mot}}(k)$. By [1], $\mathcal{DM}(k)$ inherits the structure of a triangulated tensor category from $D^b_{\mathrm{mot}}(k)$.

Gysin isomorphism

Let $i : W \to X$ be a codimension d closed embedding in $\mathbf{Sm}_k$. If i is split by a smooth morphism $p : X \to W$, one uses the Gysin isomorphism of Definition 4.18(5) to define $i_* : \mathbb{Z}_W(-d)[-2d] \to \mathbb{Z}_X(0)$; in general one uses the standard method of deformation to the normal bundle to reduce to this case.

Duality

Assuming that k admits resolution of singularities, the category $\mathcal{DM}(k)$ has a an exact pseudo-tensor duality involution $\vee : \mathcal{DM}(k) \to \mathcal{DM}(k)^{\mathrm{op}}$; for smooth projective X of dimension d over k one has

$$(\mathbb{Z}_X(n))^\vee = \mathbb{Z}_X(d-n)[2d].$$

To construct $\vee$, the method of [63] is to note that, in a tensor category $\mathcal{C}$, the dual of an object X can be viewed as a triple $(X^\vee, \delta, \epsilon)$ with $\delta : 1 \to X \otimes X^\vee$, $\epsilon : X^\vee \otimes X \to 1$, and with

$$(\epsilon \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \delta) = \mathrm{id}_X.$$

In $\mathcal{DM}(k)$, for X smooth and projective of dimension d over k, the diagonal $[\Delta] : 1 \to \mathbb{Z}_{X\times X}(d)[2d]$ gives δ, and ϵ is the composition $p_{X*} \circ \delta_X^*$, where $\delta_X : X \to X \times X$ is the diagonal inclusion and $p_X : X \to \operatorname{Spec} k$ is the structure morphism. One then shows

Lemma 4.20 ([63, Part 1, Chap. IV, lemma 1.2.3]). *Let D be a triangulated tensor category, S a collection of objects of D. Suppose that*

1. *There is a tensor category $\mathcal{C}$ such that D is the localization of $K^b(\mathcal{C})$ (as a triangulated tensor category.*
2. *Each $X \in S$ admits a dual $(X^\vee, \delta, \epsilon)$.*

Then the smallest triangulated tensor subcategory of D containing S, $D(S)$, admits a duality involution $\vee : D(S) \to D(S)^{\mathrm{op}}$, extending the given duality on S.

If k satisfies resolution of singularities, the motives $\mathbb{Z}_X(n)$, X smooth and projective over k, $n \in \mathbb{Z}$, generate $D^b_{\mathrm{mot}}(k)$ as a tensor triangulated category, so the given duality extends to $D^b_{\mathrm{mot}}(k)$, and then to the pseudo-abelianization $\mathcal{DM}(k)$. As for Hanamura's category, the functor $h : \mathbf{Sm}_k^{\mathrm{op}} \to \mathcal{DM}(k)$, $h(X) := \mathbb{Z}_X(0)$, extends to

$$h : \mathbf{Sch}_k^{\mathrm{op}} \to \mathcal{DM}(k),$$

assuming$k$ satisfies resolution of singularities.

Summing up, the category $\mathcal{DM}(k)$ is fine cohomological triangulated category of motives over k with $\mathbb{Z}$-coefficients. $\mathcal{DM}(k)$ has duality if k admits resolution of singularities.

4.5 Voevodsky's construction

Voevodsky constructs a number of categories: the category of geometric motives $DM_{gm}(k)$ with its effective subcategory $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, as well as a sheaf-theoretic construction DM^{eff}_{-}, containing $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ as a full dense subcategory. In contrast to almost all other constructions, these are based on *homology* rather than cohomology as the starting point, in particular, the motives functor from $\mathbf{Sm}_k$ to these categories is covariant.

To solve the problem of the partially defined composition of correspondences, Voevodsky introduces the notion of *finite* correspondences, for which all compositions are defined.

Finite correspondences and geometric motives

Definition 4.21. Let X and Y be in $\mathbf{Sm}_k$. The group $c(X, Y)$ is the subgroup of $z^{\dim_k X}(X \times_k Y)$ generated by integral closed subschemes $W \subset X \times_k Y$ such that

1. the projection $p_1 : W \to X$ is finite

2. the image $p_1(W) \subset X$ is an irreducible component of X.

The elements of $c(X,Y)$ are called the *finite* correspondences from X to Y.

The following basic lemma is easy to prove:

Lemma 4.22. *Let X, Y and Z be in $\mathbf{Sm}_k$, $W \in c(X,Y)$, $W' \in c(Y,Z)$. Suppose that X and Y are irreducible. Then each irreducible component C of $|W| \times Z \cap X \times |W'|$ is finite over X (via the projection p_1) and $p_1(C) = X$.*

It follows from this lemma that, for $W \in c(X,Y)$, $W' \in c(Y,Z)$, we may define the composition $W' \circ W \in c(X,Z)$ by

$$W' \circ W := p_*(p_3^*(W) \cdot p_1^*(W')),$$

where $p_1 : X \times_k Y \times_k Z \to X$ and $p_3 : X \times_k Y \times_k Z \to Z$ are the projections, and $p : |W| \times Z \cap X \times |W'| \to X \times_k Z$ is the map induced by the projection $p_{13} : X \times_k Y \times_k Z \to X \times_k Z$. The associativity of cycle-intersection implies that this operation yields an associative bilinear composition law

$$\circ : c(Y,Z) \times c(X,Y) \to c(X,Z).$$

Definition 4.23. The category $\mathrm{Cor}(k)$ is the category with the same objects as $\mathbf{Sm}_k$, with

$$\mathrm{Hom}_{\mathrm{Cor}(k)}(X,Y) := c(X,Y),$$

and with the composition as defined above.

For a morphism $f : X \to Y$ in $\mathbf{Sm}_k$, the graph $\Gamma_f \subset X \times_k Y$ is in $c(X,Y)$, so sending f to Γ_f defines a faithful functor

$$\mathbf{Sm}_k \to \mathrm{Cor}(k).$$

We write the morphism corresponding to Γ_f as f_*, and the object corresonding to $X \in \mathbf{Sm}_k$ as $[X]$.

The operation $\times_k$ (on smooth k-schemes and on cycles) makes $\mathrm{Cor}(k)$ a tensor category. Thus, the bounded homotopy category $K^b(\mathrm{Cor}(k))$ is a triangulated tensor category.

Definition 4.24. The category $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of *effective geometric motives* is the localization of $K^b(\mathrm{Cor}(k))$, as a triangulated tensor category, by

1. *Homotopy.* For $X \in \mathbf{Sm}_k$, invert $p_* : X \times \mathbb{A}^1 \to X$
2. *Mayer-Vietoris.* Let X be in $\mathbf{Sm}_k$, with Zariski open subschemes U, V such that $X = U \cup V$. Let $i_U : U \cap V \to U$, $i_V : U \cap V \to V$, $j_U : U \to X$ and $j_V : V \to X$ be the inclusions. Since $(j_{U*} + j_{V*}) \circ (i_{U*}, -i_{V*}) = 0$, we have the canonical map

$$\mathrm{Cone}([U \cap V] \xrightarrow{(i_{U*}, -i_{V*})} [U] \oplus [V]) \xrightarrow{(j_{U*}+j_{V*})} [X].$$

Invert this map.

To define the category of geometric motives, $DM_{\mathrm{gm}}(k)$, we invert the Lefschetz motive. For $X \in \mathbf{Sm}_k$, the reduced motives $\widetilde{[X]}$ is defined as

$$\widetilde{[X]} := \mathrm{Cone}(p_* : [X] \to [\mathrm{Spec}\, k]).$$

Set $\mathbb{Z}(1) := \widetilde{[\mathbb{P}^1]}[2]$, and set $\mathbb{Z}(n) := \mathbb{Z}(1)^{\otimes n}$ for $n \geq 0$.

Definition 4.25. The category $DM_{\mathrm{gm}}(k)$ is defined by inverting the functor $\otimes\mathbb{Z}(1)$ on $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, i.e.,

$$\mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(X, Y) := \lim_{n} \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(X \otimes \mathbb{Z}(n), Y \otimes \mathbb{Z}(n)).$$

Remark 4.26. In order that $DM_{\mathrm{gm}}(k)$ be again a triangulated category, it suffices that the commutativity involution $\mathbb{Z}(1) \otimes \mathbb{Z}(1) \to \mathbb{Z}(1) \otimes \mathbb{Z}(1)$ be the identity, which is in fact the case.

Of course, there arises the question of the behavior of the evident functor $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)$. Here we have

Theorem 4.27 (Voevodsky [97]). *The functor $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM^{\mathrm{gm}}(k)$ is a fully faithful embedding.*

The first proof of this result (in [100]) used resolution of singularities, but the later proof in [97] does not, and is valid in all characteristics.

Sheaves with transfer

The sheaf-theoretic construction of mixed motives is based on the notion of a Nisnevic sheaf with transfer.

Let X be a k-scheme of finite type. A *Nisnevic cover* $\mathcal{U} \to X$ is an étale morphism of finite type such that, for each finitely generated field extension F of k, the map on F-valued points $\mathcal{U}(F) \to X(F)$ is surjective. This small Nisnevic site of X, X_{Nis} has underlying category finite type étale X-schemes with covering families finite families $U_i \to X$ such that $\amalg_i U_i \to X$ is a Nisnevic cover The big Nisnevic site over k, with underlying category $\mathbf{Sm}_k$, is defined similarly. We let $\mathrm{Sh}^{\mathrm{Nis}}(k)$ denote the categories of Nisnevic sheaves of abelian groups on $\mathbf{Sm}_k$, and $\mathrm{Sh}^{\mathrm{Nis}}(X)$ the category of Nisnevic sheaves on X. For a presheaf $\mathcal{F}$ on $\mathbf{Sm}_k$ or X_{Nis}, we let $\mathcal{F}_{\mathrm{Nis}}$ denote the associated sheaf. We often denote $H^*(X_{\mathrm{Nis}}, \mathcal{F}_{\mathrm{Nis}})$ by $H^*(X_{\mathrm{Nis}}, \mathcal{F})$.

For a category $\mathcal{C}$, we have the category of presheaves of abelian groups on $\mathcal{C}$, i.e., the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$.

Definition 4.28. (1) The category $\mathrm{PST}(k)$ of presheaves with transfer is the category of presheaves of abelian groups on $\mathrm{Cor}(k)$. The category of Nisnevic sheaves with transfer on $\mathbf{Sm}_k$, $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$, is the full subcategory of $\mathrm{PST}(k)$ with objects those F such that, for each $X \in \mathbf{Sm}_k$, the restriction of F to X_{Nis} is a sheaf.

(2) Let F be a presheaf of abelian groups on $\mathbf{Sm}_k$. We call F *homotopy invariant* if for all $X \in \mathbf{Sm}_k$, the map

$$p^* : F(X) \to F(X \times \mathbb{A}^1)$$

is an isomorphism.

(3) Let F be a presheaf of abelian groups on $\mathbf{Sm}_k$. We call F *strictly homotopy invariant* if for all $q \geq 0$, the cohomology presheaf $X \mapsto H^q(X_{\mathrm{Nis}}, F_{\mathrm{Nis}})$ is homotopy invariant.

The category $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ is an abelian category with enough injectives, and we have the derived category $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$. For $F^* \in D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$, we have the cohomology sheaf, $\mathcal{H}^q(F^*)$, i.e., the Nisnevic sheaf with transfer associated to the presheaf

$$X \mapsto (\ker d^q : F^q(X) \to F^{q+1}(X))/d^{q-1}(F^{q-1}(X)).$$

Definition 4.29. The category $DM_-^{\mathrm{eff}}(k)$ is the full subcategory of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ consisting of those F^* whose cohomology sheaves are strictly homotopy invariant.

The localization theorem

The category $DM_-^{\mathrm{eff}}(k)$ is a localization of $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$. To show this, one uses the *Suslin complex* of a sheaf with transfers.

Definition 4.30. Let F be in $\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$. Define $C^{\mathrm{Sus}}(F)$ to be the sheafification of the complex of presheaves

$$X \mapsto (\ldots \to F(X \times \Delta^n) \to F(X \times \Delta^{n-1}) \to \ldots \to F(X)),$$

where the differentials are the usual alternating sum of restriction maps, and $F(X \times \Delta^n)$ is in degree $-n$. For $F^* \in D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$, define $L_{\mathbb{A}^1}(F^*)$ in $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ as the total derived functor of $F \mapsto C^{\mathrm{Sus}}(F)$.

For $F \in C^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$, we let $F^{\mathbb{A}^1}$ be the sheafification of the complex of presheaves $X \mapsto F(X \times \mathbb{A}^1)$; the projection $X \times \mathbb{A}^1 \to X$ defines the natural map $p^* : F \to F^{\mathbb{A}^1}$.

Definition 4.31. Let $D^-_{\mathbb{A}^1}(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ be the localization of the triangulated category $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$ with respect to the localizing subcategory generated by objects of the form $\mathrm{Cone}(p^* : F \to F^{\mathbb{A}^1})$.

Theorem 4.32 ([100, Chap. 5, Prop. 3.2.3]). *(1) For each $F \in D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k)))$, $L_{\mathbb{A}^1}(F)$ is in $DM_-^{\mathrm{eff}}(k)$. The resulting functor*

$$L_{\mathbb{A}^1} : D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))) \to DM_-^{\mathrm{eff}}(k)$$

is exact and is left-adjoint to the inclusion

$$DM_-^{\text{eff}}(k) \to D^-(\text{Sh}^{\text{Nis}}(\text{Cor}(k))).$$

(2) The functor $L_{\mathbb{A}^1}$ descends to an equivalence of triangulated categories

$$L_{\mathbb{A}^1} : D^-_{\mathbb{A}^1}(\text{Sh}^{\text{Nis}}(\text{Cor}(k))) \to DM_-^{\text{eff}}(k).$$

This result enables one to make $DM_-^{\text{eff}}(k)$ into a tensor category as follows. Let $\mathbb{Z}_{\text{tr}}(X)$ denote the representable Nisnevic sheaf with transfers $Y \mapsto c(Y, X)$. Define $\mathbb{Z}_{\text{tr}}(X) \otimes \mathbb{Z}_{\text{tr}}(Y) := \mathbb{Z}_{\text{tr}}(X \times_k Y)$. One shows that this operation extends to give $D^-(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$ the structure of a triangulated tensor category; the localizing functor $L_{\mathbb{A}^1}$ then induces a tensor operation on $D^-_{\mathbb{A}^1}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$, making $D^-_{\mathbb{A}^1}(\text{Sh}^{\text{Nis}}(\text{Cor}(k)))$ a triangulated tensor category.

Explicity, in $DM_-^{\text{eff}}(k)$, this gives us the functor

$$m : \mathbf{Sm}_k \to DM_-^{\text{eff}}(k),$$

defined by $m(X) := C^{\text{Sus}}(\mathbb{Z}_{\text{tr}}(X))$, and the formula

$$m(X \times_k Y) = m(X) \otimes m(Y).$$

The embedding theorem

We now have the two functors

$$\begin{array}{ccc} \mathbf{Sm}_k & \xrightarrow{[-]} & DM_{\text{gm}}^{\text{eff}}(k) \\ & \searrow^{m} & \\ & & DM_-^{\text{eff}}(k). \end{array} \qquad (6)$$

Theorem 4.33. *There is a unique exact functor $i : DM_{\text{gm}}^{\text{eff}}(k) \to DM_-^{\text{eff}}(k)$ filling in the diagram (6). i is a fully faithful embedding and a tensor functor. In addition, $DM_{\text{gm}}^{\text{eff}}$ is dense in $DM_-^{\text{eff}}(k)$.*

Here "dense" means that every object X in $DM_-^{\text{eff}}(k)$ fits in a distinguished triangle

$$\oplus_\alpha i(A_\alpha) \to \oplus_\beta i(B_\beta) \to X \to \oplus_\alpha i(A_\alpha)[1],$$

where the A_α and the B_β are in $DM_{\text{gm}}^{\text{eff}}(k)$, and the direct sums exist in $DM_-^{\text{eff}}(k)$.

Applications of the embedding theorem

The embedding theorem allows one to apply sheaf-theoretic constructions to $DM^{gm}(k)$, with some restrictions. As an example, the bi-functor $R\mathcal{H}om(-,-)$ on $D^-(\mathrm{Sh}^{\mathrm{Nis}}(\mathrm{Cor}(k))$ induces an internal Hom in $DM_-^{\mathrm{eff}}(k)$ by restriction. One then gets an internal Hom in $DM_{\mathrm{gm}}(k)$ (assuming resolution of singularities) by setting

$$\mathcal{H}om_{DM_{\mathrm{gm}}(k)}(A,B) := \mathcal{H}om_{DM_-^{\mathrm{eff}}(k)}(A\otimes\mathbb{Z}(n), B\otimes\mathbb{Z}(n+m))\otimes\mathbb{Z}(-m)$$

for n,m sufficiently large. Also, using the embedding theorem, one has

Theorem 4.34. *For $X\in\mathbf{Sm}_k$, there is a natural isomorphism*

$$\mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}(k)}(m(X),\mathbb{Z}(q)[p]) \cong H^p_{\mathrm{FS}}(X,\mathbb{Z}(q)) := \mathbb{H}^p(X_{\mathrm{Nis}},\mathbb{Z}_{\mathrm{FS}}(q)).$$

From this, Theorem 4.27 and Corollary 2.21, we have

Corollary 4.35. *For $X\in\mathbf{Sm}_k$, there is a natural isomorphism*

$$\mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(m(X),\mathbb{Z}(q)[p]) \cong \mathrm{CH}^q(X,2q-p).$$

Once one has this description of the morphisms in DM_{gm}, it follows easily that DM_{gm} is a fine homological triangulated category of motives over k with $\mathbb{Z}$-coefficients, and that DM_{gm} has duality if k admits resolution of singularities.

Comparison results

We state the main comparison theorem relating Levine's $\mathcal{DM}(k)$ and Voevodsky's $DM^{gm}(k)$. We note that replacing the functor $m:\mathbf{Sm}_k\to DM_{\mathrm{gm}}(k)$ with

$$h:\mathbf{Sm}_k^{\mathrm{op}}\to DM_{\mathrm{gm}}(k)$$
$$h(X) := \mathcal{H}om_{DM_{\mathrm{gm}}(k)}(m(X),\mathbb{Z})$$

changes $DM_{\mathrm{gm}}(k)$ from a homological category of motives to a cohomological category of motives.

Theorem 4.36 ([63, Part 1, Chap. VI, Theorem 2.5.5]). *Let k be a field admitting resolution of singularities. Sending $\mathbb{Z}_X(n)$ in $\mathcal{DM}(k)$ to $\mathcal{H}om_{DM_{\mathrm{gm}}(k)}(m(X),\mathbb{Z}(n))$ in $DM_{gm}(k)$, for $X\in\mathbf{Sm}_k$, extends to a pseudo-tensor equivalence of cohomological categories over motives over k*

$$\mathcal{DM}(k)\to DM^{gm}(k),$$

i.e., an equivalence of the underlying triangulated tensor categories, compatible with the respective functors h on $\mathbf{Sm}_k^{\mathrm{op}}$.

5 Mixed Tate Motives

Let $G = \mathrm{Gal}(\bar{k}/k)$ for some field k, and let ℓ be a prime not dividing the characteristic. In the category of continuous representations of G in finite dimensional $\mathbb{Q}_\ell$-vector spaces, one has the Tate objects $\mathbb{Q}_\ell(n)$; the subcategory formed by the successive extension of the Tate objects turns out to contain a surprising amount of information. Analogously, one has the Tate Hodge structure $\mathbb{Q}(n)$ and the subcategory of the category of admissible variations of mixed Hodge structures over a base-scheme B consisting of successive extensions of the $\mathbb{Q}(n)$; this subcategory gives rise, for example, to all the multiple polylogarithm functions. In this section, we consider the motivic version of these constructions, looking at categories of mixed motives generated by Tate objects.

We begin with an abstract approach by considering the triangulated subcategory of $DM_{\mathrm{gm}}(k) \otimes \mathbb{Q}$ generated by the Tate objects $\mathbb{Q}(n)$. Via the Tannakian formalism, this quickly leads to the search for a concrete description of this category as a category of representations of the *motivic Lie algebra* or dually, co-representations of the motivic co-Lie algebra. We outline constructions in this direction due to Bloch [11], Bloch-Kriz [18] and Kriz-May [61], in which the *motivic cycle algebra*, built out of the alternating version of Bloch cycle complex described in §2.5, plays a central role. The work of Kriz and May shows how all the representation-theoretic constructions are related and a theorem of Spitzweck [85] allows us to relate all these to the abstract construction inside DM_{gm}.

5.1 The triangulated category of mixed Tate motives

Since we now have a reasonable definition of "the" triangulated category of mixed motives over a field k, especially with $\mathbb{Q}$-coefficients, it makes sense to define the triangulated category of mixed Tate motives as the full triangulated subcategory generated by the (rational) Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$. Since the cohomological formulation has been used most often in the literature, we will do so as well. We will assume in this section that the base field k admits resolution of singularities, for simplicity.

Concretely, this means we replace the functor $m : \mathbf{Sm}_k \to DM^{gm}(k)$ with the functor $h : \mathbf{Sm}_k^{\mathrm{op}} \to DM^{gm}(k)$,

$$h(X) := \mathcal{H}om(m(X), \mathbb{Z}).$$

To give an example to fix ideals, the projective bundle formula gives the isomorphism

$$h(\mathbb{P}^n) \cong \oplus_{j=0}^n \mathbb{Z}(-j)[-2j].$$

Definition 5.1. Let k be a field. The triangulated category of Tate motives, $\mathrm{DTM}(k)$, is the full triangulated subcategory of $DM^{gm}(k)^{\mathrm{op}} \otimes \mathbb{Q}$ generated by the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.

As the duality involution $\vee : DM^{gm}(k) \to DM^{gm}(k)^{\mathrm{op}}$ is an equivalence of triangulated tensor categories, and $\mathbb{Q}(n)^\vee = \mathbb{Q}(-n)$, we have a duality involution on $\mathrm{DTM}(k)$, giving an equivalence

$$\vee : \mathrm{DTM}(k) \to \mathrm{DTM}(k)^\vee.$$

The weight-structure

The category $\mathrm{DTM}(k)$ admits a natural weight-filtration: Let $\mathrm{DTM}(k)_{\leq n}$ be the full triangulated subcategory of $\mathrm{DTM}(k)$ generated by the Tate objects $\mathbb{Z}(-m)$ with $m \leq n$. This gives the tower of subcategories

$$\ldots \subset \mathrm{DTM}(k)_{\leq n} \subset \mathrm{DTM}(k)_{\leq n+1} \subset \ldots \subset \mathrm{DTM}(k)$$

Dually, let $\mathrm{DTM}(k)^{>n}$ be the full triangulated subcategory of $\mathrm{DTM}(k)$ generated by the Tate objects $\mathbb{Z}(-m)$ with $m > n$.

The basic fact upon which the subsequenct construction rests is:

Lemma 5.2. *For $X \in \mathrm{DTM}(k)_{\leq n}$, $Y \in \mathrm{DTM}(k)^{>n}$, we have*

$$\mathrm{Hom}_{\mathrm{DTM}(k)}(X, Y) = 0.$$

Proof. For generators $X = \mathbb{Q}(-a)[s]$, $Y = \mathbb{Q}(-b)[t]$, $a \leq n < b$, this follows from

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}(-a)[s], \mathbb{Q}(-b)[t]) &= \mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}, \mathbb{Q}(a-b)[t-s]) \\ &= \mathrm{CH}^{a-b}(\mathrm{Spec}\, k, 2(a-b) - t + s) \otimes \mathbb{Q}\end{aligned}$$

which is zero since $a - b < 0$. The general result follows easily from this. □

By various methods (see, e.g., [65] or [57]), one can use the lemma to show that the inclusion $i_n : \mathrm{DTM}(k)_{\leq n} \to \mathrm{DTM}(k)$ admits a right adjoint $r_n : \mathrm{DTM}(k) \to \mathrm{DTM}(k)_{\leq n}$. This gives us the functor

$$W_n : \mathrm{DTM}(k) \to \mathrm{DTM}(k),$$

$W_n := i_n \circ r_n$, and the canonical map $\iota_n : W_n X \to X$ for X in $\mathrm{DTM}(k)$. One shows as well that $\mathrm{Cone}(\iota_n)$ is in $\mathrm{DTM}(k)^{>n}$, giving the canonical distinguished triangle

$$W_n X \to X \to W^{>n} X \to W_n X[1]$$

where $W^{>n} X := \mathrm{Cone}(\iota_n)$.

Remark 5.3. As pointed out in [57], one can perfectly well define an integral version $\mathrm{DTM}(k)_{\mathbb{Z}}$ of $\mathrm{DTM}(k)$ as the triangulated subcategory of $DM_{\mathrm{gm}}(k)$ generated by the Tate objects $\mathbb{Z}(n)$. The argument for weight filtration in $\mathrm{DTM}(k)$ works perfectly well to give a weight filtration in $\mathrm{DTM}(k)_{\mathbb{Z}}$.

The t-structure and vanishing conjecture

One can ask if the Beilinson formulation for mixed motives holds at least for mixed Tate motives. The first obstruction is the so-called *Beilinson-Soulé vanishing conjecture* (see [84]):

Conjecture 5.4. Let F be a field. Then $K_p(F)^{(q)} = 0$ if $2q \leq p$ and $p > 0$.

Translating to motivic cohomology, this says

Conjecture 5.5. Let F be a field. Then $H^p(F, \mathbb{Q}(q)) = 0$ if $p \leq 0$ and $q > 0$.

Since we have

$$\mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}, \mathbb{Q}(q)[p]) = H^p(k, \mathbb{Q}(q)),$$

we find a relation between the vanishing conjecture and the structure of the triangulated Tate category.

For example, if there were an abelian category $\mathrm{TM}(k)$ with $\mathrm{DTM}(k)$ equivalent to the derived category $D^b(\mathrm{TM}(k))$, in such a way that the Tate objects $\mathbb{Q}(n)$ were all in $\mathrm{TM}(k)$, then we would have

$$\mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}, \mathbb{Q}(q)[p]) = \mathrm{Ext}^p_{\mathrm{TM}(k)}(\mathbb{Q}, \mathbb{Q}(q)),$$

which would thus be zero for $p < 0$.

Suppose further that $\mathrm{TM}(k)$ is a rigid tensor category, inducing the tensor and duality on $\mathrm{DTM}(k)$, with functorial exact weight filtration W_*, inducing the functors W_n on $\mathrm{DTM}(k)$, and that taking the associated graded with respect to W_* induces a faithful exact functor to $\mathbb{Q}$-vector-spaces

$$\mathrm{gr}^W_* : \mathrm{TM}(k) \to \mathrm{Vec}_{\mathbb{Q}}.$$

Then, as each map $f : \mathbb{Q} \to \mathbb{Q}(a)$, $a \neq 0$ has $\mathrm{gr}^* f = 0$, it follows that

$$\mathrm{Hom}_{\mathrm{TM}(k)}(\mathbb{Q}, \mathbb{Q}(q)) = 0$$

for $q \neq 0$ as well.

In short, the existence of an abelian category of mixed Tate motives $\mathrm{TM}(k)$ with good properties implies the vanishing conjectures of Beilinson and Soulé.

There is a partial converse to this, namely,

Theorem 5.6 ([65]). *Let k be a field, and suppose that the Beilinson-Soulé vanishing conjectures hold for k. Then there is a t-structure on* $\mathrm{DTM}(k)$ *with heart* $\mathrm{TM}(k)$ *satisfying:*

1. $\mathrm{TM}(k)$ *contains all the Tate objects* $\mathbb{Q}(n)$. *The* $\mathbb{Q}(n)$ *generate* $\mathrm{TM}(k)$ *as an abelian category, closed under extensions in* $\mathrm{DTM}(k)$.
2. *The tensor operation and duality on* $\mathrm{DTM}(k)$ *restrict to* $\mathrm{TM}(k)$, *making* $\mathrm{TM}(k)$ *a rigid tensor category*

3. *The funtors W_n map* TM(k) *into itself, giving each object X of* TM(k) *a functorial finite weight-filtration*

$$0 = W_{M+1}X \subset W_M(X) \subset \ldots \subset W_N(X) = X.$$

4. *Taking associated graded for the weight-filtration gives a faithful exact tensor functor*

$$\oplus_n \mathrm{gr}_n^W : \mathrm{TM}(k) \to \mathbb{Q}\text{-mod}$$

to finite dimensional $\mathbb{Q}$*-vector spaces, making* TM(k) *a neutral* $\mathbb{Q}$*-Tannakian category.*

5. *There are canonical natural maps*

$$\phi_p(X,Y) : \mathrm{Ext}^p_{\mathrm{TM}(k)}(X,Y) \to \mathrm{Hom}_{\mathrm{DTM}(k)}(X, Y[p])$$

for X, Y in TM(k)*. $\phi_p(X,Y)$ is an isomorphism for $p = 0, 1$, and an injection for $p = 2$.*

We will describe below a criterion for DTM(k) to be the derived category $D^b(\mathrm{TM}(k))$.

Looking at part (4) above, the Tannakian formalism as explained in §3.1 gives an identification of TM(k) with the category of graded representations of a graded pro-unipotent affine algebraic group over $\mathbb{Q}$, or what amounts to the same thing, a graded pro-nilpotent Lie algebra over $\mathbb{Q}$, called the *motivic Lie algebra*. There have been a number of constructions of candidates for the motivic Lie algebra, or the associated Hopf algebra, which we will discuss below.

Remark 5.7. The works of Terasoma [91], Deligne-Goncharov [27], Goncharov [37], [36], Goncharov-Manin [40] and others has drawn a close connection between the mixed Tate category and values of the Riemann zeta function, polylogarithms and multi-zeta functions; due to lack of space, we will not discuss these works here. See the article [35] in this volume for further details.

5.2 The Bloch cycle algebra and Lie algebra

Definition 5.8. (1) Let F be a field. A *cdga* $(A^*, d, \cdot)$ over F consists of a unital, graded-commutative F-algebra $(A^* := \oplus_{n\in\mathbb{Z}} A^n, \cdot)$ together with a graded homomorphism $d = \oplus_n d^n$, $d^n : A^n \to A^{n+1}$, such that

1. $d^{n+1} \circ d^n = 0$.
2. $d^{n+m}(a \cdot b) = d^n a \cdot b + (-1)^n a \cdot d^m b$; $a \in A^n$, $b \in A^m$.

A^* is called *connected* if $A^n = 0$ for $n < 0$ and $A^0 = F \cdot 1$, *cohomologically connected* if $H^n(A^*) = 0$ for $n < 0$ and $H^0(A^*) = F \cdot 1$.

(2) An *Adams-graded* cdga is a cdga A together with a direct sum decomposition into subcomplexes $A^* := \oplus_{r\geq 0} A^*(r)$ such that $A^*(r) \cdot A^*(s) \subset A^*(r+s)$.

An Adams-graded cdga is (cohomologically) connected if the underlying cdga is (cohomologically) connected.

For $x \in A^n(r)$, we called n the *cohomological degree* of x, $n := \deg x$, and r the *Adams degree* of x, $r := |x|$.

Example 5.9. Let k be a field. Recall from §2.5 the alternating cycle complexes $z^p(k,*)^{\mathrm{Alt}}$ with commutative associative product

$$\cup^{\mathrm{Alt}} : z^p(k,*) \otimes z^q(k,*) \to z^{p+q}(k,*).$$

Bloch [11] has defined the *motivic cdga* over k, $\mathcal{N}_k^*(*)$, as the Adams-graded cdga over $\mathbb{Q}$ with

$$\mathcal{N}_k^m(r) := \begin{cases} z^r(k, 2r-m)^{\mathrm{Alt}} & \text{for } r > 0 \\ z^0(k,0)^{\mathrm{Alt}} (= \mathbb{Q} \cdot [\operatorname{Spec} k]) & \text{for } r = 0. \end{cases}$$

and product

$$\cdot : \mathcal{N}_k^m(r) \otimes \mathcal{N}_k^n(s) \to \mathcal{N}_k^{m+n}(r+s)$$

given by $\cup^{\mathrm{Alt}}$. The unit is $1 \cdot [\operatorname{Spec} k] \in \mathcal{N}^0(0)$.

Remark 5.10. The Beilinson-Soulé vanishing conjecture for the field k is exactly the statement that $\mathcal{N}_k^*(*)$ is cohomologically connected.

Bloch defines a graded co-Lie algebra $\mathcal{M}(*) = \oplus_{r>0}\mathcal{M}(r)$ as follows: Start with the cycle cdga $\mathcal{N}$. Let $\mathcal{N}_+^0 := \oplus_{r\geq 0}\mathcal{N}^0(r)$, and let $\mathcal{J} \subset \mathcal{N}$ be the differential ideal generated by $\oplus_{n<0,r}\mathcal{N}^n(r) \oplus \mathcal{N}_+^0$. Let $\bar{\mathcal{N}}$ be the quotient cdga $\mathcal{N}/\mathcal{J}$. Bloch shows

Lemma 5.11. *(1) The product $\Lambda^2\bar{\mathcal{N}}^1 \to \bar{\mathcal{N}}^2$ is injective*

(2) Let $\mathcal{M}_k = \{x \in \bar{\mathcal{N}}^1 \mid dx \text{ is in } \Lambda^2\bar{\mathcal{N}}^1 \subset \bar{\mathcal{N}}^2\}$. Then

$$d\mathcal{M}_k \subset \Lambda^2\mathcal{M}_k \subset \Lambda^2\bar{\mathcal{N}}^1 \subset \bar{\mathcal{N}}^2.$$

(3) The map $d : \mathcal{M}_k \to \Lambda^2\mathcal{M}_k$ makes $\mathcal{M}_k$ into an Adams graded co-Lie algebra over $\mathbb{Q}$.

Definition 5.12. The category of Bloch-Tate mixed motives over k, BTM_k, is the category of graded co-representations of $\mathcal{M}_k$ in $\mathbb{Q}$-mod, that is, the category of finite-dimensional graded $\mathbb{Q}$-vector spaces $V(*) = \oplus_r V(r)$ together with a graded, degree zero $\mathbb{Q}$-linear map

$$\rho : V(*) \to \mathcal{M}_k \otimes_{\mathbb{Q}} V(*)$$

satisfying the co-associativity condition $(\mathrm{id} \wedge \rho) \circ \rho = (\partial \otimes \mathrm{id}) \circ \rho$ as maps $V(*) \to \Lambda^2\mathcal{M}_k \otimes V(*)$.

BTM_k contains the Tate-objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, where $\mathbb{Q}(n)$ is the vector space $\mathbb{Q}$ supported in degree $-n$, with zero co-action ρ. There is a map $H^1(\mathcal{N}^*(r)) \to \mathrm{Ext}^1_{\mathrm{BTM}_k}(\mathbb{Q}(0), \mathbb{Q}(r))$ by sending $\eta \in Z^1(\mathcal{N}^*(r))$ to the class of the extension

$$0 \to \mathbb{Q}(r) \to V_\eta \to \mathbb{Q}(0) \to 0,$$

where $V_\eta = \mathbb{Q}(0) \oplus \mathbb{Q}(r)$ as a graded $\mathbb{Q}$-vector space, with co-action ρ given by

$$\rho(a, b) = \eta \otimes (0, a).$$

One checks that changing η by a co-boundary does not affect the extension class.

5.3 Categories arising from a cdga

As we will see below, the construction of mixed Tate motives via co-representations of the Bloch co-Lie algebra $\mathcal{M}_k$ is reasonable only under the so-called 1-minimal conjecture. Bloch and Kriz [18] have given another construction of a co-Lie algebra, and at the same time the associated Hopf algebra, by using the bar construction of $\mathcal{N}_k$. Kriz and May [61] have given a construction of a triangulated category, which derives more directly from the cdga $\mathcal{N}_k$; in case $\mathcal{N}_k$ is cohomologically connected, this triangulated category has a heart which turns out to be equivalent to the category of graded co-modules over the Bloch-Kriz co-Lie algebra.

Before we go into this, we discuss some of the general theory of the bar construction of a cdga and related constructions. We have taken this material from [61].

The bar construction

We recall the definition of the reduced bar construction of an augmented cdga $\epsilon : A^* \to k$ over a field F of characteristic zero. Let $\bar{A}^*$ be the kernel of ϵ, and form the tensor algebra

$$T_k^* \bar{A}^* = \oplus_{(n_1, \ldots, n_m)} \bar{A}^{n_1} \otimes_F \ldots \otimes_F \bar{A}^{n_m}$$

with $\bar{A}^{n_1} \otimes_F \ldots \otimes_F \bar{A}^{n_m}$ in total degree $\sum_j n_j - m$, together with a copy of F in degree 0, corresponding to the empty tensor product, which we write as $F \cdot 1$. Denote a decomposable element of $\bar{A}^{n_1} \otimes_F \ldots \otimes_F \bar{A}^{n_m}$ in $T_k^* \bar{A}^*$ as $[x_1 | \ldots | x_m]$, $x_j \in A^{n_j}$, and define the map d by

$$d([x_1 | \ldots | x_m]) = \sum_j (-1)^{\sum_{i=1}^{j-1} \deg(x_i)} [x_1 | \ldots | dx_j | \ldots | x_m] + \sum_j (-1)^j [x_1 | \ldots | x_j x_{j+1} | \ldots | x_m].$$

Set $d(F \cdot 1) = 0$. This forms the complex $(\bar{B}(A), d)$.

The *shuffle product*

$$[x_1|\ldots|x_m] \cup [x_{n+1}|\ldots|x_{m+n}] := \frac{m!n!}{(m+n)!} \sum_\sigma \operatorname{sgn}(\sigma)[x_{\sigma(1)}|\ldots|x_{\sigma(n+m)}],$$

where $\sigma \in \Sigma_{n+m}$ ranges over all permutations with $\sigma(1) < \ldots < \sigma(n)$ and $\sigma(n+1) < \ldots < \sigma(n+m)$, defines a product on $\bar{B}(A)$, satisfying the Leibniz rule with respect to d. The map

$$\delta : \bar{B}(A) \to \bar{B}(A) \otimes \bar{B}(A)$$
$$\delta([x_1|\ldots|x_m]) := \sum_{i=0}^{m} [x_1|\ldots|x_i] \otimes [x_{i+1}|\ldots|x_m]$$

(the empty tensor being 1) defines a coproduct on $\bar{B}(A)$.

This all makes $(\bar{B}(A), d, \cup, \delta)$ into a differential graded Hopf algebra over k, which is graded-commutative with respect to the product $\cup$. The cohomology $H^*(\bar{B}(A))$ is thus a graded Hopf algebra over k, in particular $H^0(\bar{B}(A))$ is a commutative Hopf algebra over k.

Let $\mathcal{I}(A)$ be the kernel of the augmentation $H^0(\bar{B}(A)) \to k$. The coproduct δ on $H^0(\bar{B}(A))$ induces the structure of a co-Lie algebra on $\gamma_A := \mathcal{I}(A)/\mathcal{I}(A)^2$.

Suppose $A = \oplus_{r\geq 0} A^*(r)$ is an Adams-graded cdga over k. We give $\bar{B}(A)$ the Adams grading $\bar{B}(A) = \oplus_{r\geq 0} \bar{B}(A)(r)$ where the Adams degree of $[x_1|\ldots|x_m]$ is

$$|[x_1|\ldots|x_m]| := \sum_j |x_j|.$$

Thus $H^0(\bar{B}(A)) = \oplus_{r\geq 0} H^0(\bar{B}(A)(r))$ becomes a graded Hopf algebra over k, commutative as a k-algebra. We also have the Adams-graded co-Lie algebra $\gamma_A = \oplus_{r>0} \gamma_A(r)$.

Remark 5.13. Let A be an Adams-graded cdga over a field F of characteristic zero. The Adams grading makes $G := \operatorname{Spec} H^0(\bar{B}(A))$ into a graded pro-unipotent affine group-scheme (i.e., G comes equipped with an action of $\mathbb{G}_m$). Thus γ_A is a graded nilpotent co-Lie algebra, and there is an equivalence of categories between the graded co-representations of $H^0(\bar{B}(A))$ in finite dimensional graded F-vector spaces, $\text{co-rep}_F(H^0(\bar{B}(A)))$, and the graded co-representations of γ_A in finite dimensional graded F-vector spaces, $\text{co-rep}_F(\gamma_A)$.

Weight filtrations and Tate objects

Let A be an Adams-graded cdga over a field F of characteristic zero, and let $M = \oplus_r M(r)$ be a graded co-module for γ_A, finite dimensional as an F-vector space. Let $W_nM = \oplus_{r\le n}M(r)$. As γ_A is positively graded, W_nM is a γ_A-subcomodule of M. Thus, each M has a finite functorial weight filtration, and the functor gr^n_W is exact. We say that M has *pure weight* n if $W_nM = M$ and $W_{n-1}M = 0$.

We have the Tate object $F(n)$, being the 1-dimensional F-vector space, concentrated in Adams degree $-n$, and with trivial (i.e. 0) co-action $F(n) \to \gamma_A \otimes F(n)$. Clearly $\mathrm{gr}^{-n}_W M \cong F(n)^a$ for some a, so all objects in $\text{co-rep}_F(\gamma_A)$ are successive extensions of Tate objects. The full subcategory of objects of pure weight n is equivalent to F-mod.

Sending M to $\mathrm{gr}^*_W M := \oplus_n \mathrm{gr}^n_W M$ defines a fiber functor

$$\mathrm{gr}^*_W : \text{co-rep}_F(\gamma_A) \to F\text{-mod}$$

making $\text{co-rep}_F(\gamma_A)$ a neutral F-Tannakian category.

The category of cell-modules

The approach of Kriz and May [61] is to define a triangulated category directly from the Adams graded cdga $\mathcal{N}$ without passing to the bar construction or using a co-Lie algebra, by considering a certain type of dg-modules over $\mathcal{N}$. We recall some of their work here.

Let A^* be a graded algebra over a field F. We let $A[n]$ be the left A^*-module which is A^{m+n} in degree m, with the A^*-action given by left multiplication. If $A^*(*) = \oplus_{n,r}A^n(r)$ is a bi-graded F-algebra, we let $A{<}r{>}[n]$ be the left $A^*(*)$-module which is $A^{m+n}(r+s)$ in bi-degree (m,s), with action given by left multiplication.

Definition 5.14. Let A be a cdga over a field F of characteristic zero.

(1) A dg-A-module (M^*, d) consists of a complex $M^* = \oplus_n M^n$ with differential d, together with a graded, degree zero map $\rho : A^* \otimes_F M^* \to M^*$ which makes $M*$ into a graded A^*-module, and satisfies the Leibniz rule

$$d(a\cdot m) = da\cdot m + (-1)^{\deg a} a\cdot m;\; a \in A^*, m \in M^*,$$

where we write $a \cdot m$ for $\rho(a\otimes m)$.

(2) If $A = \oplus_{r\ge 0}A^*(r)$ is an Adams-graded cdga, an Adams-graded dg-A-module is a dg-A-module M^* together with a decomposition into subcomplexes $M^* = \oplus_s M^*(s)$ such that $A^*(r)\cdot M^*(s) \subset M^*(r+s)$.

(3) An Adams-graded dg-A-module M is a *cell module* if M is free and finitely generated as a bi-graded A-module, where we forget the differential structure. That is, there are elements $b_j \in M^{n_j}(r_j)$, $j = 1, \ldots, s$, such that the maps $a \mapsto a\cdot b_j$ induces an isomorphism of bi-graded A-modules

$$\oplus_{j=0}^s A{<}r_j{>}[n_j] \to M.$$

The derived category

Let A be an Adams-graded cdga over a field F, and let M and N be Adams-graded dg-A-modules. Let $\mathcal{H}om(M,N)$ be the Adams-graded dg-A-module with $\mathcal{H}om(M,N)^n(r)$ the A-module maps $f : M \to N$ with $f(M^a(s)) \subset N^{a+n}(r+s)$, and differential d defined by $df(m) = d(f(m)) + (-1)^{n+1}f(dm)$ for $f \in \mathcal{H}om(M,N)^n(r)$. Similarly, let $M \otimes N$ be the Adams-graded dg-A-module

$$(M \otimes N)^n(r) = \oplus_{a+b=n,s+t=r} M^a(s) \otimes_F N^b(t),$$

with differential $d(m \otimes n) = dm \otimes n + (-1)^{\deg m} m \otimes dn$.

For $f : M \to N$ a morphism of Adams-graded dg-A-modules, we let $\mathrm{Cone}(f)$ be the Adams-graded dg-A-module with

$$\mathrm{Cone}(f)^n(r) := N^n(r) \oplus M^{n+1}(r)$$

and differential $d(n,m) = (dn + f(m), -dm)$. Let $M[1]$ be the Adams-graded dg-A-module with $M[1]^n(r) := M^{n+1}(r)$ and differential $-d$, where d is the differential of M. A sequence of the form

$$M \xrightarrow{f} N \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{j} M[1]$$

where i and j are the evident inclusion and projection, is called a *cone sequence*.

Definition 5.15. (1) The category $\mathrm{KCM}(A)$ is the F-linear triangulated category with objects the cell-A-modules M, morphisms

$$\mathrm{Hom}_K(M,N) := H^0(\mathcal{H}om(M,N))$$

with evident composition law, translation $M \mapsto M[1]$ and distinguished triangles those sequences isomorphic to a cone sequence.

(2) The category $\mathrm{DCM}(A)$ is the localization of $\mathrm{KCM}(A)$ with respect to quasi-isomorphisms, that is, invert the maps $f : M \to N$ which induce an isomorphism on cohomology $H^*(M) \to H^*(N)$.

We note that the tensor product and internal Hom of cell modules gives $\mathrm{KCM}(A)$ and $\mathrm{DCM}(A)$ the structure of rigid triangulated tensor categories.

The following is a useful result (see [61, Proposition 4.2]):

Proposition 5.16. *Let $\phi : A \to A'$ be a quasi-isomorphism of Adams graded cdgas. Then ϕ induces an equivalence of triangulated tensor categories* $\mathrm{DCM}(A) \to \mathrm{DCM}(A')$.

The weight and t-structures

It is easy to describe the weight filtration in $\mathrm{DCM}(A)$. Indeed, let $M = \oplus_j A{<}r_j{>}[n_j]$ be a cell A-module with basis $\{b_j\}$. The differential d is determined by

$$db_j = \sum_i a_{ij} b_i.$$

As $A^*(r) = 0$ if $r < 0$, and d is of weight 0 with respect to the Adams grading, it follows that $|b_i| \leq |b_j|$ if $a_{ij} \neq 0$. We may thus set

$$W_n M := \oplus_{r_j \leq n} A{<}r_j{>}[n_j] \subset M,$$

with differential the restriction of d. One shows that this gives a well-defined exact functor $W_n : \mathrm{DCM}(A) \to \mathrm{DCM}(A)$, and a natural finite weight filtration

$$0 = W_{n-1} \to W_n M \to \ldots \to W_{m-1} M \to W_m M = M$$

for M in $\mathrm{DCM}(A)$. Let $F(n)$ be the "Tate object " $A{<}{-n}{>}$.

For the t-structure, one needs to assume that A is cohomologically connected; by Proposition 5.16 we may assume that A is connected. Let $\epsilon : A \to k$ be the augmentation given by projection on $A^0(0)$, and define

$$\begin{aligned}
&\mathrm{DCM}(A)_{\leq 0} := \{M \mid H^n(M \otimes_A k) = 0 \text{ for } n > 0\} \\
&\mathrm{DCM}(A)_{\geq 0} := \{M \mid H^n(M \otimes_A k) = 0 \text{ for } n < 0\} \\
&\mathcal{H}(A) := \{M \mid H^n(M \otimes_A k) = 0 \text{ for } n \neq 0\}
\end{aligned}$$

One shows that this defines a t-structure $(\mathrm{DCM}(A)_{\leq 0}, \mathrm{DCM}(A)_{\geq 0})$ on $\mathrm{DCM}(A)$ with heart $\mathcal{H}(A)$. Also, the $F(n)$ are in $\mathcal{H}(A)$ and these generate $\mathcal{H}(A)$, oin that the smallest full abelian subcategory of $\mathcal{H}(A)$ containing all the $F(n)$ and closed under subquotients and extensions is all of $\mathcal{H}(A)$.

The subcategory of $\mathcal{H}(A)$ consisting of objects of pure weight n is equivalent to F-mod with generator the Tate object $F(-n)$, giving us the fiber functor

$$\mathrm{gr}^*_W : \mathcal{H}(A) \to F\text{-mod}$$

which makes $\mathcal{H}(A)$ a neutral F-Tannakian category.

Minimal models

A cdga A over a field F of characteristic zero is said to be *generalized nilpotent* if

1. As a graded F-algebra, $A = \mathrm{Sym}^* E$ for some $\mathbb{Z}$-graded F-vector space E, i.e., $A = \Lambda^* E_{\mathrm{odd}} \otimes \mathrm{Sym}^* E_{\mathrm{ev}}$. In addition, $E_n = 0$ for $n \leq 0$.
2. E is an increasing union of graded subspaces

$$0 = E^{-1} \subset E^0 \subset \ldots \subset E^n \subset \ldots \subset E$$

with $dE^n \subset \mathrm{Sym}^* E^{n-1}$.

Note that a generalized nilpotent cdga is automatically connected.

Let A be a cohomologically connected cdga. An *n-minimal model* of A is a map of cdgas

$$s : \mathcal{M}\{n\} \to A,$$

with $\mathcal{M}\{n\}$ generalized nilpotent and generated (as an algebra) in degrees $\le n$, such that s induces an isomorphism on H^m for $m \le n$ and an injection on H^{n+1}. One shows that this characterizes $s : \mathcal{M}\{n\} \to A$, up to unique isomorphism, so we may speak of the n-minimal model of A. Similarly, the minimal model of A is a quasi-isomorphism $\mathcal{M}\{\infty\} \to A$ with $\mathcal{M}\{\infty\}$ generalized nilpotent; we can recover $\mathcal{M}\{n\}$ as the sub-cdga of $\mathcal{M}\{\infty\}$ generated by $\oplus_{0\le i\le n}\mathcal{M}\{\infty\}^i$. We call A n-minimal if $\mathcal{M}\{n\} = \mathcal{M}\{\infty\}$. With the obvious changes, we have all these notions in the Adams-graded setting.

Remark 5.17. In rational homotopy theory, the rational homotopy type corresponding to a cdga A is a $K(\pi, 1)$ if and only if A is 1-minimal, so a 1-minimal cdga is often called a $K(\pi, 1)$.

Let A be a cohomologically connected cdga with 1-minimal model $\mathcal{M}\{1\}$, let $QA = \mathcal{M}\{1\}^1$ with map $\partial : QA \to \Lambda^2 QA$ the differential $d : \mathcal{M}\{1\}^1 \to \Lambda^2\mathcal{M}\{1\}^1 = \mathcal{M}\{1\}^2$. Then (QA, ∂) is a co-Lie algebra over F. If A is an Adams-graded cdga, then QA becomes an Adams-graded co-Lie algebra. We can also form the co-Lie algebra γ_A as in §5.3.

Putting it all together

In [61] the relations between the various constructions we have presented above are discussed. We recall the main points here.

Theorem 5.18. *Let A be an Adams graded cdga over a field F of characteristic zero. Suppose the A is cohomologically connected.*

(1) There is a functor $\rho : D^b(\text{co-rep}_F(H^0(\bar{B}(A)))) \to \mathrm{DCM}(A)$. ρ respects the weight filtrations and sends Tate objects to Tate objects. ρ induces a functor on the hearts

$$\mathcal{H}(\rho) : \text{co-rep}_F(H^0(\bar{B}(A))) \to \mathcal{H}(A)$$

which is an equivalence of filtered Tannakian categories, respecting the fiber functors gr^*_W.

(2) Let $\mathcal{M}_A\{1\}$ be the 1-minimal model of A. Then there are isomorphisms of graded Hopf algebras $H^0(\bar{B}(A)) \cong H^0(\bar{B}(\mathcal{M}_A\{1\}))$ and graded co-Lie algebras

$$QA \cong \gamma_{\mathcal{M}_A\{1\}} \cong \gamma_A.$$

(3) The functor ρ is an equivalence of triangulated categories if and only if A is a $K(\pi, 1)$ (i.e., A is 1-minimal).

5.4 Categories of mixed Tate motives

We are now ready to apply the machinery of §5.3.

Tate motives as modules

Definition 5.19. Let k be a field.

(1) The Bloch-Kriz category of mixed Tate motives over k, BKTM_k, is the category $\text{co-rep}_{\mathbb{Q}}(H^0(\bar{B}(\mathcal{N}_k)))$ of graded co-representations of the Hopf algebra $H^0(\bar{B}(\mathcal{N}_k))$ in finite dimensional graded $\mathbb{Q}$-vector spaces, equivalently, the category $\text{co-rep}_{\mathbb{Q}}(\gamma_{\mathcal{N}_k})$ of graded co-representations of the co-Lie algebra $\gamma_{\mathcal{N}_k}$ in finite dimensional graded $\mathbb{Q}$-vector spaces.

(2) The Kriz-May triangulated category of mixed Tate motives over k, DT_k, is the derived category $\mathrm{DCM}(\mathcal{N}_k)$ of cell modules over $\mathcal{N}_k$.

Remark 5.20. One can show that, assuming $\mathcal{N}_k$ is 1-minimal, the co-Lie algebra $\mathcal{M}_k$ is $Q\bar{\mathcal{N}}_k$. In general, there is a map of co-Lie algebras

$$\phi : \gamma_{\mathcal{N}_k} \to \mathcal{M}_k,$$

and hence a functor

$$\phi_* : \mathrm{BKTM}_k = \text{co-rep}(\gamma_{\mathcal{N}_k}) \to \text{co-rep}(\mathcal{M}_k) = \mathrm{BTM}_k.$$

Applying Theorem 5.18 to the situation at hand, we have

Theorem 5.21. *Let k be a field. Suppose $\mathcal{N}_k$ is cohomologically connected, i.e., the Beilinson-Soulé vanishing conjecture holds for k.*

1. *There is an exact tensor functor $\rho : D^b(\mathrm{BKTM}_k) \to \mathrm{DT}_k$, preserving the weight-filtrations and sending Tate objects to Tate objects.*
2. *The functor ρ induces an equivalence of filtered $\mathbb{Q}$-Tannakian categories $\mathrm{BKTM}_k \to \mathcal{H}(\mathcal{N}_k)$, respecting the fiber functors gr^*_W.*
3. *The functor ρ is an equivalence of triangulated categories if and only if $\mathcal{N}_k$ is a $K(\pi,1)$. In particular, if $\mathcal{N}_k$ is a $K(\pi,1)$, then*

$$\mathrm{Ext}^p_{\mathrm{BKTM}_k}(\mathbb{Q}, \mathbb{Q}(q)) = H^p(k, \mathbb{Q}(q)) = K_{2q-p}(k)^{(q)}$$

for all p and q.

The very last assertion follows from the identities (assuming ρ an equivalence)

$$\begin{aligned}
\mathrm{Ext}^p_{\mathrm{BKTM}_k}(\mathbb{Q}, \mathbb{Q}(q)) &= \mathrm{Hom}_{\mathrm{DT}_k}(\mathbb{Q}, \mathbb{Q}(q)[p]) \\
&= H^p(\mathcal{N}^*(q)) \\
&= H^p(k, \mathbb{Q}(q)).
\end{aligned}$$

Remark 5.22. If the Beilinson-Soulé vanishing conjecture fails to hold for k, then there is no hope of an equivalence of triangulated categories $D^b(\mathrm{BKTM}_k) \to \mathrm{DT}_k$, as the lack of cohomological connectness for $\mathcal{N}_k$ is equivalent to having $\mathrm{Hom}_{\mathrm{DT}_k}(\mathbb{Q}, \mathbb{Q}(q)[p]) \neq 0$ for some $q > 0$ and $p < 0$.

It is not clear if the lack of cohomological connectness of $\mathcal{N}_k$ gives an obstruction to the existence of a reasonable functor $\rho : D^b(\mathrm{BKTM}_k) \to \mathrm{DT}_k$ (say, with $\rho(\mathbb{Q}(n)) = \mathbb{Q}(n)$).

Tate motives as Voevodsky motives

The following result, extracted from [85, Theorem 2], shows how the Kriz-May triangulated category serves as a bridge between the Bloch-Kriz category of co-modules, and the more natural, but also more abstract, category of Tate motives sitting inside of Voevodsky's category DM_{gm}.

Theorem 5.23. *Let k be a field. There is a natural exact tensor functor*

$$\phi : \mathrm{DT}_k \to DM_{\mathrm{gm}}(k)$$

which induces an equivalence of triangulated tensor categories $\mathrm{DT}_k \to \mathrm{DTM}(k)$. The functor ϕ is compatible with the weight filtrations in DT_k and $\mathrm{DTM}(k)$. If $\mathcal{N}_k$ is cohomologically connected, then ϕ induces an equivalence of abelian categories

$$\mathcal{H}(\mathcal{N}_k) \to \mathrm{TM}(k).$$

Note that this gives a module-theoretic description of $\mathrm{DTM}(k)$ for all fields k, without assuming any conjectures. This result also gives a context for the $K(\pi, 1)$-*conjecture*:

Conjecture 5.24. Let k be a field. Then the cycle cdga $\mathcal{N}_k$ is a $K(\pi, 1)$.

Indeed the conjecture would imply that all the different candidates for an abelian category of mixed Tate motives over k agree: If $\mathcal{N}_k$ is 1-minimal, the $\mathcal{N}_k$ is cohomologically connected. By Theorem 5.23, the abelian categories $\mathcal{H}(\mathcal{N}_k)$ and $\mathrm{TM}(k)$ are equivalent, as well as the triangulated categories $\mathrm{DT}_k = \mathrm{DCM}(\mathcal{N}_k)$ and $\mathrm{DTM}(k)$. By Theorem 5.21, we have an equivalence of triangulated categories $D^b(\mathcal{H}(\mathcal{N}_k))$ and DT_k, and $\mathcal{H}(\mathcal{N}_k)$ is equivalent to the Bloch-Kriz category BKTM_k. By Remark 5.20, the graded co-Lie algebras $Q\bar{\mathcal{N}}_k$ and $\mathcal{M}_k$ agree, so we have equivalences of abelian categories

$$\mathrm{BTM}(k) \sim \mathrm{BKTM}(k) \sim \mathrm{TM}(k) \sim \mathcal{H}\mathrm{DT}_k$$

and triangulated categories

$$D^b(\mathrm{TM}(k)) \sim \mathrm{DTM}(k).$$

All these equivalences respect the tensor structure, the weight filtrations and duality.

5.5 Spitzweck's representation theorem

We sketch a proof of Thereom 5.23.

Cubical complexes in $DM^{\mathrm{eff}}_{-}(k)$

To give a representation of DT_k into DM_{gm}, it is convenient to use a cubical version of the Suslin-complex C_*.

Definition 5.25. Let $\mathcal{F}$ be presheaf on $\mathbf{Sm}_k$. Let $C^{\mathrm{cb}}_n(\mathcal{F})$ be the presheaf

$$C^{\mathrm{cb}}_n(\mathcal{F})(X) := \mathcal{F}(X \times \square^n)/\sum_{j=1}^{n} \pi_j^*(\mathcal{F}(X \times \square^{n-1})).$$

and let $C^{\mathrm{cb}}_*(\mathcal{F})$ be the complex with differential

$$d_n = \sum_{j=1}^{n}(-1)^{j-1}F(\iota_{j,1}) - \sum_{j=1}^{n}(-1)^{j-1}F(\iota_{j,0}).$$

If $\mathcal{F}$ is a Nisnevic sheaf, then $C^{\mathrm{cb}}_*(\mathcal{F})$ is a complex of Nisnevic sheaves, and if $\mathcal{F}$ is a Nisnevic sheaf with transfers, then $C^{\mathrm{cb}}_*(\mathcal{F})$ is a complex of Nisnevic sheaves with transfers. We extend the construction to bounded above complexes of sheaves (with transfers) by taking the total complex of the evident double complex.

For a presheaf $\mathcal{F}$, let $C^{\mathrm{Alt}}_n(\mathcal{F}) \subset C^{\mathrm{cb}}_n(\mathcal{F})_{\mathbb{Q}}$ denote as above the subspace of alternating elements with respect to the action of Σ_n on $\square^n$, forming the subcomplex $C^{\mathrm{Alt}}_*(\mathcal{F}) \subset C^{\mathrm{cb}}_*(\mathcal{F})_{\mathbb{Q}}$. We extend this to bounded above complexes of presheaves as well.

The arguments used in §2.5 to compare Bloch's cycle complex with the cubical version show

Lemma 5.26. *Let $\mathcal{F}$ be a bounded above complex of presheaves on $\mathbf{Sm}_k$.*

1. *There is a natural isomorphism $C^{\mathrm{Sus}}_*(\mathcal{F}) \cong C^{\mathrm{cb}}_*(\mathcal{F})$ in the derived category of presheaves on $\mathbf{Sm}_k$. If $\mathcal{F}$ is a presheaf with transfer, we have an isomorphism $C^{\mathrm{Sus}}_*(\mathcal{F}) \cong C^{\mathrm{cb}}_*(\mathcal{F})$ in the derived category $D^-(\mathrm{PST}(k))$.*
2. *The inclusion $C^{\mathrm{Alt}}_*(\mathcal{F})(Y) \subset C^{\mathrm{cb}}_*(\mathcal{F})_{\mathbb{Q}}(Y)$ is a quasi-isomorphism for all $Y \in \mathbf{Sm}_k$.*

In particular, $C^{\mathrm{cb}}_*(\mathcal{F})$ has homotopy invariant cohomology sheaves, so we have the functors

$$C^{\mathrm{cb}}_* : C^-(\mathrm{Sh}^{\mathrm{Nis}}(k)) \to DM^{\mathrm{eff}}_-(k).$$
$$C^{\mathrm{Alt}}_* : C^-(\mathrm{Sh}^{\mathrm{Nis}}(k)) \to DM^{\mathrm{eff}}_-(k) \otimes \mathbb{Q}.$$

Taking the usual Suslin complex also gives us a functor

$$C^{\mathrm{Sus}}_* : C^-(\mathrm{Sh}^{\mathrm{Nis}}(k)) \to DM^{\mathrm{eff}}_-(k).$$

and we thus have the isomorphism of functors $C^{\mathrm{Sus}}_* \to C^{\mathrm{cb}}_*$ and $C^{\mathrm{Alt}}_* \to (C^{\mathrm{cb}}_*)_{\mathbb{Q}}$.

The cycle cdga in $DM_-^{\text{eff}}(k)$

We apply this construction to $\mathcal{F} = \mathbb{Z}_{\text{q.fin}}(\mathbb{A}^q)$. The symmetric group Σ_q acts on this sheaf by permuting the coordinates in $\mathbb{A}^q$, we let $\mathcal{N}_k^{\text{gm}}(q) \subset C_*^{\text{Alt}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^q))$ be the subsheaf of *symmetric* sections with respect to this action.

Lemma 5.27. *The inclusion $\mathcal{N}_k^{\text{gm}}(q) \subset C_*^{\text{Alt}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^q))$ is an isomorphism in $DM_-^{\text{eff}}(k)$*

Proof. Roughly speaking, it follows from Theorem 2.25, Lemma 2.28 and Lemma 5.26 that the inclusion

$$C_*^{\text{Alt}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^q))(Y) \to z^q(Y \times \mathbb{A}^q, *)^{\text{Alt}}$$

is a quasi-isomorphism for each $Y \in \mathbf{Sm}_k$. As the pull-back

$$z^q(Y \times \mathbb{A}^q, *)^{\text{Alt}} \to z^q(Y \times \mathbb{A}^q, *)^{\text{Alt}}$$

is also a quasi-isomorphism by the homotopy property, Σ_q acts trivially on $z^q(Y \times \mathbb{A}^q, *)^{\text{Alt}}$, in $D^-(\mathbf{Ab})$. □

For $X, Y \in \mathbf{Sm}_k$, the external product of correspondences gives the associative external product

$$C_n^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^q))(X) \otimes C_m^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^p))(Y) \to C_{n+m}^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^{p+q}))(X \times_k Y)$$

Taking $X = Y$ and pulling back by the diagonal $X \to X \times_k X$ gives the cup product of complexes of sheaves

$$\cup : C_*^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^p) \otimes C_*^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^q)) \to C_*^{\text{cb}}(\mathbb{Z}_{\text{q.fin}}(\mathbb{A}^{p+q})).$$

Taking the alternating projection with respect to the $\square^*$ and symmetric projection with respect to the $\mathbb{A}^*$ yields the associative, commutative product

$$\cdot : \mathcal{N}_k^{\text{gm}}(p) \otimes \mathcal{N}_k^{\text{gm}}(q) \to \mathcal{N}_k^{\text{gm}}(p+q),$$

which makes $\mathcal{N}_k^{\text{gm}} := \oplus_{r \geq 0} \mathcal{N}_k^{\text{gm}}(r)$ into an Adams-graded cdga object in $C^-(\text{Sh}^{\text{Nis}}(k))$.

In particular, if a is in $\mathcal{N}_k^{\text{gm}}(k)$, multiplication by a gives an endomorphism $a \cdot - : \mathcal{N}_k^{\text{gm}} \to \mathcal{N}_k^{\text{gm}}$.

A replacement for $\mathcal{N}_k$

Let

$$\mathcal{N}_k(\mathbb{A}^*)^n(r) := (z^r(\mathbb{A}^r, 2r-n)^{\text{Alt}})^{\text{sym}},$$

where sym means the symmetric subspace with respect to the Σ_r-action on $\mathbb{A}^r$ by permuting the coordinates. Taking the external product and the alternating and symmetric projections defines an Adams-graded cdga $\mathcal{N}_k(\mathbb{A}^*)$. We have the evident inclusion

$$i : \mathcal{N}_k^{\mathrm{gm}}(k) \to \mathcal{N}_k(\mathbb{A}^*),$$

and the pull-back via the maps $\pi_r : \mathbb{A}^r \to \operatorname{Spec} k$ defines

$$\pi^* : \mathcal{N}_k \to \mathcal{N}_k(\mathbb{A}^*).$$

As above, i and π^* are both quasi-isomorphisms of cdgas. Thus, we have the equivalence of triangulated tensor categories

$$\mathrm{DT}_k := \mathrm{DCM}(\mathcal{N}_k) \sim \mathrm{DCM}(\mathcal{N}_k(\mathbb{A}^*)) \sim \mathrm{DCM}(\mathcal{N}_k^{\mathrm{gm}}(k)).$$

The functor $\mathbf{DT}_k \to DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$

We are now ready to define our representation of $\mathrm{DT}_k := \mathrm{DCM}(\mathcal{N}_k)$ into $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$. Let $\mathcal{N} = \mathcal{N}_k^{\mathrm{gm}}(k)$. We actually define a functor

$$\phi : \mathrm{DCM}(\mathcal{N})' \to DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$$

where $\mathrm{DCM}(\mathcal{N})'$ is the category of cell-$\mathcal{N}$-modules with a choice of basis. As this is equivalent to $\mathrm{DCM}(\mathcal{N})$, which in turn is equivalent to DT_k, the functor ϕ suffices for our purposes.

Let $M = \oplus_j \mathcal{N} m_j$ be a cell $\mathcal{N}$-module, with basis $\{m_j\}$ and differential d given by

$$dm_j = \sum_i a_{ij} m_i.$$

Encoding d as the matrix (a_{ij}), the condition $d^2 = 0$ translates as

$$(a_{ij}) \cdot (a_{ij}) = (da_{ij}), \tag{7}$$

where da_{ij} is the differential in $\mathcal{N}$. Let $\phi(M, d)$ be the complex of sheaves $\oplus_j \mathcal{N}_k^{\mathrm{gm}}(r_j)[n_j]\mu_j$, where μ_j is a formal basis element. The differential δ in $\phi(M, d)$ characterized by

$$\delta(\mu_j) := \sum_{ij} a_{ij}\mu_i,$$

and the requirement that δ satisfy the Leibniz rule

$$\delta(a \cdot \mu_j) = da \cdot \mu_j + (-1)^{\deg(a)} a \cdot \delta(\mu_j)$$

for a a local section of $\mathcal{N}_k^{\mathrm{gm}}(r_j)[n_j]$. The matrix equation (7) ensures that $\delta^2 = 0$, giving a well-defined object of $DM_-^{\mathrm{eff}}(k)$.

If $f : M \to N$ is a morphism of cell $\mathcal{N}$-modules, , we choose bases $\{m_j\}$ for M and $\{n_j\}$ for N, let $\{\mu_j\}$ and $\{\nu_j\}$ be the corresponding bases for $\phi(M)$ and $\phi(N)$. If $f(m_j) = \sum_i f_{ij} n_i$, then define $\phi(f)$ by $\phi(f)(\mu_j) = \sum_i f_{ij}$.

One easily checks that ϕ respects tensor products, the translation functor and cone sequences, so yields a well-defined exact tensor functor

$$\phi : \mathrm{DCM}(\mathcal{N}_k^{\mathrm{gm}}(k))' \to DM_{\mathrm{gm}}(k)_{\mathbb{Q}}.$$

By construction, $\phi(\mathbb{Q}(n))$ is the object $\mathcal{N}_k^{\mathrm{gm}}(n)$ of $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$, which by Lemma 5.26 and Lemma 5.27 is isomorphic to $C_*^{\mathrm{Sus}}(\mathbb{Z}_{\mathrm{q.fin}}(\mathbb{A}^n))_{\mathbb{Q}} \cong \mathbb{Q}(n)$ in $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$. Furthermore, we have

$$\mathrm{Hom}_{\mathrm{DT}_k}(\mathbb{Q}(0), \mathbb{Q}(n)[m]) = H^m(\mathcal{N}_k(n)) \cong \mathrm{CH}^n(k, 2n-m),$$

which agrees with $\mathrm{Hom}_{\mathrm{DTM}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)[m])$; it is not hard to see that ϕ induces the identity maps between these two Hom-groups. Since the $\mathbb{Q}(n)$'s are generators of DT_k, it follows that ϕ is fully faithful; since $\mathrm{DTM}(k)$ is generated by the $\mathbb{Q}(n)$'s, ϕ is therefore an equivalence. This completes the proof of the representation theorem 5.23.

6 Cycle classes, regulators and realizations

If one uses the axioms of §3.1 for a Bloch-Ogus cohomology theory, motivic cohomology becomes the universal Bloch-Ogus theory on $\mathbf{Sm}_k$. The various regulators on higher K-theory can then be factored through the Chern classes with values in motivic cohomology. Pushing this approach a bit further gives rise to "realization functors" from the triangulated category of mixed motives to the category of sheaves of abelian groups on $\mathbf{Sm}_k^{\mathrm{Zar}}$. In this section, we give a sketch of these constructions. See also the article of Goncharov [35] in this volume.

There are other methods available for defining realization functors which we will mention as well.

6.1 Cycle classes

We fix a Bloch-Ogus cohomology theory Γ on $\mathbf{Sm}_k$. In this section, we describe how one constructs functorial cycle classes

$$\mathrm{cl}_\Gamma^{q,p} : \mathrm{CH}^q(X, 2q-p) \to H_\Gamma^p(X, q),$$

and describe some of their basic properties. We refer to §3.1 for the notation.

Relative cycle classes

The main point of the construction is to use the purity property of Γ to extend the cycle classes to the relative case. Let $D = \sum_{i=1}^m D_i$ be a strict normal crossing divisor on some $Y \in \mathbf{Sm}_k$, that is, for each subset $I \subset \{1, \ldots, m\}$, the subscheme $D_I := \cap_{j\in I} D_j$ of Y is smooth over k and of pure codimension $|I|$ on Y. We include the case $I = \emptyset$ in the notation; explicitly $D_\emptyset = Y$.

Let $\tilde{\Gamma}(*)$ be a flasque model for $\Gamma(*)$, e.g., for each $X \in \mathbf{Sm}_k$, $\tilde{\Gamma}(n)(X)$ is the complex of global sections of the Godement resolution of the restriction of $\Gamma(n)$ to X_{Zar}; in particular, we have

$$H^n_\Gamma(X, q) = H^n(\tilde{\Gamma}(q)(X)).$$

Let $\tilde{\Gamma}(q)(X; D)$ be the iterated shifted cone of the restriction maps for the inclusions $D_i \to X$, that is, if $m = 1$, $D = D_1$, then

$$\tilde{\Gamma}(q)(X; D) := \mathrm{Cone}\big(i_D^* : \tilde{\Gamma}(q)(X) \to \tilde{\Gamma}(q)(D)\big)[-1]$$

and in general, $\tilde{\Gamma}(q)(X; D)$ is defined inductively as

$$\tilde{\Gamma}(q)(X; D) := \mathrm{Cone}\big(\tilde{\Gamma}(q)(X; \sum_{i=1}^{m-1} D_i) \xrightarrow{i_{D_1}^*} \tilde{\Gamma}(q)(D_1; \sum_{i=1}^{m-1} D_1 \cap D_i)\big)[-1].$$

One can also define $\tilde{\Gamma}(q)(X; D)$ as the total complex associated to the m-cube of complexes

$$I \mapsto \tilde{\Gamma}(q)(D_I),$$

from which one sees that the definition of $\tilde{\Gamma}(q)(X; D)$ is independent of the ordering of the D_i. Define the relative cohomology by

$$H^*_\Gamma(X; D, q) := H^*(\tilde{\Gamma}(q)(X; D)).$$

For $W \subset X$ a closed subset, we have relative cohomology with supports, defined as

$$H^*_{\Gamma,W}(X; D, q) := H^*(\tilde{\Gamma}^W(q)(X; D)),$$

where

$$\tilde{\Gamma}^W(q)(X; D) = \mathrm{Cone}\big(j^* : \tilde{\Gamma}(q)(X; D) \to \tilde{\Gamma}(q)(X \setminus W; j^*D)\big)[-1],$$

and $j : X \setminus W \to X$ is the inclusion.

Let $D' = \sum_{i=1}^r D'_i$ be a SNC divisor on X containing D. Let $z^q(X)_{D'}$ denote the subgroup of $z^q(X)$ generated by integral codimension q subschemes W such that

$$\mathrm{codim}_{D'_I}(W \cap D'_I) \geq q$$

for all $I \subset \{1, \ldots, r\}$, and let $z^q(X; D)_{D'}$ denote the kernel of the restriction map

$$z^q(X)_{D'} \xrightarrow{\sum_j i_j^*} \oplus_{j=1}^m z^q(D_j).$$

If $W \subset X$ is a closed subset, let $z^q_W(X; D)_{D'}$ be the subgroup of $z^q(X; D)_{D'}$ consisting of cycles supported in W; we write $z^q_W(X; D)$ for $z^q_W(X; D)_D$.

Lemma 6.1. *Let $W \subset X$ be a closed subset, $D = \sum_{i=1}^m D_i$ a strict normal crossing divisor on $X \in \mathbf{Sm}_k$. Let A be the ring $H^0_\Gamma(\mathrm{Spec}\, k, 0)$.*

1. *If $\mathrm{codim}_{D_I}(W \cap D_I) > q$ for all I, then $H^*_{\Gamma,W}(Y; D, q) = 0$. If $\mathrm{codim}_{D_I}(W \cap D_I) \geq q$ for all I, then $H^*_{\Gamma,W}(Y; D, q) = 0$ for all $p < 2q$*
2. *Suppose that $\mathrm{codim}_{D_I}(W \cap D_I) \geq q$ for all I. Then the cycle class map* cl *define an isomorphism*

$$\mathrm{cl} : z^q_W(X; D) \otimes A \to H^{2q}_W(X; D, q).$$

Proof. For $m = 0$, (1) is just the purity property of Definition 3.1(3). The property (5) and the Gysin isomorphism (4)(b) of 3.1 give the isomorphism of (2) for W smooth, and one uses purity again to extend to arbitrary W. In general, one uses the long exact cohomology sequences associated to a cone and induction on m. □

Higher Chow groups and relative Chow groups

Identifying the higher Chow groups with "relative Chow groups", making a similar identification for Γ-cohomology, and using the relative cycle map completes the construction.

For $m \geq 0$, let ∂^n_X and Λ^n_X be the SNC divisors $\sum_{i=0}^n (t_i = 0)$ and $\sum_{i=0}^{n-1} (t_i = 0)$ on $X \times \Delta^n$, respectively. For a commutative ring A, we have the higher Chow groups with A-coefficients

$$\mathrm{CH}^q(X, n; A) := H_n(z^q(X, *) \otimes A).$$

Define motivic cohomology with A-coefficients, $H^p(X, A(q))$, by

$$H^p(X, A(q)) := \mathrm{CH}^q(X, 2q - p; A).$$

Lemma 6.2. *There is an exact sequence*

$$z^q(X \times \Delta^{n+1}, \Lambda^{n+1}_X)_{\partial^{n+1}_X} \otimes A \xrightarrow{\mathrm{res}_{t_{n+1}=0}} z^q(X \times \Delta^n, \partial^n_X) \otimes A \to \mathrm{CH}^q(X, n; A) \to 0.$$

Proof. By the Dold-Kan theorem [29], the inclusion of the normalized subcomplex

$$N z^q(X, *) \to z^q(X, *)$$

is a quasi-isomorphism. Since $N z^q(X, n) = z^q(X \times \Delta^n, \Lambda^n_X)_{\partial^n_X}$ with differential

$$\mathrm{res}_{t_{n+1}=0} : z^q(X \times \Delta^{n+1}, \Lambda^{n+1}_X)_{\partial^{n+1}_X} \to z^q(X \times \Delta^n, \Lambda^n_X)_{\partial^n_X}$$

the result follows. □

Lemma 6.3. *Let X be in $\mathbf{Sm}_k$. Then $H^*_\Gamma(X\times\Delta^n;\Lambda^n_X,q)=0$ for all q and there is a natural isomorphism*

$$H^p_\Gamma(X\times\Delta^n;\partial^n_X,q)\cong H^{p-n}_\Gamma(X,q)$$

Proof. This follows from the homotopy property of Γ and induction on n. □

We can now define the cycle class map

$$\mathrm{CH}^q(X,n;A)\xrightarrow{\mathrm{cl}^q(n)}H^{2q-n}_\Gamma(X,q),$$

where A is the coefficient ring $H^0_\Gamma(\operatorname{Spec}k,0)$; we then have

$$\mathrm{cl}^{q,p}:H^p(X,A(q))\to H^p_\Gamma(X,q)$$

by $\mathrm{cl}^{q,p}:=\mathrm{cl}^q(2q-p)$.

Indeed, from Lemma 6.1, we have natural isomorphisms

$$z^q(X\times\Delta^n,\partial^n_X)\otimes A\cong\varinjlim_W H^{2q}_{\Gamma,W}(X\times\Delta^n;\partial^n_X,q)$$
$$z^q(X\times\Delta^{n+1},\Lambda^{n+1}_X)_{\partial^{n+1}_X}\otimes A\cong\varinjlim_{W'} H^{2q}_{\Gamma,W'}(X\times\Delta^{n+1};\Lambda^{n+1}_X,q)$$

where W runs over codimension q closed subsets of $X\times\Delta^n$, "in good position" with respect to the faces of Δ^n, and W' runs over codimension q closed subsets of $X\times\Delta^{n+1}$, "in good position" with respect to the faces of Δ^{n+1}. "Forgetting the supports" gives maps

$$\varinjlim_W H^{2q}_{\Gamma,W}(X\times\Delta^n;\partial^n_X,q)\to H^{2q}_\Gamma(X\times\Delta^n;\partial^n_X,q)$$
$$\varinjlim_{W'} H^{2q}_{\Gamma,W'}(X\times\Delta^{n+1};\Lambda^{n+1}_X,q)\to H^{2q}_\Gamma(X\times\Delta^{n+1};\Lambda^{n+1}_X,q)$$

Putting these together and using and Lemma 6.2 and Lemma 6.3 gives the desired cycle class maps

$$\mathrm{cl}^q(n):\mathrm{CH}^q(X,n;A)\to H^{2q-n}_\Gamma(X,q).$$

Remark 6.4. With a bit more work, one can achieve the maps $\mathrm{cl}^{q,p}$ as maps

$$\mathrm{cl}^q:\mathbb{Z}_{\mathrm{FS}}(q)\otimes^L A\to\Gamma(q)\tag{8}$$

in $D(\mathrm{Sh}^{\mathrm{Zar}}(k))$, compatible with the multiplicative structure. Using Remark 2.22, we have the structure map

$$\mathrm{cl}^1\circ u:\mathbb{G}_m[-1]\to\Gamma(1)$$

promised in §3.1.

For additional details, we refer the reader to [33] ([33] considers cl^q as a map from the cycle complexes $\mathbb{Z}_{\mathrm{Bl}}(q)$ instead, but one can easily recover the statements made above from this).

In any case, we have:

Theorem 6.5. *Fix a coefficient ring A. Motivic cohomology with A-coefficients, $H^*(-, A(*))$, as the Bloch-Ogus theory on $\mathbf{Sm}_k$ represented by $\mathbb{Z}_{\mathrm{FS}}(*) \otimes^L A$, is the universal Bloch-Ogus cohomology theory with coefficient ring A, in the sense of Definition 3.1.*

Remark 6.6. With minor changes, the cycle classes described here extend to the case of scheme smooth and quasi-projective over a Dedekind domain, for example, over a localization of a ring of integers in a number field, using the extension of the cycle complexes described in Remark 2.9. For instance, we have cycle classes

$$\mathrm{cl}^{q,p} : H^p(X, \mathbb{Z}/n(q)) := \mathrm{CH}^q(X, 2q-p; \mathbb{Z}/n) \to H^p_{\text{ét}}(X, \mathbb{Z}/n(q))$$

for $X \to \mathrm{Spec}\,(\mathcal{O}_F[1/n])$ smooth and quasi-projective, F a number field.

Explicit formulas

The abstract approach outlined above does not lend itself to easy computations in explicit examples, except perhaps for the case of units and Milnor K-theory. Goncharov explains in his article [35] how one can give a fairly explicit formula for the cycle class map to real Deligne cohomology; this has been refined recently in [59] and [60] to give formulas for the map to integral Deligne cohomology. Although this search for explicit formulas may at first seem to be merely a computational convenience, in fact such formulas lie at the heart of some important conjectures, for instance, Zagier's conjecture on relating values of L-functions to polylogarithms [102].

Regulators

The classical case of a regulator is the Dirichlet regulator, which is the co-volume of the lattice of units of a number field under the embedding given by the logarithm of the various absolute values. The term "regulator" now generally refers to a real-valued invariant of some K-group, especially if there is some link with the classical case.

The Dirichlet construction was first generalized to higher K-theory of number rings by Borel [20] using group cohomology, and was later reinterpreted by Beilinson [5] as a lattice co-volume arising from a Gillet-type Chern class to real Deligne cohomology. In the context of the cycle class maps described above, we only wish to remark that it is easy to show that Gillet's Chern class $c^{p,q}_\Gamma : K_{2q-p}(X) \to H^p_\Gamma(X, q)$ factors as

$$K_{2q-p}(X) \xrightarrow{c^{p,q}} H^p(X, \mathbb{Z}(q)) \xrightarrow{\mathrm{cl}^q_\Gamma} H^p_\Gamma(X, q).$$

for $\Gamma(*)$ a Bloch-Ogus cohomology theory.

6.2 Realizations

Extending the cycle class map

In this section, we describe the method used by Levine [63, Part 1, Chap. V] for defining a realization functor on $\mathcal{DM}(k)$ associated to a Bloch-Ogus cohomology theory Γ (see [63, Part 1, Chap. V, Theorem 1.3.1] for a precise statement, but note the remark below). We retain the notation of §6.1.

Remark 6.7. There is an error in the statement of [63, Part 1, Chap. V, Definition 1.1.6 and Theorem 1.3.1]: The graded complex of sheaves $\mathcal{F}$ should be of the form $\mathcal{F} = \oplus_{q\in\mathbb{Z}}\mathcal{F}(q)$, not $\oplus_{qge0}\mathcal{F}(q)$, as it is stated in *loc. cit.*. In Definition 1.1.6, the axioms (ii) and (iii) are for $q \geq 0$, whereas the axiom (iv) is for all q_1, q_2 and axiom (v) is for all q. I am grateful to Bruno Kahn for pointing out this error.

One would at first like to extend the assignment $\mathbb{Z}_X(q) \mapsto \tilde{\Gamma}(q)(X)$ to a functor

$$\Re_\Gamma : \mathcal{DM}(k) \to D(\mathbf{Ab}).$$

There are essentially two obstructions to doing this:

1. In $\mathcal{DM}(k)$, we have the isomorphism

   $$\mathbb{Z}_X(q) \otimes \mathbb{Z}_Y(q') \cong \mathbb{Z}_{X\times Y}(q+q'),$$

   but there is no requirement that the external products for Γ induce an analogous isomorphism in $D(\mathbf{Ab})$,

   $$\tilde{\Gamma}(q)(X) \otimes^L \tilde{\Gamma}(q')(Y) \cong \tilde{\Gamma}(q+q')(X \times Y).$$

   In fact, in many naturally occuring examples, the above map is *not* an isomorphism.
2. The object $\Gamma(*) = \oplus_q \Gamma(q)$ is indeed a commutative ring-object in the derived category of sheaves on $\mathbf{Sm}_k^{\mathrm{Zar}}$, but the commutativity and asscia-tivity properties of the product may not lift to similar properties on the level of the representing complexes $\tilde{\Gamma}(q)(X)$.

To avoid these problems, one considers a refinement of a Bloch-Ogus theory, namely a *geometric* cohomology theory on $\mathbf{Sm}_k^?$ [63, Part 1, Chap. V, Def. 1.1.6], where ? is a Grothendieck topology, at least as fine as the Zariski topology, having enough points (e.g., the étale, Zariski or Nisnevic topologies). Let A be a commutative ring and let $\mathrm{Sh}_A^?(\mathbf{Sm}_k)$ be the category of sheaves of A-modules on $\mathbf{Sm}_k^?$. Essentially, a geometric cohomology Γ is given by a graded commutative ring object $\hat{\Gamma}(*) = \oplus_{q\in\mathbb{Z}}\hat{\Gamma}(q)$ in $C(\mathrm{Sh}_A^?(\mathbf{Sm}_k))$, such that

1. All stalks of the sheaves $\hat{\Gamma}(q)^n$ are flat A-modules.

2. For X in $\mathbf{Sm}_k$, let $p_X : X \to \operatorname{Spec} k$ denote the projection. Then for X and Y in $\mathbf{Sm}_k$, the product map
$$Rp_{X*}(\hat{\Gamma}(q)_{|X}) \otimes^L Rp_{Y*}(\hat{\Gamma}(q')_{|Y}) \to Rp_{X\times Y*}(\hat{\Gamma}(q+q')_{|X\times Y})$$
is an isomorphism in $D(\mathrm{Sh}^?_A(\operatorname{Spec} k))$.
3. Let $\alpha : \mathbf{Sm}^?_k \to \mathbf{Sm}^{\mathrm{Zar}}_k$ be the change of topology morphism, and let $\Gamma(n) := R\alpha_*\hat{\Gamma}(n)$. Then $\Gamma(*) := \oplus_{n\geq 0}\Gamma(n)$ defines a Bloch-Ogus cohomology theory on $\mathbf{Sm}_k$, in the sense of §3.1.
4. Let 1 denote the unit in $\mathrm{Sh}^?_A(\operatorname{Spec} k)$ and $[\widehat{\operatorname{Spec} k}] : 1 \to \hat{\Gamma}(0)_{|\operatorname{Spec} k}$ the map in $D(\mathrm{Sh}^?_A(\operatorname{Spec} k))$ corresponding to the cycle class $[\operatorname{Spec} k] \in H^0_\Gamma(\operatorname{Spec} k, 0)$. Then $[\widehat{\operatorname{Spec} k}]$ is an isomorphism.

Examples of such theories include: de Rham cohomology, singular cohomology, étale cohomology with mod n coefficients.

Having made this refinement, one is able to extend to assignment $\mathbb{Z}_X(q) \mapsto Rp_{X*}(\hat{\Gamma}(q)_{|X})$ to a good realization functor:

Theorem 6.8 ([63, Part 1, Chap. V, Thm. 1.3.1]). *Let Γ be a geometric cohomology theory on $\mathbf{Sm}^?_k$, and let $A := H^0_\Gamma(\operatorname{Spec} k, 0)$. Then sending $\mathbb{Z}_X(q)$ to $Rp_{X*}(\hat{\Gamma}(q)_{|X})$ extends to an exact pseudo-tensor functor*
$$\Re_\Gamma : \mathcal{DM}(k)_A \to D(\mathrm{Sh}^?_A(\operatorname{Spec} k)).$$

Here $\mathrm{Sh}^?_A(\operatorname{Spec} k)$ is the category of sheaves of A-modules on $\operatorname{Spec} k$, for the ?-topology. $\mathcal{DM}(k)_A$ is the extension of $\mathcal{DM}(k)$ to an A-linear triangulated category formed by taking the A-extension of the additive category $\bar{\mathcal{A}}_{\mathrm{mot}}(k)$ and applying the construction used in §4.4 to form $\mathcal{DM}(k)$ (this is not the same as the standard A-extension $\mathcal{DM}(k)\otimes A$ if for instance A is not flat over $\mathbb{Z}$).

The rough idea is to first extend the assignment
$$\mathbb{Z}_X(q) \mapsto Rp_{X*}(\hat{\Gamma}(q)_{|X})$$
to the additive category $\bar{\mathcal{A}}_{\mathrm{mot}}(k) \otimes A$ (notation as in §4.4) by sending the cycle map $[Z] : * \to \mathbb{Z}_X(d)[2d]$ to a representative of the cycle class with supports in codimension q for the cohomology theory Γ. The lack of a canonical representative creates problems, so we replace $\bar{\mathcal{A}}_{\mathrm{mot}}(k)$ with a DG-category $\mathcal{A}_{\mathrm{mot}}(k)$ for which the relations among the cycle maps $[Z]$ are only satisfied up to homotopy and "all higher homotopies". Proceeding along this line, one constructs a functor
$$\Re^{(*)}_\Gamma : K^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A) \to K(\mathrm{Sh}^?_A(\operatorname{Spec} k)).$$
One then "forgets supports" in the theory Γ and passes to the derived category
$$\Re^K_\Gamma : K^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A) \to D(\mathrm{Sh}^?_A(\operatorname{Spec} k)).$$

Now let $D^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A)$ be the localization of $K^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A)$ as a triangulated tensor category, formed by inverting the same generating set of maps we used to form $D^b(\bar{\mathcal{A}}_{\mathrm{mot}}(k))$ from $K^b(\bar{\mathcal{A}}_{\mathrm{mot}}(k))$. The Bloch-Ogus axioms for Γ imply that $\Re^K_\Gamma$ extends to a functor on $D^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A)$; one then extends to the pseudo-abelian hull of $D^b(\mathcal{A}_{\mathrm{mot}}(k) \otimes A)$ and proves that this pseudo-abelian hull is equivalent to our original category $\mathcal{DM}(k)_A$.

Remark 6.9. We take this opportunity to correct an error in [63], pointed out to us by Bruno Kahn: In [63], we only required that a geometric cohomology be non-negatively graded: $\hat{\Gamma}(*) = \oplus_{q\geq 0}\hat{\Gamma}(q)$. This of course leaves nowhere to send $\mathbb{Z}_X(q)$ for $q < 0$, so the full $\mathbb{Z}$-grading, as described above, is required.

Remark 6.10. Although theories such as Beilinson's absolute Hodge cohomology, Deligne cohomology, or ℓ-adic étale cohomology do not fit into the framework of a geometric cohomology theory, the method of construction of the realization functor does go through to give realization functors for these theories as well. We refer the reader to [63, Part 1, Chap. V, §2] for these constructions.

We would like to correct an error in our construction of the absolute Hodge realization, pointed out to us by Pierre Deligne: In diagram (2.3.8.1), pg. 279, defining the object $D[X, \bar{X}]$, the operation Dec is improperly applied, and the functor $p_{(X,\bar{X})*}$ (top of page 278) is incorrectly defined. To correct this, one changes $p_{(X,\bar{X})*}$ by first taking global sections as indicated in diagram (2.3.6,8), and then applying the operation Dec to all the induced W-filtrations on the global sections. One also deletes the operation Dec from all applications in the diagram (2.3.8.1) defining $D[X, \bar{X}]$. With these changes, the construction goes through as described in [63].

Huber's method

Huber constructs realizations for the rational Voevodsky category $DM_{\mathrm{gm}}(k)_{\mathbb{Q}}$ in ([49, 50]) using a method very similar to the construction used by Nori to prove Proposition 3.17. The idea is the following: Suppose the base field is $\mathbb{C}$. Let $W \to X$ be a finite dominant morphism, with $X \in \mathbf{Sm}_k$, and W and X irreducible. Let $W' \to X$ be the normalization of X in the Galois closure of $k(W)/k(X)$, let $G = \mathrm{Gal}(k(W')/k(X))$, and let $C^*(X)$ denote the singular cochain complex of $X(\mathbb{C})$ with $\mathbb{Q}$-coefficients. Then G acts on $C^*(W')$, and in fact the natural map $C^*(X) \to C^*(W')$ gives a quasi-isomorphism

$$C^*(X) \to C^*(W')^G,$$

where $C^*(W')^G$ is the subcomplex of $C^*(W')$ of G-invariant cochains. Also, since we have $\mathbb{Q}$-coefficients, there is a projection $\pi : C^*(W') \to C^*(W')^G$. Thus, one can define the pushforward $\pi_{W/X*} : C^*(W) \to C^*(X)$ as the composition in $D^+(\mathbb{Q}\text{-mod})$

$$C^*(W) \xrightarrow{\frac{1}{d}p^*} C^*(W') \xrightarrow{\pi} C^*(W')^G \xleftarrow{\sim} C^*(X)$$

where $p : W' \to W$ is the projection and d is the degree of p. Now, if $W = \sum_i n_i W_i$ is in $\mathrm{Cor}(X, Y)$, we have the map

$$W_* : C^*(Y) \to C^*(X)$$

in $D^+(\mathbb{Q}\text{-mod})$ defined as the sum $\sum_i n_i W_{i*}$, where W_{i*} is the composition

$$C^*(Y) \xrightarrow{\pi^*_{W_i/Y}} C^*(W_i) \xrightarrow{\pi_{W_i/X*}} C^*(X)$$

where $\pi_{W_i/Y} : W_i \to Y$ is the evident map.

Refining this to give maps on the level of complexes, the assignment $X \mapsto C^*(X)$ extends to a functor

$$R_{\text{sing}} : \mathrm{Cor}(\mathbb{C})^{\text{op}}_{\mathbb{Q}} \to C^+(\mathbb{Q}\text{-Vec});$$

the properties of singular cohomology as a Bloch-Ogus theory imply that R_{sing} extends to an exact functor

$$\Re_{\text{sing}} : DM_{\text{gm}}(\mathbb{C})_{\mathbb{Q}} \to D^+(\mathbb{Q}\text{-mod}).$$

Two essential problems occur in this approach:

1. For many interesting theories Γ (e.g. de Rham cohomology), even though there are extensions of Γ to complexes on all reduced normal quasi-projective k-schemes, it is often not the case that $\Gamma(q)(X) \to \Gamma(q)(W')^G$ is a quasi-isomorphism, as was the case for singular cohomology.
2. It is not so easy (even in the case of singular cohomology) to refine the map $\pi_{W/X*}$ to give a functorial map on the level of complexes.

Huber overcomes these difficulties to give realizations for singular cohomology, as described above, as well as for $\mathbb{Q}_\ell$-étale cohomology, and rational Deligne cohomology.

Nori's realizations

Using the functor (4) (see just below Definition 3.20)

$$\Pi : DM_{\text{gm}}(k) \to D^b(\mathrm{NMM}(k)),$$

and the universal property of the category $\mathrm{NMM}(k)$ (derived from the universal property of $\mathrm{ECM}(k)$), one has integral realization functors from $DM_{\text{gm}}(k)$ for: singular cohomology, ℓ-adic étale cohomology, de Rham cohomology, and Beilinson's absolute Hodge cohomology. These do not seem to have been used at all in the literature up to now, so we hope that a good version of Nori's work will appear soon.

Bloch-Kriz realizations

We conclude our overview of realizations by briefly discussing the method used in [18] for constructing realizations of the Tate category BKTM_k. Denote the motivic Hopf algebra $H^0(\bar{B}(\mathcal{N}_k))$ by $\chi_{\mathrm{mot}}(k)$ (see Definition 5.19 for the notation).

One can consider for instance the category of continuous $G_k := \mathrm{Gal}(\bar{k}/k)$ representations M in finite dimensional $\mathbb{Q}_\ell$-vector spaces, such that M has a finite filtration W_*M with quotients $\mathrm{gr}_n^W M$ being given by the nth power of the cyclotomic character. This forms a Tannakian $\mathbb{Q}_\ell$-category, classified by an Adams-graded Hopf algebra $\chi_{\text{ét},\ell}(k)$. Thus, in order to define an étale realization of the category $\mathrm{BKTM}_k = \text{co-rep}(\chi_{\mathrm{mot}}(k))$, it suffices to give a homomorphism of Hopf algebras

$$\phi_{\text{ét}} : \chi_{\mathrm{mot}}(k) \to \chi_{\text{ét},\ell}(k).$$

Using a modification of the cycle-class method discussed in §6.1, they show that the cycle class map (8), for $\Gamma(*) = \mathbb{Q}_\ell$-étale cohomology, can be refined to give rise to such a homomorphism $\phi_{\text{ét}}$, and hence a realization functor

$$\phi_{\text{ét}} : \mathrm{BKTM}_k \to \text{co-rep}(\chi_{\text{ét},\ell}(k)) \to \mathbb{Q}_\ell[G_k]\text{-mod}.$$

It would be interesting to compare this realization functor with the one given by Spitzweck's representation theorem and Nori's realization functor.

A similar method yields a description of real mixed Hodge structures as the Tannakian category of co-representations of an Adams graded Hopf algebra χ_{Hdg} over $\mathbb{R}$, and a realization homomorphism

$$\phi_{\mathrm{Hdg}} : \chi_{\mathrm{mot}}(\mathbb{C}) \to \chi_{\mathrm{Hdg}}.$$

Again, it would be interesting to compare this with Nori's approach, and to see if the refined cycle classes of [59, 60] allow one to give a more explicit description of ϕ_{Hdg}.

References

1. Balmer, Paul; Schlichting, Marco. *Idempotent completion of triangulated categories*. J. Algebra **236** (2001), no. 2, 819–834.
2. Beilinson, A. A. *Height pairing between algebraic cycles*. K-theory, arithmetic and geometry (Moscow, 1984–1986), 1–25, Lecture Notes in Math., Vol. 1289, Springer, Berlin-New York, 1987.
3. Beilinson, A. A. *On the derived category of perverse sheaves*. K-theory, arithmetic and geometry (Moscow, 1984–1986), 27–41, Lecture Notes in Math., Vol. 1289, Springer, Berlin-New York, 1987.
4. Beilinson, A. A. *Notes on absolute Hodge cohomology*. Applications of algebraic K-theory to algebraic geometry and number theory, Part I (Boulder, Colo., 1983), 35–68, Contemp. Math., **55**, Amer. Math. Soc., Providence, R.I., 1986.

5. Beilinson, A. A. *Higher regulators and values of L-functions*. Current problems in mathematics, Vol. 24, 181–238, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.
6. Beilinson, A. A. *Letter to Soulé*, 1982.
7. Beilinson, A.A.; Deligne, Pierre. *Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs*. Motives (Seattle, WA, 1991), 97–121, Proc. Sympos. Pure Math., **55**, Part 2, Amer. Math. Soc., Providence, RI, 1994.
8. Bloch, Spencer. **Higher regulators, algebraic K-theory and zeta functions of elliptic curves**. CRM monograph series **11**, Amer. Math. Soc., Prov. RI, 2000.
9. Bloch, Spencer. *Lectures on mixed motives*. Algebraic geometry—Santa Cruz 1995, 329–359, Proc. Sympos. Pure Math., **62**, Part 1, Amer. Math. Soc., Providence, RI, 1997.
10. Bloch, Spencer. *The moving lemma for higher Chow groups*. J. Algebraic Geom. **3** (1994), no. 3, 537–568.
11. Bloch, Spencer. *Algebraic cycles and the Lie algebra of mixed Tate motives*. J. Amer. Math. Soc. **4** (1991), no. 4, 771–791.
12. Bloch, Spencer. *Algebraic K-theory, motives, and algebraic cycles*. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 43–54, Math. Soc. Japan, Tokyo, 1991.
13. Bloch, Spencer. *Algebraic cycles and higher K-theory*. Adv. in Math. **61** (1986), no. 3, 267–304.
14. Bloch, Spencer. *Algebraic cycles and values of L-functions. II*. Duke Math. J. **52** (1985), no. 2, 379–397.
15. Bloch, Spencer. *Algebraic cycles and values of L-functions*. J. Reine Angew. Math. 350 (1984), 94–108.
16. Bloch, Spencer; Kato, Kazuya. *L-functions and Tamagawa numbers of motives*. The Grothendieck Festschrift, Vol. I, 333–400, Progr. Math., **86**, Birkhäuser Boston, Boston, MA, 1990.
17. Bloch, Spencer; Kato, Kazuya. *p-adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. **63** (1986), 107-152.
18. Bloch, Spencer; Kriz, Igor. *Mixed Tate motives*. Ann. of Math. (2) **140** (1994), no. 3, 557–605.
19. Bloch, Spencer; Ogus, Arthur. *Gersten's conjecture and the homology of schemes*. Ann. Sci. École Norm. Sup. (4) **7** (1974), 181–201.
20. Borel, Armand. *Stable real cohomology of arithmetic groups*. Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272.
21. Deligne, Pierre. *À quoi servent les motifs?* Motives (Seattle, WA, 1991), 143–161, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994.
22. Deligne, Pierre. *Catégories tannakiennes*. The Grothendieck Festschrift, Vol. II, 111–195, Progr. Math., **87**, Birkhäuser Boston, Boston, MA, 1990.
23. Deligne, Pierre. *Le groupe fondamental de la droite projective moins trois points*. Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 79–297, Math. Sci. Res. Inst. Publ., **16**, Springer, New York-Berlin, 1989.
24. Deligne, Pierre. *Letter to C. Soulé*. (1985).
25. Deligne, Pierre. *Valeurs de fonctions L et périodes d'intégrales*. With an appendix by N. Koblitz and A. Ogus. Proc. Sympos. Pure Math., **XXXIII**, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math.,

Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 313–346, Amer. Math. Soc., Providence, R.I., 1979.

26. Deligne, Pierre. *Théorie de Hodge. II.* Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 5–57.
27. Deligne, Pierre; Goncharov, A.B. *Groupes fondamentaux motiviques de Tate mixte*. Preprint, Feb 2003, math.NT/0302267, http://lanl.arXiv.org.
28. Deligne, Pierre; Milne, James S.; Ogus, Arthur; Shih, Kuang-yen. **Hodge cycles, motives, and Shimura varieties**. Lecture Notes in Mathematics, Vol. 900. Philosophical Studies Series in Philosophy, **20**. Springer-Verlag, Berlin-New York, 1982.
29. Dold, Albrecht. *Homology of symmetric products and other functors of complexes*. Ann. of Math. (2) **68** (1958) 54–80.
30. Dupont, Johan L.; Sah, Chih Han. *Scissors congruences. II.* J. Pure Appl. Algebra **25** (1982), no. 2, 159–195.
31. Fakhruddin, Najmuddin. *Notes of Nori's lectures on mixed motives*. T.I.F.R., Mumbai, 2000.
32. Geisser, Thomas. *Motivic cohomology, K-theory and topological cyclic homology*. This volume.
33. Geisser, Thomas; Levine, Marc. *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, Jour. f.d. Reine u. Ang. Math **530** (2001), 55-103.
34. Gillet, Henri. *Riemann-Roch theorems for higher algebraic K-theory*. Adv. in Math. **40** (1981), no. 3, 203–289.
35. Goncharov, A. B. *Regulators*. This volume.
36. Goncharov, A. B. *Periods and mixed motives*. Preprint May 2002, math.AG/0202154, http://lanl.arXiv.org.
37. Goncharov, A. B. *Multiple polylogarithms and mixed Tate motives*. Preprint, May 2001, math.AG/0103059, http://lanl.arXiv.org.
38. Goncharov, A. B., *Geometry of configurations, polylogarithms, and motivic cohomology*. Adv. Math. **114** (1995), no. 2, 197–318.
39. Goncharov, A. B. *Polylogarithms and motivic Galois groups*. Motives (Seattle, WA, 1991), 43–96, Proc. Sympos. Pure Math., **55**, Part 2, Amer. Math. Soc., Providence, RI, 1994.
40. Goncharov, A. B.; Manin, Yu. *Multiple zeta-motives and moduli spaces $M_{0,n}$*. Preprint April 2002, math.AG/0204102, http://lanl.arXiv.org.
41. Guillén, Francisco; Navarro Aznar, Vicente. *Un critère d'extension des foncteurs définis sur les schémas lisses*. Publ. Math. Inst. Hautes Études Sci. No. **95** (2002), 1–91.
42. Friedlander, E.M. *Motivic complexes of Suslin and Voevodsky*, this volume.
43. Hain, Richard; Matsumoto, Makoto. *Tannakian Fundamental Groups Associated to Galois Groups*. Preprint May 2002, math.AG/0010210, http://lanl.arXiv.org.
44. Hanamura, Masaki. *Homological and cohomological motives of algebraic varieties*. Invent. Math. **142** (2000), no. 2, 319–349.
45. Hanamura, Masaki. *Mixed motives and algebraic cycles. III.* Math. Res. Lett. **6** (1999), no. 1, 61–82.
46. Hanamura, Masaki. *Mixed motives and algebraic cycles, II.* Preprint, 1996.
47. Hanamura, Masaki. *Mixed motives and algebraic cycles, I.* Math. Res. Lett. **2** (1995), no. 6, 811–821.
48. Huber, Annette. **Mixed motives and their realization in derived categories**. Lecture Notes in Mathematics, Vol. 1604. Springer-Verlag, Berlin, 1995.

49. Huber, Annette. *Realization of Voevodsky's motives*. J. Algebraic Geom. **9** (2000), no. 4, 755–799.
50. Huber, Annette. *Erratum*. To appear, J. Algebraic Geom.
51. Huber, Annette; Wildeshaus, Jörg. *Classical motivic polylogarithm according to Beilinson and Deligne*. Doc. Math. **3** (1998), 27–133.
52. Jannsen, Uwe. *Mixed motives, motivic cohomology, and Ext-groups*. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 667–679, Birkhäuser, Basel, 1995.
53. Jannsen, Uwe. *Motivic sheaves and filtrations on Chow groups*. Motives (Seattle, WA, 1991), 245–302, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994.
54. Jannsen, Uwe. **Mixed motives and algebraic K-theory**. With appendices by S. Bloch and C. Schoen. Lecture Notes in Mathematics, Vol. 1400. Springer-Verlag, Berlin, 1990.
55. Kahn, Bruno. *La conjecture de Milnor (d'àpres V. Voevodsky)*. This volume.
56. Kahn, Bruno. *Algebraic K-theory, algebraic cycles and arithmetic geometry*. This volume.
57. Kahn, Bruno. *Weight filtration and mixed Tate motives*, preprint Feb. 2001, http://www.math.uiuc.edu/K-theory/0467/
58. Kapranov, M. M. *On the derived categories of coherent sheaves on some homogeneous spaces*. Invent. Math. **92** (1988), no. 3, 479–508.
59. Kerr, Matt. *Geometric construction of regulator currents with applications to algebraic cycles*. Princeton University Ph.D. thesis, 2003.
60. Kerr, Matt; Lewis, James D.; Müller-Stach, Stefan. *The Abel-Jacobi map for higher Chow groups*. Preprint, 2003.
61. Kriz, Igor; May, J. P. **Operads, algebras, modules and motives**. Astérisque No. 233 (1995)
62. Levine, Marc. *Techniques of localization in the theory of algebraic cycles*, J. Alg. Geom. **10** (2001) 299-363.
63. Levine, Marc. **Mixed Motives**, Math. Surveys and Monographs **57** AMS, Prov. 1998.
64. Levine, Marc. *Bloch's higher Chow groups revisited.* K-theory (Strasbourg, 1992). Astérisque No. 226 (1994), 10, 235–320.
65. Levine, Marc. *Tate motives and the vanishing conjectures for algebraic K-theory*. Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), 167–188, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **407**, Kluwer Acad. Publ., Dordrecht, 1993.
66. Levine, Marc. *The indecomposable K_3 of fields*, Ann. Sci. Ecole Norm. Sup.(4) **22** No. 2(1989) 255-344.
67. Lichtenbaum, Stephen. *Motivic complexes*. Motives (Seattle, WA, 1991), 303–313, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994.
68. Lichtenbaum, Stephen. *The construction of weight-two arithmetic cohomology*. Invent. Math. **88** (1987), no. 1, 183–215.
69. Lichtenbaum, Stephen. *Values of zeta-functions at nonnegative integers*. Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), 127–138, Lecture Notes in Math., Vol. 1068, Springer, Berlin, 1984.
70. Mazza, C.; Voevodsky, V.; Weibel, C. *Notes on motivic cohomology*, Beta.1 version, Fall 2003. http://math.rutgers.edu/ weibel/motiviclectures.html

71. Merkurjev, A. S.; Suslin, A. A. *The group K_3 for a field*, Math. USSR-Izv. **36** (1991), no. 3, 541–565.
72. Merkurjev, A. S.; Suslin, A. A. *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*. Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 1011–1046, 1135–1136.
73. Milnor, John. *Algebraic K-theory and quadratic forms*. Invent. Math. **9** (1969/1970) 318–344.
74. Murre, J. P. *On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples*. Indag. Math. (N.S.) **4** (1993), no. 2, 177–188.
75. Nekovář, Jan. *Beilinson's conjectures*. Motives (Seattle, WA, 1991), 537–570, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994.
76. Nesterenko, Yu. P.; Suslin, A. A. *Homology of the general linear group over a local ring, and Milnor's K-theory*. Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146.
77. Nori, Mahdav. *A modified Hodge conjecture*. Text of a lecture given at Duke University, March 2003.
78. Nori, Mahdav. *Constructible sheaves*. Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000), 471–491, Tata Inst. Fund. Res. Stud. Math., **16**, Tata Inst. Fund. Res., Bombay, 2002.
79. Ojanguren, Manuel; Panin, Ivan. *Rationally trivial Hermitian spaces are locally trivial*. Math. Z. **237** (2001), no. 1, 181–198.
80. Ramakrishnan, Dinakar. *Regulators, algebraic cycles, and values of L-functions*. Algebraic K-theory and algebraic number theory, 183–310, Contemp. Math., **83**, Amer. Math. Soc., Providence, RI, 1989.
81. Scholl, Anthony J. *Remarks on special values of L-functions*. L-functions and arithmetic (Durham, 1989), 373–392, London Math. Soc. Lecture Note Ser., **153**, Cambridge Univ. Press, Cambridge, 1991.
82. Scholl, Anthony J. *Extensions of motives, higher Chow groups and special values of L-functions*. Séminaire de Théorie des Nombres, Paris, 1991–92, 279–292, Progr. Math., **116**, Birkhäuser Boston, Boston, MA, 1993.
83. Saavedra Rivano, Neantro. **Catégories Tannakiennes**. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York, 1972.
84. Soulé, Christophe. *Opérations en K-théorie algébrique*. Canad. J. Math. **37** (1985), no. 3, 488–550.
85. Spitzweck, Marcus. *Some constructions for Voevodsky's triangulated categories of motives*. Preprint.
86. Suslin, Andrei; Voevodsky, Vladimir. *Singular homology of abstract algebraic varieties*. Invent. Math. **123** (1996), no. 1, 61–94.
87. Suslin, A. A. *K_3 of a field, and the Bloch group*. Galois theory, rings, algebraic groups and their applications (Russian). Trudy Mat. Inst. Steklov. **183** (1990), 180–199, 229.
88. Suslin, A. A. *Algebraic K-theory of fields*. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 222–244, Amer. Math. Soc., Providence, RI, 1987.
89. Suslin, A. A. *Homology of GL_n, characteristic classes and Milnor K-theory*. Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), 357–375, Lecture Notes in Math., **1046**, Springer, Berlin, 1984.

90. Suslin, Andrei; Voevodsky, Vladimir, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*. The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.
91. Terasoma, Tomohide. *Mixed Tate motives and multiple zeta values*. Invent. Math. **149** (2002), no. 2, 339–369.
92. Totaro, Burt. *Milnor K-theory is the simplest part of algebraic K-theory*. K-Theory **6** (1992), no. 2, 177–189.
93. Voevodsky, Vladimir. *Motivic cohomology with $\mathbb{Z}/l$-coefficients*. Preprint June 2003, http://www.math.uiuc.edu/K-theory/0639/
94. Voevodsky, Vladimir. *Motives over simplicial schemes*. Preprint June 2003, http://www.math.uiuc.edu/K-theory/0638/
95. Voevodsky, Vladimir. *Cross functors*. Lecture ICTP, Trieste, July 2002.
96. Voevodsky, Vladimir. *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*. Int. Math. Res. Not. 2002, no. 7, 351–355.
97. Voevodsky, Vladimir. *Cancellation theorem*. Preprint Jan. 2002, http://www.math.uiuc.edu/K-theory/0541/
98. Voevodsky, Vladimir. *On 2-torsion in motivic cohomology*. Preprint July 2001, http://www.math.uiuc.edu/K-theory/0502/
99. Voevodsky, Vladimir. *The Milnor Conjecture*. Preprint Dec. 1996, http://www.math.uiuc.edu/K-theory/0170/
100. Voevodsky, Vladimir; Suslin, Andrei; Friedlander, Eric M. **Cycles, transfers, and motivic homology theories**. Annals of Mathematics Studies, **143**. Princeton University Press, Princeton, NJ, 2000.
101. Weibel, Charles. *Mennicke type symbols for relative K_2*, Lecture Notes in Math. **1046** 451-464.
102. Zagier, Don. *Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields*. Arithmetic algebraic geometry (Texel, 1989), 391–430, Progr. Math., **89**, Birkhäuser Boston, Boston, MA, 1991.
103. **Beilinson's conjectures on special values of L-functions**. Edited by M. Rapoport, N. Schappacher and P. Schneider. Perspectives in Mathematics, **4**. Academic Press, Inc., Boston, MA, 1988.
104. **Motives**. Summer Research Conference on Motives, U. Jannsen, S. Kleiman, J.-P. Serre, ed., Proc. of Symp. in Pure Math. **55** part 1, AMS, Providence, R.I., 1994.

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