

A COMPARISON OF MOTIVIC AND CLASSICAL STABLE HOMOTOPY THEORIES

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ABSTRACT. Let k be an algebraically closed field of characteristic zero. Let $c : \mathcal{SH} \rightarrow \mathcal{SH}(k)$ be the functor induced by sending a space to the constant presheaf of spaces on \mathbf{Sm}/k . We show that c is fully faithful. In particular, c induces an isomorphism

$$c_* : \pi_n(E) \rightarrow \Pi_{n,0}(c(E))$$

for all spectra E .

Fix an embedding $\sigma : k \rightarrow \mathbb{C}$ and let $Re_B : \mathcal{SH}(k) \rightarrow \mathcal{SH}$ be the associated Betti realization. Let \mathbb{S}_k be the motivic sphere spectrum. We show that the Tate-Postnikov tower for \mathbb{S}_k

$$\dots \rightarrow f_{n+1}\mathbb{S}_k \rightarrow f_n\mathbb{S}_k \rightarrow \dots \rightarrow f_0\mathbb{S}_k = \mathbb{S}_k$$

has Betti realization which is strongly convergent, in fact $Re(f_n\mathbb{S}_k)$ is $n - 1$ connected. This gives a spectral sequence “of algebro-geometric origin” converging to the homotopy groups of \mathbb{S} ; this spectral sequence at E_2 agrees with the E_2 terms in the Adams-Novikov spectral sequence.

CONTENTS

Introduction	2
1. Cellular model structures	5
2. Right Bousfield localization	9
3. A two-variable Postnikov tower	12
4. Connectedness	14
5. A detour: Morel’s homotopy t -structure	16
6. The Betti realization	17
7. Slices of the sphere spectrum	22
8. Proof of the main theorem	23
9. The Suslin-Voevodsky theorem for homotopy	25
10. Slices of the sphere spectrum	26
References	31

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INTRODUCTION

Our main object in this paper is to use Voevodsky’s slice tower [28] and its Betti realization to prove two comparison results between the classical stable homotopy category \mathcal{SH} and the motivic version $\mathcal{SH}(k)$, for k an algebraically closed field of characteristic zero.

For $\mathcal{E} \in \mathcal{SH}(k)$, we have the bi-graded homotopy sheaf $\Pi_{a,b}\mathcal{E}$, which is the Nisnevich sheaf on \mathbf{Sm}/k associated to the presheaf

$$U \mapsto [\Sigma_{S^1}^a \Sigma_{G_m}^b \Sigma_T^\infty U_+, \mathcal{E}]_{\mathcal{SH}(k)}$$

(note the perhaps non-standard indexing).

Our first result is concerned with the exact symmetric monoidal functor

$$c : \mathcal{SH} \rightarrow \mathcal{SH}(k).$$

The functor c is derived from the constant presheaf functor from pointed spaces (i.e. pointed simplicial sets) to presheaves of pointed spaces over \mathbf{Sm}/k . Given an embedding of k into \mathbb{C} , Ayoub [3] has defined a “Betti realization functor”

$$Re_B^\sigma : \mathcal{SH}(k) \rightarrow \mathcal{SH}$$

which gives a left inverse to c . In particular, c is faithful. We will improve this by showing

Theorem 1. *Let k be an algebraically closed field of characteristic zero with an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then the “constant presheaf” functor*

$$c : \mathcal{SH} \rightarrow \mathcal{SH}(k)$$

is fully faithful.

We note that, as a special case, theorem 1 implies

Corollary 2. *Let k be an algebraically closed field of characteristic zero with an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Let \mathbb{S}_k be the motivic sphere spectrum in $\mathcal{SH}(k)$ and \mathbb{S} the classical sphere spectrum in \mathcal{SH} . Then the Betti realization functor gives an isomorphism*

$$Re_{B^*}^\sigma : \Pi_{n,0}\mathbb{S}_k(k) \rightarrow \pi_n(\mathbb{S})$$

for all $n \in \mathbb{Z}$.

In fact, the corollary implies the theorem, by a limit argument (see lemma 8.2).

We have as well a homotopy analog of the theorem of Suslin-Voevodsky comparing Suslin homology and singular homology with mod N coefficients [26, theorem 8.3]:

Theorem 3. *Let k be an algebraically closed field of characteristic zero with an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then for all $X \in \mathbf{Sm}/k$, all integers $N > 1$ and $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\Pi_{n,0}(\Sigma_T^\infty X_+; \mathbb{Z}/N)(k) \cong \pi_n(\Sigma^\infty X_+^{\text{an}}; \mathbb{Z}/N)$$

See corollary 6.9 for a more precise statement.

The idea for the proof of theorem 1 is as follows: As mentioned above, we reduce by a limit argument to proving corollary 2. We consider Voevodsky’s slice tower for the sphere spectrum

$$\dots \rightarrow f_{n+1}\mathbb{S}_k \rightarrow f_n\mathbb{S}_k \rightarrow \dots \rightarrow f_0\mathbb{S}_k = \mathbb{S}_k$$

and its Betti realization. Let $s_n\mathbb{S}_k$ be the n th layer in this tower. This gives us a spectral sequence starting with $\Pi_{*,0}s_n\mathbb{S}_k(k)$, which should converge to $\Pi_{*,0}\mathbb{S}_k(k)$. Similarly, we have a spectral sequence starting with $\pi_*(Re_B^\sigma(s_n\mathbb{S}_k))$, which should converge to $\pi_*\mathbb{S}$ (since $Re_B^\sigma(\mathbb{S}_k) = \mathbb{S}$). By a theorem of Pelaez [24], the layers $s_n\mathbb{S}_k$ are effective motives. Some computations found in our paper [16] show that $s_n\mathbb{S}_k$ is in fact a torsion effective motive for $n > 0$. On the other hand, Voevodsky [29] has computed the 0th layer $s_0\mathbb{S}_k$, and shows that this is the motivic Eilenberg-MacLane spectrum $M\mathbb{Z}$. The theorem of Suslin-Voevodsky *loc. cit.* shows that the Betti realization gives an isomorphism from the Suslin homology of a torsion effective motive to the singular homology of its Betti realization; one handles the 0th slice by a direct computation.

Thus, if we knew the two spectral sequences were strongly convergent, we would be done. The strong convergence of the motivic version was settled in [16], so the main task in this paper is to show that the Betti realization of the slice tower also yields a strongly convergent spectral sequence.

We accomplish this by introducing a second truncation variable into the story, namely we consider a motivic version of the classical Postnikov tower, filtering by “topological connectivity”. Our results along this line can be viewed as a refinement of Morel’s construction of the homotopy t -structure on $\mathcal{SH}(k)$ cf. [20]. In fact, Morel’s \mathbb{A}^1 -connectedness theorem shows that, for instance $\Pi_{a,b}\mathbb{S}_k = 0$ for $a < 0$, $b \in \mathbb{Z}$. Our extension of this is our result that this same connectedness in the topological variable a passes to all the terms $f_n\mathbb{S}_k$ in the slice tower (this is of course a general phenomenon, not restricted to the sphere spectrum, see lemma 4.3).

In order to translate this connectedness in the homotopy sheaves into connectedness in the Betti realization, we adapt the method employed by Pelaez in [24], using the technique of right Bousfield localization. This has the advantage of constructing the necessary truncation functors on the level of the underlying model category, and in addition giving a set of generating cofibrations for constructing the truncations. Using this approach, we are able to show that the $f_n\mathbb{S}_k$ are built out of objects of the form $\Sigma_{S^1}^a \Sigma_{\mathbb{G}_m}^b \Sigma_T^\infty X_+$ with $b \geq n$ and, what is new, $a \geq 0$ (and $X \in \mathbf{Sm}/k$). As both \mathbb{G}_m and S^1 realize to S^1 , this shows that $f_n\mathbb{S}_k$ has Betti realization which is $n - 1$ connected.

The proof of theorem 3 runs along the same lines as that of theorem 1, except that we start from the beginning with a torsion object, so we omit the ad hoc computation of the 0th layer that occurred in the proof of theorem 1.

We conclude the paper with a closer look at the layers in the slice tower for \mathbb{S}_k . Voevodsky has given a conjectural formula for these, generalizing his computation of $s_0\mathbb{S}_k$. The conjecture gives a connection of the layer $s_q\mathbb{S}_k$ with the complex of homotopy groups (in degree $-2q$) arising from the Adams-Novikov spectral sequence. Relying on an as yet unpublished result of Hopkins-Morel (see however the preprint of M. Hoyois [12]), we give a sketch of the proof of Voevodsky’s conjecture.

Via our main result, the Betti realization of the slice tower for \mathbb{S}_k gives a tower converging to \mathbb{S} in \mathcal{SH} . Voevodsky’s conjecture shows that the associated spectral sequence converging to the homotopy groups of \mathbb{S} has E_2 term closely related to the E_2 -terms in the Adams-Novikov spectral sequence. Our results and Voevodsky’s conjecture lead to the following:

Theorem 4. *Let k be an algebraically closed field of characteristic zero. Let $E_2^{p,2q}(AN)$ be the $E_2^{p,2q}$ term in the Adams-Novikov spectral sequence, i.e.,*

$$E_2^{p,2q}(AN) = \text{Ext}_{MU_*(MU)}^{p,2q}(MU_*, MU_*),$$

and let $E_2^{p,q}(AH)$ be the $E_2^{p,q}$ term in the Atiyah-Hirzebruch spectral sequence for $\Pi_{,0}\mathbb{S}_k(k)$, associated to the slice tower for \mathbb{S}_k , i.e.,*

$$E_2^{p,q}(AH) = \Pi_{-p-q,0}(s_{-q}\mathbb{S}_k)(k) \implies \Pi_{-p-q,0}\mathbb{S}_k(k) = \pi_{-p-q}(\mathbb{S}),$$

Then

$$E_2^{p,q}(AH) = E_2^{p-q,2q}(AN) \otimes \hat{\mathbb{Z}}(q),$$

where $\hat{\mathbb{Z}}(q) = \varprojlim_N \mu_N^{\otimes q}$.

See theorem 10.3 for the details and proof of this result.

It would be interesting to see if there were a deeper connection relating the Atiyah-Hirzebruch spectral sequence (for $k = \bar{k}$ of characteristic zero) and the Adams-Novikov spectral sequence via our theorem 1 identifying $\Pi_{-p-q,0}(\mathbb{S}_k)(k)$ with $\pi_{-p-q}(\mathbb{S})$. Formulated another way, although the Betti realization of the slice tower for \mathbb{S}_k gives a tower converging to \mathbb{S} in \mathcal{SH} and the associated spectral sequence converging to the homotopy groups of \mathbb{S} has E_2 term the same (up to reindexing) as the E_2 -terms in the Adams-Novikov spectral sequence, we do not know if the two spectral sequences continue to be the same or are in any other way related. For instance, since $E_2^{p,q}(AN)$ is concentrated in even q degrees, we have $E_2^{p,q}(AN) = E_3^{p,q}(AN)$. The d_3 differential goes from $E_2^{p-q,2q}(AN)$ to $E_2^{p-q+3,2(q-1)}(AN)$, and the d_2 differential in the AH spectral sequence similarly goes from $E_2^{p,q}(AH)$ to $E_2^{p+2,q-1}(AH)$, but we don't know if $d_2(AH) = d_3(AN) \otimes \text{id}$. In any case, we raise the question: is $E_r^{p,q}(AH) = E_{2r-1}^{p-q,2q}(AN) \otimes \hat{\mathbb{Z}}(q)$ and $d_r^{p,q}(AH) = d_{2r-1}^{p-q,2q}(AN) \otimes \text{id}$ for all $r \geq 2$?

Dugger and Isaksen [6] and independently Hu, Kriz and Ormsby [13] have constructed motivic versions of the Adams and Adams-Novikov spectral sequences, and have made explicit computations. Dugger and Isaksen note [6, remark 4.3] that their computations give an isomorphism between the weight zero part of the motivic Adams spectral sequence and the topological version, within the range of their computations. It would be interesting to see what connections the slice tower for \mathbb{S}_k has with the motivic Adams or motivic Adams-Novikov spectral sequences. Additionally, the fact that for k algebraically closed, the weight 0 pieces of these spectral sequences converge to the (2-completed) homotopy groups of \mathbb{S} (using our theorem 1), and that fact that the computations of Dugger-Isaksen show that the weight 0 part of the motivic Adams spectral sequence agrees with the topological version in low degree leads us to the

Conjecture 5. *For an algebraically closed field k of characteristic zero, the Betti realization induces an isomorphism of the weight 0 parts of the motivic Adams, resp. motivic Adams-Novikov, spectral sequence with its topological counterpart.*

Remark 1. Since posting the first version of this paper, Dan Isaksen (private communication) has pointed out that, at least after 2-completion, this conjecture for the Adams-Novikov spectral sequence follows from results in [6] and [13].

Finally, if we take $k = \bar{\mathbb{Q}}$ we have the isomorphism

$$\Pi_{n,0}(\mathbb{S}_{\bar{\mathbb{Q}}})(\bar{\mathbb{Q}}) \cong \pi_n(\mathbb{S}).$$

We can rewrite $\Pi_{n,0}(\mathbb{S}_{\bar{\mathbb{Q}}})(\bar{\mathbb{Q}})$ as

$$\Pi_{n,0}(\mathbb{S}_{\bar{\mathbb{Q}}})(\bar{\mathbb{Q}}) = \mathrm{Hom}_{\mathcal{SH}(\bar{\mathbb{Q}})}(\mathbb{S}_{\bar{\mathbb{Q}}}, \mathbb{S}_{\bar{\mathbb{Q}}}) = \Pi_{n,0}(\mathbb{S}_{\bar{\mathbb{Q}}})(\bar{\mathbb{Q}}).$$

Thus, there is a natural $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -action on $\Pi_{n,0}(\mathbb{S}_{\bar{\mathbb{Q}}})(\bar{\mathbb{Q}}) \cong \pi_n(\mathbb{S})$. This may of course be the trivial action. Mike Hopkins (private communication) has pointed out to me that multiplication by k^q on $E_2^{p,-2q}(AN) \otimes \mathbb{Z}_{(\ell)}$ is trivial for all k prime to ℓ . As $k \in \mathbb{Z}_{\ell}^{\times} \mapsto \times k^q$ is also the action by the cyclotomic character on $\mathbb{Z}_{\ell}(q) := \varprojlim_{\nu} \mu_{\ell^{\nu}}^{\otimes q} \cong \mathbb{Z}_{\ell}$, this (together with theorem 4) suggests that the Galois action on the E_2 -term for the slice tower is trivial as well. In any case, it would be nice to settle the question of whether this $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -action on $\pi_n(\mathbb{S})$ is trivial or not.

The paper is organized as follows. The first three sections deal with the construction of the two-variable Postnikov tower and a discussion of its properties. In §1 we recall some of the background on cofibrantly generated and cellular model categories. In §2 we discuss some facts about right Bousfield localization and we apply this machinery to give the construction of the two-variable tower in §3. We prove our main connectedness results in §4. In §5 we make some comments relating the two-variable Postnikov tower with Morel's homotopy t -structure on $\mathcal{SH}(k)$ (this section is included for completeness sake and is not used in the remainder of the paper). We recall some facts about Ayoub's Betti realization in §6, prove our main theorem on the connectedness of the Betti realization (theorem 6.1) and make a few simple computations. We also describe the consequences for torsion effective motives and their Betti realizations (corollary 6.9) of the Suslin-Voevodsky theorem. Besides the Suslin-Voevodsky result, this relies heavily on the theorem of Röndigs-Østvær, giving an equivalence of the homotopy category of modules over the motivic Eilenberg-MacLane spectrum and a version of Voevodsky's triangulated category of motives [25, theorem 1.1].

The next three sections, §7, §8 and §9 assemble all the pieces to prove theorems 1 and 3. We conclude with a discussion of Voevodsky's conjecture on the slices of the sphere spectrum in §10.

I would like to thank Ivan Panin for discussions that encouraged me to look at the possibility of extending the Suslin-Voevodsky theorem to the Betti realization for $\mathcal{SH}(k)$. I would also like to thank Pablo Pelaez for discussing aspects of Bousfield localization with me and pointing out that this is an effective way of defining Postnikov towers.

1. CELLULAR MODEL STRUCTURES

In section 3, we apply the method used by Pelaez [23, 24], in his study of the slice filtration in $\mathcal{SH}(k)$, to define a two-variable Postnikov tower in $\mathcal{SH}(k)$. The method relies on the fact that motivic model structure on $\mathbf{Spt}_T(k)$ is cellular and we require a bit of information about this structure to make our construction. To describe this, we first recall a few notions about cellular model categories. For details on cofibrantly generated and cellular model categories, we refer the reader to [10].

To make the necessary definitions, we need to say what a λ -sequence is and what a regular cardinal is. Let λ be a cardinal, where by definition a cardinal is the least ordinal γ among the set of ordinals with equal cardinality. The set of ordinals $\beta < \lambda$ is a well-ordered set, which we consider as a category in the usual way, with a unique morphism $\beta_1 \rightarrow \beta_2$ if and only if $\beta_1 \leq \beta_2$.

Let \mathcal{C} be a cocomplete category. A functor

$$X : \{\beta \mid \beta < \lambda\} \rightarrow \mathcal{C}$$

is a λ -sequence if for all limit ordinals β the natural map

$$\varinjlim_{\gamma < \beta} X_\gamma \rightarrow X_\beta$$

is an isomorphism.

Given a λ -sequence $X_0 \rightarrow \dots \rightarrow X_\beta \rightarrow \dots$ one defines $X_\lambda := \varinjlim_{\beta} X_\beta$ and the induced morphism $X_0 \rightarrow X_\lambda$ is said to be constructed from the sequence by *transfinite composition*.

A cardinal λ is *regular* if given a set A with cardinality $|A| < \lambda$ and a collection of sets S_α , $\alpha \in A$, with $|S_\alpha| < \lambda$, then $|\cup_\alpha S_\alpha| < \lambda$. There are lots of regular cardinals, for instance, \aleph_0 is regular and if γ is a cardinal, the smallest cardinal λ with $\lambda > \gamma$ is also regular.

Definition 1.1. Let I be a set of morphisms in a cocomplete category \mathcal{C} . Let λ be a cardinal and let

$$(1.1) \quad X_0 \rightarrow \dots \rightarrow X_\beta \rightarrow \dots; \quad \beta < \lambda$$

be a λ -sequence in \mathcal{C} such that each map $X_\beta \rightarrow X_{\beta+1}$ is a pushout in a diagram of the form

$$(1.2) \quad \begin{array}{ccc} \coprod_{i \in C_\beta} A_i & \longrightarrow & X_\beta \\ \coprod_i f_i \downarrow & & \\ \coprod_i B_i & & \end{array}$$

with each $f_i : A_i \rightarrow B_i$ in I . The transfinite composition $X_0 \rightarrow X_\lambda$ is called a *relative I -complex*; if X_0 is the initial object \emptyset in \mathcal{C} , X_λ is an *I -cell complex*.

Suppose we have a relative I -complex given by the data (1.1), (1.2). Suppose we have for each $\beta < \lambda$ a subset $C'_\beta \subset C_\beta$ and a λ -sequence

$$X'_0 \rightarrow \dots \rightarrow X'_\beta \rightarrow \dots; \quad \beta < \lambda$$

in \mathcal{C} such that each map $X'_\beta \rightarrow X'_{\beta+1}$ is a pushout in a diagram of the form

$$\begin{array}{ccc} \coprod_{i \in C'_\beta} A_i & \longrightarrow & X'_\beta \\ \coprod_i f_i \downarrow & & \\ \coprod_i B_i & & \end{array}$$

and there is a map of λ sequences

$$\begin{array}{ccccccc} X'_0 & \longrightarrow & \dots & \longrightarrow & X'_\beta & \longrightarrow & \dots \\ \downarrow & & & & \downarrow & & \\ X_0 & \longrightarrow & \dots & \longrightarrow & X'_\beta & \longrightarrow & \dots \end{array}$$

such that for each $i \in C'_\beta$, the diagram

$$\begin{array}{ccc} A_i & \longrightarrow & X'_\beta \\ & \searrow & \downarrow \\ & & X_\beta \end{array}$$

commutes. The induced map $X'_\lambda \rightarrow X_\lambda$ is called an *I-subcomplex* of the relative *I-complex* $X_0 \rightarrow X_\lambda$.

Definition 1.2. We recall that a model category \mathcal{M} is *cofibrantly generated* [10, definition 11.1.2] if it admits sets I, J of morphisms such that

- (1) a map is a trivial fibration if and only if it has the right lifting property with respect to all morphisms in I .
- (2) a map is a fibration if and only if it has the right lifting property with respect to all morphisms in J .
- (3) Both I and J permit the small object argument

The last condition is detailed in [10, definition 10.5.15]. Roughly speaking, a set of morphisms I in a cocomplete category \mathcal{C} permits the small object argument if there is a cardinal κ such that, given a regular cardinal $\lambda \geq \kappa$ and a λ -sequence in \mathcal{C} as in (1.1),

$$X_0 \rightarrow \dots \rightarrow X_\beta \rightarrow \dots; \quad \beta < \lambda$$

in \mathcal{C} such that each map $X_\beta \rightarrow X_{\beta+1}$ is a pushout in a diagram of the form (1.2) with each $f_i : A_i \rightarrow B_i$ in I , then for each $W \in \mathcal{C}$ which is the domain of some map in I , the natural map

$$\varinjlim_{\beta} \text{Hom}_{\mathcal{C}}(W, X_\beta) \rightarrow \text{Hom}_{\mathcal{C}}(W, \varinjlim_{\beta} X_\beta)$$

is an isomorphism.

It follows from conditions (1) and (2) above that each morphism in I is a cofibration and each morphism in J is a trivial cofibration. I is called the set of *generating cofibrations* and J the set of *generating trivial cofibrations*.

A cofibrantly generated model category \mathcal{M} is *cellular* if some additional conditions on the generating cofibrations I and the generating trivial cofibrations J are satisfied. For our purposes, we do not need to specify these conditions; we refer the interested reader to [10, definition 12.1.1].

All this machinery is useful due to the following theorem of Hirschhorn:

Theorem 1.3 ([10, theorems 4.1.1 and 5.1.1]). *Let \mathcal{M} be a cellular model category with generating cofibrations I and generating trivial cofibrations J .*

- 1. *Suppose \mathcal{M} is a left proper model category. Let \mathcal{S} be a set of maps in \mathcal{M} . Then the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ exists.*
- 2. *Suppose \mathcal{M} is a right proper model category. Let \mathcal{S} be a set of maps in \mathcal{M} . Then the right Bousfield localization $R_{\mathcal{S}}\mathcal{M}$ exists.*

We will only be using the right Bousfield localization and will give the relevant details in theorem 2.4. For the complete story, we refer the reader to [10].

We consider these notions for the category of simplicial presheaves on \mathbf{Sm}/k , endowed with the injective model structure with respect to the Nisnevich topology, and denoted $\Delta^{\text{op}}\text{PreSh}(\mathbf{Sm}/k)_{\text{Nis}}$. This is in fact a cellular model category (see

Jardine [14]). The weak equivalences are the maps $f : \mathcal{X} \rightarrow \mathcal{Y}$ which is a weak equivalence on all Nisnevich stalks. The set of generating cofibrations I is $\{i : Y \rightarrow U \times \Delta^n\}$ with i a monomorphism, $Y, U \in \mathbf{Sm}/k$, $n \geq 0$. The set of generating trivial cofibrations J_{Nis} is the set of trivial cofibrations, i.e., monomorphisms $i : A \rightarrow B$ of simplicial presheaves which are a stalkwise weak equivalence, which satisfy an additional cardinality boundedness condition (see [14]).

$\Delta^{\text{op}}\text{PreSh}(\mathbf{Sm}/k)_{Nis}$ has as left Bousfield localization the *motivic model structure*. Let $\mathcal{V}_{\mathcal{M}}$ be the set of maps

$$\mathcal{V}_{\mathcal{M}} := \{p_U : U \times_k \mathbb{A}^1 \rightarrow U \mid U \in \mathbf{Sm}/k\}$$

where p_U is the projection. Let $J_{\mathcal{M}}$ be the set of maps of simplicial presheaves $j : A \rightarrow B$ such that

- a) j is a monomorphism of simplicial presheaves
- b) j is a $\mathcal{V}_{\mathcal{M}}$ weak equivalence (in $\Delta^{\text{op}}\text{PreSh}(\mathbf{Sm}/k)_{Nis}$ with the Nisnevich-injective model structure).
- c) A cardinality boundedness condition, which we omit.

Let $I_{\mathcal{M}} = I$. Then $\Delta^{\text{op}}\text{PreSh}(\mathbf{Sm}/k)_{Nis}$ is a cellular model category with generating cofibrations $I_{\mathcal{M}}$, generating trivial cofibrations $J_{\mathcal{M}}$ and weak equivalences the $\mathcal{V}_{\mathcal{M}}$ weak equivalences. We denote this model category by \mathcal{M} . In particular, the homotopy category of \mathcal{M} is the Morel-Voevodsky unstable motivic homotopy category $\mathcal{H}(k)$.

Replacing $I_{\mathcal{M}}$ with $I_{\mathcal{M}_*} := \{i_+ : Y_+ \rightarrow U \times \Delta_+^n, i \in I_{\mathcal{M}}\}$, and $J_{\mathcal{M}}$ with $J_{\mathcal{M}_*} := \{j_+ : A_+ \rightarrow B_+, j \in J_{\mathcal{M}}\}$ gives the pointed version \mathcal{M}_* of \mathcal{M} (i.e., the category of presheaves of pointed simplicial sets on \mathbf{Sm}/k) a cellular model structure. \mathcal{M}_* is in fact a proper, cellular, simplicial symmetric model category [24, proposition 2.3.7].

We pass to the stable setting. Let $T = S^1 \wedge G_m$ and let $\mathbf{Spt}_T(k)$ be the category of T -spectra in $\Delta^{\text{op}}\text{PreSh}(\mathbf{Sm}/k)$, i.e., objects are sequences $\mathcal{E} := (E_0, E_1, \dots, E_n, \dots)$, $E_n \in \mathcal{M}_*$, together with bonding maps $\epsilon_n : E_n \wedge T \rightarrow E_{n+1}$. Morphisms are sequences of maps compatible with the bonding.

For $\mathcal{X} \in \mathcal{M}_*$, $a, b \geq 0$, we have the Nisnevich sheaf $\Pi_{a,b}(\mathcal{X})$ associated to the presheaf

$$U \mapsto [\Sigma_{S^1}^a \Sigma_{G_m}^b U_+, \mathcal{X}]$$

For $\mathcal{E} = (E_0, E_1, \dots)$ in $\mathbf{Spt}_T(k)$, we have the Nisnevich sheaf $\Pi_{a,b}(\mathcal{E})$, $a, b \in \mathbb{Z}$ defined by

$$\Pi_{a,b}(\mathcal{E}) := \varinjlim_n \Pi_{a+n, b+n}(E_n)$$

with the bonding maps giving the transition maps needed to define the colimit. A map $f : \mathcal{E} \rightarrow \mathcal{F}$ in $\mathbf{Spt}_T(k)$ is a *stable \mathbb{A}^1 weak equivalence* if

$$f_* : \Pi_{a,b}(\mathcal{E}) \rightarrow \Pi_{a,b}(\mathcal{F})$$

is an isomorphism for all $a, b \in \mathbb{Z}$.

For $n \geq 0$, let $F_n : \mathcal{M}_* \rightarrow \mathbf{Spt}_T(k)$ be the functor with

$$F_n(\mathcal{X}) := (F_n(\mathcal{X})_0, F_n(\mathcal{X})_1, \dots, F_n(\mathcal{X})_m, \dots),$$

where

$$F_n(\mathcal{X})_m := \begin{cases} * & \text{if } m < n \\ \Sigma_T^{m-n} \mathcal{X} & \text{if } m \geq n \end{cases}$$

The bonding maps ϵ_m are the identity if $m \geq n$, the basepoint map if $m < n$.

Theorem 1.4 ([24, theorem 2.5.4]). *There is a cellular model structure on $\mathbf{Spt}_T(k)$ such that the weak equivalences are the stable \mathbb{A}^1 weak equivalences, the generating cofibrations $I_{\mathcal{M}_*}^T$ are*

$$I_{\mathcal{M}_*}^T := \{F_n(i_+) : F_n(Y_+) \rightarrow F_n(U \times \Delta_+^n) \mid i \in I\}$$

and the generating trivial cofibrations are

$$J_{\mathcal{M}_*}^T := \{j : A \rightarrow B\}$$

such that

- i) B is an $I_{\mathcal{M}_*}^T$ -cell complex and j is an $I_{\mathcal{M}_*}^T$ subcomplex of B .
- ii) j is a stable \mathbb{A}^1 weak equivalence
- iii) a cardinality condition on B , which we omit.

With this model structure, $\mathbf{Spt}_T(k)$ is a proper simplicial \mathcal{M}_* model category.

2. RIGHT BOUSFIELD LOCALIZATION

We recall material from [10].

Definition 2.1 ([10, definition 3.1.8]). Let K be a set of objects in a model category \mathcal{M}_* . A morphism $f : X \rightarrow Y$ in \mathcal{M} is a K -colocal weak equivalence if for each $A \in K$, the induced map on the homotopy function complexes

$$f_* : \mathcal{H}om_{\mathcal{M}_*}(A, X) \rightarrow \mathcal{H}om_{\mathcal{M}}(A, Y)$$

is a weak equivalence in \mathbf{Spc}_* . An object B is K -colocal if B is cofibrant and for every K -colocal weak equivalence $f : X \rightarrow Y$, the induced map on the homotopy function complexes

$$f_* : \mathcal{H}om_{\mathcal{M}_*}(B, X) \rightarrow \mathcal{H}om_{\mathcal{M}_*}(B, Y)$$

is a weak equivalence in \mathbf{Spc}_* .

Definition 2.2. Let \mathcal{M} be a model category, K a set of cofibrant objects of \mathcal{M} . The class of K -cellular objects is the smallest class of cofibrant objects of \mathcal{M} containing K and closed under homotopy colimits and weak equivalences.

Remark 2.3. Suppose that \mathcal{M} is a stable model category such that \mathbf{HoM} becomes a triangulated category with shift functor equal suspension and the distinguished triangles the mapping cone (Puppe) sequences. Let K be a set of cofibrant objects of \mathcal{M} . Then the image of the class of K -cellular objects in \mathbf{HoM} is the class of objects in smallest full subcategory \mathcal{C} of \mathbf{HoM} containing K , closed under arbitrary small coproducts and with the property that, if $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle with A and B in \mathcal{C} , then C is in \mathcal{C} .

Indeed, each such distinguished triangle exhibits C as the homotopy colimit of $A \rightarrow B$. Conversely, if

$$F : I \rightarrow \mathcal{M}$$

is a functor from a small category I , then $\mathrm{hocolim}_I F$ can be expressed as a colimit of a sequence of cofibrations

$$C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n \rightarrow \dots$$

with each map $C_n \rightarrow C_{n+1}$ given by a pushout diagram

$$\begin{array}{ccc} \coprod F(\alpha) \otimes S^n & \longrightarrow & C_n \\ \downarrow & & \downarrow \\ \coprod F(\alpha) \otimes D^{n+1} & \longrightarrow & C_{n+1} \end{array}$$

with the coproduct over a suitable index set. Thus in \mathbf{HoM} , we have the distinguished triangle

$$\oplus F(\alpha)[n] \rightarrow C_n \rightarrow C_{n+1} \rightarrow \oplus_\alpha F(\alpha)[n+1]$$

and since each map $C_n \rightarrow C_{n+1}$ is a cofibration, we have the distinguished triangle

$$\oplus_n C_n \rightarrow \oplus_n C_n \rightarrow \operatorname{hocolim}_I F \rightarrow \oplus_n C_n[1]$$

Theorem 2.4 ([10, theorem 5.1.1, theorem 5.1.5]). *Let K be a set of objects in a right proper cellular model category \mathcal{M} .*

1. *The right Bousfield localization of \mathcal{M} with respect to the class of K -colocal weak equivalences exists. That is, there is a model structure $\mathcal{R}_K\mathcal{M}$ on the underlying category of \mathcal{M} in which*

- a) *the class of weak equivalences in $\mathcal{R}_K\mathcal{M}$ are the K -colocal weak equivalences*
- b) *the class of fibrations in $\mathcal{R}_K\mathcal{M}$ are the fibrations in \mathcal{M}*
- c) *the cofibrations in $\mathcal{R}_K\mathcal{M}$ are the maps satisfying the left lifting property with respect to maps which are fibrations and K -colocal weak equivalences in \mathcal{M} .*

2. *The cofibrant objects are the K -colocal objects of \mathcal{M}*

3. *$\mathcal{R}_K\mathcal{M}$ is a right proper model category*

4. *If \mathcal{M} is a simplicial model category, then the simplicial structure on \mathcal{M} gives $\mathcal{R}_K\mathcal{M}$ the structure of a simplicial model category.*

5. *If the objects in K are all cofibrant, then the class of K -colocal objects is the same as the class of K -cellular objects.*

Let \mathcal{M} be a model category, K a set of objects in \mathcal{M} and suppose the right Bousfield localization $\mathcal{R}_K\mathcal{M}$ exists (e.g., \mathcal{M} is cellular and right proper). By the description (a)-(c) above of the weak equivalences, fibrations and cofibrations in $\mathcal{R}_K\mathcal{M}$, it follows that the identity on the underlying category of \mathcal{M} is a right Quillen functor

$$\operatorname{id}_r : \mathcal{M} \rightarrow \mathcal{R}_K\mathcal{M}$$

and thus has as left adjoint the identity functor

$$\operatorname{id}_l : \mathcal{R}_K\mathcal{M} \rightarrow \mathcal{M}$$

which is a left Quillen functor. Since \mathcal{M} and $\mathcal{R}_K\mathcal{M}$ have the same fibrations, the right derived functor of id_r is just the localization

$$q : \mathbf{HoM} \rightarrow \mathbf{HoR}_K\mathcal{M}$$

of \mathbf{HoM} with respect to the K -colocal weak equivalences. The left derived functor of id_l

$$i := L\operatorname{id}_l : \mathbf{HoR}_K\mathcal{M} \rightarrow \mathbf{HoM}$$

is thus the left adjoint to q . By definition, i is defined on the objects of \mathcal{M} by $i(X) = \tilde{X}_K$, where $p : \tilde{X}_K \rightarrow X$ is a cofibrant replacement of X with respect to

model structure $\mathcal{R}_K\mathcal{M}$, that is, \tilde{X}_K is K -colocal and p is a fibration and a K -colocal weak equivalence.

Proposition 2.5. *The functor $i : \mathbf{Ho}\mathcal{R}_K\mathcal{M} \rightarrow \mathbf{Ho}\mathcal{M}$ is an equivalence of $\mathbf{Ho}\mathcal{R}_K$ with the full subcategory of $\mathbf{Ho}\mathcal{M}$ with objects the K -colocal objects of \mathcal{M} .*

Proof. Let A and B be K -colocal objects of \mathcal{M} . Let $i_A : A \rightarrow \tilde{A}$, $i_B : B \rightarrow \tilde{B}$ be fibrant replacements of A and B in \mathcal{M} . As A and B are cofibrant in both \mathcal{M} and $\mathcal{R}_K\mathcal{M}$, \tilde{A} and \tilde{B} are fibrant cofibrant objects in \mathcal{M} and in $\mathcal{R}_K\mathcal{M}$. Thus

$$\mathrm{Hom}_{\mathbf{Ho}\mathcal{M}}(\tilde{A}, \tilde{B}) = \pi_0(\mathcal{H}om_{\mathcal{M}}(\tilde{A}, \tilde{B})) = \pi_0(\mathcal{H}om_{\mathcal{R}_K\mathcal{M}}(\tilde{A}, \tilde{B})) = \mathrm{Hom}_{\mathbf{Ho}\mathcal{R}_k\mathcal{M}}(\tilde{A}, \tilde{B}).$$

Furthermore $i_A : A \rightarrow \tilde{A}$, $i_B : B \rightarrow \tilde{B}$ are fibrant replacements of A and B , respectively, in both \mathcal{M} and $\mathcal{R}_K\mathcal{M}$, so

$$\mathrm{Hom}_{\mathbf{Ho}\mathcal{M}}(\tilde{A}, \tilde{B}) = \mathrm{Hom}_{\mathbf{Ho}\mathcal{M}}(A, B) \quad \mathrm{Hom}_{\mathbf{Ho}\mathcal{R}_K\mathcal{M}}(\tilde{A}, \tilde{B}) = \mathrm{Hom}_{\mathbf{Ho}\mathcal{R}_k\mathcal{M}}(A, B)$$

Finally, if \bar{A}, \bar{B} are arbitrary objects of $\mathcal{R}_K\mathcal{M}$, with cofibrant replacements $A \rightarrow \tilde{A}$, $B \rightarrow \tilde{B}$ (in $\mathcal{R}_K\mathcal{M}$), then A and B are K -colocal, $i(\bar{A}) \cong A \cong \tilde{A}$, $i(\bar{B}) \cong B \cong \tilde{B}$ (isomorphisms in $\mathbf{Ho}\mathcal{R}_K\mathcal{M}$) so

$$\mathrm{Hom}_{\mathbf{Ho}\mathcal{R}_k\mathcal{M}}(\bar{A}, \bar{B}) = \mathrm{Hom}_{\mathbf{Ho}\mathcal{R}_K\mathcal{M}}(\tilde{A}, \tilde{B}).$$

Thus i is fully faithful and has essential image the full subcategory of $\mathbf{Ho}\mathcal{M}$ with objects the K -colocal objects of \mathcal{M} , proving the result. \square

We collect these results in the following useful form, without any claim to originality:

Theorem 2.6. *Let K be a set of cofibrant objects in a right proper cellular model category \mathcal{M} and let $\mathbf{Ho}\mathcal{M}(K)$ be the full subcategory of $\mathbf{Ho}\mathcal{M}$ with objects the K -cellular objects of \mathcal{M} . Then*

1. *the inclusion $i : \mathbf{Ho}\mathcal{M}(K) \rightarrow \mathbf{Ho}\mathcal{M}$ admits a right adjoint $r : \mathbf{Ho}\mathcal{M} \rightarrow \mathbf{Ho}\mathcal{M}(K)$.*
2. *For an object $X \in \mathcal{M}$, $i \circ r(X)$ is the image in $\mathbf{Ho}\mathcal{M}$ of a cofibrant replacement $A \rightarrow X$ with respect to the model structure $\mathcal{R}_K\mathcal{M}$.*
3. *$r : \mathbf{Ho}\mathcal{M} \rightarrow \mathbf{Ho}\mathcal{M}(K)$ identifies $\mathbf{Ho}\mathcal{M}(K)$ with the localization of $\mathbf{Ho}\mathcal{M}$ with respect to the K -colocal weak equivalences.*

Proof. This follows directly from theorem 2.4 and proposition 2.5 \square

We need one last result in this section, namely a more explicit description of the K -colocal objects in \mathcal{M} . To simplify the situation a bit, we will assume that K is a set of cofibrant objects in a cofibrantly generated model category \mathcal{M} , with a given set of generating trivial cofibrations J . Let $\Lambda(K)$ be the set of morphisms of the form $A \otimes \partial\Delta[n] \rightarrow A \otimes \Delta[n]$, with $A \in K$, and let $\overline{\Lambda(K)} = \Lambda(K) \cup J$. This gives us the notions of a relative $\overline{\Lambda(K)}$ -complex and a $\overline{\Lambda(K)}$ -cell complex (see definition 1.1).

The following result is a weakening of [10, corollary 5.3.7].

Proposition 2.7. *Let K be a set of cofibrant objects in a right proper cellular model category \mathcal{M} , and let A be a K -colocal object of \mathcal{M} . Then there is a $\overline{\Lambda(K)}$ -cell complex Y and a weak equivalence $X \rightarrow Y$ in \mathcal{M} such that A is a retract of X .*

3. A TWO-VARIABLE POSTNIKOV TOWER

Following a suggestion of P. Pelaez, we refine the construction of Voevodsky's slice filtration to a two-variable version which measures both S^1 -connectedness and \mathbb{G}_m -connectedness.

We consider $\mathbf{Spt}_T(k)$ with its motivic stable model structure. For integers a, b , let

$$K_{a,b} := \{F_n(\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q X_+) \mid X \in \mathbf{Sm}/k, p - n \geq a, q - n \geq b\}.$$

We have as well the sets

$$K_{a,-\infty} := \{F_n(\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q X_+) \mid X \in \mathbf{Sm}/k, p - n \geq a\}$$

and

$$K_{-\infty,b} := \{F_n(\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q X_+) \mid X \in \mathbf{Sm}/k, q - n \geq b\}.$$

This gives us the full subcategories of $\mathcal{SH}(k) = \mathbf{HoSpt}_T(k)$

$$\tau^{a,b}\mathcal{SH}(k) := \mathbf{HoSpt}_T(k)(K_{a,b}).$$

By remark 2.3, $\tau^{a,b}\mathcal{SH}(k)$ is the smallest full subcategory of $\mathcal{SH}(k)$ containing the set of objects $K_{a,b}$, closed under small coproducts and taking ‘‘cones’’ of morphisms. As $F_n(\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q X_+) \cong \Sigma_{S^1}^{p-n} \Sigma_{\mathbb{G}_m}^{q-n} X_+$ in $\mathcal{SH}(k)$, $\tau^{a,b}\mathcal{SH}(k)$ can also be described as the smallest full subcategory of $\mathcal{SH}(k)$ containing $\{\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q X_+ \mid X \in \mathbf{Sm}/k, p \geq a, q \geq b\}$ and closed under small coproducts and taking ‘‘cones’’ of morphisms.

In addition, $\tau^{-\infty,b}\mathcal{SH}(k)$ is closed under $\Sigma_{S^1}^n$ for $n \in \mathbb{Z}$, hence $\tau^{-\infty,b}\mathcal{SH}(k)$ is a localizing subcategory of $\mathcal{SH}(k)$, indeed, $\tau^{-\infty,b}\mathcal{SH}(k)$ is the localizing category generated by the objects $\{\Sigma_T^q X_+ \mid X \in \mathbf{Sm}/k, p \geq a, q \geq b\}$, which category is used to define the Tate-Postnikov tower in $\mathcal{SH}(k)$.

Write $\bar{\mathbb{Z}}$ for $\mathbb{Z} \cup \{-\infty\}$. Giving $\bar{\mathbb{Z}}^2$ the partial order $(a, b) \leq (a', b')$ iff $a \leq a'$ and $b \leq b'$, we have

$$\tau^{a',b'}\mathcal{SH}(k) \subset \tau^{a,b}\mathcal{SH}(k) \text{ if } (a, b) \leq (a', b').$$

Let

$$i_{a,b} : \tau^{a,b}\mathcal{SH}(k) \rightarrow \mathcal{SH}(k)$$

be the inclusion.

Theorem 3.1. *For each $(a, b) \in \bar{\mathbb{Z}}^2$, the inclusion functor $i_{a,b}$ admits a right adjoint $r_{a,b} : \mathcal{SH}(k) \rightarrow \tau^{a,b}\mathcal{SH}(k)$. In addition*

- (1) $r_{a,b}$ identifies $\tau^{a,b}\mathcal{SH}(k)$ with the localization of $\mathcal{SH}(k)$ with respect to the $K_{a,b}$ -colocal weak equivalences.
- (2) For $(a, b) \leq (a', b')$, the inclusion $i_{a',b'}^{a,b} : \tau^{a',b'}\mathcal{SH}(k) \rightarrow \tau^{a,b}\mathcal{SH}(k)$ admits a right adjoint $r_{a',b'}^{a,b} : \tau^{a,b}\mathcal{SH}(k) \rightarrow \tau^{a',b'}\mathcal{SH}(k)$. We have $r_{a',b'} = r_{a',b'}^{a,b} \circ r_{a,b}$ and $r_{a',b'}^{a,b}$ identifies $\tau^{a',b'}\mathcal{SH}(k)$ with the localization of $\tau^{a,b}\mathcal{SH}(k)$ with respect to the $K_{a',b'}$ -colocal weak equivalences.
- (3) a morphism $f : X \rightarrow Y$ in $\tau^{a,b}\mathcal{SH}(k)$ is an isomorphism if and only if for each $A \in K_{a,b}$ the map

$$f_* : [A, X]_{\mathcal{SH}(k)} \rightarrow [A, Y]_{\mathcal{SH}(k)}$$

is an isomorphism.

Proof. (2) follows directly from (1) and the uniqueness of adjoints. (1) follows from theorem 1.4, theorem 2.4 and theorem 2.6.

For (3), the isomorphisms in $\tau^{a,b}\mathcal{SH}(k)$ are given by the $K_{a,b}$ -colocal weak equivalences in $\mathcal{R}_{K_{a,b}}\mathbf{Spt}_T(k)$. Choosing fibrant-cofibrant replacements \tilde{X}, \tilde{Y} for X, Y , and lifting f to a map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$, it suffices to show that if

$$f_* : [A, X]_{\mathcal{SH}(k)} \rightarrow [A, Y]_{\mathcal{SH}(k)}$$

is an isomorphism for all $A \in K_{a,b}$, then \tilde{f} is a $K_{a,b}$ -colocal weak equivalence. But if A is in $K_{a,b}$, so is $\Sigma_{S^1}^n A$ for all $n \geq 0$. We have

$$\begin{aligned} \pi_n(\mathcal{H}om(A, \tilde{X})) &= [A[n], X]_{\mathcal{SH}(k)} \\ &\cong [A[n], Y]_{\mathcal{SH}(k)} \\ &= \pi_n(\mathcal{H}om(A, \tilde{Y})) \end{aligned}$$

i.e.

$$f_* : \pi_n(\mathcal{H}om(A, \tilde{X})) \rightarrow \pi_n(\mathcal{H}om(A, \tilde{Y}))$$

is an isomorphism for all $n \geq 0$. Thus $f_* : \mathcal{H}om(A, \tilde{X}) \rightarrow \mathcal{H}om(A, \tilde{Y})$ is a simplicial weak equivalence for all $A \in K_{a,b}$ and hence f is a $K_{a,b}$ -colocal weak equivalence. \square

For $(a, b) \in \bar{Z}^2$, define the endofunctor

$$f_{a,b} : \mathcal{SH}(k) \rightarrow \mathcal{SH}(k)$$

as the composition $i_{a,b} \circ r_{a,b}$. We write f_n^t for $f_{-\infty, n}$ and f_n^s for $f_{n, -\infty}$. By theorem 3.1(2), we have the lattice of natural transformations

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & f_{a+1, b+1} & \longrightarrow & f_{a, b+1} & \longrightarrow & \cdots \longrightarrow f_{b+1}^t \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & f_{a+1, b} & \longrightarrow & f_{a, b} & \longrightarrow & \cdots \longrightarrow f_b^t \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \\ \cdots & \longrightarrow & f_{a+1}^s & \longrightarrow & f_a^s & \longrightarrow & \cdots \longrightarrow \text{id} \end{array}$$

Remark 3.2. In number of papers on Voevodsky's slice tower, one considers the sequence of localizing subcategories

$$\cdots \subset \tau^{-\infty, b}\mathcal{SH}(k) \subset \tau^{-\infty, b}\mathcal{SH}(k) \subset \cdots \subset \mathcal{SH}(k),$$

($\tau^{-\infty, b}\mathcal{SH}(k)$ was usually denoted $\Sigma_T^b\mathcal{SH}^{eff}(k)$, with $\mathcal{SH}^{eff}(k) := \tau^{-\infty, 0}\mathcal{SH}(k)$). The existence of the right adjoint to the inclusion $\Sigma_T^b\mathcal{SH}^{eff}(k) \rightarrow \mathcal{SH}(k)$ follows from Neeman's Brown representability theorem. What we are now writing as f_b^t was usually denoted f_b .

Pelaez introduced the approach via right Bousfield localization to better understand multiplicative properties of Voevodsky's slice tower.

Remark 3.3 (The S^1 and unstable theory). The above construction goes through with minor changes if we replace $\mathcal{SH}(k)$ with $\mathcal{SH}_{S^1}(k)$. For $n \geq 0$, let $F_n^s : \mathbf{Spc}_\bullet(k) \rightarrow \mathbf{Spt}_{S^1}(k)$ be the functor

$$F_n^s(\mathcal{X}) = (F_n^s(\mathcal{X})_0, \dots, F_n^s(\mathcal{X})_m, \dots)$$

with $F_n^s(\mathcal{X})_m = pt$ for $m < n$, $F_n^s(\mathcal{X})_m = \Sigma_{S^1}^{m-n} \mathcal{X}$ for $m \geq n$ and with identity bonding maps. For $a \in \mathbb{Z}$, $b \geq 0$, let $K_{a,b}^s$ be the set of objects $F_n^s(\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q X_+)$, with $p \geq a + n$, $q \geq b$. With its motivic model structure, $\mathbf{Spt}_{S^1}(k)$ is a cellular proper simplicial model category and we can form the right Bousfield localizations $\mathcal{R}_{K_{a,b}^s} \mathbf{Spt}_{S^1}$. This gives us the subcategories $\tau_s^{a,b} \mathcal{SH}_{S^1}(k)$ of $K_{a,b}^s$ -colocal objects, adjoint functors $i_{a,b}^s, r_{a,b}^s, f_{a,b}^s$, $a \in \mathbb{Z}$, $b \geq 0$, with properties exactly analogous to those listed in theorem 3.1. Defining the truncation functors $f_{a,b}^s := i_{a,b}^s \circ r_{a,b}^s$ gives us the two-variable Postnikov tower in $\mathcal{SH}_{S^1}(k)$.

As $\mathbf{Spc}_\bullet(k)$ with its the motivic model structure is also a cellular proper simplicial model category, the same approach, with

$$K_{a,b}^{un} := \{\Sigma_{S^1}^{a'} \Sigma_{\mathbb{G}_m}^{b'} X_+ \mid a' \geq a, b' \geq b, X \in \mathbf{Sm}/k\}, a, b \geq 0,$$

defines a two variable Postnikov (really Whitehead) tower in $\mathcal{H}_\bullet(k)$, again with properties analogous to those listed in theorem 3.1.

4. CONNECTEDNESS

Definition 4.1. Let $\mathcal{E} \in \mathcal{SH}(k)$. We say that \mathcal{E} is *topologically N -connected* if $\Pi_{a,b}(\mathcal{E}) = 0$ for $a \leq N$, $b \in \mathbb{Z}$. For $E \in \mathcal{SH}_{S^1}(k)$, we call E *topologically N -connected* if $\Pi_{a,b}(E) = 0$ for $a \leq N$, $b \geq 0$.

Remark 4.2. We call an S^1 spectrum E *N -connected* if the homotopy sheaf $\pi_m E$ is zero for all $m \leq N$. Let $E_m = \Omega_{\mathbb{G}_m}^\infty(\Sigma_{\mathbb{G}_m}^m \mathcal{E}) \in \mathcal{SH}_{S^1}(k)$. Then \mathcal{E} is topologically N -connected if and only if E_m is N -connected for all $m \in \mathbb{Z}$. Indeed, as $\Omega_{\mathbb{G}_m}^\infty : \mathcal{SH}(k) \rightarrow \mathcal{SH}_{S^1}(k)$ is right adjoint to the infinite suspension functor $\Sigma_{\mathbb{G}_m}^\infty : \mathcal{SH}_{S^1}(k) \rightarrow \mathcal{SH}(k)$, we have

$$\pi_a \Omega_{\mathbb{G}_m}^\infty(\Sigma_{\mathbb{G}_m}^m \mathcal{F}) = \Pi_{a,0}(\Sigma_{\mathbb{G}_m}^m \mathcal{F}) = \Pi_{a,-m} \mathcal{F}$$

for $\mathcal{F} \in \mathcal{SH}(k)$, $a, m \in \mathbb{Z}$.

Lemma 4.3. *If \mathcal{E} is topologically N -connected, then so is $f_n^t \mathcal{E}$ for all $n \in \mathbb{Z}$.*

Proof. As in the above remark, let $E_m = \Omega_T^\infty(\Sigma_{\mathbb{G}_m}^m \mathcal{E})$. Then E_m is N -connected and by [17, proposition 3.2], $f_q^t E_m$ is N -connected for all q . But

$$f_{m+n}^t E_m = \Omega_T^\infty(f_{m+n}^t \Sigma_{\mathbb{G}_m}^m \mathcal{E}) = \Omega_T^\infty(\Sigma_{\mathbb{G}_m}^m f_n^t \mathcal{E});$$

the first identity is [16, lemma 2.2]. Thus $\Omega_T^\infty(\Sigma_{\mathbb{G}_m}^m f_n^t \mathcal{E})$ is N -connected for all m , hence $f_n^t \mathcal{E}$ is topologically N -connected. \square

Lemma 4.4. *Let X be in \mathbf{Sm}/k . Then for $p \geq a$, and all $q \in \mathbb{Z}$, $\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q \Sigma_T^\infty X_+$ is topologically $a - 1$ connected.*

Proof. This is [16, proposition 5.7(1)]. \square

Lemma 4.5. *Take $\mathcal{E} \in \mathcal{SH}(k)$. Then $f_{a,b} \mathcal{E}$ is topologically $a - 1$ connected.*

Proof. By proposition 2.5 and theorem 2.6, $\tau^{a,b}\mathcal{SH}(k)$ is the full subcategory of $K_{a,b}$ -colocal objects of $\mathcal{SH}(k)$. Each element of $K_{a,b}$ is isomorphic in $\mathcal{SH}(k)$ to $\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q \Sigma_T^\infty X_+$ for some $X \in \mathbf{Sm}/k$, $p \geq a$, $q \geq b$. By lemma 4.4 $\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q \Sigma_T^\infty X_+$ is $p + n - 1$ connected in $\mathcal{H}_\bullet(k)$, and thus $\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q \Sigma_T^\infty X_+$ is topologically $p - 1$ connected in $\mathcal{SH}(k)$.

If $\mathcal{F} \in \mathcal{SH}(k)$ is topologically $a - 1$ connected, then for each $U \in \mathbf{Sm}/k$, we have $[\Sigma_{S^1}^n \Sigma_{\mathbb{G}_m}^m \Sigma_T^\infty U_+, \mathcal{F}]_{\mathcal{SH}(k)} = 0$ for all $n \geq a$. This follows from the Gersten spectral sequence on U .

Since each $\Sigma_{S^1}^n \Sigma_{\mathbb{G}_m}^m U_+$ is compact in $\mathcal{SH}(k)$, it follows that if $\mathcal{E} = \varinjlim \mathcal{E}_\beta$ for some λ -sequence $\beta \rightarrow \mathcal{E}_\beta$ in $\mathbf{Spt}_T(k)$, with each \mathcal{E}_β topologically $a - 1$ connected, then \mathcal{E} is topologically $a - 1$ connected. Also if $A \rightarrow B$ is a cofibration in $\mathbf{Spt}_T(k)$ and we have a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ B & \longrightarrow & \mathcal{F} \end{array}$$

with A , B and \mathcal{E} topologically $a - 1$ connected, then the distinguished triangle

$$A \rightarrow \mathcal{E} \oplus B \rightarrow \mathcal{F} \rightarrow A[1]$$

allows us to conclude that \mathcal{F} is also topologically $a - 1$ connected. Similarly, if $A \rightarrow B$ is a trivial cofibration, and \mathcal{F} is a pushout as above, then \mathcal{F} is topologically $a - 1$ connected if \mathcal{E} is.

From this it follows that each $\overline{\Lambda(K_{a,b})}$ -cell complex Y in $\mathbf{Spt}_T(k)$ is topologically $a - 1$ connected. Clearly a retract of a topologically $a - 1$ connected object of $\mathbf{Spt}_T(k)$ is topologically $a - 1$ connected. By proposition 2.7, this implies that every $K_{a,b}$ -colocal object of $\mathbf{Spt}_T(k)$ is topologically $a - 1$ connected. In particular, each object of $\tau^{a,b}\mathcal{SH}(k)$ is topologically $a - 1$ connected. Since $f_{a,b}\mathcal{E} = i_{a,b}r_{a,b}\mathcal{E}$, it follows that $f_{a,b}\mathcal{E}$ is topologically $a - 1$ connected. \square

Lemma 4.6. *Let $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a morphism in $\Sigma_T^n \mathcal{SH}^{eff}(k)$. Then f induces an isomorphism*

$$f_* : \Pi_{a,b}\mathcal{F}_1 \rightarrow \Pi_{a,b}\mathcal{F}_2$$

for all $a \in \mathbb{Z}$ and all $b \geq n$ if and only if f is an isomorphism.

Proof. The implication “ f an isomorphism $\implies f_* : \Pi_{a,b}\mathcal{F}_1 \rightarrow \Pi_{a,b}\mathcal{F}_2$ an isomorphism” is evident. For the other direction, by theorem 3.1(3), it suffices to show that

$$f_* : [\Sigma_{S^1}^a \Sigma_T^m \Sigma_T^\infty X_+, \mathcal{F}_1] \rightarrow [\Sigma_{S^1}^a \Sigma_{\mathbb{G}_m}^m \Sigma_T^\infty X_+, \mathcal{F}_2]$$

is an isomorphism for all $X \in \mathbf{Sm}/k$, $a \in \mathbb{Z}$, $m \geq n$. Filtering X by closed subsets of codimension i for $i = 0, \dots, \dim X$ gives the “Gersten” spectral sequence converging to $[\Sigma_{S^1}^a \Sigma_{\mathbb{G}_m}^m \Sigma_T^\infty X_+, \mathcal{F}]$, with E_1 term

$$\bigoplus_{x \in X^{(i)}} \Pi_{a+i, m+i}(\mathcal{F})(k(x))$$

By assumption, the map f induces an isomorphism on the E_1 terms and hence an isomorphism on the abutment. \square

Proposition 4.7. *Take $\mathcal{E} \in \mathcal{SH}(k)$. If \mathcal{E} is topologically $a - 1$ connected, then the canonical map*

$$f_{a,b}\mathcal{E} \rightarrow f_b^t \mathcal{E}$$

is an isomorphism in $\mathcal{SH}(k)$.

Proof. Since both $f_{a,b}\mathcal{E}$ and $f_b^t\mathcal{E}$ are in $\Sigma_T^b\mathcal{SH}^{eff}(k)$, it suffices to show that $f_{a,b}\mathcal{E} \rightarrow f_b^t\mathcal{E}$ is an isomorphism in $\Sigma_T^b\mathcal{SH}^{eff}(k)$. Thus we need only see that

$$\Pi_{p,q}f_{a,b}\mathcal{E} \rightarrow \Pi_{p,q}f_b^t\mathcal{E}$$

is an isomorphism for all $p \in \mathbb{Z}$, $q \geq b$. But since \mathcal{E} is topologically $a-1$ connected, so is $f_b^t\mathcal{E}$ (lemma 4.3) and $f_{a,b}\mathcal{E}$ is also topologically $a-1$ connected (lemma 4.5). Thus we need only see that $\Pi_{p,q}f_{a,b}\mathcal{E} \rightarrow \Pi_{p,q}f_b^t\mathcal{E}$ is an isomorphism for $p \geq a$, $q \geq b$. By the universal properties for $f_{a,b}\mathcal{E} \rightarrow \mathcal{E}$, $f_b^t\mathcal{E} \rightarrow \mathcal{E}$, these maps induce isomorphisms on $[\Sigma_{S^1}^p \Sigma_{\mathbb{G}_m}^q X_+, -]$ for all $p \geq a$, $q \geq b$, hence induce isomorphisms

$$\Pi_{p,q}f_{a,b}\mathcal{E} \rightarrow \Pi_{p,q}\mathcal{E}; \quad \Pi_{p,q}f_b^t\mathcal{E} \rightarrow \Pi_{p,q}\mathcal{E}.$$

As the diagram

$$\begin{array}{ccc} \Pi_{p,q}f_{a,b}\mathcal{E} & \longrightarrow & \Pi_{p,q}f_b^t\mathcal{E} \\ & \searrow & \downarrow \\ & & \Pi_{p,q}\mathcal{E} \end{array}$$

commutes, we see that $\Pi_{p,q}f_{a,b}\mathcal{E} \rightarrow \Pi_{p,q}f_b^t\mathcal{E}$ is an isomorphism for $p \geq a$, $q \geq b$, completing the proof. \square

5. A DETOUR: MOREL'S HOMOTOPY t -STRUCTURE

We pause to make some comments relating the results of §4 to Morel's homotopy t -structure in $\mathcal{SH}(k)$. For the facts on t -structures, we refer the reader to [4].

Definition 5.1. Let $\mathcal{SH}(k)_{\leq 0}$ be the full subcategory of $\mathcal{SH}(k)$ consisting of objects \mathcal{E} with $\Pi_{a,b}\mathcal{E} = 0$ for $a < 0$, $b \in \mathbb{Z}$, and let $\mathcal{SH}(k)_{\geq 0}$ be the full subcategory of $\mathcal{SH}(k)$ consisting of objects \mathcal{E} with $\Pi_{a,b}\mathcal{E} = 0$ for $a > 0$, $b \in \mathbb{Z}$.

Theorem 5.2 ([20, theorem 5.2.3]). *The triple $(\mathcal{SH}(k), \mathcal{SH}(k)_{\geq 0}, \mathcal{SH}(k)_{\leq 0})$ is a non-degenerate t -structure on $\mathcal{SH}(k)$.*

As usual, define $\mathcal{SH}(k)_{\geq N} := \Sigma_{S^1}^{-N}\mathcal{SH}(k)_{\geq 0}$, $\mathcal{SH}(k)_{\leq N} := \Sigma_{S^1}^{-N}\mathcal{SH}(k)_{\leq 0}$ and let $\tau_{\geq N} : \mathcal{SH}(k) \rightarrow \mathcal{SH}(k)_{\geq N}$, $\tau_{\leq N} : \mathcal{SH}(k) \rightarrow \mathcal{SH}(k)_{\leq N}$ be the truncation functors. These may be defined as follows: given $\mathcal{E} \in \mathcal{SH}(k)$ and $N \in \mathbb{Z}$, there is a distinguished triangle in $\mathcal{SH}(k)$

$$\mathcal{E}_1 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_1[1]$$

with $\mathcal{E}_1 \in \mathcal{SH}(k)_{\leq N}$, $\mathcal{E}_2 \in \mathcal{SH}(k)_{\geq N+1}$; this triangle is then uniquely determined by \mathcal{E} , up to unique isomorphism. Setting $\tau_{\leq N}\mathcal{E} \rightarrow \mathcal{E}$ to be the map $\mathcal{E}_1 \rightarrow \mathcal{E}$ and $\mathcal{E} \rightarrow \tau_{\geq N+1}\mathcal{E}$ to be the map $\mathcal{E} \rightarrow \mathcal{E}_2$ defines the functors $\tau_{\leq N}$ and $\tau_{\geq N+1}$ and shows that $\tau_{\leq N}$, $\tau_{\geq N}$ are the right, resp. left, adjoints to the respective inclusion functors

$$i_{\leq N} : \mathcal{SH}(k)_{\leq N} \rightarrow \mathcal{SH}(k); \quad i_{\geq N} : \mathcal{SH}(k)_{\geq N} \rightarrow \mathcal{SH}(k).$$

Proposition 5.3. $\mathcal{SH}(k)_{\leq N} = \tau^{N, -\infty}\mathcal{SH}(k)$ and $\tau_{\leq N} \cong r_{N, -\infty}$

Proof. We note that $\mathcal{SH}(k)_{\leq N}$ is exactly the full subcategory of topologically $N-1$ -connected objects of $\mathcal{SH}(k)$. For $\mathcal{E} \in \mathcal{SH}(k)$, $\tau_{\leq N}\mathcal{E} \rightarrow \mathcal{E}$ is universal for maps from

an object of $\mathcal{SH}(k)_{\leq N}$ to \mathcal{E} . By lemma 4.5 $f_{N,-\infty}\mathcal{E}$ is in $\mathcal{SH}(k)_{\leq N}$, hence we have a commutative triangle

$$\begin{array}{ccc} \tau_{\leq N}\mathcal{E} & \xrightarrow{\alpha} & \mathcal{E} \\ \beta \uparrow & \nearrow \gamma & \\ f_{N,-\infty}\mathcal{E} & & \end{array}$$

Taking the long exact sequence of homotopy sheaves associated to the distinguished triangle

$$\tau_{\leq N}\mathcal{E} \rightarrow \mathcal{E} \rightarrow \tau_{\geq N+1}\mathcal{E} \rightarrow \tau_{\leq N}\mathcal{E}[1]$$

we see that the induced map $\Pi_{a,b}\tau_{\leq N}\mathcal{E} \rightarrow \Pi_{a,b}\mathcal{E}$ is an isomorphism for $a \geq N$, $b \in \mathbb{Z}$. The same is true for $\Pi_{a,b}f_{N,-\infty}\mathcal{E} \rightarrow \Pi_{a,b}\mathcal{E}$ by the universal property of $f_{N,-\infty}\mathcal{E} \rightarrow \mathcal{E}$. Thus β gives an isomorphism on $\Pi_{a,b}$ for all $a, b \in \mathbb{Z}$, hence is an isomorphism in $\mathcal{SH}(k)$, and thus $r_{N,-\infty}\mathcal{E} \rightarrow \tau_{\leq N}\mathcal{E}$ is an isomorphism in $\mathcal{SH}(k)_{\leq N} = \tau^{N,-\infty}\mathcal{SH}(k)$. \square

6. THE BETTI REALIZATION

Ayoub [3, definition 2.1] has constructed a Betti realization functor as an exact symmetric monoidal functor

$$Betti_{X,\mathcal{M}} : \mathcal{SH}_{\mathcal{M}}(X) \rightarrow D_{\mathcal{M}}(X^{\text{an}})$$

Here \mathcal{M} is a model category of coefficients (which is required to satisfy certain axioms). We refer the reader to [3] for details. For us, we take $X = \text{Spec } k$ with a given embedding into \mathbb{C} , and \mathcal{M} the category of symmetric spectra with the projective stable model structure. Then $\mathcal{SH}_{\mathcal{M}}(\text{Spec } k)$ is equivalent to the category $\mathcal{SH}(k)$ and by [3, remark 1.9] $D_{\mathcal{M}}(\text{Spec } k^{\text{an}}) = D_{\mathcal{M}}(\text{pt})$ is equivalent to the stable homotopy category $\mathcal{SH} \cong \mathbf{Ho}(\mathcal{M})$. In this case, we denote $Betti_{X,\mathcal{M}}$ by Re_B^σ . For X a finite type k -scheme, we write X^{an} for $X(\mathbb{C})$ with the classical topology.

The realization functor is induced by the functor

$$An : \mathbf{Sm}/X \rightarrow \mathbf{SmAn}/X^{\text{an}}$$

sending a smooth X -scheme $Y \rightarrow X$ to the smooth map of analytic spaces $Y^{\text{an}} \rightarrow X^{\text{an}}$. This induces the adjoint pair on the presheaf categories (An^*, An_*) ; in particular, one has for Y a finite type X -scheme the natural transformation

$$(6.1) \quad \epsilon_Y : An^*(Y) \rightarrow Y^{\text{an}}.$$

By [3, lemma 1.10], we have the natural isomorphisms

$$(6.2) \quad \begin{aligned} Betti_{X,\mathcal{M}}(\Sigma_T\mathcal{E}) &\cong \Sigma_{S^1}^2 Betti_{X,\mathcal{M}}(\mathcal{E}) \\ Betti_{X,\mathcal{M}}(\Sigma_{\mathbb{G}_m}\mathcal{E}) &\cong \Sigma_{S^1} Betti_{X,\mathcal{M}}(\mathcal{E}) \end{aligned}$$

Here is our main theorem on the connectedness of the Betti realization of the slice tower.

Theorem 6.1. *Suppose that k has characteristic zero and let $\sigma : k \hookrightarrow \mathbb{C}$ be an embedding. Let $Re_B^\sigma : \mathcal{SH}(k) \rightarrow \mathcal{SH}$ be the associated realization functor. Take $\mathcal{E} \in \mathcal{SH}(k)$ and suppose that \mathcal{E} is topologically $N - 1$ -connected. Then $Re_B^\sigma(f_q^t\mathcal{E})$ is $q + N - 1$ connected for all $q \in \mathbb{Z}$.*

Proof. For $\mathcal{X} := \Sigma_{S^1}^m \Sigma_{\mathbb{G}_m}^n \Sigma_T^\infty X_+ \in K_{N,q}$, $Re_B^\sigma(\mathcal{X}) = \Sigma^{m+n} \Sigma^\infty X(\mathbb{C})_+^{an}$, hence $Re_B(\mathcal{X})$ is $m+n-1$ connected. We note that Re_B^σ is constructed on the model category level as a left Quillen functor (together with various equivalences of categories) and thus commutes with colimits of pushouts by cofibrations. Following the same argument as used in the proof of lemma 4.5, it follows that for each $\mathcal{X} \in \tau^{N,q} \mathcal{SH}(k)$, $Re_B^\sigma(\mathcal{X})$ is $N+q-1$ connected. Thus, for each $\mathcal{E} \in \mathcal{SH}(k)$, $Re_B^\sigma(f_{N,q}\mathcal{E})$ is $N+q-1$ connected. But by proposition 4.7, $f_{N,q}\mathcal{E} \rightarrow f_q^t \mathcal{E}$ is an isomorphism, completing the proof. \square

We proceed to analyze the Betti realization in some detail.

Lemma 6.2. *For $X \in \mathbf{Sm}/k$, the map $Re_B^\sigma(\Sigma_T^\infty X_+) \rightarrow \Sigma^\infty X_+^{an}$ induced by (6.1) and (6.2) is an isomorphism in \mathcal{SH} .*

Proof. $Betti_{X,\mathcal{M}}$ is unital and one has

$$f_{\#}^{an} \circ Betti_{Y'} \cong Betti_Y \circ f_{\#}$$

for $f : Y' \rightarrow Y$ a smooth morphism of finite type k -schemes [3, proposition 2.5]. We apply this to the structure morphism $p : X \rightarrow \text{Spec } k$. As $f_{\#} 1_{\mathcal{SH}(X)} = \Sigma_T^\infty X_+$ and $f_{\#}^{an} 1_{D_{\text{Spt}\Sigma}(X^{an})} = \Sigma_T^\infty X_+^{an}$, this proves the lemma. \square

Lemma 6.3. *Let X be a finite type k -scheme. Then the map $Re_B^\sigma(\Sigma_T^\infty X_+) \rightarrow \Sigma^\infty X_+^{an}$ induced by (6.1) and (6.2) is an isomorphism in \mathcal{SH} .*

Proof. Let $X_\bullet \rightarrow X$ be a cdh hypercover with each X_n smooth over k ; such X_\bullet exists since k admits resolution of singularities. We consider X as the object in $\mathcal{H}(k)$ given by the \mathbb{A}^1 -localization of the presheaf on \mathbf{Sm}/k , $Y \mapsto \text{Hom}_{\mathbf{Sch}_k}(Y, X)$. By Voevodsky's theorem comparing the unstable motivic homotopy categories for the Nisnevich and *cdh* topologies, we have

$$\text{Tot} \Sigma_T^\infty X_\bullet \cong \Sigma_T^\infty X$$

in $\mathcal{SH}(k)$ and hence we have an isomorphism after applying Re_B^σ . Furthermore, Re_B is a left adjoint hence

$$Re_B^\sigma(\text{Tot} \Sigma_T^\infty X_\bullet)_+ \cong \text{Tot} Re_B^\sigma(\Sigma_T^\infty X_\bullet)_+$$

By lemma 6.2, we have

$$Re_B^\sigma(\Sigma_T^\infty X_\bullet)_+ \cong \Sigma^\infty X_\bullet^{an}$$

hence

$$Re_B^\sigma(\text{Tot} \Sigma_T^\infty X_\bullet)_+ \cong \Sigma^\infty (\text{Tot} X_\bullet^{an})_+$$

Since $X_\bullet \rightarrow X$ is a cdh hypercover, it follows that $X_\bullet^{an} \rightarrow X^{an}$ is a hypercover for the classical topology. By the argument used in the proof of [3, proposition 1.4] (see particularly Étape 1) the map

$$\Sigma^\infty (\text{Tot} X_\bullet^{an})_+ \rightarrow \Sigma^\infty X_+^{an}$$

is an isomorphism in \mathcal{SH} , which completes the proof. \square

For $X \in \mathbf{Sm}/k$, we have the symmetric motivic spectrum

$$(\Sigma_T^\infty X_+)^{tr} := (\text{Sym}^\infty X_+, \text{Sym}^\infty \Sigma_T X_+, \dots, \text{Sym}^\infty \Sigma_T^n X_+, \dots).$$

The bonding maps are defined by sending $s \wedge (\sum_i t_1^i \wedge \dots \wedge t_n^i \wedge x_i)$ to $\sum_i s \wedge t_1^i \wedge \dots \wedge t_n^i \wedge x_i$.

Let $M\mathbb{Z} \in \mathcal{SH}(k)$ denote the motivic Eilenberg-MacLane spectrum $(\Sigma_T^\infty \text{Spec } k_+)^{tr}$. The maps

$$\begin{aligned} & (\text{Sym}^M \Sigma_T^m \text{Spec } k_+) \wedge (\text{Sym}^N \Sigma_T^n X_+) \rightarrow \text{Sym}^{NM} \Sigma_T^{n+m} X_+ \\ & \left(\sum_{j=1}^M s_1^j \wedge \dots \wedge s_m^j \right) \wedge \left(\sum_{i=1}^N t_1^i \wedge \dots \wedge t_n^i \wedge x_i \right) \mapsto \sum_{i=1}^N \sum_{j=1}^M s_1^j \wedge \dots \wedge s_m^j \wedge t_1^i \wedge \dots \wedge t_n^i \wedge x_i \end{aligned}$$

make $(\Sigma_T^\infty X_+)^{tr}$ into an $M\mathbb{Z}$ -module.

Similarly, for S a pointed space, one has the symmetric spectrum

$$(\Sigma^\infty S)^{tr} := (\text{Sym}^\infty S, \text{Sym}^\infty \Sigma S, \dots, \text{Sym}^\infty \Sigma^n S, \dots).$$

with bonding maps defined as above. We let $H\mathbb{Z} := (\Sigma^\infty S^0)^{tr}$. The Dold-Thom theorem can be phrased as

Theorem 6.4 ([7]). *Suppose S has the homotopy type of a countable CW complex. Then $\pi_n(\Sigma^\infty S)^{tr} = H_n(S, \mathbb{Z})$ for $n \in \mathbb{Z}$.*

In particular, $H\mathbb{Z}$ is isomorphic in \mathcal{SH} to the Eilenberg-MacLane spectrum $EM(\mathbb{Z})$.

Proposition 6.5. *For $X \in \mathbf{Sm}/k$, there is a natural isomorphism in \mathcal{SH}*

$$Re_B^\sigma((\Sigma_T^\infty X_+)^{tr}) \cong (\Sigma^\infty X_+^{\text{an}})^{tr}$$

Proof. Up to stable weak equivalence, we can represent $(\Sigma_T^\infty X_+)^{tr}$ by the spectrum

$$(X_+, \text{Sym}^2 T \wedge X_+, \dots, \text{Sym}^{2n} T^{\wedge n} \wedge X_+, \dots)$$

using the same formula for the bonding maps. This in turn is isomorphic in $\mathcal{SH}(k)$ to

$$\varinjlim_n \Omega_T^n \Sigma_T^\infty \text{Sym}^{2n} T^{\wedge n} \wedge X_+,$$

where the map $\Omega_T^n \Sigma_T^\infty \text{Sym}^{2n} T^{\wedge n} \wedge X_+ \rightarrow \Omega_T^{n+1} \Sigma_T^\infty \text{Sym}^{2n+2} T^{\wedge n+1} \wedge X_+$ is the adjoint of

$$\begin{aligned} \Sigma_T \Omega_T^n \Sigma_T^\infty \text{Sym}^{2n} T^{\wedge n} \wedge X_+ & \rightarrow \Omega_T^n \Sigma_T \Sigma_T^\infty \text{Sym}^{2n} T^{\wedge n} \wedge X_+ \\ & \xrightarrow{\Omega_T^n \epsilon_n} \Omega_T^{n+1} \Sigma_T^\infty \text{Sym}^{2n+2} T^{\wedge n+1} \wedge X_+. \end{aligned}$$

Similarly, we may use the model

$$\varinjlim_n \Omega^{2n} \Sigma^\infty \text{Sym}^{2n} S^{2n} \wedge X_+^{\text{an}},$$

for $(\Sigma^\infty X_+^{\text{an}})^{tr}$.

Since Re_B^σ is the left derived functor of a left Quillen functor, Re_B^σ commutes with colimits in the model categories. From lemma 6.3 we have

$$\begin{aligned} Re_B^\sigma(\Sigma_T^\infty \text{Sym}^{2n} T^{\wedge n} \wedge X_+) & \cong Re_B^\sigma(\Sigma_T^\infty \text{Sym}^{2n} (\mathbb{P}^1, \infty)^{\wedge n} \wedge X_+) \\ & \cong \Sigma^\infty \text{Sym}^{2n} S^{2n} \wedge X_+^{\text{an}}. \end{aligned}$$

We note that $\Sigma_T^\infty \text{Sym}^{2n} T^{\wedge n} \wedge X_+$ is a compact object of $\mathcal{SH}(k)$. Indeed, $\Sigma_T^\infty Y_+$ is compact for all $Y \in \mathbf{Sm}/k$, $\Sigma_T^\infty \text{Sym}^{2n} T^{\wedge n} \wedge X_+$ is isomorphic to $\Sigma_T^\infty \text{Sym}^{2n} (\mathbb{P}^1, \infty)^{\wedge n} \wedge X_+$ and $\text{Sym}^{2n} (\mathbb{P}^1)^n \times X$ admits a finite cubical hyperresolution by objects in \mathbf{Sm}/k (see [9]). Thus the object $\Sigma_T^\infty \text{Sym}^{2n} (\mathbb{P}^1, \infty)^{\wedge n} \wedge X_+$ lies in the triangulated subcategory generated by the $\Sigma_T^\infty Y_+$ and hence, by [3, théorème 3.19(C)], the natural map

$$Re_B^\sigma(\Omega_T^n \Sigma_T^\infty \text{Sym}^{2n} T^{\wedge n} \wedge X_+) \rightarrow \Omega^{2n} Re_B^\sigma(\Sigma_T^\infty \text{Sym}^{2n} T^{\wedge n} \wedge X_+)$$

is an isomorphism in \mathcal{SH} . Passing to the colimit again, we see that the natural map

$$Re_B^\sigma(\varinjlim_n \Omega_T^n \Sigma_T^\infty \mathrm{Sym}^{2n} T^{\wedge n} \wedge X_+) \rightarrow \varinjlim_n \Omega^{2n} Re_B^\sigma(\Sigma_T^\infty \mathrm{Sym}^{2n} T^{\wedge n} \wedge X_+)$$

is an isomorphism in \mathcal{SH} . Finally, the natural map

$$Re_B^\sigma(\Sigma_T^\infty \mathrm{Sym}^{2n} T^{\wedge n} \wedge X_+) \rightarrow \Sigma^\infty \mathrm{Sym}^{2n} S^{2n} \wedge X_+^{\mathrm{an}}$$

is an isomorphism in \mathcal{SH} , by lemma 6.3, giving the natural isomorphism

$$Re_B^\sigma(\varinjlim_n \Omega_T^n \Sigma_T^\infty \mathrm{Sym}^{2n} T^{\wedge n} \wedge X_+) \rightarrow \varinjlim_n \Omega^{2n} \Sigma^\infty \mathrm{Sym}^{2n} S^{2n} \wedge X_+^{\mathrm{an}}$$

in \mathcal{SH} . Putting this all together yields the isomorphism

$$Re_B^\sigma((\Sigma_T^\infty X_+)^{\mathrm{tr}}) \cong (\Sigma^\infty X_+^{\mathrm{an}})^{\mathrm{tr}}$$

in \mathcal{SH} . □

Remark 6.6. Similar results to the above have been proved by different methods by Voevodsky in [30].

Roendigs-Østvær [25] consider the category $\mathbf{ChSS}_{\mathbb{G}_m^{\mathrm{tr}}[1]}^{\mathrm{tr}}$ of symmetric $\mathbb{G}_m^{\mathrm{tr}}[1]$ -spectra in (unbounded) chain complexes of presheaves with transfer on \mathbf{Sm}/k . They define a model structure on $\mathbf{ChSS}_{\mathbb{G}_m^{\mathrm{tr}}[1]}^{\mathrm{tr}}$; the homotopy category $\mathbf{HoChSS}_{\mathbb{G}_m^{\mathrm{tr}}[1]}^{\mathrm{tr}}$ is denoted $DM(k)$. One has as well the category of (unbounded) effective motives, $DM^{\mathrm{eff}}(k)$, defined as the localization of the unbounded derived category of Nisnevich sheaves with transfers with respect to the localizing category generated by objects $\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{A}^1) \rightarrow \mathbb{Z}^{\mathrm{tr}}(X)$. $DM^{\mathrm{eff}}(k)$ contains Voevodsky's category $DM_-^{\mathrm{eff}}(k)$ as a full triangulated subcategory, which in turn contains Voevodsky's category of effective geometric motives $DM_{gm}^{\mathrm{eff}}(k)$ as a full triangulated subcategory. One has the Spanier-Whitehead category of geometric motives $DM_{gm}^{\mathrm{eff}}(k)[- \otimes \mathbb{Z}(1)^{-1}] =: DM_{gm}(k)$; by Voevodsky's cancellation theorem, the functors in the diagram of triangulated tensor categories

$$\begin{array}{ccccc} DM_{gm}^{\mathrm{eff}} & \longrightarrow & DM_-^{\mathrm{eff}}(k) & \longrightarrow & DM^{\mathrm{eff}}(k) \\ \downarrow & & & & \downarrow \\ DM_{gm}(k) & \longrightarrow & & \longrightarrow & DM(k) \end{array}$$

are all fully faithful embeddings. For details we refer the reader to [25, section 2.3].

One may also consider the category $\mathrm{Mod} - M\mathbb{Z}$ of modules in symmetric motivic spectra for $M\mathbb{Z}$. Let $F : \mathrm{Mod} - M\mathbb{Z} \rightarrow \mathbf{Spt}^\Sigma(k)$ be the forgetful functor. Defining a morphism f in $\mathrm{Mod} - M\mathbb{Z}$ to be a fibration, resp. weak equivalence, if $F(f)$ is so in $\mathbf{Spt}^\Sigma(k)$ gives $\mathrm{Mod} - M\mathbb{Z}$ a model category structure for which F becomes a right Quillen functor, with left adjoint the free $M\mathbb{Z}$ -module functor $\mathcal{E} \mapsto M\mathbb{Z} \wedge \mathcal{E}$. This yields the faithful functor $RF : \mathbf{Ho}(\mathrm{Mod} - M\mathbb{Z}) \rightarrow \mathcal{SH}(k)$.

The main result of [25] is

Theorem 6.7 ([25, theorem 1.1]). *Let k be a field of characteristic zero. Then there is an equivalence of $DM(k)$ with $\mathbf{Ho}(\mathrm{Mod} - M\mathbb{Z})$ as triangulated tensor categories, sending $\mathbb{Z}^{\mathrm{tr}}(X)$ to $(\Sigma_T^\infty X_+)^{\mathrm{tr}}$ for $X \in \mathbf{Sm}/k$.*

We write

$$EM_{\mathbb{A}^1} : DM(k) \rightarrow \mathcal{SH}(k)$$

for the faithful functor induced by the equivalence $\mathbf{Ho}(\text{Mod} - M\mathbb{Z}) \cong DM(k)$ and the forgetful functor $RF : \mathbf{Ho}(\text{Mod} - M\mathbb{Z}) \rightarrow \mathcal{SH}(k)$.

Remark 6.8. The functor $EM_{\mathbb{A}^1}$ preserves arbitrary coproducts. Indeed, for $M = \bigoplus_{\alpha} M_{\alpha}$, we have

$$\begin{aligned} \text{Hom}_{\mathcal{SH}(k)}(\Sigma_{S^1}^a \Sigma_{\mathbb{G}_m}^b U_+, EM_{\mathbb{A}^1}(M)) &\cong \text{Hom}_{DM(k)}(\mathbb{Z}^{tr}(U)(b)[a+b], M) \\ &\cong \bigoplus_{\alpha} \text{Hom}_{DM(k)}(\mathbb{Z}^{tr}(U)(b)[a+b], M_{\alpha}) \\ &\cong \bigoplus_{\alpha} \text{Hom}_{\mathcal{SH}(k)}(\Sigma_{S^1}^a \Sigma_{\mathbb{G}_m}^b U_+, EM_{\mathbb{A}^1}(M_{\alpha})) \end{aligned}$$

Thus, the canonical map $\bigoplus_{\alpha} EM_{\mathbb{A}^1}(M_{\alpha}) \rightarrow EM_{\mathbb{A}^1}(\bigoplus_{\alpha} M_{\alpha})$ induces an isomorphism on the homotopy sheaves $\Pi_{a,b}$, and is hence an isomorphism in $\mathcal{SH}(k)$.

For a triangulated category \mathcal{T} , we let $\mathcal{T}_{\mathbb{Q}}$ be the localization with respect to the localizing subcategory \mathcal{T}_{tor} of objects \mathcal{E} such that

$$\text{Hom}_{\mathcal{T}}(A, \mathcal{E}) \otimes \mathbb{Q} = 0$$

for all compact objects A . We let $\mathcal{E}_{\mathbb{Q}}$ denote the image of $\mathcal{E} \in \mathcal{T}$ in $\mathcal{T}_{\mathbb{Q}}$.

We require the following result, which is essentially a rephrasing of the theorem of Suslin-Voevodsky [26, theorem 8.3].

Corollary 6.9. *Take $M \in DM^{eff}(k)_{tor}$ and let $\sigma : k \hookrightarrow \mathbb{C}$ be an embedding. Suppose k is algebraically closed. Then the Betti realization induces an isomorphism*

$$Re_{B^*}^{\sigma} : \Pi_{n,0} EM_{\mathbb{A}^1}(M)(k) \rightarrow \pi_n(Re_B^{\sigma}(EM_{\mathbb{A}^1}(M))).$$

for all $n \in \mathbb{Z}$.

Proof. Let $\mathcal{C} \subset DM^{eff}(k)_{tor}$ be the full subcategory of objects M for which the result holds. Then \mathcal{C} is a triangulated subcategory of $DM^{eff}(k)_{tor}$. We have already noted that the functor $EM_{\mathbb{A}^1}$ preserves arbitrary coproducts; since $Re_{B^*}^{\sigma}$ is a left adjoint, this functor preserves arbitrary coproducts as well. Since both $\Pi_{n,0}(-)$ and $\pi_n(-)$ commute with arbitrary coproducts, \mathcal{C} is closed under arbitrary coproducts.

As $DM_{-}^{eff}(k)$ is by definition the full subcategory of $D^{-}(Shv_{\text{Nis}}^{tr}(\mathbf{Sm}/k))$ with cohomology sheaves in the abelian category of homotopy invariant Nisnevich sheaves with transfer, it suffices to prove the result for M a homotopy invariant Nisnevich sheaves with transfer such that $M_{\mathbb{Q}} = 0$.

For such an M , we have the canonical surjection

$$\mathcal{L}_0(M)_{tor} := \bigoplus_{s \in M(X), Ns=0} \mathbb{Z}^{tr}(X)/N \rightarrow M$$

Clearly the kernel of this map is again a torsion sheaf, giving the canonical left resolution

$$\dots \rightarrow \mathcal{L}_n(M)_{tor} \rightarrow \dots \rightarrow \mathcal{L}_0(M)_{tor} \rightarrow M.$$

This reduces us to the case $M = \mathbb{Z}^{tr}(X)/N$.

In this case, $EM_{\mathbb{A}^1}(M) = (\Sigma_T^{\infty} X_+)^{tr}/N$ and $Re_B^{\sigma}(EM_{\mathbb{A}^1}(M)) \cong (\Sigma_+^{\infty} X_+^{\text{an}})^{tr}/N$. As $EM_{\mathbb{A}^1}$ has left adjoint $M\mathbb{Z} \wedge (-)$ (and similarly for $EM : D(\mathbf{Ab}) \rightarrow \mathcal{SH}$), we have natural isomorphisms

$$(6.3) \quad \begin{aligned} \Pi_{n,0}((\Sigma_T^{\infty} X_+)^{tr}/N)(k) &\cong H_n^{Sus}(X, \mathbb{Z}/N) \\ \pi_n((\Sigma_+^{\infty} X_+^{\text{an}})^{tr}/N) &\cong H_n^{sing}(X^{\text{an}}, \mathbb{Z}/N) \end{aligned}$$

Each element $\alpha \in H_n^{Sus}(X, \mathbb{Z}/N)$ is represented by a map of pairs of schemes $\tilde{\alpha} : (\Delta^n, \partial\Delta^n) \rightarrow (\text{Sym}^{MN} X_+, N \times (\text{Sym}^M X_+))$, where

$$N \times : \text{Sym}^M X_+ \rightarrow \text{Sym}^{MN} X_+$$

is the multiplication map. From this representation, one sees that the map

$$Re_{B_*}^\sigma : \Pi_{n,0}((\Sigma_T^\infty X_+)^{tr}/N) \rightarrow \pi_n((\Sigma^\infty X_+^{\text{an}})^{tr}/N)$$

is compatible, via the isomorphisms (6.3), with the map

$$H_n^{Sus}(X, \mathbb{Z}/N) \rightarrow H_n^{sing}(X^{\text{an}}, \mathbb{Z}/N)$$

sending $\tilde{\alpha}$ to $\tilde{\alpha}^{\text{an}} \in \pi_n((\Sigma^\infty X_+^{\text{an}})^{tr}/N) \cong H_n^{sing}(X^{\text{an}}, \mathbb{Z}/N)$. By the Suslin-Voevodsky theorem [26, theorem 8.3], this latter map induces an isomorphism $H_n^{Sus}(X, \mathbb{Z}/N) \rightarrow H_n^{sing}(X^{\text{an}}, \mathbb{Z}/N)$, which completes the proof. \square

7. SLICES OF THE SPHERE SPECTRUM

We have the canonical distinguished triangle of endofunctors on $\mathcal{SH}(k)$

$$f_{q+1}^t \rightarrow f_q^t \rightarrow s_q \rightarrow f_{q+1}^t[1].$$

Lemma 7.1. *Suppose k has finite Galois cohomological dimension for torsion modules. Then for $X \in \mathbf{Sm}/k$ of dimension d over k ,*

- (1) *for $q \geq d + 1$, $f_q^t(\Sigma_T^\infty X_+)$ goes to zero in $\mathcal{SH}(k)_\mathbb{Q}$.*
- (2) *for $q \geq d + 1$, $s_q(\Sigma_T^\infty X_+)$ goes to zero in $\mathcal{SH}(k)_\mathbb{Q}$.*

In particular, $f_q^t(\mathbb{S}_k)$ and $s_q(\mathbb{S}_k)$ goes to zero in $\mathcal{SH}(k)_\mathbb{Q}$ for $q \geq 1$.

Proof. As in the proof of lemma 4.6, it suffices to show that the homotopy sheaves $\Pi_{a,b} f_q^t(\Sigma_T^\infty X_+)$ are torsion for $a \in \mathbb{Z}$, $b \geq q \geq d + 1$. In this case, the universal property of $f_q^t(\Sigma_T^\infty X_+) \rightarrow \Sigma_T^\infty X_+$ gives us an isomorphism

$$\Pi_{a,b} f_q^t(\Sigma_T^\infty X_+) \rightarrow \Pi_{a,b} \Sigma_T^\infty X_+.$$

so it suffices to see that $\Pi_{a,b} \Sigma_T^\infty X_+$ is a torsion sheaf for $b \geq d + 1$, $a \in \mathbb{Z}$. If $y \in Y \in \mathbf{Sm}/k$ is a point, then since the $\Pi_{a,b} \Sigma_T^\infty X_+$ are strictly \mathbb{A}^1 -invariant sheaves [19, corollary 6.2.9], the restriction map

$$\Pi_{a,b} \Sigma_T^\infty X_+(\mathcal{O}_{Y,y}) \rightarrow \Pi_{a,b} \text{Sigma}_T^\infty X_+(k(Y))$$

is injective (a consequence of [20, lemma 3.3.4]), so it suffices to see that $\Pi_{a,b} \Sigma_T^\infty X_+(F)$ is torsion for all fields finitely generated over k , $a \in \mathbb{Z}$ and $b \geq d + 1$. This is [16, proposition 5.7(1)]. \square

We recall that, by Pelaez's theorem, $s_q(\mathbb{S}_k)$ is the motivic Eilenberg-Maclean spectrum of a motive $\pi_q^\mu(\mathbb{S}_k)(q)[2q]$:

$$s_q(\mathbb{S}_k) \cong EM_{\mathbb{A}^1}(\pi_q^\mu(\mathbb{S}_k)(q)[2q]).$$

We also know that $\pi_q^\mu(\mathbb{S}_k)(q)[2q]$ is in $\mathbb{Z}(q) \otimes DM^{eff}(k)$, hence $\pi_q^\mu(\mathbb{S}_k)$ is in $DM^{eff}(k)$.

Lemma 7.2. *Suppose k has finite Galois cohomological dimension for torsion modules. Then for $X \in \mathbf{Sm}/k$ of dimension d over k , $\pi_q^\mu(\Sigma_T^\infty X_+)_\mathbb{Q}$ in $DM(k)_\mathbb{Q}$ is zero for $q > d$. In particular, $\pi_q^\mu(\mathbb{S}_k)_\mathbb{Q} = 0$ in $DM(k)_\mathbb{Q}$ for $q > 0$.*

Proof. By lemma 7.1, $s_q(\Sigma_T^\infty X_+)_\mathbb{Q} = 0$ for $q > d$. As the Eilenberg-Maclean functor $DM(k)_\mathbb{Q} \rightarrow \mathcal{SH}(k)_\mathbb{Q}$ is faithful, it follows that $\pi_q^\mu(\Sigma_T^\infty X_+)(q)[2q]_\mathbb{Q}$ is zero in $DM(k)_\mathbb{Q}$ for $q > d$. \square

8. PROOF OF THE MAIN THEOREM

We fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$, with k algebraically closed, and write Re for $Re_B^\sigma : \mathcal{SH}(k) \rightarrow \mathcal{SH}$.

Proposition 8.1. *The map*

$$Re_* : \Pi_{n,0}(s_q(\mathbb{S}_k))(k) \rightarrow \pi_n(Re(s_q(\mathbb{S}_k)))$$

is an isomorphism for all q and n .

Proof. We note that \mathbb{S}_k is effective, hence for $q < 0$, $s_q(\mathbb{S}_k) = 0$. For $q = 0$, $s_0(\mathbb{S}_k) \cong M\mathbb{Z}$ by Voevodsky's theorem [29], hence $Re(s_0(\mathbb{S}_k)) = H\mathbb{Z}$. We thus have

$$\Pi_{n,0}s_0(\mathbb{S}_k)(k) = \Pi_{n,0}M\mathbb{Z} = H_n^{Sus}(\mathrm{Spec} k, \mathbb{Z}) = \begin{cases} 0 & \text{for } n \neq 0 \\ \mathbb{Z} & \text{for } n = 0. \end{cases}$$

Similarly,

$$\pi_n H\mathbb{Z} = \begin{cases} 0 & \text{for } n \neq 0 \\ \mathbb{Z} & \text{for } n = 0. \end{cases}$$

and it is easy to see that the Betti realization gives an isomorphism $H_0^{Sus}(\mathrm{Spec} k, \mathbb{Z}) \rightarrow \mathbb{Z}$. This handles the cases $q \leq 0$.

For $q > 0$, lemma 7.2 tells us that π_q^μ is in $DM(k)_{tor}^{eff}$. As $s_q(\mathbb{S}_k) = EM_{\mathbb{A}^1}(\pi_q^\mu(q)[2q])$, corollary 6.9 shows that the Betti realization gives an isomorphism

$$Re_* : \pi_n(s_q(\mathbb{S}_k))(k) \rightarrow \pi_n(Re_B(s_q(\mathbb{S}_k))).$$

for all $n \in \mathbb{Z}$. □

Lemma 8.2. *Suppose that Re induces an isomorphism*

$$Re_* : \Pi_{n,0}(\mathbb{S}_k)(k) \rightarrow \pi_n(\mathbb{S})$$

for all n . Then the constant presheaf functor $c : \mathcal{SH} \rightarrow \mathcal{SH}(k)$ is fully faithful.

Proof. As $Re(\Sigma_{\mathbb{S}^1}^n \mathbb{S}_k) = \Sigma^n \mathbb{S}$, our hypothesis on Re_* can be expressed in another way as saying that

$$Re : [\Sigma_{\mathbb{S}^1}^n \mathbb{S}_k, \mathbb{S}_k]_{\mathcal{SH}(k)} \rightarrow [Re(\Sigma_{\mathbb{S}^1}^n \mathbb{S}_k), Re(\mathbb{S}_k)]_{\mathcal{SH}}$$

is an isomorphism for all n . An elementary induction, using the fact that Re is an exact functor, implies that for $E, F \in \mathcal{SH}_{\mathrm{fin}}$ finite spectra, the map

$$Re : [E \wedge \mathbb{S}_k, F \wedge \mathbb{S}_k]_{\mathcal{SH}(k)} \rightarrow [Re(E \wedge \mathbb{S}_k), Re(F \wedge \mathbb{S}_k)]_{\mathcal{SH}}$$

is an isomorphism. Since $Re \circ c \cong \mathrm{id}$, and $c(E) = E \wedge \mathbb{S}_k$, this shows that c is fully faithful on $\mathcal{SH}_{\mathrm{fin}}$.

For $F \in \mathbf{Spt}$, we can write F as a colimit of finite subspectra. Since $E \in \mathcal{SH}_{\mathrm{fin}}$ is compact, a limit argument extends the above isomorphism to show that

$$c_* : [E, F]_{\mathcal{SH}} \rightarrow [c(E), c(F)]_{\mathcal{SH}}$$

is an isomorphism for $E \in \mathcal{SH}_{\mathrm{fin}}$, $F \in \mathcal{SH}$. Now take $E \in \mathbf{Spt}$. Write $E = \varinjlim_n E_n$, for a tower of cofibrations

$$0 = E_0 \subset E_1 \subset \dots \subset E$$

with $E_n \in \mathcal{SH}_{\mathrm{fin}}$ for all n . Then we have the exact sequence

$$0 \rightarrow R^1 \varprojlim_n [E_n, F]_{\mathcal{SH}} \rightarrow [E, F]_{\mathcal{SH}} \rightarrow \varprojlim_n [E_n, F]_{\mathcal{SH}} \rightarrow 0.$$

The functor c is compatible with cofibrations and colimits, giving the exact sequence

$$0 \rightarrow R^1 \varprojlim [c(E_n), c(F)]_{\mathcal{SH}(k)} \rightarrow [c(E), c(F)]_{\mathcal{SH}(k)} \rightarrow \varprojlim_n [c(E_n), c(F)]_{\mathcal{SH}(k)} \rightarrow 0.$$

Since c_* maps the first sequence to the second one, the map

$$c_* : [E, F]_{\mathcal{SH}} \rightarrow [c(E), c(F)]_{\mathcal{SH}}$$

is an isomorphism for $E, F \in \mathcal{SH}$. \square

Thus, our main theorem 1 follows from

Theorem 8.3. *For k algebraically closed of characteristic zero, with embedding $\sigma : k \hookrightarrow \mathbb{C}$, the map $Re_* : \Pi_{n,0}(\mathbb{S}_k)(k) \rightarrow \pi_n(\mathbb{S})$ is an isomorphism for all n .*

Proof. First we consider the case $n = 0$. By Morel's theorem, $\Pi_{0,0}(\mathbb{S}_k)(k) = \text{GW}(k)$, which is isomorphic to \mathbb{Z} via the dimension function, as k is algebraically closed. This shows that the map $\mathbb{S}_k \rightarrow s_0(\mathbb{S}_k) \cong M\mathbb{Z}$ induces an isomorphism

$$\Pi_{0,0}(\mathbb{S}_k)(k) \rightarrow \Pi_{0,0}(s_0\mathbb{S}_k)(k) = \Pi_{0,0}M\mathbb{Z}(k) = \mathbb{Z}$$

Similarly, the first Postnikov layer for \mathbb{S} , $\mathbb{S} \rightarrow H\mathbb{Z}$, arises from the isomorphism $\pi_0(\mathbb{S}) \cong \mathbb{Z}$. This gives us the commutative diagram

$$\begin{array}{ccccc} \Pi_{0,0}\mathbb{S}_k(k) & \xlongequal{\quad} & \Pi_{0,0}s_0\mathbb{S}_k(k) & \xlongequal{\quad} & \mathbb{Z} \\ Re_* \downarrow & & Re_* \downarrow & & \parallel \\ \pi_0(\mathbb{S}) & \xlongequal{\quad} & \pi_0(H\mathbb{Z}) & \xlongequal{\quad} & \mathbb{Z} \end{array}$$

from which it follows that $Re_* : \Pi_{0,0}(\mathbb{S}_k)(k) \rightarrow \pi_0(\mathbb{S})$ is an isomorphism.

Next, consider the Tate-Postnikov tower for \mathbb{S}_k . We have the distinguished triangle

$$f_1^t\mathbb{S}_k \rightarrow \mathbb{S}_k \rightarrow s_0\mathbb{S}_k \rightarrow f_1^t\mathbb{S}_k[1]$$

with $s_0\mathbb{S}_k \cong M\mathbb{Z}$.

We have already seen that the map

$$\Pi_{0,0}\mathbb{S}_k(k) \rightarrow \Pi_{0,0}M\mathbb{Z}(k) = \mathbb{Z}$$

is an isomorphism. Using Morel's connectedness theorem [20, theorem 4.2.10], \mathbb{S}_k is topologically -1 connected, hence $f_1^t\mathbb{S}_k$ is also topologically -1 connected (lemma 4.3). From the long exact sequence

$$\dots \rightarrow \Pi_{a+1,0}M\mathbb{Z}(k) \rightarrow \Pi_{a,0}f_1^t\mathbb{S}_k(k) \rightarrow \Pi_{a,0}\mathbb{S}_k(k) \rightarrow \Pi_{a,0}M\mathbb{Z}(k) \rightarrow \dots$$

and the fact that $\Pi_{a,0}M\mathbb{Z}(k) = H^{-a}(k, \mathbb{Z}(0)) = 0$ for $a \neq 0$, we see that

$$\Pi_{a,0}f_1^t\mathbb{S}_k(k) = \begin{cases} \Pi_{a,0}\mathbb{S}_k(k) & \text{for } a > 0 \\ 0 & \text{for } a \leq 0. \end{cases}$$

Finally, $Re(M\mathbb{Z})$ is the usual Eilenberg-MacLane spectrum $H\mathbb{Z}$, hence

$$\pi_a(Re(M\mathbb{Z})) = \begin{cases} 0 & \text{for } a \neq 0 \\ \mathbb{Z} & \text{for } a = 0. \end{cases}$$

As Re is exact, it suffices to show that

$$Re_* : \Pi_{a,0}f_1^t\mathbb{S}_k(k) \rightarrow \pi_a(Re(f_1^t\mathbb{S}_k))$$

is an isomorphism for all a .

For this we use the spectral sequence associated to the Tate-Postnikov tower

$$\dots \rightarrow f_{n+1}^t \mathbb{S}_k \rightarrow f_n^t \mathbb{S}_k \rightarrow \dots \rightarrow f_1^t \mathbb{S}_k$$

and its Betti realization

$$\dots \rightarrow Re(f_{n+1}^t \mathbb{S}_k) \rightarrow Re(f_n^t \mathbb{S}_k) \rightarrow \dots \rightarrow Re(f_1^t \mathbb{S}_k).$$

By [16, theorem 3], the first tower gives a strongly convergent spectral sequence

$$E_{p,q}^2 = \Pi_{p+q,0}(s_q \mathbb{S}_k)(k) \implies \Pi_{p+q,0} f_1^t \mathbb{S}_k(k)$$

By theorem 6.1, $Re(f_q^t(\mathbb{S}_k))$ is $q-1$ connected, hence the Betti tower gives us the strongly convergent spectral sequence

$$E_{p,q}^2 = \pi_{p+q}(Re(s_q(\mathbb{S}_k))) \implies \pi_{p+q} Re(f_1^t \mathbb{S}_k).$$

Thus, it suffices to show that

$$Re_* : \Pi_{n,0}(s_q \mathbb{S}_k)(k) \rightarrow \pi_n(Re(s_q \mathbb{S}_k))$$

is an isomorphism for all $q > 0$ and all n . This is proposition 8.1. \square

9. THE SUSLIN-VOEVODSKY THEOREM FOR HOMOTOPY

Let $\mathcal{SH}(k)_{\text{fin}} \subset \mathcal{SH}(k)$ be the thick subcategory of $\mathcal{SH}(k)$ generated by the suspension spectra $\Sigma_T^\infty X_+$, for X smooth and projective over k . This is the same as the pseudo-abelianization of the full triangulated subcategory of $\mathcal{SH}(k)$ generated by the suspension spectra $\Sigma_T^\infty X_+$, for X smooth and projective over k ; in characteristic zero, this is the same as the pseudo-abelianization of the full triangulated subcategory of $\mathcal{SH}(k)$ generated by the suspension spectra $\Sigma_T^\infty X_+$, for X smooth over k . We have the effective subcategory $\mathcal{SH}^{eff}(k) = \tau^{-\infty,0} \mathcal{SH}(k) \subset \mathcal{SH}(k)$ and let $\mathcal{SH}^{eff}(k)_{\text{tor}} \subset \mathcal{SH}^{eff}(k)$ be the full subcategory with objects those \mathcal{E} such that $\mathcal{E}_{\mathbb{Q}} = 0$ in $\mathcal{SH}(k)_{\mathbb{Q}}$.

Theorem 9.1. *Suppose k is algebraically closed of characteristic zero, with an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then for $\mathcal{E} \in \mathcal{SH}(k)_{\text{fin}} \cap \mathcal{SH}^{eff}(k)_{\text{tor}}$, the map*

$$Re_{B*}^\sigma : \Pi_{n,0} \mathcal{E}(k) \rightarrow \pi_n(Re_B^\sigma)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Proof. Since \mathcal{E} is in $\mathcal{SH}^{eff}(k)$, we have $\mathcal{E} = f_0^t \mathcal{E}$. By [16, theorem 3], as \mathcal{E} is in $\mathcal{SH}_{\text{fin}}(k)$, the tower

$$\dots \rightarrow f_{n+1}^t \mathcal{E} \rightarrow f_n^t \mathcal{E} \rightarrow \dots \rightarrow f_0 \mathcal{E} = \mathcal{E}$$

gives rise to a strongly convergent spectral sequence

$$E_{p,q}^2 = \Pi_{p+q,0} s_q \mathcal{E}(k) \implies \Pi_{p+q,0} \mathcal{E}(k).$$

Futhermore, by [16, proposition 5.7(3)] there is an integer N such that \mathcal{E} is topologically $N-1$ -connected. By theorem 6.1, $Re_B^\sigma(f_n^t \mathcal{E})$ is $n+N-1$ connected for all $n \in \mathbb{Z}$, and hence the tower

$$\dots \rightarrow Re_B^\sigma(f_{n+1}^t \mathcal{E}) \rightarrow Re_B^\sigma(f_n^t \mathcal{E}) \rightarrow \dots \rightarrow Re_B^\sigma(f_0^t \mathcal{E}) = Re_B^\sigma(\mathcal{E})$$

defines a strongly convergent spectral sequence

$$E_{p,q}^2 = \pi_{p+q} Re_B^\sigma(s_q \mathcal{E}) \implies \pi_{p+q} Re_B^\sigma \mathcal{E}.$$

Since $f_0\mathcal{E} = \mathcal{E}$, it follows that $s_q\mathcal{E} = 0$ for $q < 0$; as $\mathcal{E}_{\mathbb{Q}} = 0$, we have $(s_q\mathcal{E})_{\mathbb{Q}} = 0$ as well, that is, each $\pi_q^\mu\mathcal{E}$ is in $DM^{eff}(k)_{tor}$. By corollary 6.9, the map

$$Re_{B*}^\sigma : \Pi_{p+q,0}s_q\mathcal{E}(k) \rightarrow \pi_{p+q}Re_B^\sigma(s_q\mathcal{E})$$

is an isomorphism for all p, q ; as both spectral sequences are strongly convergent,

$$Re_{B*}^\sigma : \Pi_{n,0}\mathcal{E}(k) \rightarrow \pi_n(Re_B^\sigma)$$

is an isomorphism for all $n \in \mathbb{Z}$, as desired. \square

As a special case, we have the homotopy analog of the theorem of Suslin-Voevodsky promised in the introduction (theorem 3):

Corollary 9.2. *Let k be an algebraically closed field of characteristic zero with an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then for all $X \in \mathbf{Sm}/k$, all integers $N > 1$ and $n \in \mathbb{Z}$, the map*

$$Re_{B*}^\sigma : \Pi_{n,0}(\Sigma_T^\infty X_+; \mathbb{Z}/N)(k) \rightarrow \pi_n(\Sigma^\infty X_+^{\text{an}}; \mathbb{Z}/N)$$

is an isomorphism. Here $\Pi_{n,0}(-; \mathbb{Z}/N)$ and $\pi_n(-; \mathbb{Z}/N)$ are the homotopy sheaves, resp. homotopy groups, with mod N coefficients.

Proof. We note that $\Pi_{n,0}(\mathcal{E}; \mathbb{Z}/N)$ is by definition $\Pi_{n,0}(\mathcal{E}/N)$, and similarly for $\pi_n(E; \mathbb{Z}/N)$. We may apply theorem 9.1 to the object $\Sigma_T^\infty X_+/N$, which is in $\mathcal{SH}(k)_{\text{fin}} \cap \mathcal{SH}^{eff}(k)_{\text{tor}}$; we need only note that, by lemma 6.2, $Re_B^\sigma(\Sigma_T^\infty X_+/N) \cong \Sigma^\infty X_+^{\text{an}}/N$. \square

10. SLICES OF THE SPHERE SPECTRUM

Voevodsky has formulated a conjecture [27, conjecture 9] giving a formula for the slices of \mathbb{S}_k in terms of the Adams-Novikov spectral sequence for the homotopy groups of \mathbb{S} . This conjecture follows from properties of the motivic Thom spectrum MGL , together with an as yet unpublished result of Hopkins-Morel [11] on the slices of MGL . As we are hopeful that the Hopkins-Morel result will appear soon (a preprint by M. Hoyois [12] is now available), we give some of the details of the proof of Voevodsky's conjecture, without any claim to originality. Our main object in presenting this material is to raise some questions on the Betti realization of the slice tower for \mathbb{S}_k and its possible connection with the classical Adams-Novikov spectral sequence.

We first recall Voevodsky's conjecture. Consider the cosimplicial spectrum

$$(10.1) \quad \begin{array}{ccccccc} & \leftarrow & & \leftarrow & & \leftarrow & & \leftarrow \\ & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\ & & \leftarrow & & \leftarrow & & \leftarrow & & \leftarrow \\ MU^{\wedge*} := \dots & \vdots & MU^{\wedge n+1} & \vdots & MU^{\wedge n} & \vdots & \dots & \leftarrow & MU^{\wedge 2} & \rightarrow & MU \\ & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & \leftarrow \\ & \leftarrow & & \leftarrow & & \leftarrow & & \leftarrow & & \leftarrow & \leftarrow \end{array}$$

with $MU^{\wedge n}$ in degree $n - 1$. The maps \leftarrow insert the unit in the various factors, and the maps \rightarrow are multiplication maps.

Applying π_* and taking the usual alternating sum of the coface maps gives the complex of graded abelian groups

$$\pi_*(MU^{\wedge*}) = \pi_*(MU) \rightarrow \pi_*(MU \wedge MU) \rightarrow \dots \rightarrow \pi_*(MU^{\wedge n}) \rightarrow \dots$$

Let $p : MU \rightarrow \overline{MU}$ be the homotopy cofiber of the unit map $\mathbb{S} \rightarrow MU$. We have the canonical isomorphism

$$MU \wedge MU \cong MU[b_1, b_2, \dots]$$

where for a monomial b^I , $I = (i_1, \dots, i_r)$, we take MUb^I to mean $\Sigma^{\sum_j j \cdot i_j} MU$. The unit map $MU \wedge \mathbb{S} \rightarrow MU \wedge MU$ is thus identified with the summand $MU \cdot 1$ and thus the map $MU \wedge MU \rightarrow MU \wedge \overline{MU}$ is canonically split.

Let $\pi_*(NMU)^*$ be the complex of graded abelian groups

$$\pi_*(NMU)^* := \pi_*(MU) \rightarrow \pi_*(MU \wedge \overline{MU}) \rightarrow \dots \rightarrow \pi_*(MU \wedge \overline{MU}^{\wedge n-1}) \rightarrow \dots$$

where the differential $\pi_*(MU \wedge \overline{MU}^{\wedge n-1}) \rightarrow \pi_*(MU \wedge \overline{MU}^{\wedge n})$ is induced by the map inserting the unit in the first factor and mapping MU to \overline{MU} via p . The splitting mentioned above identifies $\pi_*(NMU)^*$ with the normalized subcomplex of $\pi_*(MU^{\wedge *})$; in particular, we have an inclusion $\pi_*(NMU)^* \hookrightarrow \pi_*(MU^{\wedge *})$ which is a quasi-isomorphism.

Furthermore, via this inclusion $\pi_*(MU \wedge \overline{MU}^{\wedge n})$ is identified with an ideal in a polynomial algebra over the Lazard ring $\mathbb{L} = \pi_*(MU)$:

$$\pi_*(MU \wedge \overline{MU}^{\wedge n}) = \mathbb{L} \otimes (\mathbb{Z}[b_1, b_2, \dots]_+)^{\otimes n}$$

where $\mathbb{Z}[-]_+$ means the ideal generated by all the variables b_i . The grading is given by setting $\deg b_m^j = -2m$ and using the grading in \mathbb{L} induced by the isomorphism $\pi_*(MU) \cong \mathbb{L}$. In particular, we have for each $q \geq 0$ the degree $-2q$ summand of the above complex

$$\pi_{-2q}(NMU)^* := [\mathbb{L} \rightarrow \mathbb{L} \otimes \mathbb{Z}[b_1, b_2, \dots]_+ \rightarrow \dots \rightarrow \mathbb{L} \otimes (\mathbb{Z}[b_1, b_2, \dots]_+)^{\otimes n} \rightarrow \dots]_{-2q};$$

note that $\mathbb{L} \otimes (\mathbb{Z}[b_1, b_2, \dots]_+)^{\otimes m} \rightarrow \dots \rightarrow \mathbb{L} \otimes (\mathbb{Z}[b_1, b_2, \dots]_+)^{\otimes n}$ is zero for $m > n$, so $\pi_{-2q}(NMU)^*$ is supported in cohomological degrees $[0, q]$.

Conjecture 10.1 (Voevodsky [27, conjecture 9]). *There is a natural isomorphism in $\mathcal{SH}(k)$*

$$s_q(\mathbb{S}_k) \cong \Sigma_T^q M\mathbb{Z} \wedge EM(\pi_{-2q}(NMU)^*).$$

The conjecture immediately implies

Corollary 10.2. 1. $\pi_q^\mu(\mathbb{S}_k) \cong \pi_{-2q}(NMU)^* \otimes \mathbb{Z}^{tr}$.

2. The cohomology sheaves $\mathcal{H}^p(\pi_q^\mu(\mathbb{S}_k))$ of the effective motive $\pi_q^\mu(\mathbb{S}_k)$ are zero for $p < 0$, $p > q$.

3. For each $q > 0$ and each p , $0 \leq p \leq q$ there is a finite abelian group $A_{p,q}$ with $\mathcal{H}^p(\pi_q^\mu(\mathbb{S}_k)) \cong A_{p,q} \otimes \mathbb{Z}^{tr}$.

4. $\pi_0^\mu(\mathbb{S}_k) = \mathbb{Z}^{tr}$.

The group $A_{p,q}$ is just the $E_2^{p,2q}$ term in the Adams-Novikov spectral sequence

$$A_{p,q} = \text{Ext}_{MU_*(MU)}^{p,-2q}(MU_*, MU_*) = E_2^{p,2q}(AN).$$

This follows directly from the identification of $E_2^{p,q}(AN)$ with $H^p(\pi_{-q}(NMU)^*)$ (see e.g. [1, III, §15], here we use the standard conventions for indexing a spectral sequence). The computation (4) above recovers Voevodsky's computation of $s_0\mathbb{S}_k$ (but this result is possibly used somewhere in the proof of the Hopkins-Morel theorem, so this might not be a new proof of Voevodsky's result).

Corollary 10.2 hints at a possible connection between the Atiyah-Hirzebruch spectral sequence associated to the slice tower for \mathbb{S}_k :

$$E_2^{p,q}(AH) = H^{p-q}(\mathrm{Spec} k, \pi_{-q}^\mu \mathbb{S}_k(-q)) \implies \Pi_{-p-q,0}(\mathbb{S}_k)(k),$$

and the Adams-Novikov spectral sequence. In fact, we have

Theorem 10.3. *For k algebraically closed of characteristic zero we have*

$$E_2^{p,q}(AH) = E_2^{p-q,2q}(AN) \otimes \hat{\mathbb{Z}}(q)$$

where $\hat{\mathbb{Z}}(q) = \varprojlim_N \mu_N^{\otimes q}$.

This is theorem 4, announced in the introduction; we reiterate that we do not know if $d_3(AN) = d_2(AH)$, even though these two differentials have isomorphic source and target.

Proof. Since k is algebraically closed and of characteristic zero, the Suslin-Voevodsky theorem [26, theorem 8.3] implies (for $q \geq 0$)

$$H^n(\mathrm{Spec} k, \mathbb{Z}/N(q)) = \begin{cases} 0 & \text{for } n \neq 0 \\ \mu_N^{\otimes q} & \text{for } n = 0. \end{cases}$$

Thus the spectral sequence

$$E_2^{a,b} = H^a(\mathrm{Spec} k, \mathcal{H}^b(\pi_q^\mu \mathbb{S}_k)(q)) \implies H^{a+b}(\mathrm{Spec} k, \pi_q^\mu \mathbb{S}_k(q))$$

degenerates at E_2 , $E_2^{a,b} = 0$ for $a \neq 0$, and we have

$$E_2^{p,-q}(AH) = H^{p+q}(\mathrm{Spec} k, \pi_q^\mu \mathbb{S}_k(q)) = A_{p+q,q} \otimes \hat{\mathbb{Z}}(q) = E_2^{p+q,-2q}(AN) \otimes \hat{\mathbb{Z}}(q). \quad \square$$

Proof of conjecture 10.1. We adapt the construction of the Adams-Novikov spectral sequence given in [1, loc. cit.]. Consider the distinguished triangle

$$(10.2) \quad \overline{MGL}[-1] \rightarrow \mathbb{S}_k \rightarrow MGL \rightarrow \overline{MGL}$$

Using the cell structure of MGL , it is easy to see that the unit map $\mathbb{S}_k \rightarrow MGL$ induces an isomorphism $s_0 \mathbb{S}_k \rightarrow s_0 MGL$. Since MGL and \mathbb{S}_k are both in $\mathcal{SH}^{eff}(k)$, it follows that \overline{MGL} also in $\mathcal{SH}^{eff}(k)$ and that $s_0 \overline{MGL} = 0$. Thus \overline{MGL} is in $\Sigma_T \mathcal{SH}^{eff}(k)$ and hence $\overline{MGL}^{\wedge N}$ is in $\Sigma_T^N \mathcal{SH}^{eff}(k)$ for each $N \geq 1$.

We use the following result

Lemma 10.4. *Let k be a field of characteristic zero. For a complex C_* of abelian groups, we let $EM(C_*)$ be the associated Eilenberg-MacLane spectrum. Fix an integer $q \geq 0$. Sending a bounded complex of free \mathbb{Z} -modules C_* to $M\mathbb{Z} \wedge EM(C_*)$ defines a fully faithful embedding*

$$\Sigma_T^q EM : D^b(\mathbf{Ab}) \rightarrow \mathcal{SH}(k)$$

Proof. This follows from the computation

$$\mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_T^q M\mathbb{Z}, \Sigma_{S^1}^a \Sigma_T^q M\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } a = 0 \\ 0 & \text{else.} \end{cases}$$

To make the computation, we first note that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_T^q M\mathbb{Z}, \Sigma_{S^1}^a \Sigma_T^q M\mathbb{Q}) &\cong \mathrm{Hom}_{\mathcal{SH}(k)_{\mathbb{Q}}}(\Sigma_T^q M\mathbb{Q}, \Sigma_{S^1}^a \Sigma_T^q M\mathbb{Q}) \\ &\cong \mathrm{Hom}_{\mathcal{SH}(k)_{\mathbb{Q}^+}}(\Sigma_T^q M\mathbb{Q}, \Sigma_{S^1}^a \Sigma_T^q M\mathbb{Q}) \\ &\cong \mathrm{Hom}_{DM(k)_{\mathbb{Q}}}(\mathbb{Q}(q)[2q], \mathbb{Q}(a)) \end{aligned}$$

The first isomorphism is the adjoint property of \mathbb{Q} -localization, the second follows from the fact that $M\mathbb{Z}$ is orientable hence $M\mathbb{Q}_- = 0$ and the last follows from the theorem of Deglise-Morel identifying $DM(k)_{\mathbb{Q}}$ with $\mathcal{SH}(k)_{\mathbb{Q}^+}$.

This gives us

$$\mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_T^q M\mathbb{Z}, \Sigma_{S^1}^a \Sigma_T^q M\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } a = 0 \\ 0 & \text{else.} \end{cases}$$

We then use the sequence

$$\Sigma_T^q M\mathbb{Z} \rightarrow \Sigma_T^q M\mathbb{Q} \rightarrow \Sigma_T^q M(\mathbb{Q}/\mathbb{Z});$$

this reduces us to showing that

$$\mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_T^q M\mathbb{Z}, \Sigma_{S^1}^a \Sigma_T^q M(\mathbb{Z}/N)) = \begin{cases} \mathbb{Z}/N & \text{for } a = 0 \\ 0 & \text{else.} \end{cases}$$

For this, we use the sequence

$$M\mathbb{Z} \xrightarrow{\times N} M\mathbb{Z} \rightarrow M\mathbb{Z}/N$$

and reduce to showing that

$$\mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma_T^q M(\mathbb{Z}/N), \Sigma_{S^1}^a \Sigma_T^q M(\mathbb{Z}/N)) = \begin{cases} \mathbb{Z}/N & \text{for } a = 0 \\ \mathrm{Ext}_{\mathbf{Ab}}^1(\mathbb{Z}/N, \mathbb{Z}/N) & \text{for } a = 1 \\ 0 & \text{else.} \end{cases}$$

This follows from Voevodsky's computation of the motivic Steenrod algebra [30, theorem 3.49], in particular, that the only weight 0 mod ℓ operation is the Bockstein map. \square

By an n -cube in a category \mathcal{C} , we mean a functor from the partially ordered set $\{0, 1\}^n$ to \mathcal{C} . For an n -cube $(i_1, \dots, i_n) \mapsto \mathcal{E}_{(i_1, \dots, i_n)}$ in $\mathbf{Spt}_T^{\Sigma}(k)$, we have the map of $n-1$ -cubes $\mathcal{E}_{(*, \dots, *, 0)} \rightarrow \mathcal{E}_{(*, \dots, *, 1)}$ and we form the T -spectrum $\mathrm{Tot}_n \mathcal{E}_*$ inductively in n as the homotopy cofiber of $\mathrm{Tot}_{n-1} \mathcal{E}_{(*, \dots, *, 0)} \rightarrow \mathrm{Tot}_{n-1} \mathcal{E}_{(*, \dots, *, 1)}$. We make a similar definition for n -cubes in $\mathbf{Spt}_T(k)$, \mathbf{Spt} or $C(\mathbf{Ab})$.

Form the product $[\mathbb{S}_k \rightarrow MGL]^{\wedge n}$ as an n -cube in $\mathbf{Spt}_T^{\Sigma}(k)$, giving us the object $\mathrm{Tot}[\mathbb{S}_k \rightarrow MGL]^{\wedge n}$ in $\mathbf{Spt}_T^{\Sigma}(k)$. The distinguished triangle (10.2) defines an isomorphism of $\mathrm{Tot}[\mathbb{S}_k \rightarrow MGL]^{\wedge n}$ with $\overline{MGL}^{\wedge n}$ in $\mathcal{SH}(k)$. In particular, we have

$$s_q \mathrm{Tot}[\mathbb{S}_k \rightarrow MGL]^{\wedge n} = 0$$

for $0 \leq q \leq n-1$.

Let $[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}$ be the n -cube formed from $[\mathbb{S}_k \rightarrow MGL]^{\wedge n}$ by replacing the \mathbb{S}_k located at the vertex $(0, \dots, 0)$ with the 0-object. We thus have the homotopy cofiber sequence

$$\mathrm{Tot}[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n} \rightarrow \mathrm{Tot}[\mathbb{S}_k \rightarrow MGL]^{\wedge n} \rightarrow \mathrm{Tot}[\mathbb{S}_k \rightarrow 0]^{\wedge n}$$

As $\text{Tot}[\mathbb{S}_k \rightarrow 0]^{\wedge n}$ is isomorphic in $\mathcal{SH}(k)$ to $(\mathbb{S}_k[1])^{\wedge n} = \mathbb{S}_k[n]$, this gives us the distinguished triangle in $\mathcal{SH}(k)$

$$\mathbb{S}_k \rightarrow \text{Tot}[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}[-n+1] \rightarrow \overline{MGL}^{\wedge n}[-n+1] \rightarrow \mathbb{S}_k[1].$$

In particular, we have the isomorphism

$$(10.3) \quad s_q \mathbb{S}_k \cong s_q \text{Tot}[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}[-n+1]$$

for $0 \leq q < n$.

As s_q is exact, we have

$$(10.4) \quad s_q \text{Tot}[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}[-n+1] \cong \text{Tot} s_q[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}[-n+1]$$

Here the notation $s_q[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}$ means we apply s_q to each term in the n -cube $[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}$, where we use the functorial model for s_q in $\mathbf{Spt}_T^{\Sigma}(k)$ furnished by Pelaez's construction [24]. Furthermore, the vertex $(i_1, \dots, i_n) \neq (0, \dots, 0)$ in $[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}$ is $MGL^{\wedge \sum_j i_j}$, so the corresponding vertex in $s_q[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}$ is $s_q(MGL^{\wedge \sum_j i_j})$.

We now apply the theorem of Hopkins-Morel [11, 12]

Theorem 10.5 (Hopkins-Morel). $s_q MGL \cong \Sigma_T^q MZ \otimes MU_{-2q}$.

Using the fact that MGL is oriented, the Hopkins-Morel theorem generalizes immediately to give the isomorphism

$$(10.5) \quad s_q MGL^{\wedge j} \cong \Sigma_T^q MZ \otimes \pi_{-2q}(MU^{\wedge j}).$$

In addition, this shows that $s_q MGL^{\wedge j}$ is in the essential image of the functor $\Sigma_T^q EM : D^b(\mathbf{Ab}) \rightarrow \mathcal{SH}(k)$. Applying lemma 10.4, it follows that the isomorphism (10.5) is natural with respect to the maps in the n -cube $[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}$ and the n -cube $\pi_{-2q}[\mathbb{S} \rightarrow MU]_0^{\wedge n}$, where $\pi_{-2q}[\mathbb{S} \rightarrow MU]_0^{\wedge n}$ is the n -cube in \mathbf{Ab} formed by applying π_{-2q} termwise to the n -cube $[\mathbb{S} \rightarrow MU]_0^{\wedge n}$ in \mathbf{Spt} .

Thus the isomorphisms (10.5) yield an isomorphism

$$(10.6) \quad s_q \text{Tot}[\mathbb{S}_k \rightarrow MGL]_0^{\wedge n}[-n+1] \cong \Sigma_T^q MZ \wedge EM(\text{Tot}(\pi_{-2q}[\mathbb{S} \rightarrow MU]_0^{\wedge n})[-n+1]).$$

We now show that $\text{Tot}(\pi_{-2q}[\mathbb{S} \rightarrow MU]_0^{\wedge n})[-n+1]$ and $\pi_{-2q}(NMU)^*$ are quasi-isomorphic complexes. For this, let $\Delta_{inj} \subset \Delta$ be the subcategory of injective maps $[n] \rightarrow [m]$ in Δ . The comma category $\Delta_{inj}/[n-1]$ is isomorphic to an n -cube with the vertex $(0, \dots, 0)$ deleted, by sending an injective map $f : [i] \rightarrow [n-1]$ to the element $\epsilon(f) \in \{0, 1\}^n$ with $\epsilon(f)_j = 1$ if and only if $j-1$ is in the image of f . Given a cosimplicial abelian group $p \mapsto A^p$ and an integer $n \geq 1$, we may then form the n -cube of abelian groups by sending each injective map $f : [i] \rightarrow [n-1]$ to A^i and each morphism $\varphi : f \rightarrow g$ in $\Delta_{inj}/[n-1]$, $g : [j] \rightarrow [n-1]$ to $A([\varphi]) : A^i \rightarrow A^j$, where $[\varphi] : [i] \rightarrow [j]$ is the unique injective map with $g \circ [\varphi] = f$. We fill in the value at the vertex 0 to be 0, giving the n -cube $\square^n(A^*)$.

For a cosimplicial abelian group $p \mapsto A^p$, let (A^*, d) be associated complex, with differential the usual alternating sum of the coface maps. We have as well the quasi-isomorphic normalized subcomplex $NA^* \subset A^*$, with

$$NA^p = \bigcap_{i=1}^{p+1} \ker \delta_i^p$$

and differential $\delta_0^p : NA^p \rightarrow NA^{p+1}$.

The following result is standard

Lemma 10.6. *1. The collection of identity maps on A^i , $i = 0, \dots, n-1$ defines a quasi-isomorphism*

$$\sigma_{\leq n-1}A^* \rightarrow \text{Tot}_n \square^n(A^*)[-n+1].$$

Here $\sigma_{\leq n-1}$ is the “stupid truncation”.

2. The inclusion $\sigma_{\leq n-1}NA^ \hookrightarrow \sigma_{\leq n-1}A^*$ is an isomorphism on H^p for $0 \leq p < n-1$.*

Fix an integer $q \geq 0$ and take n to be any integer $n \geq q+2$. By lemma 10.6, we have a map of complexes

$$\sigma_{\leq n-1}\pi_{-2q}(NMU)^* \rightarrow \sigma_{\leq n-1}\pi_{-2q}(MU^{\wedge*}) \rightarrow \text{Tot}\pi_{-2q}[\mathbb{S} \rightarrow MU]_0^{\wedge n}[-n+1]$$

which is an isomorphism on cohomology in degree $\leq n-2$. Also, $\pi_{-2q}(NMU)^m = 0$ for $m > q$, so $\sigma_{\leq n-1}\pi_{-2q}(NMU)^* = \sigma_{\leq n-2}\pi_{-2q}(NMU)^* = \pi_{-2q}(NMU)^*$. Letting $e_i : \pi_{-2q}MU^{\wedge n-1} \rightarrow \pi_{-2q}MU^{\wedge n}$ be the map induced by inserting the unit in the i th factor, one sees by reason of degree that the map

$$\sum_{i=0}^n e_i : \oplus \pi_{-2q}MU^{\wedge n-1} \rightarrow \pi_{-2q}MU^{\wedge n}$$

is surjective, and thus the map

$$\pi_{-2q}(NMU)^* \rightarrow \text{Tot}\pi_{-2q}[\mathbb{S} \rightarrow MU]_0^{\wedge n}[-n+1]$$

is a quasi-isomorphism.

Combining this with (10.3), (10.4) and (10.6) completes the proof. □

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