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Smooth motives

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Abstract. Following ideas of Bondarko [4], we construct a DG category whose homotopy category is equivalent to the full subcategory of motives over a base-scheme S generated by the motives of smooth projective S-schemes, assuming that S is itself smooth over a perfect field.

Contents

Introduction

Recently, Bondarko [4] has given a construction of a DG category of motives over a field k, built out of "higher finite correspondences" between smooth projective varieties over k. Assuming resolution of singularities, the homotopy category of this DG category is equivalent to Voevodsky's category of effective geometric motives. The main goal in this paper is to extend this construction to the case of motives over

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a base-scheme S. For simplicity, we restrict to the case of a regular S, essentially of finite type over a field, although many aspects of the construction should be possible in a more general setting.

If S has positive Krull dimension, one would not expect that the motive of an arbitrary smooth S-scheme be expressible in terms of motives of smooth projective S-schemes. Thus, the category of motives we construct represents a special type of motive over S. Since the Betti realization of our motives will land in the derived category of local systems on S^{an} rather than in the derived category of constructible sheaves, we call our category $SmMot^{\mathrm{eff}}_{gm}(S)$ the category of *smooth effective geometric motives over* S. We have as well a version with the Tate motive inverted, $SmMot_{gm}(S)$. Both $SmMot^{\mathrm{eff}}_{gm}(S)$ and $SmMot_{gm}(S)$ are constructed by taking the homotopy category of suitable DG categories, and then taking an idempotent completion.

We do not construct a tensor structure on $SmMot^{\mathrm{eff}}_{gm}(S)$ or $SmMot_{gm}(S)$. However, after passing to $\mathbb{Q}$-coefficients, we replace our cubical construction with alternating cubes, which makes possible a tensor structure on DG categories whose homotopy categories are equivalent to $SmMot^{\mathrm{eff}}_{gm}(S)_{\mathbb{Q}}$ and $SmMot_{gm}(S)_{\mathbb{Q}}$ (up to idempotent completion).

Our main comparison result involves the categories of motives $DM^{\mathrm{eff}}(S)$ and $DM(S)$ constructed by Cisinski-Déglise. We construct exact functors

$$\rho^{\mathrm{eff}}_S : SmMot^{\mathrm{eff}}_{gm}(S) \to DM^{\mathrm{eff}}(S);\ \rho_S : SmMot_{gm}(S) \to DM(S)$$

and we show

Theorem 1 *Let k be a field. Suppose that S is a smooth k-scheme, essentially of finite type over k. Then ρ_S is a fully faithful embedding.*

This of course implies that ρ^{eff}_S is a faithful embedding, but due to the lack of a cancellation theorem in $DM^{\mathrm{eff}}(S)$, we do not know if ρ^{eff}_S is full.

Our main technical tool is an extension of the Friedlander-Lawson-Voevodsky moving lemmas to the case of cycles on a smooth projective scheme over a regular semi-local base scheme B, with $\mathcal{O}_B$ containing a field. This enables us to extend the fundamental duality theorem for equi-dimensional cycles on smooth projective varieties to smooth projective schemes over a regular semi-local base (over a field). We pass from the semi-local case to an arbitrary regular base (over a field) by making a Zariski sheafification; we were not able to extend the available techniques beyond the semi-local case, so the Zariski sheafification is forced upon us. We do not know if the duality theorem over a general base holds before making the Zariski sheafification.

As hinted above, our interest in these constructions arose from our desire to construct a refined realization functor on the subcategory of $DM(S)$ generated by smooth projective S-schemes. One example, given above, is that we should have a realization functor

$$\Re_B : SmMot_{gm}(S) \to D^b(\mathrm{Loc}/S^{\mathrm{an}}),$$

where $\mathrm{Loc}/S^{\mathrm{an}}$ is the abelian category of local systems of abelian groups on S^{an}, refining the usual Betti realization of $DM_{gm}(S)$ into the derived category of constructible sheaves. Similarly, one should have realizations of $SmMot_{gm}(S)$ to the derived categories of smooth l-adic étale sheaves on $S^{\text{ét}}$ or variations of mixed Hodge structures on S^{an}. By our main theorem, we can view the triangulated category $DTM(S)$ of mixed Tate motives over S as the full subcategory of $SmMot_{gm}(S)$

generated by the Tate twists of the motive of S. Our construction of $SmMot_{gm}(S)$ as the homotopy of a DG category (after taking an idempotent completion) gives a similar DG description of $DTM(S)$. Thus, we can hope to refine the realization functors for $SmMot_{gm}(S)$ even further if we restrict to $DTM(S)$. This should give us a Betti realization functor on $DTM(S)$ to the derived category of uni-potent local systems on $S^{\rm an}$, an étale realization functor to relatively uni-potent étale sheaves on $S^{\rm ét}$ and a Hodge realization to uni-potent variations of mixed Hodge structures on $S^{\rm an}$. The paper of Deligne-Goncharov [6] and our work with Esnault [9], giving constructions of the mixed Tate fundamental group for some types of schemes S, gave us the motivation for the construction of categories of smooth motives and refined realization functors. As this paper is long enough already, we will postpone the construction of these realization functors to a future work.

The paper is organized as follows. We begin with a resumé of the cubical category and cubical constructions. This is a more convenient setting for constructing commutative DG structures than the simplicial one; we took the opportunity here of collecting a number of useful results on cubical constructions that are scattered throughout the literature. We also discuss a variant of cubical structures involving the extended cubical category. This variation on the cubical theme adds the cubical analog of the simplicial degeneracy maps; many of the most useful results on cubical objects that arise in nature actually use the extended cubical structure, so we thought it would be useful to give an abstract discussion.

The next section recalls from [8, 15, 17, 22] various versions of complexes over a DG category and the associated triangulated derived categories, as well as some useful foundational results.

In section §4 we apply this machinery to the category of correspondences, endowed with the algebraic n-cubes as a cubical object. This leads to our construction of the DG category of higher correspondences, $dgCor_S$, the full DG subcategory $dgPrCor_S$ of correspondences on smooth projective S-scheme, the Zariski sheafified version $R\Gamma(S, \underline{dgPrCor}_S)$, the DG category of motivic complexes $dgSmMot_S^{\rm eff} := \mathcal{C}^b(R\Gamma(S, \underline{dgPrCor}_S))$, and finally the triangulated category of smooth effective motives over S, $SmMot_{gm}^{\rm eff}(S)_S$, defined by taking the idempotent completion of the homotopy category $\mathcal{K}^b(R\Gamma(S, \underline{dgPrCor}_S))$. We also define the $\mathbb{Q}$-version with alternating cubes, $dgSmMot_{S\mathbb{Q}}^{\rm eff}$, which is a DG tensor category, leading to the tensor triangulated category $SmMot_{gm}^{\rm eff}(S)_{\mathbb{Q}}$. Finally, we consider versions of these categories, $dgSmMot_S$, $dgSmMot_{S\mathbb{Q}}$, $SmMot_{gm}(S)$ and $SmMot_{gm}(S)_{\mathbb{Q}}$, formed by inverting the Lefschetz motive.

In §5 we state our main duality theorem for equi-dimensional cycles over a semi-local base (theorem 5.4), as well as the the projective bundle formula (theorem 5.5). We derive the consequences of these results for duality in the categories $SmMot_{gm}^{\rm eff}(S)$ and $SmMot_{gm}(S)$. In §6, we briefly recall some aspects of the definition of the Cisinski-Déglise categories of motives over a base, $DM^{\rm eff}(S)$ and $DM(S)$, define exact functors

$$\begin{aligned}
&\rho_S^{\rm eff} : SmMot_{gm}^{\rm eff}(S) \to DM^{\rm eff}(S)\\
&\rho_S : SmMot_{gm}(S) \to DM(S),
\end{aligned}$$

and prove our main result (corollary 6.14). Finally, in section §7 we prove our extension of the Friedlander-Lawson-Voevodsky moving lemmas and give the proofs of theorems 5.4 and 5.5.

As I have already mentioned, a major motivation for this paper arose out of the joint work [9] with Hélène Esnault, whom I would like to thank for her encouragement and suggestions. Also, among a number of helpful comments of the referee, the suggestion that I should use the work of Keller and Toën on DG categories for the discussion in §2 gave me a chance to broaden my DG horizons, for which I am very grateful. Finally, I want to express my heartfelt gratitude to Spencer Bloch. As so much of my work has been inspired by Spencer's remarkably original ideas, his steady encouragement over the years has been more important to me than I can possibly put into words.

1 Cubical objects and DG categories

1.1 Cubical objects. We recall some notions discussed in e.g. [18]. We introduce the "cubical category" **Cube**. This is the subcategory of **Sets** with objects $\underline{n} := \{0,1\}^n$, $n = 0, 1, 2, \ldots$, and morphisms generated by

1. Inclusions: $\eta_{n,i,\epsilon} : \underline{n} \to \underline{n+1}$, $\epsilon = 0, 1$, $i = 1, \ldots, n+1$
$$\eta_{n,i,\epsilon}(y_1, \ldots, y_{n-1}) = (y_1, \ldots, y_{i-1}, \epsilon, y_i, \ldots, y_{n-1})$$
2. Projections: $p_{n,i} : \underline{n} \to \underline{n-1}$, $i = 1, \ldots, n$,
$$p_{n,i}(y_1, \ldots, y_n) = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n).$$
3. Permutations of factors: $(\epsilon_1, \ldots, \epsilon_n) \mapsto (\epsilon_{\sigma(1)}, \ldots, \epsilon_{\sigma(n)})$ for $\sigma \in S_n$.
4. Involutions: $\tau_{n,i}$ exchanging 0 and 1 in the ith factor of $\underline{n}$.

Clearly all the Hom-sets in **Cube** are finite. For a category $\mathcal{A}$, we call a functor $F : \mathbf{Cube}^{\text{op}} \to \mathcal{A}$ a *cubical object* of $\mathcal{A}$ and a functor $F : \mathbf{Cube} \to \mathcal{A}$ a *co-cubical object* of $\mathcal{A}$.

Remark 1.1 The permutations and involutions in (3) and (4) give rise to a subgroup of $\text{Aut}_{\mathbf{Sets}}(\underline{n})$ isomorphic to the semi-direct product $F_n := (\mathbb{Z}/2)^n \ltimes \Sigma_n$, where Σ_n acts on $(\mathbb{Z}/2)^n$ by permuting the factors.

We extend the standard sign representation of Σ_n to the sign representation
$$\text{sgn} : F_n \to \{\pm 1\}$$
by
$$\text{sgn}(\epsilon_1, \ldots, \epsilon_n, \sigma) := (-1)^{\sum_j \epsilon_j} \text{sgn}(\sigma).$$

Example 1.2 Let S be a scheme, set $\mathbb{A}^1_S := \text{Spec}_S\, \mathcal{O}_S[y]$. We set $\square^n_S := (\mathbb{A}^1_S)^n$. S_n acts on $\square^n_S$ by permuting the factors. We let $\mathbb{Z}/2$ act on $\mathbb{A}^1_S$ by $x \mapsto 1 - x$. This gives us an action of F_n on $\square^n_S$.

Let $p_{n,i} : (\mathbb{A}^1_S)^n \to (\mathbb{A}^1_S)^{n-1}$ be the projection omitting the ith factor. Letting $y_{n,i} : (\mathbb{A}^1_S)^n \to \mathbb{A}^1_S$ be the projection on the ith factor, we use the coordinate system $(y_1, \cdots, y_n)$ on $\square^n_S$, with $y_i := y \circ y_{n,i}$.

Let $\eta_{n,i,\epsilon} : \square^{n-1}_S \to \square^n_S$ be the inclusion
$$\eta_{n,i,\epsilon}(y_1, \ldots, y_{n-1}) = (y_1, \ldots, y_{i-1}, \epsilon, y_i, \ldots, y_{n-1})$$
This gives us the co-cubical object $n \mapsto \square^n_S$ in $\mathbf{Sm}/S$.

A *face* of $\square^n_S$ is a subscheme F defined by equations of the form

$$y_{i_1} = \epsilon_1, \dots, y_{i_s} = \epsilon_s;\ \epsilon_j \in \{0,1\}.$$

1.2 Cubical objects in a pseudo-abelian category. Let $\mathcal{A}$ be a pseudo-abelian category, $\underline{A} : \mathbf{Cube}^{\mathrm{op}} \to \mathcal{A}$ a cubical object. For $\epsilon \in \{0,1\}$, let $\pi^\epsilon_{n,i} : \underline{A}(\underline{n}) \to \underline{A}(\underline{n})$ be the endomorphism $p^*_{n,i} \circ \eta^*_{n-1,i,\epsilon}$, and set

$$\pi_n := (\mathrm{id} - \pi^1_{n,n}) \circ \dots \circ (\mathrm{id} - \pi^1_{n,1}).$$

Note that the $\pi^\epsilon_{n,i}$ are commuting idempotents, and that the subobject $(\mathrm{id} - \pi^\epsilon_{n,i})^*(\underline{A}(\underline{n})) \subset \underline{A}(\underline{n})$ is a kernel for $\eta_{n,i,\epsilon}$. Since $\mathcal{A}$ is pseudo-abelian, the objects

$$\underline{A}(\underline{n})^0 := \cap^n_{i=1} \ker \eta^*_{n-1,i,1} \subset \underline{A}(\underline{n})$$

and

$$\underline{A}(\underline{n})^{\mathrm{degn}} := \sum_{i=1}^n p^*_{n,i}(\underline{A}(\underline{n-1})) \subset \underline{A}(\underline{n})$$

are well-defined.

Let $(\underline{A}_*, d)$ be the complex with $\underline{A}_n := \underline{A}(\underline{n})$ and with

$$d_n := \sum_{i=1}^n (-1)^i (\eta^*_{n,i,1} - \eta^*_{n,i,0}) : \underline{A}_{n+1} \to \underline{A}_n.$$

Write $\underline{A}^0_n, \underline{A}^{\mathrm{degn}}_n$ for $\underline{A}^0(\underline{n}), \underline{A}^{\mathrm{degn}}(\underline{n})$, respectively.

The following result is the basis of all "cubical" constructions; the proof is elementary and is left to the reader.

Lemma 1.3 *Let $\underline{A} : \mathbf{Cube}^{\mathrm{op}} \to \mathcal{A}$ be a cubical object in a pseudo-abelian category $\mathcal{A}$. Then*

1. *For each n, π_n maps $\underline{A}_n$ to $\underline{A}^0_n$ and defines a splitting*

$$\underline{A}_n = \underline{A}^{\mathrm{degn}}_n \oplus \underline{A}^0_n.$$

2. *$d_n(\underline{A}^{\mathrm{degn}}_n) = 0$, $d_n(\underline{A}^0_n) \subset \underline{A}^0_{n-1}$*

Definition 1.4 Let $\underline{A} : \mathbf{Cube}^{\mathrm{op}} \to \mathcal{A}$ be a cubical object in a pseudo-abelian category $\mathcal{A}$. Define the complex (A_*, d) to be the quotient complex

$$\underline{A}_* / \underline{A}^{\mathrm{degn}}_*$$

of $\underline{A}_*$.

Lemma 1.3 shows that A_* is well-defined and is isomorphic to the subcomplex $\underline{A}^0_*$ of $\underline{A}_*$. We often use cohomological notation, with $A^n := A_{-n}$, etc.

1.3 Products. Suppose we have two cubical objects

$$\underline{A}, \underline{B} : \mathbf{Cube}^{\mathrm{op}} \to \mathcal{A}$$

in a tensor category $(\mathcal{A}, \otimes)$. Form the diagonal cubical object $\underline{A \otimes B}$ by

$$\underline{A \otimes B}(\underline{n}) := \underline{A}(\underline{n}) \otimes \underline{B}(\underline{n})$$

and on morphisms by

$$\underline{A \otimes B}(f) := \underline{A}(f) \otimes \underline{B}(f).$$

Let $p^1_{n,m}:\underline{n+m}\to\underline{n}$, $p^2_{n,m}:\underline{n+m}\to\underline{m}$ be the projections on the first n and last m factors, respectively. Let

$$\cup^{n,m}_{A,B}:\underline{A}(\underline{n})\otimes\underline{B}(\underline{m})\to\underline{A}(\underline{n+m})\otimes\underline{B}(\underline{n+m})$$

be the map $\underline{A}(p^1_{n,m})\otimes\underline{B}(p^2_{n,m})$. One easily checks that the direct sum of the maps $\cup^{n,m}_{A,B}$ defines a map of complexes

$$\cup_{A,B}:\underline{A}^*\otimes\underline{B}^*\to\underline{A\otimes B}^*. \tag{1.1}$$

It is easy to see that we have an associativity property

$$\cup_{A\otimes B,C}\circ(\cup_{A,B}\otimes\mathrm{id}_{\underline{C}^*})=\cup_{A,B\otimes C}\circ(\mathrm{id}_{\underline{A}^*}\otimes\cup_{B,C}) \tag{1.2}$$

but not in general a commutativity property.

1.4 Alternating cubes. Recall the semi-direct product $F_n:=(\mathbb{Z}/2)^n\ltimes\Sigma_n$ and the sign representation $\mathrm{Sgn}:F_n\to\{\pm1\}$. If $\mathcal{A}$ is a pseudo-abelian category and M an F_n-module in $\mathcal{A}$ (i.e., we are given a homomorphism $F_n\to\mathrm{Aut}_{\mathcal{A}}(M)$), we let M^{Sgn} be the largest subobject of M on which F_n acts by the sign representation:

$$M^{\mathrm{Sgn}}:=\cap_{g\in F_n}\ker((g-\mathrm{Sgn}(g)\mathrm{id}_M).$$

Similarly, if Σ_n acts on M, we let M^{sgn} be the subobject of M

$$M^{\mathrm{sgn}}:=\cap_{g\in \Sigma_n}\ker((g-\mathrm{sgn}(g)\mathrm{id}_M).$$

Let $\underline{A}:\mathbf{Cube}^{\mathrm{op}}\to\mathcal{A}$ be a cubical object in a pseudo-abelian category $\mathcal{A}$. For each $n=0,1,2,\ldots$, define the subobject $\underline{A}^{\mathrm{Alt}}(\underline{n})$ of $\underline{A}(\underline{n})$ by

$$\underline{A}^{\mathrm{Alt}}(\underline{n}):=\underline{A}(\underline{n})^{\mathrm{Sgn}}$$

Similarly, let $A^{\mathrm{alt}}(n):=A(\underline{n})^{\mathrm{sgn}}$.

Lemma 1.5 *1. $n\mapsto\underline{A}^{\mathrm{Alt}}(\underline{n})$ defines a sub-cubical object of $\underline{A}$.*

2. $n\mapsto A^{\mathrm{alt}}(\underline{n})$ defines a sub-complex of A_*.

3. Suppose $2\times\mathrm{id}$ is invertible on all the objects $\underline{A}(\underline{n})$. Then the map $\underline{A}_*\to A_*$ induces an isomorphism of complexes $\underline{A}^{\mathrm{Alt}}_*\to A^{\mathrm{alt}}_*$.

Proof This is straightforward, noting that the degenerate subcomplex is killed by the idempotent

$$\frac{1}{2^n}\prod_{i=1}^n(1-\tau_i)$$

where τ_i is the involution in the ith factor of $\underline{n}$. □

1.5 Extended cubes. We note that product of sets makes **Sets** a symmetric monoidal category, and that **Cube** is a symmetric monoidal subcategory. Let **ECube** be the smallest symmetric monoidal subcategory of **Sets** having the same objects as **Cube**, containing **Cube** and containing the morphism

$$\mu:\underline{2}\to\underline{1}$$

defined by the multiplication of integers:

$$\mu((1,1))=1;\ \mu(a,b)=0\text{ for }(a,b)\neq(1,1).$$

An *extended cubical object* in a category $\mathcal{C}$ is a functor $F:\mathbf{ECube}^{\mathrm{op}}\to\mathcal{C}$.

Let $\underline{F} : \mathbf{Cube}^{\mathrm{op}} \to \mathcal{A}$ be a cubical object in a pseudo-abelian category. Let $NF(\underline{n}) \subset \underline{F}(\underline{n})$ be the subobject

$$NF(\underline{n}) := \cap_{i=2}^{n} \ker(\eta^*_{n,i,0}) \cap \cap_{i=1}^{n} \ker(\eta^*_{n,i,1}).$$

This defines the *normalized subcomplex* NF^* of $\underline{F}^*$. Note that NF^* is a subcomplex of $\underline{F}(\underline{*})_0$.

Lemma 1.6 *Let $\underline{F} : \mathbf{ECube}^{\mathrm{op}} \to \mathcal{A}$ be an extended cubical object in a pseudo-abelian category $\mathcal{A}$. Then the inclusion $i : NF^* \to \underline{F}(\underline{*})_0$ is a homotopy equivalence.*

Proof Let

$$N^M F_n := \begin{cases} \cap_{i=n-M}^{n} \ker(\eta^*_{n,i,0}) \cap \cap_{i=1}^{n} \ker(\eta^*_{n,i,1}) & \text{for } n - M > 2 \\ NF^n & \text{for } n - M \leq 2 \end{cases}$$

For each M, the subobjects $N^M F_n \subset \underline{F}^0_n$ form a subcomplex $N^M F_*$ of $\underline{F}^0_*$ which contains NF_* and agrees with NF_* in degrees $n \leq M+2$. Since $N^{-1}F_* = \underline{F}^0_*$, it thus suffices to show that the inclusion

$$i^M : N^M F_* \to N^{M-1} F_*$$

is a homotopy equivalence, such that the chosen homotopy inverse p^M and chosen homotopy h^M between $i^M \circ p^M$ and id satisfy:

1. $p^M \circ i^M = \mathrm{id}$
2. $h^M_n : N^{M-1}F_{n-1} \to N^{M-1}F_n$ is the zero map for $n \leq M$

Indeed, in this case, the infinite composition

$$p := \ldots p^M \circ p^{M-1} \circ \ldots \circ p^0$$

makes sense, as does the infinite sum

$$h := \sum_M i^0 \circ \ldots i^{M-1} \circ h^M \circ p^{M-1} \circ \ldots p^0.$$

The map p gives a homotopy inverse to i, with $pi = \mathrm{id}$ and h defines a homotopy between ip and the identity. We proceed to define the maps p^M and h^M.

Define the map $q : \underline{2} \to \underline{1}$ by

$$q(x, y) = 1 - \mu(1-x, 1-y) = 1 - (1-x)(1-y).$$

For $1 \leq i \leq n-1$, let $q_{n,i} : \underline{n} \to \underline{n-1}$ be the map

$$q_{n,i}(x_1, \ldots, x_n) := (x_1, x_2, \ldots, x_{i-1}, q(x_i, x_{i+1}), x_{i+2}, \ldots, x_n).$$

Then

$$\begin{aligned} &q_{n,i} \circ \eta_{n,i,0} = q_{n,i} \circ \eta_{n,i+1,0} = \mathrm{id}_{\underline{n-1}} \\ &q_{n,i} \circ \eta_{n,i,1} = q_{n,i} \circ \eta_{n,i+1,1} = \eta_{n,i,1} \circ p_{n-1,i} \\ &q_{n,i} \circ \eta_{n,j,\epsilon} = \begin{cases} \eta_{n-1,j,\epsilon} \circ q_{n-1,i-1} & \text{for } 1 \leq j < i \\ \eta_{n-1,j-1,\epsilon} \circ q_{n-1,i} & \text{for } i+1 < j \leq n \end{cases} \end{aligned} \tag{1.3}$$

Defining $q^*_{n,j} = 0$ for $j \leq 0$ or $j \geq n$, this implies that, for $i \geq 1$, the maps

$$p^M_n := \mathrm{id} - q^*_{n,n-M-1} \circ \eta^*_{n-1,n-M,0}$$

define a map of complexes $p^M : \underline{F}^0_* \to \underline{F}^0_*$ which restricts to the inclusion $N^M F_* \to \underline{F}^0_*$ on $N^M F_*$, and maps $N^{M-1}F_*$ to $N^M F_*$. We let

$$p^M : N^{M-1}F_* \to N^M F_*$$

be the restriction.

Let $h_n^M : \underline{F}^0_{n-1} \to \underline{F}^0_n$ be the map $(-1)^{n-M} q^*_{n,n-M-1}$. The relations (1.3) imply that h_n^M restricts to a map

$$h_n^M : N^{M-1}F_{n-1} \to N^{M-1}F_n$$

and, on $\underline{F}^0_n$, we have

$$\begin{aligned}
&d_n h_M^{n+1} + h_M^n d_{n-1}\\
&\quad = (-1)^{n-M+1}\left[\sum_{j=1}^{n+1}(-1)^j \eta^*_{n,j,0}\circ q^*_{n+1,n-M} - \sum_{l=1}^{n}(-1)^l q^*_{n,n-M-1}\circ \eta^*_{n-1,l,0}\right]\\
&\quad = (-1)^{n-M+1}\sum_{j=1}^{n-M-1}(-1)^j q^*_{n,n-M-1}\circ\eta^*_{n-1,j,0}\\
&\qquad + (-1)^{n-M+1}\sum_{j=n-M+2}^{n+1}(-1)^j q^*_{n,n-M}\circ\eta^*_{n-1,j-1,0}\\
&\qquad\quad - (-1)^{n-M+1}\sum_{l=1}^{n}(-1)^l q^*_{n,n-M-1}\circ\eta^*_{n-1,l,0}\\
&\quad = q^*_{n,n-M-1}\eta^*_{n-1,n-M,0}\\
&\qquad + (-1)^{n-M}\sum_{j=n-M+1}^{n}(-1)^j (q^*_{n,n-M} + q^*_{n,n-M-1})\circ\eta^*_{n-1,j,0}
\end{aligned}$$

Since $\eta^*_{n-1,j,0} = 0$ on $N^{M-1}F_n$ for $j \geq n-M+1$, the h_n^M give the desired homotopy. □

Let $\underline{F} : \mathbf{ECube}^{\mathrm{op}} \to \mathcal{A}$ be an extend cubical object in an abelian category $\mathcal{A}$. Then we have the following description of $H_n(NF_*)$:

$$H_n(NF_*) = \frac{\cap_{i=1}^n \ker \eta^*_{n-1,i,0} \cap \cap_{i=1}^n \ker \eta^*_{n-1,i,1}}{\eta^*_{n,1,0}[\cap_{i=2}^{n+1} \ker \eta^*_{n,i,0} \cap \cap_{i=1}^{n+1} \ker \eta^*_{n,i,1}]}$$

From this description, we see that the symmetric group S_n acts on $H_n(NF_*)$ through the permutation action on $\underline{n}$ and the action $\sigma \mapsto \mathrm{id} \times F(\sigma)$ on $\underline{F}(\underline{n+1})$. Via lemma 1.6, this gives us an S_n-action on $H_n(F_*)$.

Proposition 1.7 $\underline{F} : \mathbf{ECube}^{\mathrm{op}} \to \mathcal{A}$ *be an extended cubical object in an abelian category $\mathcal{A}$. Suppose that the Hom-groups in $\mathcal{A}$ are $\mathbb{Q}$-vector spaces. Then the inclusion*

$$F_*^{\mathrm{alt}} \to F_*$$

is a quasi-isomorphism.

Proof Since the Hom-groups in $\mathcal{A}$ are $\mathbb{Q}$-vector spaces, the natural map

$$H_n(F_*^{\mathrm{alt}}) \to H_n(F_*)^{\mathrm{alt}}$$

is an isomorphism for all n. Thus, we need only show that the symmetric group S_n acts on $H_n(F_*)$ by the sign representation.

Fix an element in $Z_n(NF_*)$ representing a class $[z] \in H_n(NF_*)$, i.e.

$$\eta^*_{n-1,i,\epsilon}(z) = 0$$

for all i and for $\epsilon = 0, 1$. Let $\tau : \underline{n} \to \underline{n}$ be the permutation exchanging the first two factors. Let $h_n : \underline{n+1} \to \underline{n}$ be the map

$$h_n(x_1, x_2, x_3, \ldots, x_{n+1}) := (x_2, q(x_1, x_3), x_4, \ldots, x_{n+1}),$$

and let $b := h_n^*(z)$. Then

$$\begin{aligned}
&h_n \circ \eta_{n,1,0} = \mathrm{id} \\
&h_n \circ \eta_{n,2,0}(x_1, \ldots, x_n) = (0, q(x_1, x_2), x_3, \ldots, x_n) = \\
&h_n \circ \eta_{n,3,0} = \tau \\
&h_n \circ \eta_{n,j,0} = \eta_{n-1,j-1,0} \circ h_{n-1} \text{ for } j \geq 4.
\end{aligned}$$

Similarly, $h_n \circ \eta_{n,j,1} = \eta_{n-1,j',1} \circ f_{n,j}$ for some j' and some map $f_{n,j} : \underline{n} \to \underline{n-1}$. Thus

$$db = z + \tau^*(z)$$

proving the result. □

1.6 Cubical enrichments and DG categories. The category of cubical abelian groups $\mathbf{Ab}^{\mathbf{Cube}^{\mathrm{op}}}$ inherits the structure of a symmetric monoidal category from $\mathbf{Ab}$. Explicitly, if we have two cubical abelian groups $n \mapsto A(n)$, $n \mapsto B(n)$, the tensor product $A \otimes B$ is the cubical abelian group $n \mapsto A(n) \otimes B(n)$, with morphisms acting by

$$g(a \otimes b) = g(a) \otimes g(b).$$

A *cubical category* is a category $\mathcal{C}$ enriched in cubical abelian groups, that is, for each pair of objects X, Y of $\mathcal{C}$, we have a cubical abelian group

$$\begin{aligned}
\underline{\mathcal{H}om}(X, Y, -) :&\mathbf{Cube}^{\mathrm{op}} \to \mathbf{Ab}, \\
&n \mapsto \underline{\mathcal{H}om}_{\mathcal{C}}(X, Y, n)
\end{aligned}$$

for each object X of $\mathcal{C}$, an identity element $\mathrm{id}_X \in \underline{\mathcal{H}om}_{\mathcal{C}}(X, X, 0)$ and for objects X, Y, Z an associative composition law

$$\underline{\circ}_{X,Y,Z} : \underline{\mathcal{H}om}_{\mathcal{C}}(Y, Z, -) \otimes \underline{\mathcal{H}om}_{\mathcal{C}}(X, Y, -) \to \underline{\mathcal{H}om}_{\mathcal{C}}(X, Z, -),$$

with $f \underline{\circ}_{X,Y,Z} \mathrm{id}_Z = f$, $\mathrm{id}_X \underline{\circ}_{X,Y,Z} g = g$.

We have the functor $\mathcal{C} \mapsto \mathcal{C}_0$ from cubical categories to pre-additive categories, where $\mathcal{C}_0$ has the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{\mathcal{C}_0}(X, Y) := \underline{\mathcal{H}om}(X, Y, 0).$$

A *cubical enrichment* of a pre-additive category $\mathcal{C}$ is a cubical category $\tilde{\mathcal{C}}$ together with an isomorphism $\mathcal{C} \cong \tilde{\mathcal{C}}_0$.

Next, we recall some basic facts about DG categories. For a complex $C \in C(\mathbf{Ab})$, we have the group of cycles in degree n, $Z^n C$ and the cohomology $H^n C$. For complexes C, C', we have the Hom-complex $\mathcal{H}om_{C(\mathbf{Ab})}(C, C')^*$, with

$$\mathcal{H}om_{C(\mathbf{Ab})}(C, C')^n := \prod_p \mathrm{Hom}_{\mathbf{Ab}}(C^p, C'^{n+p},$$

and with differential

$$d_{C',C} f := d_{C'} \circ f - (-1)^{\deg f} f \circ d_C.$$

We have as well as the group of maps of complexes

$$\mathrm{Hom}_{C(\mathbf{Ab})}(C, C') := Z^0 \mathcal{H}om_{C(\mathbf{Ab})}(C, C')^*.$$

This forms the additive category $C(\mathbf{Ab})$.

$C(\mathbf{Ab})$ has the *shift* functor $C \mapsto C[1]$ with $C[1]^n := C^{n+1}$ and differential $d^n_{C[1]} := -d^{n+1}_C$; for a morphism $f = \{f^n : C^n \to D^n\}$, $f[1] : C[1] \to D[1]$ is the collection $f[1]^n := f^{n+1}$. For a morphism $f : A \to B$, we have the *cone* complex $\mathrm{Cone}(f)$ and the *cone sequence*

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} A[1] \tag{1.4}$$

with $\mathrm{Cone}(f)^n := B^n \oplus A^{n+1}$ and differential $d(b,a) := (db + f(a), -da)$; i and p are the evident inclusion and projection. Clearly this sequence is natural with respect to morphisms $(u,v) : f \to g$.

Tensor product of complex $A^* \otimes B^*$ is defined as usual by $(A^* \otimes B^*)^n := \oplus_{i+j=n} A^i \otimes B^j$, with differential given by the Leibniz rule

$$d(a \otimes b) = da \otimes b + (-1)^{\deg a} a \otimes db.$$

The commutativity constraint $\tau_{A,B} : A^* \otimes B^* \to B^* \otimes A^*$ is

$$\tau_{A,B}(a \otimes b) = (-1)^{\deg a \cdot \deg b} b \otimes a.$$

This makes $C(\mathbf{Ab})$ into a symmetric monoidal category (in fact, a tensor category).

More generally, for a commutative ring k, we have the k-linear tensor category of complexes $C(k)$, and complexes of k-modules $\mathcal{H}om_{C(k)}(A,B)^*$.

A *DG category* is a category enriched in complexes of abelian groups. Concretely, for objects X, Y in a DG category $\mathcal{C}$, one has the *Hom complex*

$$\mathcal{H}om_{\mathcal{C}}(X,Y)^* \in C(\mathbf{Ab}),$$

and for X, Y, Z in $\mathcal{C}$, a composition law

$$\circ_{X,Y,Z} : \mathcal{H}om_{\mathcal{C}}(Y,Z)^* \otimes \mathcal{H}om_{\mathcal{C}}(X,Y)^* \to \mathcal{H}om_{\mathcal{C}}(X,Z)$$

The map $\circ_{X,Y,Z}$ is a map of complexes; equivalently, we have the Leibniz rule:

$$d(f \circ g) = df \circ g + (-1)^{\deg f} f \circ dg.$$

One has associativity of composition and an identity morphism $\mathrm{id}_X \in \mathcal{H}om_{\mathcal{C}}(X,X)^0$ with $d\mathrm{id}_X = 0$.

Each DG category $\mathcal{C}$ has an underlying pre-additive category $\mathrm{frg}\,\mathcal{C}$ with the same objects and with morphisms the underlying group of the complex $\mathcal{H}om_{\mathcal{C}}(X,Y)^*$, i.e., forget the grading and differential. A functor of DG categories $F : \mathcal{A} \to \mathcal{B}$ is a functor of the underlying pre-additive categories such that $F : \mathcal{H}om_{\mathcal{A}}(X,Y)^* \to \mathcal{H}om_{\mathcal{B}}(F(X),F(Y))^*$ is a map of complexes for each X, Y in $\mathcal{A}$. This defines the category of (small) DG categories, with forgetful functor frg to pre-additive categories.

For a DG category $\mathcal{C}$, one has the additive category $Z^0\mathcal{C}$, with the same objects as $\mathcal{C}$ and with

$$\mathrm{Hom}_{Z^0\mathcal{C}}(X,Y) := Z^0\mathcal{H}om_{\mathcal{C}}(X,Y).$$

We have the category $H^0\mathcal{C}$, with the same objects as $\mathcal{C}$ and with

$$\mathrm{Hom}_{H^0\mathcal{C}}(X,Y) := H^0\mathcal{H}om_{\mathcal{C}}(X,Y),$$

and the graded version $H^*\mathcal{C}$ with graded group of morphisms

$$\mathrm{Hom}_{H^*\mathcal{C}}(X,Y)^* := H^*\mathcal{H}om_{\mathcal{C}}(X,Y).$$

Clearly each DG functor $F : \mathcal{C} \to \mathcal{C}'$ induces functors of additive categories $H^0F : H^0\mathcal{C} \to H^0\mathcal{C}'$ and $Z^0F : H^0\mathcal{C} \to Z^0\mathcal{C}'$, making H^0 and Z^0 functors from

DG categories to pre-additive categories; similarly, $\mathcal{C} \mapsto H^*\mathcal{C}$ defines a functor from DG categories to graded pre-additive categories;

More generally, for a commutative ring k, we have the notion of a k-DG category, this being a category enriched in complexes $C(k)$, with functors Z^0 and H^0 to k-additive categories.

Example 1.8 With the Hom-complexes described above and the evident composition law, we have the structure of a DG category on $C(\mathbf{Ab})$, which we denote by $C_{dg}(\mathbf{Ab})$; we recover $C(\mathbf{Ab})$ as $Z^0C_{dg}(\mathbf{Ab})$. More generally, for a commutative ring k, we have the k-tensor category of complexes $C(k)$ and the k-DG category $C_{dg}(k)$.

We now show how to associate a DG category to a cubical category. Let $\mathcal{C}$ be a cubical category. For objects X, Y in $\mathcal{C}$, let $\mathcal{H}om_{dg\mathcal{C}}(X,Y)^*$ be the complex $\underline{\mathcal{H}om}_{\mathcal{C}}(X,Y)^*/\underline{\mathcal{H}om}_{\mathcal{C}}(X,Y)^*_{\mathrm{degn}}$ associated to the cubical abelian group

$$n \mapsto \underline{\mathcal{H}om}_{\mathcal{C}}(X,Y,n)$$

following definition 1.4. For objects X, Y, Z in $\mathcal{C}$, let

$$\circ_{X,Y,Z} : \mathcal{H}om_{dg\mathcal{C}}(Y,Z)^* \otimes \mathcal{H}om_{dg\mathcal{C}}(X,Y)^* \to \mathcal{H}om_{dg\mathcal{C}}(X,Z)^*$$

be the map of complexes induced by the composition $\underline{\circ}_{X,Y,Z}$ and the product (1.1).

Lemma 1.9 *The complexes $\mathcal{H}om_{dg\mathcal{C}}(X,Y)^*$ and composition law $\circ_{X,Y,Z}$ define a DG category $dg\mathcal{C}$.*

The proof is easy and is left to the reader. We sometimes write $\mathcal{C}$ for the DG category $dg\mathcal{C}$, if the context makes the meaning clear.

Here is a method for constructing a cubical category from a tensor category which we will use to construct our DG category of motives.

Definition 1.10 Let $n \mapsto \square^n$ be a co-cubical object (denoted $\square^*$) in a tensor category $(\mathcal{C}, \otimes)$ such that $\square^0$ is the unit object with respect to $\otimes$.

A *co-multiplication* δ^* on $\square^*$ is a morphism of co-cubical objects

$$\delta^* : \square^* \to \square^* \otimes \square^*,$$

where $\square^* \otimes \square^*$ is the diagonal co-cubical object $n \mapsto \square^n \otimes \square^n$, which is

1. *co-associative*: $(\delta^* \otimes \mathrm{id}_{\square^*}) \circ \delta^* = (\mathrm{id}_{\square^*} \otimes \delta^*) \circ \delta^*$.
2. *co-unital*: $\square^0 \xrightarrow{\delta(0)} \square^0 \otimes \square^0 \xrightarrow{\mu} \square^0 = \mathrm{id}_{\square^0}$ where μ is the unit isomorphism for $\otimes$.
3. *symmetric*: Let t be the commutativity constraint in $(\mathcal{C}, \otimes)$. Then $t_{\square^*,\square^*} \circ \delta^* = \delta^*$.

Given a co-multiplication on a co-cubical object $\square^*$, define

$$\underline{\mathcal{H}om}(X,Y,n) := \mathcal{H}om_{\mathcal{C}}(X \times \square^n, Y),$$

giving us the cubical object $n \mapsto \underline{\mathcal{H}om}(X,Y,n)$ of $\mathbf{Ab}$; we denote the associated complex by $\underline{\mathcal{H}om}(X,Y)^*$. The co-multiplication gives us the map of cubical objects

$$\hat{\circ}_{X,Y,Z} : \underline{\mathcal{H}om}(Y,Z,*) \otimes \underline{\mathcal{H}om}(X,Y,*) \to \underline{\mathcal{H}om}(X,Z,*)$$

sending $f : X \otimes \square^n \to Y$ and $g : Y \otimes \square^n \to Z$ to the composition

$$X \otimes \square^n \xrightarrow{\delta(n)} X \otimes \square^n \otimes \square^n \xrightarrow{f \otimes \mathrm{id}} Y \otimes \square^n \xrightarrow{g} Z$$

The following proposition is proved by a straightforward computation.

Proposition 1.11 *Let $\square^*$: **Cube** $\to \mathcal{C}$ be a co-cubical object in a tensor category $\mathcal{C}$, with a co-multiplication δ. Then $(X, Y, n) \mapsto \mathcal{H}om_{\mathcal{C}}(X \times \square^n, Y)$, with the composition law $\underline{\circ}_{X,Y}$ defined above, defines a cubical enrichment of $\mathcal{C}$.*

We denote the DG category formed by the cubical enrichment described above by $(\mathcal{C}, \otimes, \square^*, \delta)$, or just $(\mathcal{C}, \square^*)$ when the context makes the meaning clear.

Suppose now that, in addition to the assumptions used above, the Hom-groups in $\mathcal{C}$ are $\mathbb{Q}$-vector spaces, i.e., $\mathcal{C}$ is $\mathbb{Q}$-linear. We may then define the *alternating projection*

$$\mathrm{Alt} : \mathcal{H}om_{\mathcal{C}}(X \times \square^n, Y) \to \mathcal{H}om_{\mathcal{C}}(X \times \square^n, Y)^{\mathrm{alt}}$$

by applying the idempotent Alt in the rational group ring $\mathbb{Q}[F_n]$ corresponding to the sign representation.

Definition 1.12 Suppose that $\mathcal{C}$ is $\mathbb{Q}$-linear. Define the sub-DG category $(\mathcal{C}, \otimes, \square^*, \delta)^{\mathrm{alt}}$ of $(\mathcal{C}, \otimes, \square^*, \delta)$, with the same objects as $(\mathcal{C}, \otimes, \square^*, \delta)$, and with complex of morphisms given by the subcomplex

$$\mathcal{H}om_{\mathcal{C}}(X, Y)^{\mathrm{alt}n} := \mathcal{H}om_{\mathcal{C}}(X \times \square^n, Y)^{\mathrm{Alt}} \subset \mathcal{H}om_{\mathcal{C}}(X, Y, n).$$

The composition law is defined by the composition

$$\mathcal{H}om_{\mathcal{C}}(Y, Z)^{\mathrm{alt}*} \otimes \mathcal{H}om_{\mathcal{C}}(X, Y)^{\mathrm{alt}*} \xrightarrow{\circ} \mathcal{H}om_{\mathcal{C}}(X, Z)^{*} \xrightarrow{\mathrm{alt}} \mathcal{H}om_{\mathcal{C}}(X, Z)^{\mathrm{alt}*}$$

Proposition 1.13 *Let $\square^*$: **Cube** $\to \mathcal{C}$ be a co-cubical object in a tensor category $\mathcal{C}$, with a co-multiplication δ. Suppose that $\square^*$ extends to a functor*

$$\square^* : \mathbf{ECube} \to \mathcal{C}$$

and that $\mathcal{C}$ is $\mathbb{Q}$-linear. Then the natural inclusion functor

$$(\mathcal{C}, \square^*)^{\mathrm{alt}} \to (\mathcal{C}, \square^*)$$

is a quasi-equivalence[1] of $\mathbb{Q}$-DG categories.

Proof This follows immediately from proposition 1.7. $\square$

1.7 Tensor structure. There is a natural tensor structure on the $\mathbb{Q}$-DG category $(\mathcal{C}, \otimes, \square^*, \delta)^{\mathrm{alt}}$, which we now describe.

Given $f : X \times \square^n \to Y$, $f' : X' \times \square^{n'} \to Y'$, define

$$f \tilde{\otimes} g : X \otimes X' \otimes \square^{n+n'} \to Y \otimes Y'$$

as the composition

$$X \otimes X' \otimes \square^{n+n'} \xrightarrow{\mathrm{id} \otimes \delta_{n,n'}} X \otimes X' \otimes \square^n \otimes \square^{n'}$$
$$\xrightarrow{\tau_{X', \square^n}} X \otimes \square^n \otimes X' \otimes \square^{n'} \xrightarrow{f \otimes f'} Y \otimes Y'.$$

Assuming that $\mathcal{C}$ is $\mathbb{Q}$-linear, we define $f \otimes_{\square} g$ by applying the alternating projection:

$$f \otimes_{\square} g := (f \tilde{\otimes} g) \circ (\mathrm{id}_{X \otimes X'} \otimes \mathrm{alt}_{n+n'}).$$

Proposition 1.14 *Let $\mathcal{C}$ be a $\mathbb{Q}$-tensor category, $\square^*$ a co-cubical object of $\mathcal{C}$ and $\delta : \square \to \square \otimes \square$ a co-multiplication. Then $((\mathcal{C}, \otimes, \square^*, \delta)^{\mathrm{alt}}, \otimes_{\square})$ is a $\mathbb{Q}$-DG tensor category, with commutativity constraints induced by the commutativity constraints in $\mathcal{C}$.*

---

[1] for the definition of the term " quasi-equivalence", see definition 2.13

Proof One checks easily that the integral operation $\tilde{\otimes}$ satisfies the Leibniz rule:

$$\partial(f\tilde{\otimes}g) = \partial f\tilde{\otimes}g + (-1)^{\deg f} f \otimes \partial g.$$

Let $\delta_{n,m} : \square^{n+m} \to \square^n \otimes \square^m$ denote the composition $(p_1^{n,m} \otimes p_2^{n,m}) \circ \delta^{n+m}$. Let $\pi_{\mathrm{Alt}}^N : \square^N \to \square^N$ be the map induced by the alternating idempotent in $\mathbb{Q}[F_N]$.

It follows from the properties of co-associativity and symmetry of δ, together with the fact that δ is a map of co-cubical objects, that

$$\begin{aligned}(\mathrm{id}_{\square^n} \otimes t_{m,n'} \otimes \mathrm{id}_{\square^{m'}})&(\delta_{n,m} \otimes \delta_{n',m'}) \circ \delta_{n+m,n'+m'} \\ &= (\delta_{n,n'} \otimes \delta_{m,m'}) \circ \delta_{n+n',m+m'} \circ \square(\mathrm{id}_{\underline{n}} \times \tau_{m,n'} \times \mathrm{id}_{m'}),\end{aligned}$$

where $\tau_{m,n'} : \underline{m} \times \underline{n'} \to \underline{n'} \times \underline{m}$ is the symmetry in **Cube**, and t_{**} is the symmetry in $\mathcal{C}$. Composing on the right with $\pi_{\mathrm{Alt}}^{n+m+n'+m'}$ yields the identity

$$\begin{aligned}(\mathrm{id}_{\square^n} \otimes t_{m,n'} \otimes \mathrm{id}_{\square^{m'}})&(\delta_{n,m} \otimes \delta_{n',m'}) \circ \delta_{n+m,n'+m'} \circ \pi_{\mathrm{Alt}}^{n+m+n'+m'} \\ &= (-1)^{mn'}(\delta_{n,n'} \otimes \delta_{m,m'}) \circ \delta_{n+n',m+m'} \circ \pi_{\mathrm{Alt}}^{n+m+n'+m'}.\end{aligned}$$

The identity

$$(f \otimes_\square g) \circ (f' \otimes_\square g') = (-1)^{\deg g \deg f'} ff' \otimes_\square gg'$$

follows directly from this.

One shows by a similar argument that, for $f \in \mathrm{Hom}_{\mathcal{C}}(X \otimes \square^p, Y)^{\mathrm{Alt}}$, $g \in \mathrm{Hom}_{\mathcal{C}}(X' \otimes \square^q, Y')^{\mathrm{Alt}}$, we have

$$t_{Y,Y'} \circ (f \otimes_\square g) = (-1)^{pq}(g \otimes_\square f) \circ (t_{X,X'} \otimes \mathrm{id}_{\square^{p+q}}),$$

completing the proof. $\square$

Remark 1.15 One could hope that the operations $\tilde{\otimes}$ define at least a monoidal structure on $(\mathcal{C}, \otimes, \square^*, \delta)$, but this is in general not the case. In fact, as we noted in the proof of proposition 1.14, sending f, g to $f\tilde{\otimes}g$ does satisfy the Leibniz rule, but we do not have the identity

$$(f\tilde{\otimes}g) \circ (f'\tilde{\otimes}g') = (-1)^{\deg f' \deg g} ff'\tilde{\otimes}gg'$$

in general: the cubes on the two sides of this equation are in a different order.

In spite of this, the operation $\otimes$ *does* extend to an action of $\mathcal{C}$ on $(\mathcal{C}, \otimes, \square^*, \delta)$. Consider $\mathcal{C}$ as a DG category with all morphisms of degree zero (and zero differential). Define

$$\otimes : \mathcal{C} \otimes (\mathcal{C}, \otimes, \square^*, \delta) \to (\mathcal{C}, \otimes, \square^*, \delta)$$

to be the same as $\otimes$ on objects, and on morphisms by

$$\otimes : \mathrm{Hom}_{\mathcal{C}}(A, B) \otimes \mathrm{Hom}_{\mathcal{C}}(X \otimes \square^n, Y) \to \mathrm{Hom}_{\mathcal{C}}(A \otimes X \otimes \square^n, B \otimes Y)$$

2 Complexes over a DG category

We recall various constructions of categories of complexes over a DG category, the triangulated categories arising from these DG categories, and their main properties.

2.1 The category of $\mathcal{A}^{\mathrm{op}}$-modules and the derived category. We begin with an abstract approach, following [16, 17]. Let $\mathcal{C}$ be a DG category. The opposite category $\mathcal{C}^{\mathrm{op}}$ is the DG category with Hom complexes

$$\mathcal{H}om_{\mathcal{C}^{\mathrm{op}}}(X,Y)^* := \mathcal{H}om_{\mathcal{C}}(Y,X)^*$$

and with composition law given by

$$\begin{aligned}\mathcal{H}om_{\mathcal{C}^{\mathrm{op}}}(Y,Z)^* \otimes \mathcal{H}om_{\mathcal{C}^{\mathrm{op}}}(X,Y)^* = \mathcal{H}om_{\mathcal{C}^{\mathrm{op}}}(Z,Y)^* \otimes \mathcal{H}om_{\mathcal{C}^{\mathrm{op}}}(Y,X)^* \\ \xrightarrow{\tau} \mathcal{H}om_{\mathcal{C}}(Y,X)^* \otimes \mathcal{H}om_{\mathcal{C}}(Z,Y)^* \\ \xrightarrow{\circ_{Z,Y,X}} \mathcal{H}om_{\mathcal{C}}(Z,X)^* = \mathcal{H}om_{\mathcal{C}^{\mathrm{op}}}(X,Z)^*\end{aligned}$$

A *DG module* M over a (small) DG category $\mathcal{A}$ is a DG functor

$$M : \mathcal{A} \to C_{dg}(\mathbf{Ab})$$

For DG modules M,N, we have the Hom-complex $\mathrm{Hom}(M,N)^*$. $\mathrm{Hom}(M,N)^n$ is the degree n natural transformations $f : M \to N$, and with differential induced by the differential in the complexes $\mathrm{Hom}_{C(\mathbf{Ab})}(M(a),N(a))^*$, $a \in \mathcal{A}$.

We let $\mathcal{C}_{dg}(\mathcal{A})$ denote the DG category of $\mathcal{A}^{\mathrm{op}}$-modules. We have as well additive category $\mathcal{C}(\mathcal{A}) := Z^0\mathcal{C}_{dg}(\mathcal{A})$ and the homotopy category $\mathcal{K}(\mathcal{A}) := H^0(\mathcal{C}_{dg}(\mathcal{A}))$.

$\mathcal{C}(\mathcal{A})$ inherits a shift functor $M \mapsto M[1]$ from $C(\mathbf{Ab})$; similarly for each morphism $f : M \to N$, we have the cone $\mathrm{Cone}(f)$ and the sequence of morphisms in $\mathcal{C}(\mathcal{A})$

$$M \xrightarrow{f} N \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} M[1]$$

induced from the sequence (1.4).

The shift operator defines a translation functor $M \mapsto M[1]$ on $\mathcal{K}(\mathcal{A})$. One makes $\mathcal{K}(\mathcal{A})$ into a triangulated category by taking the distinguished triangles to be those isomorphic to a cone sequence (see e.g. [16, §2.2]). Note that $\mathcal{C}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ admits arbitrary direct sums.

2.2 Idempotent completion. Recall that an additive category $\mathcal{A}$ is *pseudo-abelian* if each idempotent endomorphism admits a kernel and cokernel. If $\mathcal{A}$ is an additive category, one has the *idempotent completion* $\mathcal{A} \to \mathcal{A}^\natural$ of $\mathcal{A}$, which is the universal functor of $\mathcal{A}$ to a pseudo-abelian category; $\mathcal{A}^\natural$ has objects (M,p), where M is an object of $\mathcal{A}$ and $p : M \to M$ is an idempotent endomorphism,

$$\mathrm{Hom}_{\mathcal{A}^\natural}((M,p),(N,q)) := p^*q_*\mathrm{Hom}_{\mathcal{A}}(M,N) = q_*p^*\mathrm{Hom}_{\mathcal{A}}(M,N) \subset \mathrm{Hom}_{\mathcal{A}}(M,N),$$

and the composition law in $\mathcal{A}^\natural$ is induced by that of $\mathcal{A}$. If $\mathcal{A}$ is a tensor category, $\mathcal{A}^\natural$ inherits the tensor structure, making $\mathcal{A} \to \mathcal{A}^\natural$ is a tensor functor.

One extends the definition to DG categories and DG tensor categories in the evident manner: if $\mathcal{C}$ is a DG category, then $\mathcal{C}^\natural$ has objects (M,p) with $p : M \to M$ an idempotent endomorphism in $Z^0\mathcal{C}$ The Hom-complex is given by

$$\mathcal{H}om_{\mathcal{C}^\natural}((M,p),(N,q))^* := p^*q_*\mathcal{H}om_{\mathcal{C}}(M,N) = q_*p^*\mathcal{H}om_{\mathcal{C}}(M,N).$$

A theorem of Balmer-Schlichting [1] tells us that, for $\mathcal{A}$ a triangulated category, $\mathcal{A}^\natural$ has a canonical structure of a triangulated category for which $\mathcal{A} \to \mathcal{A}^\natural$ is exact; the same holds in the setting of tensor triangulated categories.

2.3 The derived category. A morphism $f : M \to N$ in $\mathcal{K}(\mathcal{A})$ is a *quasi-isomorphism* if $H^*(f(a)) : H^*(M(a)) \to H^*(N(a))$ is an isomorphism for each $a \in \mathcal{A}$. Call M *acyclic* if the canonical map $0 \to M$ is a quasi-isomorphism; the full subcategory of acyclic complexes is a thick subcategory of $\mathcal{K}(\mathcal{A})$, closed under direct sums.

Definition 2.1 ([16, §4.1]) The *derived category* $\mathcal{D}(\mathcal{A})$ is the localization of $\mathcal{K}(\mathcal{A})$ with respect to the full subcategory of acyclic complexes.

Remark 2.2 As $\mathcal{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ admit arbitrary direct sums, these triangulated categories are idempotently complete.

In $\mathcal{C}(\mathcal{A})$, we have the representable $\mathcal{A}^{\rm op}$-modules: for $A \in \mathcal{A}$, let $A^\vee \in \mathcal{C}(\mathcal{A})$ be the $\mathcal{A}^{\rm op}$-module

$$A^\vee(X) := \mathcal{H}om_{\mathcal{A}}(X, A)$$

We let $\mathcal{C}^b_{dg}(\mathcal{A})$ denote the smallest full DG subcategory of $\mathcal{C}_{dg}(\mathcal{A})$ containing the modules $A^\vee$, $A \in \mathcal{A}$, and closed under shift and taking cones. Let $\mathcal{K}^b(\mathcal{A}) := H^0\mathcal{C}^b_{dg}(\mathcal{A})$.

Let $\mathcal{C}^\infty_{dg}(\mathcal{A}) \subset \mathcal{C}_{dg}(\mathcal{A})$ be the full DG subcategory of *semi-free* $\mathcal{A}^{\rm op}$-modules, that is, DG functors $M : \mathcal{A}^{\rm op} \to C_{dg}(\mathbf{Ab})$ such that,

1. After composing M with the forgetful functor from $C_{dg}(\mathbf{Ab})$ to graded abelian groups, $M = \oplus_{i\in I} A_i^\vee[n_i]$.
2. The index set I can be written as a nested union $I = \cup_{j\geq 0} I_j$, $I_0 \subset I_1 \subset \ldots$, such that, letting $M_a := \oplus_{i\in I_a} A_i^\vee[n_i]$, we have $d_M = 0$ on M_0 and $d_M(M_{a+1}) \subset M_a$.

It is easy to see that $\mathcal{C}^\infty_{dg}(\mathcal{A}) \supset \mathcal{C}^b_{dg}(\mathcal{A})$. Set $\mathcal{K}^\infty(\mathcal{A}) := H^0\mathcal{C}^\infty_{dg}(\mathcal{A})$.

Let $\mathrm{per}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ be the full subcategory of compact objects of $\mathcal{D}(\mathcal{A})$ (recall that an object X in an additive category is compact if $\mathrm{Hom}(X, \oplus_\alpha A_\alpha) = \oplus_\alpha \mathrm{Hom}(X, A_\alpha)$). It is easy to see that $\mathrm{per}(\mathcal{A})$ is a triangulated subcategory of $\mathcal{D}(\mathcal{A})$.

Theorem 2.3 *1. The inclusion $\mathcal{K}^\infty(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ followed by the quotient map $\mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ induces an equivalence $\varphi : \mathcal{K}^\infty(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$.*

2. $\varphi(\mathcal{K}^b(\mathcal{A})) \subset \mathrm{per}(\mathcal{A})$ and φ identifies $\mathrm{per}(\mathcal{A})$ with the pseudo-abelian hull of $\mathcal{K}^b(\mathcal{A})$.

For a proof, we refer the reader to [17, theorem 3.2, corollary 3.7].

Remark 2.4 If $\mathcal{A}$ is an additive category, we consider $\mathcal{A}$ as a DG category with all Hom-complexes supported in degree zero. Then $\mathcal{C}^b_{dg}(\mathcal{A})$ is equivalent to the usual category of bounded complexes $C^b_{dg}(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{A})$ is equivalent to $K^b(\mathcal{A}) := H^0(C^b_{dg}(\mathcal{A}))$.

2.4 Tensor structure. A *DG tensor category* is a DG category $\mathcal{A}$ together with the structure of a tensor category on the underlying pre-additive category $\mathrm{frg}\,\mathcal{A}$ which is compatible with the DG structure, meaning:

1. tensor product of morphisms defines a map of complexes
$$\otimes : \mathcal{H}om_{\mathcal{A}}(X, Y) \otimes \mathrm{Hom}_{\mathcal{A}}(X', Y') \to \mathcal{H}om_{\mathcal{A}}(X \otimes X', Y \otimes Y').$$
2. the associativity, commutativity and unit constraints are morphisms in $Z^0\mathcal{A}$.

Given a commutative ring k, we have the analogous notion of a k-DG tensor category; functors of DG tensor categories are the evident notion.

Let $\mathcal{A}$ be a DG tensor category. We explain how to extend the tensor product construction from $\mathcal{A}$ to $\mathcal{C}_{dg}(\mathcal{A})$. Start by defining the tensor product of representable functors by $A^\vee \otimes B^\vee := (A \otimes B)^\vee$. Extend to translations as follows: set $A^\vee[n] \otimes B^\vee := (A \otimes B)^\vee[n]$. For an $\mathcal{A}^{\mathrm{op}}$-module M, let $-M$ be the $\mathcal{A}$ with the same underlying functor to graded groups, and with differential $d_{-M} = -d_M$. Set $A^\vee \otimes (B^\vee[n]) := (-1)^n(A \otimes B)^\vee[n]$. Extend the tensor product to arbitrary $\mathcal{A}^{\mathrm{op}}$-modules by taking the Kan extension, that is: For an arbitrary $\mathcal{A}^{\mathrm{op}}$-module M, let I_M be the category with objects $(A, n, \varphi : A^\vee[n] \to M)$, where the morphisms are $f : A \to B$ that yield a commutative diagram

$$\begin{array}{ccc} A^\vee[n] & \xrightarrow{f[n]} & B[n] \\ & \searrow^{\varphi} & \downarrow{\scriptstyle\psi} \\ & & M \end{array}$$

Define $M \otimes N$ as the colimit over $I_M \times I_N$ of $A^\vee[n] \otimes B^\vee[m]$. The tensor product of morphisms is defined in the evident manner.

One easily shows that this does indeed define a tensor structure on $\mathcal{C}_{dg}(\mathcal{A})$, making $\mathcal{C}_{dg}(\mathcal{A})$ a DG tensor category, making $\mathcal{C}^b_{dg}(\mathcal{A})$ and $\mathcal{C}^\infty_{dg}(\mathcal{A})$ DG tensor subcategories. Similarly, one shows that the tensor product commutes with shift and is compatible with the formation of cone sequences (both up to canonical isomorphism).

Passing to the homotopy categories, we have the tensor triangulated category $\mathcal{K}(\mathcal{A})$ with tensor triangulated subcategories $\mathcal{K}^b(\mathcal{A})$ and $\mathcal{K}^\infty(\mathcal{A})$; via theorem 2.3, this makes $\mathcal{D}(\mathcal{A})$ a tensor triangulated category with tensor triangulated subcategory $\mathrm{per}(\mathcal{A})$.

Remark 2.5 In the setting of DG tensor categories, the equivalences of proposition 2.16 and remark 2.17 are equivalences of tensor triangulated categories.

2.5 The category $\mathcal{A}^{\mathrm{pretr}}$. For the reader who prefers a less abstract approach, we give the concrete version. We review a version of Kapranov's explicit construction [15] of complexes over a DG category.

Definition 2.6 Let $\mathcal{A}$ be a DG category. The DG category $\mathrm{Pre\text{-}Tr}(\mathcal{A})$ has objects $\mathcal{E}$ consisting of the following data:

1. A finite collection of objects of $\mathcal{A}$, $\{E_i, N \le i \le M\}$ (N and M depending on $\mathcal{E}$).
2. Morphisms $e_{ij} : E_j \to E_i$ in $\mathcal{A}$ of degree $j-i+1$, for $N \le j, i \le M$, satisfying

$$(-1)^i de_{ij} + \sum_k e_{ik}e_{kj} = 0.$$

For $\mathcal{E} := \{E_i, e_{ij}\}$, $\mathcal{F} := \{F_i, f_{ij}\}$, a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ of degree n is a collection of morphisms $\varphi_{ij} : E_j \to F_i$ in $\mathcal{A}$, such that φ_{ij} has degree $n+j-i$. The composition of morphisms $\varphi : \mathcal{E} \to \mathcal{F}$, $\psi : \mathcal{F} \to \mathcal{G}$ is defined by

$$(\psi \circ \varphi)_{ij} := \sum_k \psi_{ik} \circ \varphi_{kj}.$$

Given a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ of degree n, define

$$\partial_{\mathcal{F},\mathcal{E}}(\varphi) \in \mathrm{Hom}_{\text{Pre-Tr}(\mathcal{A})}(\mathcal{E}, \mathcal{F})^{n+1}$$

to be the collection $\partial_{\mathcal{F},\mathcal{E}}(\varphi)_{ij} : E_j \to F_i$ with

$$\partial_{\mathcal{F},\mathcal{E}}(\varphi)_{ij} := (-1)^i d(\varphi_{ij}) + \sum_k f_{ik}\varphi_{kj} - (-1)^n \sum_k \varphi_{ik} e_{kj}.$$

For $\mathcal{E} = \{E_i, e_{ij} : E_j \to E_i\}$ in Pre-Tr($\mathcal{A}$), and $n \in \mathbb{Z}$, define $\mathcal{E}[n]$ by

$$(\mathcal{E}[n])_i := E_{i+n},\ e[n]_{ij} := (-1)^n e_{i+n,j+n} : (\mathcal{E}[n])_j \to (\mathcal{E}[n])_i.$$

For $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})^p$, define $\varphi[n] \in \mathrm{Hom}(\mathcal{E}[n], \mathcal{F}[n])^p$ by setting

$$\varphi[n]_{ij} : (\mathcal{E}[n])_j \to (\mathcal{F}[n])_i$$

equal to $(-1)^{np}\varphi_{i+n,j+n}$. It follows directly from the definitions that $(\mathcal{E}, \varphi) \mapsto (\mathcal{E}[n], \varphi[n])$ defines a DG auto-isomorphism with inverse the shift by $-n$.

Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a degree 0 morphism in Pre-Tr($\mathcal{A}$) with $\partial\varphi = 0$. We define the *cone* of φ, Cone(φ), as the object in Pre-Tr($\mathcal{A}$) with

$$\mathrm{Cone}(\varphi)_i := F_i \oplus E_{i+1}$$

and with morphisms $\mathrm{Cone}(\varphi)_{ij} : \mathrm{Cone}(\varphi)_j \to \mathrm{Cone}(\varphi)_i$ given by the usual matrix

$$\begin{pmatrix} f_{ij} & 0 \\ \varphi_{ij} & -e_{ij} \end{pmatrix}$$

The inclusions $F_i \to F_i \oplus E_{i+1}$ define the morphism $i_\varphi : \mathcal{F} \to \mathrm{Cone}(\varphi)$ and the projections $F_i \oplus E_{i+1} \to E_{i+1}$ define the morphism $p_\varphi : \mathrm{Cone}(\varphi) \to \mathcal{E}[1]$. Note that $\partial(i_\varphi) = 0 = \partial(p_\varphi)$. This gives us the *cone sequence* in Z^0Pre-Tr($\mathcal{A}$):

$$\mathcal{E} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{i_\varphi} \mathrm{Cone}(\varphi) \xrightarrow{p_\varphi} \mathcal{E}[1].$$

For a detailed verification that the above definitions satisfy the necessary properties, see [19, §2.1, 2.2].

We have the fully faithful embedding of DG categories

$$\tilde{i} : \mathcal{A} \to \text{Pre-Tr}(\mathcal{A})$$

sending an object E of $\mathcal{A}$ to $\tilde{i}(E) := \{E_0 = E\}$; noting that

$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{pretr}}}(i(E), i(F))^* = \mathrm{Hom}_{\mathcal{A}}(i(E), i(F))^*$$

defines i on morphisms and shows that $\tilde{i}$ is a fully faithful embedding.

Definition 2.7 Let $\mathcal{A}$ be a DG category. $\mathcal{A}^{\mathrm{pretr}}$ is the smallest full *DG* subcategory of Pre-Tr($\mathcal{A}$) containing $\tilde{i}(\mathcal{A})$ and closed under translation and the formation of cones.

We let

$$i : \mathcal{A} \to \mathcal{A}^{\mathrm{pretr}}$$

be the DG functor induced by $\tilde{i}$.

Remark 2.8 $\mathcal{A}^{\mathrm{pretr}}$ is denoted Pre-Tr$^+(\mathcal{A})$ in [15]. Also, one can explicitly describe the objects in $\mathcal{A}^{\mathrm{pretr}}$ as follows: $\mathcal{E} = \{E_i, e_{ij}\}$ is in $\mathcal{A}^{\mathrm{pretr}}$ if there is a finite ordered set K such that $E_i \cong \oplus_{j\in J} E_i^k$ and the k, k' component $e_{ij}^{kk'} : E_j^{k'} \to E_i^k$ is zero unless $k' < k$. See also [8, §2.4] for a slightly different description of $\mathcal{A}^{\mathrm{pretr}}$.

As remarked in [8, Remark 2.6], the inclusion $\mathcal{A}^{\mathrm{pretr}} \to$ Pre-Tr($\mathcal{A}$) is in general *not* an equivalence of DG categories. However, we do have

Definition 2.9 (Bondarko [4, §2.7.2]) Let $\mathcal{A}$ be a DG category. Call $\mathcal{A}$ *negative* if $\mathrm{Hom}_{\mathcal{A}}(X,Y)^n = 0$ for $n > 0$.

Remark 2.10 If $\mathcal{A}$ is negative, then the inclusion $\mathcal{A}^{\mathrm{pretr}} \to \text{Pre-Tr}(\mathcal{A})$ is an isomorphism. Indeed, given $\{E_i, e_{ij}\}$ in $\text{Pre-Tr}(\mathcal{A})$, $e_{ij} : E_j \to E_i$ has degree $j - i + 1$, hence is zero if $j \geq i$.

For $\mathcal{A}$ negative, $\mathcal{A}^{\mathrm{pretr}}$ has "stupid truncations" $\sigma_{\leq n}$, $\sigma_{\geq n}$, where

$$\sigma_{\leq n}(\{E_i, e_{ij}\}) = \{E_i, e_{ij}, i, j \leq n\}; \quad \sigma_{\geq n}(\{E_i, e_{ij}\}) = \{E_i, e_{ij}, i, j \geq n\},$$

generalizing the stupid truncation functors for complexes over an additive category. Bondarko uses these to construct a "weight filtration" on his category of motives, see [4, §7].

We are only interested in the subcategory $\mathcal{A}^{\mathrm{pretr}}$, so for the remainder of this section, we will limit ourselves to constructions involving $\mathcal{A}^{\mathrm{pretr}}$, although all the constructions extend directly to $\text{Pre-Tr}(\mathcal{A})$.

Now suppose that $(\mathcal{A}, \otimes)$ is a DG tensor category. We give $\mathcal{A}^{\mathrm{pretr}}$ a tensor structure as follows. On objects $\mathcal{E} = \{E_i, e_{ij}\}$, $\mathcal{F} = \{F_i, f_{ij}\}$, let $\mathcal{E} \otimes \mathcal{F}$ be the object with terms

$$(\mathcal{E} \otimes \mathcal{F})_i := \oplus_k E_k \otimes F_{i-k}$$

and maps $g_{ij} : (\mathcal{E} \otimes \mathcal{F})_j \to (\mathcal{E} \otimes \mathcal{F})_i$ given by the sum of the maps

$$(-1)^{(l-k+1)n} e_{kl} \otimes \mathrm{id}_{F_n} \text{ or } (-1)^k \mathrm{id}_{E_k} \otimes f_{mn}.$$

We also need to define a cup product map

$$\cup : \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathrm{Hom}(\mathcal{E}', \mathcal{F}') \to \mathrm{Hom}(\mathcal{E} \otimes \mathcal{E}', \mathcal{F} \otimes \mathcal{F}')$$

For this, suppose $\mathcal{E} = \{E_i, e_{ij}\}$, $\mathcal{F} = \{F_i, f_{ij}\}$, $\mathcal{E} = \{E'_i, e'_{ij}\}$, $\mathcal{F}' = \{F'_i, f'_{ij}\}$. Given $\varphi_{ij} : E_j \to E'_i$ of degree $j - i + p$, $\psi_{kl} : F_l \to F'_k$ of degree $l - k + q$ (the degree taken in $\mathcal{C}$), define

$$\varphi_{ij} \cup \psi_{kl} := (-1)^{(j-i+p)k+qj} \varphi_{ij} \otimes \psi_{kl} : E_j \otimes F_l \to E'_i \otimes F'_k.$$

Taking the sum over all components defines the graded map

$$\cup : \mathrm{Hom}_{\mathcal{A}^{\mathrm{pretr}}}(\mathcal{E}, \mathcal{F}) \otimes \mathrm{Hom}_{\mathcal{A}^{\mathrm{pretr}})}(\mathcal{E}', \mathcal{F}') \to \mathrm{Hom}_{\mathcal{A}^{\mathrm{pretr}}}(\mathcal{E} \otimes \mathcal{E}', \mathcal{F} \otimes \mathcal{F}').$$

Finally, we need a commutativity constraint. Let $t_{E,F} : E \otimes F \to F \otimes E$ be the commutativity constraint in $\mathcal{A}$. Define

$$\tau_{\mathcal{E},\mathcal{F}} : \mathcal{E} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{E}$$

as the collection of maps

$$(-1)^{jl} t_{E_i,F_l} : E_i \otimes F_l \to F_l \otimes E_i.$$

A number of direct computations (see [19, §2.3] for details) show

Proposition 2.11 *Let $(\mathcal{A}, \otimes)$ be a DG tensor category. Then the operations $(\mathcal{E}, \mathcal{F}) \mapsto \mathcal{E} \otimes \mathcal{F}$, $(\varphi, \psi) \mapsto \varphi \cup \psi$ and commutativity constraints τ_{**} makes $\mathcal{A}^{\mathrm{pretr}}$ into a DG tensor category.*

Via the embedding $i : \mathcal{A} \to \mathcal{A}^{\mathrm{pretr}}$, we have the functor

$$i^{\mathrm{pretr}} : \mathcal{A}^{\mathrm{pretr}} \to \mathcal{C}_{dg}(\mathcal{A})$$

sending $\mathcal{E}$ in $\mathcal{A}^{\mathrm{pretr}}$ to the functor

$$i^{\mathrm{pretr}}(\mathcal{E})(X) := \mathrm{Hom}_{\mathcal{A}^{\mathrm{pretr}}}(i(X), \mathcal{E}),$$

and similarly for morphisms. One easily checks that i^{pretr} is compatible with translation and cone sequences, and that $i^{\mathrm{pretr}} \circ i$ is the functor $E \mapsto E^\vee$.

Proposition 2.12 *The functor i^{pretr} is a fully faithful embedding, inducing an isomorphism of DG categories*

$$\varphi : \mathcal{A}^{\mathrm{pretr}} \to \mathcal{C}^b_{dg}(\mathcal{A}).$$

If $\mathcal{A}$ is a DG tensor category, then i^{pretr} is a DG tensor functor and φ is a DG tensor equivalence.

Proof For a morphism $f : X \to Y$ in $\mathcal{C}(\mathcal{A})$, and for Z in $\mathcal{C}(\mathcal{A})$, we have the natural isomorphism of graded groups

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(Z, \mathrm{cone}(f))^* = \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(Z, Y)^* \otimes \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(Z, X)^*[1]$$

As the Yoneda functor $E \mapsto E^\vee$ is a fully faithful embedding $\mathcal{A} \to \mathcal{C}_{dg}(\mathcal{A})$, and i^{pretr} is compatible with translations and cone sequences it follows that i^{pretr} is fully faithful, and φ is an isomorphism. If $\mathcal{A}$ is a DG tensor category, then the Yoneda embedding $\mathcal{A} \to \mathcal{C}_{dg}(\mathcal{A})$ is compatible with tensor products, by definition; as this compatibility extends to translations, direct sums and cone sequences, it follows that i^{pretr} is a DG tensor functor and hence φ is a DG tensor equivalence. □

2.6 Quasi-equivalence. Equivalence of DG categories is too rigid a notion, instead, we have the relation of quasi-equivalence.

Definition 2.13 A DG functor $f : \mathcal{A} \to \mathcal{B}$ of small DG categories is a *quasi-equivalence* if

1. f induces a surjection on isomorphism classes.
2. $H^*f : H^*\mathcal{A} \to H^*\mathcal{B}$ is fully faithful, in other words, f induces a quasi-isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(X, Y)^* \to \mathrm{Hom}_{\mathcal{B}}(f(X), f(Y))^*$$

for all X, Y in $\mathcal{A}$.

The main result on quasi-equivalence is the following:

Theorem 2.14 *If $f : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence, then the induced functors $\mathcal{D}(f) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$, $\mathcal{K}(f) : \mathrm{per}(\mathcal{A}) \to \mathrm{per}(\mathcal{B})$, $\mathcal{K}^b(f) : \mathcal{K}^b(\mathcal{A}) \to \mathcal{K}^b(\mathcal{B})$ are equivalences of triangulated categories. If f is in addition a DG tensor functor, then all these equivalences are equivalences of tensor triangulated categories.*

The result for $\mathcal{D}(f) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ can be found in [22, proposition 3.2]. Passing to the subcategory of compact objects implies the result for the restriction to $\mathrm{per}(\mathcal{A}) \to \mathrm{per}(\mathcal{B})$. For $\mathcal{K}^b(f) : \mathcal{K}^b(\mathcal{A}) \to \mathcal{K}^b(\mathcal{B})$, we can use the fact that

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X^\vee, Y^\vee[n]) \cong H^n(\mathrm{Hom}_{\mathcal{A}}(X, Y)^*)$$

and that the subcategory of $\mathcal{K}(\mathcal{A})^2$ of pairs of objects (X, Y) for which f induces an isomorphism

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y^\vee[n]) \to \mathrm{Hom}_{\mathcal{K}(\mathcal{B})}(f(X), f(Y)[n])$$

is triangulated. One can also derive the result for $\mathrm{per}(\mathcal{A}) \to \mathrm{per}(\mathcal{B})$ from this, using theorem 2.3 to identify $\mathrm{per}(\mathcal{A})$ and $\mathrm{per}(\mathcal{B})$ with the pseudo-abelian hulls of $\mathcal{K}^b(\mathcal{A})$ and $\mathcal{K}^b(\mathcal{B})$, respectively.

2.7 Morita theory. A weaker equivalence than quasi-equivalence of DG categories, *Morita equivalence*, induces a equivalence on the derived category. This equivalence relation is induced from the relation given by the existence of a "tilting module" with some good properties. We first recall the general setting from [17, §3.5].

Let $\mathcal{A}$ and $\mathcal{B}$ be small DG categories. The DG category $\mathcal{A}\otimes\mathcal{B}$ has objects (A,B), with A in $\mathcal{A}$, $\mathcal{B}$ in $\mathcal{B}$, morphisms

$$\mathrm{Hom}((A,B),(A',B'))^* := \mathrm{Hom}_{\mathcal{A}}(A,A')^*\otimes\mathrm{Hom}_{\mathcal{B}}(B,B')^*$$

with composition law

$$(f\otimes g)\circ(f'\otimes g') = (-1)^{\deg g\cdot\deg f'}ff'\otimes gg'.$$

Let X an $\mathcal{B}$-$\mathcal{A}$-bimodule, that is, an $\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}$-module. For each $\mathcal{B}^{\mathrm{op}}$-module M, we have the $\mathcal{A}^{\mathrm{op}}$-module

$$GM := \mathrm{Hom}(X,M) : A\mapsto \mathrm{Hom}_{\mathcal{C}_{dg}(\mathcal{B})}(X(A,-),M)^*,$$

giving the DG functor $G:\mathcal{C}_{dg}(\mathcal{B})\to\mathcal{C}_{dg}(\mathcal{A})$. G has the left adjoint F, $F(L) := L\otimes_{\mathcal{A}}X$,

$$(L\otimes_{\mathcal{A}^{\mathrm{op}}}X)(M) := \oplus_N L(N)\otimes X(N,M)/\sim$$

where $\sim$ is the quotient identifying

$$\begin{array}{ccc}
\oplus_{f:N'\to N}L(N')\otimes X(N,M) & \xrightarrow{\Sigma_f f_*\otimes\mathrm{id}} & L(N)\otimes X(N,M)\\
\Big\downarrow{\scriptstyle \Sigma_f\mathrm{id}\otimes f^*} & & \\
L(N')\otimes X(N',M) & &
\end{array}$$

Taking derived functors, one has the pair of adjoint functors $LF:\mathcal{D}(\mathcal{A})\to\mathcal{D}(\mathcal{B})$, $RG:\mathcal{D}(\mathcal{B})\to\mathcal{D}(\mathcal{B})$. We recall the following result of Keller:

Lemma 2.15 (Keller [17, Lemma 3.10]) *LF is an equivalences if the following conditions hold:*

1. *$X(A,-)\in\mathcal{D}(\mathcal{B})$ is perfect for all $A\in\mathcal{A}$.*
2. *The morphism*
$$\mathrm{Hom}_{\mathcal{A}}(A,A')^*\to\mathrm{Hom}_{\mathcal{C}(\mathcal{B})}(X(A,-),X(A',-))^*$$
is a quasi-isomorphism for all A,A' in $\mathcal{A}$
3. *The DG $\mathcal{B}^{\mathrm{op}}$-modules $X(A,-)$, $A\in\mathcal{A}$, form a set of compact generators for $\mathcal{D}(\mathcal{B})$.*

We will use this general theory to prove

Proposition 2.16 *Let $\mathcal{A}_0$ be a small DG category, and let $\mathcal{A}$ be either $\mathcal{C}^b_{dg}(\mathcal{A}_0)$ or $\mathcal{C}_{dg}(\mathcal{A}_0)$. If $\mathcal{A}_0$ is an additive category, we also allow $\mathcal{A}=C^?_{dg}(\mathcal{A}_0)$, where $/ = +,-,\emptyset$. Then there are a natural equivalences*

$$\mathcal{D}(\mathcal{A})\sim\mathcal{D}(\mathcal{A}_0);\ \mathrm{per}(\mathcal{A})\sim\mathrm{per}(\mathcal{A}_0)$$

Proof As an exact equivalence $\mathcal{D}(\mathcal{A})\sim\mathcal{D}(\mathcal{A}_0)$ induces an exact equivalence on the subcategories of perfect objects, we need only verify the hypotheses of lemma 2.15.

In case $\mathcal{A}=\mathcal{C}^b_{dg}(\mathcal{A}_0)$ or $\mathcal{C}_{dg}(\mathcal{A}_0)$, let $i_0:\mathcal{A}_0\to\mathcal{A}$ be the Yoneda embedding; in case $\mathcal{A}_0$ is an additive category and $\mathcal{A}=C^?_{dg}(\mathcal{A}_0)$, let $i_0:\mathcal{A}_0\to\mathcal{A}$ be the embedding sending $A \in \mathcal{A}_0$ to A considered as a complex supported in degree zero. In all cases, let X be the $\mathcal{A}$-$\mathcal{A}_0$-bimodule $(A, B) \mapsto \mathrm{Hom}_{\mathcal{A}}(B, i_0(A))^*$. Then $X(A, -) = i_0(A)^\vee$; property (1) follows from theorem 2.3 and (2) follows from the Yoneda lemma and the fact that i_0 is fully faithful. It remains to show that the $X(A, -)$ form a set of generators for $\mathcal{D}(\mathcal{A})$. Using theorem 2.3, it suffices to show that the $X(A, -)$ form a set of generators for $\mathcal{K}^\infty(\mathcal{A})$.

For this, we first consider the case $\mathcal{A} = \mathcal{C}_{dg}(\mathcal{A}_0)$. It follows easily from the definition of semi-free modules that the smallest triangulated subcategory of $\mathcal{K}^\infty(\mathcal{A})$ closed under direct sums and containing the image of the Yoneda embedding of $\mathcal{A}$ is all of $\mathcal{K}^\infty(\mathcal{A})$. Replacing $\mathcal{A}$ with $\mathcal{A}_0$, we see that the same holds for the image of $i_0(\mathcal{A}_0)$ in $\mathcal{K}^\infty(\mathcal{A}_0)$. Combining these two facts, we see that smallest triangulated subcategory of $\mathcal{K}^\infty(\mathcal{A})$ closed under direct sums and containing the objects $X(A, -)$, A in $\mathcal{A}_0$, is all of $\mathcal{K}^\infty(\mathcal{A})$, and hence the $X(A, -)$ form a set of generators for $\mathcal{K}^\infty(\mathcal{A})$. The proof for $\mathcal{A} = C^?_{dg}(\mathcal{A}_0)$ is similar. □

Remark 2.17 Let $\mathcal{A}_0$ be the additive category of injective objects in a Grothendieck abelian category $\mathcal{C}$. Then the homotopy category $H^0(C^+_{dg}(\mathcal{A}_0))$ of $C^+_{dg}(\mathcal{A}_0)$ is equivalent to the derived category $D^+(\mathcal{C})$, and $H^0(C^+_{dg}(\mathcal{A}_0))$ is in turn equivalent to the full triangulated subcategory of $\mathcal{K}^\infty(\mathcal{A}_0) \sim \mathcal{D}(\mathcal{A}_0)$ of objects with bounded below cohomology. Using proposition 2.16, we have an equivalence

$$\mathcal{D}(C^+_{dg}(\mathcal{A}_0)) \to \mathcal{D}(\mathcal{A}_0)$$

inducing an equivalence of $\mathcal{D}^b(C^+_{dg}(\mathcal{A}_0))$ with $D^+(\mathcal{C})$.

Remark 2.18 There is a *total complex functor*

$$\mathrm{Tot} : (\mathcal{A}^{\mathrm{pretr}})^{\mathrm{pretr}} \to \mathcal{A}^{\mathrm{pretr}}$$

defined as follows: if $\mathcal{E} = \{\mathcal{E}^l, \varphi^{kl} : \mathcal{E}^l \to \mathcal{E}^k\}$, where $\mathcal{E}^l = \{E^l_i, e^l_{ij} : E^l_j \to E^l_i\}$ and $\varphi^{kl} = \{\varphi^{kl}_{ij} : E^l_j \to E^k_i\}$, then $\mathrm{Tot}(\mathcal{E})$ is given by the collection of objects

$$\mathrm{Tot}(\mathcal{E})_n := \oplus_{j+l=n} E^l_j,$$

together with the morphisms $\mathrm{Tot}(\mathcal{E})_{mn} : \mathrm{Tot}(\mathcal{E})_n \to \mathrm{Tot}(\mathcal{E})_m$ defined as the sum of terms $(-1)^l e^l_{ij}$ and φ^{kl}_{ij}.

For an additive category $\mathcal{A}_0$, exactly the same formulas define a total complex functor

$$\mathrm{Tot} : C^?_{dg}(\mathcal{A}_0)^{\mathrm{pretr}} \to C^?_{dg}(\mathcal{A}_0);\ ? = +, -, \infty$$

which is an equivalence of DG categories with inverse the evident extension of the inclusion functor i_0.

One easily checks that Tot respects translation and cone sequences, and hence induces exact equivalences of triangulated categories

$$\mathrm{Tot} : \mathcal{K}^b(C^?_{dg}(\mathcal{A}_0)) \to K^?(\mathcal{A}_0);\ ? = +, -, \infty$$

If $\mathcal{A}$ is a DG tensor category, or $\mathcal{A}_0$ is a tensor category, the above equivalences are equivalences of DG tensor categories, resp. tensor triangulated categories. Thus, we have in this case a "concrete" version of Morita equivalence.

Corollary 2.19 *Let $\mathcal{C}$ be a DG category, $\mathcal{A}$ an additive category, and*

$$F : \mathcal{C} \to C^?(\mathcal{A})$$

a DG functor, $? = b, +, -, \infty$. Then F defines a canonical exact functor

$$K^b(F) : \mathcal{K}^b(\mathcal{C}) \to K^?(\mathcal{A});$$

if F is a DG tensor functor, then $K^?(F)$ is an exact tensor functor for $? = b, +, -$.

3 Sheafification of DG categories

We show how to construct a DG category $R\Gamma\mathcal{C}$ out of a presheaf of DG categories $U \mapsto \mathcal{C}(U)$ so that the Hom complexes in $R\Gamma\mathcal{C}$ compute the hypercohomology of the presheaf of Hom complexes for $\mathcal{C}$.

3.1 Sheafifying. Fix a Grothendieck topology τ on some full subcategory Opn_S^τ of $\mathbf{Sch}_S$; we assume that Opn_S^τ is closed under fiber product over S, that τ is subcanonical, and that τ has a conservative set of points $Pt_\tau(S)$. Let $Sh_\tau(S)$ be the category of sheaves of abelian groups on Opn_S for the topology τ, and $PSh_\tau(S)$ the category of presheaves on Opn_S. We let

$$i : \prod_{x\in Pt_\tau(S)} \mathbf{Ab} \to Sh_\tau(S)$$

be the canonical map of topoi, i.e, we have the functor $i^* : Sh_\tau(S) \to \prod_{x\in Pt_\tau(S)}$ (the factor i_x^* sends f to the stalk f_x at x) and the right adjoint of i^*, $i_* : \prod_{x\in Pt_\tau(S)} \to Sh_\tau(S)$. Let $\mathcal{G}^0 := i_* \circ i^*$, $\mathcal{G}^n := (\mathcal{G}^0)^n$. Combining the unit $\epsilon : \mathrm{id} \to i_*i^*$ and the co-unit $\eta : i^*i_* \to \mathrm{id}$ of the adjunction gives us the cosimplicial object in the category of endofunctors of $Sh_\tau(S)$:

$$\begin{array}{ccccc} & & & \to & \\ & \to & & \leftarrow & \\ \mathcal{G}^0 & \leftarrow & \mathcal{G}^1 & \to & \ldots \\ & \to & & \leftarrow & \\ & & & \to & \end{array}$$

with augmentation $\epsilon : \mathrm{id} \to \mathcal{G}^0$. For a sheaf $\mathcal{F}$, we let

$$\mathcal{F} \xrightarrow{\epsilon} \mathcal{G}^*\mathcal{F}$$

be the augmented complex associated to the augmented cosimplicial sheaf $n \mapsto \mathcal{G}^n(\mathcal{F})$; we extend this construction to complexes by taking the total complex of the associated double complex.

Lemma 3.1 *For each complex of sheaves $\mathcal{F}$, the augmentation $\epsilon : \mathcal{F} \to \mathcal{G}^*(\mathcal{F})$ is a quasi-isomorphism, and $\mathcal{G}^n(\mathcal{F})$ is acyclic for the functor $R^n\Gamma(S, -)$.*

Thus, we have a functorial $\Gamma(S, -)$-acyclic resolution for complexes of sheaves of abelian groups on S. In addition, i^* is a tensor functor, and the isomorphism

$$i^*(i_*(\mathcal{A}) \otimes i_*(\mathcal{B})) \to i^*i_*(\mathcal{A}) \otimes i^*i_*(\mathcal{B})$$

followed by the co-unit

$$i^*i_*(\mathcal{A}) \otimes i^*i_*(\mathcal{B}) \to \mathcal{A} \otimes \mathcal{B}$$

induces a natural transformation $i_*(\mathcal{A}) \otimes i_*(\mathcal{B}) \to i_*(\mathcal{A} \otimes \mathcal{B})$. This gives us a natural map

$$\mathcal{G}^0(\mathcal{F}) \otimes \mathcal{G}^0(\mathcal{F}') \to \mathcal{G}^0(\mathcal{F} \otimes \mathcal{F}')$$

and thus by iteration a map of cosimplicial sheaves

$$\mu : [n \mapsto \mathcal{G}^n(\mathcal{F}) \otimes \mathcal{G}^n(\mathcal{F}')] \to [n \mapsto \mathcal{G}^n(\mathcal{F} \otimes \mathcal{F}')].$$

We have the Alexander-Whitney map

$$AW : \mathcal{G}^*(\mathcal{F}) \otimes \mathcal{G}^*(\mathcal{F}') \to [n \mapsto \mathcal{G}^n(\mathcal{F}) \otimes \mathcal{G}^n(\mathcal{F}')]^*$$

sending $\mathcal{G}^p(\mathcal{F})\otimes\mathcal{G}^q(\mathcal{F}')\to\mathcal{G}^{p+q}(\mathcal{F})\otimes\mathcal{G}^{p+q}(\mathcal{F}')$ by $\mathcal{G}(i_1^{p,q})\otimes\mathcal{G}(i_2^{p,q})$, where

$$i_1^{p,q}:\{0,\ldots,p\}\to\{0,\ldots,p+q\};\quad i_2^{p,q}:\{0,\ldots,q\}\to\{0,\ldots,p+q\}$$

are the maps

$$i_1^{p,q}(j)=j,\ i_2^{p,q}(j)=j+p.$$

The composition

$$\mu^*_{\mathcal{F},\mathcal{F}'}:=\mu\circ AW:\mathcal{G}^*(\mathcal{F})\otimes\mathcal{G}^*(\mathcal{F}')\to\mathcal{G}^*(\mathcal{F}\otimes\mathcal{F}')$$

makes $\mathcal{F}\mapsto\mathcal{G}^*(\mathcal{F})$ a weakly monoidal functor, i.e., we have the associativity

$$\mu^*_{\mathcal{F}\otimes\mathcal{F}',\mathcal{F}''}\circ(\mu^*_{\mathcal{F},\mathcal{F}'}\otimes\mathrm{id}_{\mathcal{G}^*(\mathcal{F}'')})=\mu^*_{\mathcal{F},\mathcal{F}'\otimes\mathcal{F}''}\circ(\mathrm{id}_{\mathcal{G}^*(\mathcal{F})}\otimes\mu^*_{\mathcal{F}',\mathcal{F}''}).$$

These facts enable the following construction: Let $U\mapsto\mathcal{C}(U)$ be a presheaf of DG categories on Opn_S^τ. For objects X and Y in $\mathcal{C}(S)$, we have the presheaf

$$[f:U\to S]\mapsto\mathcal{H}om_{\mathcal{C}(U)}(f^*(X),f^*(Y));$$

we denote the associated sheaf by $\underline{\mathcal{H}om}^\tau_{\mathcal{C}}(X,Y)$.

Let $R\Gamma(S,\mathcal{C})$ be the DG category with the same objects as $\mathcal{C}(S)$, and with Hom-complex

$$\mathcal{H}om_{R\Gamma(S,\mathcal{C})}(X,Y)^*:=\mathcal{G}^*(\underline{\mathcal{H}om}^\tau_{\mathcal{C}}(X,Y))(S).$$

Our remarks on the Godement resolution show that the composition law for the sheaf of DG categories $\mathcal{C}$ defines canonically an associative composition law

$$\circ:\mathcal{H}om_{R\Gamma(S,\mathcal{C})}(Y,Z)^*\otimes\mathcal{H}om_{R\Gamma(S,\mathcal{C})}(X,Y)^*\to\mathcal{H}om_{R\Gamma(S,\mathcal{C})}(X,Z)^*$$

for $R\Gamma(S,\mathcal{C})$, making $R\Gamma(S,\mathcal{C})$ a DG category. In addition we have

Proposition 3.2 *Let $\mathcal{C}$ be a presheaf of DG categories on Opn_S^τ. Then for each pair of objects X,Y of $\mathcal{C}(S)$, we have a canonical isomorphism*

$$\mathbb{H}^n(S_\tau,\underline{\mathcal{H}om}^\tau_{\mathcal{C}}(X,Y))\cong H^n(\mathcal{H}om_{R\Gamma(S,\mathcal{C})}(X,Y)).$$

Indeed, since $\mathcal{F}\to\mathcal{G}^*(\mathcal{F})$ is a $\Gamma(S,-)$-acyclic resolution of $\mathcal{F}$ for any complex of sheaves $\mathcal{F}$, we have a canonical isomorphism $\mathbb{H}^n(S_\tau,\mathcal{F})\cong H^n(\mathcal{G}^*(\mathcal{F})(S))$ for any complex of presheaves $\mathcal{F}$ on Opn_S^τ.

3.2 Thom-Sullivan co-chains. Suppose now that we are given a presheaf of DG tensor categories $U\mapsto\mathcal{C}(U)$ on Opn_S^τ. The Alexander-Whitney maps gives us a natural associative pairing of Hom-complexes

$$\otimes:\mathcal{H}om_{R\Gamma(S,\mathcal{C})}(X,Y)^*\otimes\mathcal{H}om_{R\Gamma(S,\mathcal{C})}(X',Y')^*\to\mathcal{H}om_{R\Gamma(S,\mathcal{C})}(X\otimes X',Y\otimes Y')^*$$

satisfying the Leibniz rule and compatible with the tensor product operation on $\mathcal{C}(S)$, however, this map is not commutative. For this, we need assume that $U\mapsto\mathcal{C}(U)$ is a sheaf of $\mathbb{Q}$-DG category; we can then modify the construction of $R\Gamma(S,\mathcal{C})$, giving a quasi-equivalent DG category $R\Gamma(S,\mathcal{C})^\otimes$ with a well-defined DG tensor structure. The construction uses the *Thom-Sullivan cochains*; we have taken this material from [13].

Let $|\Delta_n|$ be the real n-simplex

$$|\Delta_n|:=\{(t_0,\ldots,t_n)\in\mathbb{R}^{n+1}\mid\sum_{i=0}^n t_i=1,0\le t_i\},$$

and let Δ_n be the simplicial set $\mathrm{Hom}_\Delta(-,[n]):\Delta^{\mathrm{op}}\to\mathbf{Sets}$. For $f:[m]\to[n]$ in Δ, let $|f|:|\Delta_m|\to|\Delta_n|$ denote the corresponding affine-linear map.

For a cosimplicial abelian group G, we have the associated complex G^*, and the *normalized* subcomplex, NG^*, with NG^p the subgroup of those $g \in G([p])$ such $G(f)(g) = 0$ for all $f : [p] \to [q]$ in Δ which are not injective. The inclusion of $NG^* \hookrightarrow G^*$ is a homotopy equivalence. In particular, we have the complex of (simplicial) cochains of Δ_n, and the subcomplex of normalized cochains $Z^*(\Delta_n)$.

Let $\Omega^*(|\Delta_n|)$ denote the complex of $\mathbb{Q}$-polynomial differential forms on $|\Delta_n|$:

$$\Omega^*(|\Delta_n|) := \Omega^*_{\mathbb{Q}[t_0,\ldots,t_n]/\sum_{i=0}^n t_i - 1}.$$

Sending $[n]$ to $Z^*(\Delta_n)$, $\Omega^*(|\Delta_n|)$ determines functors

$$Z^* : \Delta^{\mathrm{op}} \to C^{\geq 0}(\mathbf{Ab}), \quad \Omega^* : \Delta^{\mathrm{op}} \to C^{\geq 0}(\mathrm{Mod}_{\mathbb{Q}}),$$

where $C^{\geq 0}(\mathrm{Mod}_{\mathbb{Q}})$ is the category of complexes of $\mathbb{Q}$-vector spaces concentrated in degrees ≥ 0, and $C^{\geq 0}(\mathbf{Ab})$ is the integral version. There is a natural homotopy equivalence

$$\int : \Omega^* \to Z^* \otimes \mathbb{Q}$$

defined by

$$\int(\omega)(\sigma) = \int_{|\sigma|} \omega$$

for $\omega \in \Omega^m(|\Delta_n|)$ and σ an m-simplex of Δ_n.

We have the category $\mathrm{Mor}(\Delta)$, with objects the morphisms in Δ, where a morphism $f \to g$ is a commutative diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ \uparrow & & \downarrow \\ \bullet & \xrightarrow[g]{} & \bullet \end{array} .$$

For functors $F : \Delta^{\mathrm{op}} \to C^{\geq 0}(\mathbf{Ab})$ and $G : \Delta \to \mathbf{Ab}$, we have the functor

$$F(\mathrm{domain}) \otimes G(\mathrm{range}) : \mathrm{Mor}(\Delta) \to C^{\geq 0}$$
$$f \mapsto F(\mathrm{domain}(f)) \otimes G(\mathrm{range}(f));$$

let $F \underset{\leftarrow}{\otimes} G$ be the projective limit

$$F \underset{\leftarrow}{\otimes} G := \varprojlim_{\mathrm{Mor}(\Delta)} F(\mathrm{domain}) \otimes G(\mathrm{range}).$$

Explicitly, an element ϵ of $(F \underset{\leftarrow}{\otimes} G)^m$ is given by a collection $p \mapsto \epsilon_p \in F^m([p]) \otimes G([p])$ such that

$$F(f) \otimes G(\mathrm{id})(\epsilon_p) = F(\mathrm{id}) \otimes G(f)(\epsilon_q)$$

for each $f : [q] \to [p]$ in Δ. The operation $\underset{\leftarrow}{\otimes}$ is functorial and respects homotopy equivalence.

For a cosimplicial abelian group G, we have the well-defined map $e : Z^* \underset{\leftarrow}{\otimes} G \to NG^*$ defined by sending $\epsilon := (\ldots \epsilon_p \ldots) \in (Z^* \underset{\leftarrow}{\otimes} G)^q$ to $\epsilon_q(\mathrm{id}_{[q]}) \in G([q])$. In [13, Lemma 3.1] it is shown that this map is well-defined, lands in NG^*, and gives an isomorphism of complexes. Thus, we have the natural homotopy equivalences

$$\int \underset{\leftarrow}{\otimes} \mathrm{id} : \Omega^* \underset{\leftarrow}{\otimes} G \to NG^* \otimes \mathbb{Q} \to G^* \otimes \mathbb{Q}.$$

The operation of wedge product makes Ω^* into a simplicial commutative differential graded algebra. Suppose we have two cosimplicial abelian groups $G, G' : \Delta \to \mathbf{Ab}$, giving us the diagonal cosimplicial abelian group $G \otimes G' : \Delta \to \mathbf{Ab}$. We have the map

$$\cup : (\Omega^* \underset{\leftarrow}{\otimes} G) \otimes (\Omega^* \underset{\leftarrow}{\otimes} G') \to \Omega^* \underset{\leftarrow}{\otimes} (G \otimes G')$$

induced by the map

$$(\omega \otimes g) \otimes (\omega' \otimes g') \mapsto (\omega \wedge \omega') \otimes (g \otimes g'),$$

for $\omega \otimes g \in \Omega^q(|\Delta_p|) \otimes G([p])$, $\omega' \otimes g' \in \Omega^{q'}(|\Delta_p|) \otimes G([p])$. It is easy to check that this gives a well-defined functorial product of cochain complexes and that $\cup$ is associative and commutative, in the evident sense.

We have as well the product map

$$\cup_{AW} : G^* \otimes G'^* \to (G \otimes G')^*$$

induced by the Alexander-Whitney maps $i_1^p : [p] \to [p+q]$, $i_2^q : [q] \to [p+q]$

$$(g \in G^p) \otimes (g' \in G'^q) \mapsto G(i_1^p)(g) \otimes G'(i_2^q)(g') \in G^{p+q} \otimes G'^{p+q}$$

It follows easily from the change of variables formula for integration that the diagram

$$\begin{array}{ccc} (\Omega^* \underset{\leftarrow}{\otimes} G) \otimes (\Omega^* \underset{\leftarrow}{\otimes} G') & \xrightarrow{\cup} & \Omega^* \underset{\leftarrow}{\otimes} (G \otimes G') \\ \downarrow{\scriptstyle \int \otimes \int} & & \downarrow{\scriptstyle \int} \\ (G^* \otimes \mathbb{Q}) \otimes (G'^* \otimes \mathbb{Q}) & \xrightarrow[\cup_{AW}]{} & (G \otimes G')^* \otimes \mathbb{Q} \end{array} \tag{3.1}$$

commutes.

We extend the operation $\Omega^* \underset{\leftarrow}{\otimes}(-)$ to functors $G^* : \Delta^{\rm op} \to C^+(\mathbf{Ab})$ by taking the total complex of the double complex

$$\ldots \to \Omega^* \underset{\leftarrow}{\otimes} G^0 \to \Omega^* \underset{\leftarrow}{\otimes} G^1 \to \ldots\ .$$

All the properties of $\Omega^* \underset{\leftarrow}{\otimes}(-)$ described above extend to the case of complexes.

3.3 Sheafifying DG tensor categories. Let $U \mapsto \mathcal{C}(U)$ be a presheaf of $\mathbb{Q}$-DG tensor categories on Opn_S^τ.

Define

$$\mathcal{H}om_{R\Gamma(S,\mathcal{C})^\otimes}(X,Y)^* := \Omega^* \underset{\leftarrow}{\otimes} \mathcal{G}^*(\mathcal{H}om^\tau_{\mathcal{C}}(X,Y))(S)$$

Using the functoriality of the Godement resolution, the composition law for $U \mapsto \mathcal{C}(U)$ together with the product $\cup$ defined in §3.2 defines a composition law

$$\circ : \mathcal{H}om_{R\Gamma(S,\mathcal{C})^\otimes}(Y,Z)^* \otimes \mathcal{H}om_{R\Gamma(S,\mathcal{C})^\otimes}(X,Y)^* \to \mathcal{H}om_{R\Gamma(S,\mathcal{C})^\otimes}(X,Z)^*$$

giving us the DG category $R\Gamma(S,\mathcal{C})^\otimes$. By the commutativity of (3.1) the maps

$$\int \underset{\leftarrow}{\otimes} \mathrm{id} : \mathcal{H}om_{R\Gamma(S,\mathcal{C})^\otimes}(X,Y)^* \to \mathcal{H}om_{R\Gamma(S,\mathcal{C})}(X,Y)^*$$

define a DG functor

$$\int : R\Gamma(S,\mathcal{C})^\otimes \to R\Gamma(S,\mathcal{C})$$

which is a quasi-equivalence.

Finally, the tensor product operation on $\mathcal{C}$ gives rise to maps of cosimplicial objects

$$\otimes : \mathcal{G}^*(\mathcal{H}om^\tau_{\mathcal{C}}(X,Y))(S)\otimes\mathcal{G}^*(\mathcal{H}om^\tau_{\mathcal{C}}(X',Y'))(S) \to \mathcal{G}^*(\mathcal{H}om^\tau_{\mathcal{C}}(X\otimes X',Y\otimes Y'))(S)$$

Applying the Thom-Sullivan cochain construction gives the map

$$\tilde{\otimes} : \mathcal{H}om_{R\Gamma(S,\mathcal{C})^\otimes}(X,Y)^*\otimes\mathcal{H}om_{R\Gamma(S,\mathcal{C})^\otimes}(X',Y')^* \to \mathcal{H}om_{R\Gamma(S,\mathcal{C})^\otimes}(X\otimes X',Y\otimes Y')^*$$

that makes $R\Gamma(S,\mathcal{C})^\otimes$ a DG tensor category, with commutativity constraint induced from $\mathcal{C}(S)$.

Remark 3.3 Let $U \mapsto \mathcal{C}(U)$ be a presheaf of $\mathbb{Q}$-DG tensor categories on Opn^τ_S. The quasi-equivalence $\int : R\Gamma(S,\mathcal{C})^\otimes \to R\Gamma(S,\mathcal{C})$ induces an equivalence of triangulated categories

$$\mathcal{K}^b(\int) : \mathcal{K}^b(R\Gamma(S,\mathcal{C})^\otimes) \to \mathcal{K}^b(R\Gamma(S,\mathcal{C}))$$

(theorem 2.14). In addition, the DG tensor structure we have defined on $R\Gamma(S,\mathcal{C})^\otimes$ endows $\mathcal{C}^b(R\Gamma(S,\mathcal{C})^\otimes)$ with the structure of a DG tensor category, and makes $\mathcal{K}^b(R\Gamma(S,\mathcal{C})^\otimes)$ a tensor triangulated category. Similarly, for the idempotent completions, we have equivalences of (tensor) triangulated categories

$$\mathcal{D}(\int) : \mathrm{per}(R\Gamma(S,\mathcal{C})^\otimes) \to \mathrm{per}(R\Gamma(S,\mathcal{C})).$$

4 DG categories of motives

Let S be a fixed base-scheme; we assume that S is a regular scheme of finite Krull dimension. Let $\mathbf{Sm}/S$ denote the category of smooth S-schemes of finite type, $\mathbf{Proj}/S \subset \mathbf{Sm}/S$ be full subcategory of $\mathbf{Sm}/S$ consisting of the smooth projective S-schemes. Our construction of the DG category of motives goes as follows: We form the DG category of correspondences $dgCor_S$ from a cubical enhancement of the category of finite correspondences Cor_S, using the algebraic n-cubes as the cubical object; restricting to smooth projective S-schemes gives us our basic DG category $dgPrCor_S$. We sheafify the construction over S, then take the homotopy category of complexes over the sheafified DG category to form the triangulated category $\mathcal{K}^b(R\Gamma(S,\underline{dgPrCor}_S))$. Taking the idempotent completion gives us our category of smooth effective motives

$$SmMot^{\mathrm{eff}}_{gm}(S) := \mathrm{per}(R\Gamma(S,\underline{dgPrCor}_S));$$

inverting tensor product with the Lefschetz motive gives us the category of smooth motives $SmMot_{gm}(S)$.

We also have a parallel version with $\mathbb{Q}$-coefficients; using alternating cubes endows everything with a tensor structure. Now for the details.

4.1 DG categories of correspondences. We begin by recalling the definition of the category of finite correspondences.

Definition 4.1 For $X, Y \in \mathbf{Sm}/S$, $Cor_S(X,Y)$ is the free abelian group on the integral closed subschemes $W \subset X \times_S Y$ such that the projection $W \to X$ is finite and surjective onto an irreducible component of X.

Now let X, Y and Z be in $\mathbf{Sm}/S$. Take generators $W \in Cor_S(X,Y)$, $W' \in Cor_S(Y,Z)$. As in [24, Chap. V], each component T of the intersection $W \times_S Z \cap X \times_S W' \subset X \times_S Y \times_S Z$ is finite over $X \times_S Z$ and over X (via the projections) and the map

$$T \to X$$

is surjective over some irreducible component of X. Thus, letting p_{XY}, p_{YZ}, etc., denote the projections from the triple product $X \times_S Y \times_S Z$, and $\cdot_{XYZ}$ the intersection product of cycles on $X \times_S Y \times_S Z$, the expression

$$W \circ W' := p_{XZ*}(p_{XY}^*(W) \cdot_{XYZ} p_{YZ}^*(W'))$$

gives a well-defined and associative composition law

$$\circ : Cor_S(X,Y) \otimes Cor_S(Y,Z) \to Cor_S(X,Z),$$

defining the pre-additive category Cor_S. Cor_S is an additive category, with disjoint union being the direct sum, and product over S makes Cor_S into a tensor category.

Sending $X \in \mathbf{Sm}/S$ to $X \in Cor_S$ and sending $f : X \to Y$ to the graph of f, $\Gamma_f \subset X \times_S Y$, defines a faithful embedding $i_S : \mathbf{Sm}/S \to Cor_S$. Making $\mathbf{Sm}/S$ a symmetric monoidal category using the product over S makes i_S a symmetric monoidal functor.

From example 1.2, we have the co-cubical object $\square_S^*$,

$$n \mapsto \square_S^n \cong \mathbb{A}_S^n.$$

In fact, $\square_S^*$ extends to a functor

$$\square_S^* : \mathbf{ECube} \to \mathbf{Sm}/S$$

by sending the multiplication map $\mu : \underline{2} \to \underline{1}$ to the usual multiplication

$$\mu_S : \square_S^2 \to \square_S^1;\ \mu_S(x,y) = xy.$$

Define the co-multiplication $\delta : \square_S^* \to \square_S^* \otimes \square_S^*$ by taking the collection of diagonal maps $\delta^n := \delta_{\square^n} : \square_S^n \to \square_S^n \times_S \square_S^n$. One easily verifies the properties of definition 1.10.

Definition 4.2 Let $dgCor_S$ denote the DG category $(Cor_S, \otimes, \square_S^*, \delta)$. We denote the DG tensor category $((Cor_{S\mathbb{Q}}, \otimes, \square_S^*, \delta)^{\mathrm{alt}}, \otimes_\square)$ by $dgCor_S^{\mathrm{alt}}$.

Proposition 4.3 *1. The DG categories $dgCor_S$ and $dgCor_S^{\mathrm{alt}}$ are negative (see definition 2.9).*

2. The functor $dgCor_S^{\mathrm{alt}} \to dgCor_{S\mathbb{Q}}$ is a quasi-equivalence.

Proof Indeed, (1) is obvious, and (2) follows from proposition 1.13. $\square$

Definition 4.4 Let $dgPrCor_S$ be the full DG subcategory of $dgCor_S$ with objects $X \in \mathbf{Proj}/S$. $dgPrCor_S^{\mathrm{alt}}$ is similarly defined as the full DG subcategory of $dgCor_S^{\mathrm{alt}}$ with objects $X \in \mathbf{Proj}/S$.

Note that $dgPrCor_S^{\mathrm{alt}}$ is a DG tensor subcategory of $dgCor_S^{\mathrm{alt}}$, and the functor $dgPrCor_S^{\mathrm{alt}} \to dgPrCor_{S\mathbb{Q}}$ induced by $dgCor_S^{\mathrm{alt}} \to dgCor_{S\mathbb{Q}}$ is a quasi-equivalence.

4.2 Functoriality. Let $f : S' \to S$ be a k-morphism of regular schemes. We have the well-defined pull-back functor $f^* : Cor_S \to Cor_{S'}$, with $f^*(X) = X \times_S S'$ for $X \in \mathbf{Sm}/S$ and using Serre's intersection multiplicity formula to define the pull-back of cycles

$$f^* : Cor_S(X,Y) \to Cor_{S'}(f^*(X), f^*(Y))$$

Note that the cycle pull-back is always well-defined for finite correspondences, using the isomorphism

$$(X \times_S S') \times_{S'} (Y \times_S S') \cong (X \times_S S') \times_S Y.$$

Since $f^*(X) \times_{S'} \square^n_{S'} \cong f^*(X \times_S \square^n_S)$, the pull-back extends to the map of complexes

$$f^* : \mathcal{H}om_{dgCor_S}(X,Y) \to \mathcal{H}om_{dgCor_{S'}}(f^*(X), f^*(Y)),$$

defining the functor $S \mapsto dgCor_S$ from regular schemes to DG categories. We have as well the sub-functor $S \mapsto dgPrCor_S$.

A similar construction defines the functor $S \mapsto dgCor_S^{\mathrm{alt}}$, from regular schemes to DG tensor categories, and the subfunctor $S \mapsto dgPrCor_S^{\mathrm{alt}}$.

4.3 Complexes and smooth motives.

Definition 4.5 Let S be a regular scheme. Let $\underline{dgPrCor}_S$ denote the Zariski presheaf

$$U \mapsto \underline{dgPrCor}_S(U) := dgPrCor_U.$$

The DG category of *smooth effective geometric motives over S*, $dgSmMot_S^{\mathrm{eff}}$, is defined as

$$dgSmMot_S^{\mathrm{eff}} := \mathcal{C}^b(R\Gamma(S, \underline{dgPrCor}_S)).$$

The triangulated category, $SmMot_{gm}^{\mathrm{eff}}(S)$, of smooth effective motives over S is defined as the idempotent completion of the homotopy category

$$\mathcal{K}^b(R\Gamma(S, \underline{dgPrCor}_S^{\mathrm{eff}})) = H^0 dgSmMot_S^{\mathrm{eff}},$$

in other words,

$$SmMot_{gm}^{\mathrm{eff}}(S) = \mathrm{per}(R\Gamma(S, \underline{dgPrCor}_S^{\mathrm{eff}})) \subset \mathcal{D}(R\Gamma(S, \underline{dgPrCor}_S^{\mathrm{eff}})).$$

The Zariski presheaf, $\underline{dgPrCor}_S^{\mathrm{alt}}$, is defined by

$$U \mapsto \underline{dgPrCor}_S^{\mathrm{alt}}(U) := dgPrCor_U^{\mathrm{alt}}.$$

The DG tensor category of smooth effective geometric motives over S with $\mathbb{Q}$-coefficients, $dgSmMot_{S\mathbb{Q}}^{\mathrm{eff}}$, is defined as

$$dgSmMot_{S\mathbb{Q}}^{\mathrm{eff}} := \mathcal{C}^b(R\Gamma(S, \underline{dgPrCor}_S^{\mathrm{alt}})^{\otimes}).$$

The tensor triangulated category, $SmMot_{gm}^{\mathrm{eff}}(S)_{\mathbb{Q}}$, of smooth effective motives over S with $\mathbb{Q}$-coefficients is defined as the idempotent completion of the homotopy category $\mathcal{K}^b(R\Gamma(S, \underline{dgSmMot}_S^{\mathrm{alt}})^{\otimes}) = H^0 dgSmMot_{S\mathbb{Q}}^{\mathrm{eff}}$,

$$SmMot_{gm}^{\mathrm{eff}}(S)_{\mathbb{Q}} := \mathrm{per}(R\Gamma(S, \underline{dgSmMot}_S^{\mathrm{alt}})^{\otimes}) \subset \mathcal{D}(R\Gamma(S, \underline{dgSmMot}_S^{\mathrm{alt}})^{\otimes}).$$

Remark 4.6 We can also define unbounded versions: the triangulated category of effective motives over S

$$SmMot^{\rm eff}(S) := \mathcal{D}(R\Gamma(S, \underline{dgPrCor}_S^{\rm eff})),$$

and the tensor triangulated category of effective motives over S with $\mathbb{Q}$-coefficients

$$SmMot^{\rm eff}(S)_{\mathbb{Q}} := \mathcal{D}(R\Gamma(S, \underline{dgSmMot}_S^{\rm alt})^{\otimes}).$$

We will mainly restrict our discussion to the geometric categories.

Proposition 4.7 *The functor* $\underline{dgPrCor}_S^{\rm alt} \to \underline{dgPrCor}_S \otimes \mathbb{Q}$ *defined by the natural functors*

$$dgPrCor_U^{\rm alt} \to dgPrCor_U \otimes \mathbb{Q}$$

gives rise to a functor of DG categories

$$R\Gamma(S, \underline{dgPrCor}_S^{\rm alt})^{\otimes} \to R\Gamma(S, \underline{dgPrCor}_S) \otimes \mathbb{Q},$$

which in turn induces an equivalence of $SmMot_{gm}^{\rm eff}(S)_{\mathbb{Q}}$ *with the idempotent completion of* $SmMot_{gm}^{\rm eff}(S) \otimes \mathbb{Q}$*, as triangulated categories.*

Proof This follows from proposition 4.3 and remark 3.3. □

Remark 4.8 The DG categories $R\Gamma(S, \underline{dgPrCor}_S)$ and $dgSmMot_S^{\rm eff}$ are functorial in the regular scheme S, as is the triangulated category $SmMot_{gm}^{\rm eff}(S)$. The same holds for the DG tensor categories $R\Gamma(S, \underline{dgPrCor}_S^{\rm alt})^{\otimes}$, $dgSmMot_{S\mathbb{Q}}^{\rm eff}$ and the tensor triangulated category $SmMot_{gm}^{\rm eff}(S)_{\mathbb{Q}}$.

4.4 Lefschetz motives. In $\mathbf{Proj}/S$, we have the idempotent endomorphism α of $\mathbb{P}^1_S$ defined as the composition

$$\mathbb{P}^1_S \xrightarrow{p} S \xrightarrow{i_\infty} \mathbb{P}^1_S.$$

Since Cor_S is a tensor category, we have the object $\mathbb{L} := (\mathbb{P}^1_S, 1-\alpha)$ in $Cor_S^\natural$, as well as the nth tensor power $\mathbb{L}^n$ of $\mathbb{L}$ and, for each $X \in \mathbf{Sm}/S$, the object $X \otimes \mathbb{L}^n$. We thus have these objects in the DG categories of correspondences $dgPrCor_S^\natural$ and $R\Gamma(S, \underline{dgPrCor}_S)^\natural$.

Definition 4.9 The DG category of Lefschetz motives over S, $dgLCor_S^{\rm eff}$, is defined to be the full DG subcategory of $R\Gamma(S, \underline{dgPrCor}_S)^\natural$ with objects $\mathbb{L}^d$, $d \geq 0$. The DG category of effective mixed Tate motives is

$$dgTMot_S^{\rm eff} := \mathcal{C}^b(dgLCor_S^{\rm eff}).$$

The triangulated category of effective mixed Tate motives over S, $DTMot^{\rm eff}(S)$, is the homotopy category $\mathcal{K}^b(dgLCor_S^{\rm eff}) = H^0 dgTMot_S^{\rm eff}$.

We have parallel definitions of DG tensor categories and a tensor triangulated category with $\mathbb{Q}$-coefficients. The DG tensor category of Lefschetz motives over S with $\mathbb{Q}$-coefficients, $dgLCor_{S\mathbb{Q}}^{\rm eff}$, is the full DG subcategory of $R\Gamma(S, \underline{dgSmMot}_S^{\rm alt})^{\otimes\natural}$ with objects finite direct sums of the $\mathbb{L}^d$, $d \geq 0$, The DG tensor category of effective mixed Tate motive with $\mathbb{Q}$-coefficients is

$$dgTMot_{S\mathbb{Q}}^{\rm eff} := \mathcal{C}^b(dgLCor_{S\mathbb{Q}}^{\rm eff}),$$

and the tensor triangulated category of effective mixed Tate motives over S with , $DTMot^{\rm eff}(S)_{\mathbb{Q}}$, is the homotopy category $\mathcal{K}^b(dgLCor_{S\mathbb{Q}}^{\rm eff}) = H^0 dgTMot_{S\mathbb{Q}}^{\rm eff}$.

5 Duality

In section 7, we will prove an extension of the moving lemmas of Friedlander-Lawson to smooth projective schemes over a regular, semi-local base. In this section, we use these moving lemmas to define a twisted duality for Hom-complexes in $R\Gamma(S, \underline{dgSmMot}_S)$, and extend this to a duality in various categories of motives.

In this section S will be a regular scheme over a fixed base-field k.

5.1 Equi-dimensional cycles.

Definition 5.1 Let X and Y be smooth over S, $r \geq 0$ an integer. The group $z^S_{equi}(Y,r)(X)$ is the free abelian group on the integral subschemes $W \subset X \times_S Y$ such that the projection $W \to X$ dominates an irreducible component X' of X, and such that, for each $x \in X$, the fiber W_x over x has pure dimension r over $k(x)$, or is empty.

We let $z^S_{equi}(Y,r)^{\text{eff}}(X) \subset z^S_{equi}(Y,r)(X)$ be the submonoid of effective cycles, that is, the free monoid on the generators W for $z^S_{equi}(Y,r)(X)$ described above.

For an S-morphism $f : X' \to X$, and for $W \in z^S_{equi}(Y,r)(X)$, the pull-back cycle $(f \times \text{id}_Y)^*(W)$ is well-defined and in $z^S_{equi}(Y,r)(X')$. Thus $z^S_{equi}(Y,r)$ is a presheaf on $\mathbf{Sm}/S$. For $x \in X \in \mathbf{Sm}/S$, and for $W \in z^S_{equi}(Y,r)(X)$, we denote the pull-back $i_x^*(W)$ by the inclusion $i_x : x \to X$ by W_x.

Suppose Y is projective over S, with a fixed embedding $Y \hookrightarrow \mathbb{P}^N_S$. Then we have a well-defined degree homomorphism

$$\deg : z^S_{equi}(Y,r)(X) \to H^0(X_{\text{Zar}}, \mathbb{Z})$$

which sends a cycle $W \in z^S_{equi}(Y,r)(X)$ to the locally constant function on X

$$x \mapsto \deg(W_x),$$

where $\deg(W_x)$ is the usual degree of the cycle W_x in $\mathbb{P}^N$.

For each integer $e \geq 1$, we let $z^S_{equi}(Y,r)^{\text{eff}}_{\leq e}(X)$ be the subset of $z^S_{equi}(Y,r)^{\text{eff}}(X)$ consisting by those W with $\deg(W) \leq e$ on X. We let

$$z^S_{equi}(Y,r)_{\leq e}(X) \subset z^S_{equi}(Y,r)(X)$$

be the subgroup generated by the set $z^S_{equi}(Y,r)^{\text{eff}}_{\leq e}(X)$. Thus we have the presheaf of sets $z^S_{equi}(Y,r)^{\text{eff}}_{\leq e}(X)$, and the presheaf of abelian groups $z^S_{equi}(Y,r)_{\leq e}(X)$.

Definition 5.2 For X, Y in $\mathbf{Sm}/S$, the *cubical Suslin complex* $C^S(Y,r)^*(X)$ is the complex associated to the cubical object

$$n \mapsto z^S_{equi}(Y,r)(X \times_S \square^n_S),$$

i.e.

$$C^S(Y,r)^n(X) := z^S_{equi}(Y,r)(X \times_S \square_S^{-n})/\text{degn}.$$

For $f : X' \to X$, define

$$f^* : C^S(Y,r)(X) \to C^S(Y,r)(X')$$

via the pull-back maps

$$(f \times \text{id}_{\square^n})^* : z^S_{equi}(Y,r)(X \times \square^n) \to z^S_{equi}(Y,r)(X' \times \square^n);$$

this defines the presheaf of complexes $C^S(Y,r)$ on $\mathbf{Sm}/S$.

Suppose that Y is in $\mathbf{Proj}/S$ and we are given an embedding $Y \hookrightarrow \mathbb{P}^N_S$ over S. Let $C^S(Y,r)_{\le e}(X) \subset C^S(Y,r)(X)$ be the subcomplex corresponding to the cubical abelian group

$$n \mapsto z^S_{equi}(Y,r)_{\le e}(X \times_S \square^n_S) \subset z^S_{equi}(Y,r)(X \times_S \square^n_S).$$

Remarks 5.3 1. Suppose that Y is in $\mathbf{Proj}/S$. Then

$$C^S(Y,0)^*(X) = \mathcal{H}om_{dgCor_S}(X,Y)^*,$$

and $C^S(Y,0)$ is the presheaf $\mathcal{H}om_{dgCor_S}(-,Y)^*$ on $\mathbf{Sm}/S$.

2. Let $f : Y \to Y'$ be a proper morphism. Push-forward by the projection $\mathrm{id} \times f : X \times \square^n \times Y \to X \times \square^n \times Y'$ defines the map of complexes

$$f_*(X) : C^S(Y,r)(X) \to C^S(Y',r)(X)$$

giving us the map of presheaves $f_* : C^S(Y,r) \to C^S(Y',r)$. Thus $Y \mapsto C^S(Y,r)$ defines a functor from $\mathbf{Proj}/S$ to complexes of sheaves on $\mathbf{Sm}/S$.

3. Correspondences act on $z^S_{equi}(Y,r)(X)$, both in Y (covariantly) and in X (contravariantly). Thus sending (Y,X) to $z^S_{equi}(Y,r)(X)$ extends to a functor

$$z^S_{equi}(-,r)(-) : Cor_S \times Cor_S^{\mathrm{op}} \to \mathbf{Ab}$$

We have a similar extension of the complexes $C^S(Y,r)(X)$ to

$$C^S(-,r)(-) : Cor_S \times Cor_S^{\mathrm{op}} \to C^-(\mathbf{Ab})$$

As $\mathbf{Ab}$ is abelian, the bi-functors $z^S_{equi}(-,r)(-)$: and $C^S(-,r)(-)$ extend to the idempotent completion of Cor_S; in particular, the presheaves $z^S_{equi}(Y \otimes \mathbb{L}^n, r)$ and $C^S(Y \otimes \mathbb{L}^n, r)$ are defined.

Now take $Y, X \in \mathbf{Sm}/S$ with $X \to S$ equi-dimensional of dimension p over S. Each integral $W \in z^S_{equi}(Y,r)(U \times_S X)$ gives us an integral subscheme W of $U \times_S X \times_S Y$ which is equi-dimensional of relative dimension $r+p$ over some component of U. This defines the map of complexes

$$\int_X : C^S(Y,r)(U \times_S X) \to C^S(X \times_S Y, r+p)(U)$$

We can now state our main result, to be proven in §7.

Theorem 5.4 *Let S be a regular semi-local k-scheme, essentially of finite type over k. Then for all $X, Y \in \mathbf{Proj}/S$, $U \in \mathbf{Sm}/S$, with $X \to S$ of relative dimension p, the map*

$$\int_X : C^S(Y,r)(U \times_S X) \to C^S(X \times_S Y, r+p)(U)$$

is a quasi-isomorphism.

In addition, we will need a computation of $C_*(Y \times \mathbb{P}^n, r)$ for $r \ge n$. Fix linear inclusions $\iota_j : \mathbb{P}^j \to \mathbb{P}^n$, $j = 0, \ldots, n$. For each i, $0 \le i \le n \le r$, define the map

$$\alpha_j : C^S(Y, r-j)(X) \to C^S(Y \times \mathbb{P}^n, r)(X)$$

by sending $W \subset Y \times_S X \times \square^n$ to $(\iota_j \times \mathrm{id})_*(\mathbb{P}^j \times W)$.

Theorem 5.5 *Let S be a regular semi-local k-scheme, essentially of finite type over k. Then for all $X, Y \in \mathbf{Proj}/S$, and for $r \geq n$, the map*

$$\sum_{j=0}^{n} \alpha_j : \oplus_{j=0}^{n} C^S(Y, r-j)(X) \to C^S(Y \times \mathbb{P}^n, r)(X)$$

is a quasi-isomorphism.

Theorem 5.5 will also be proven in §7. As consequence, we have

Corollary 5.6 *Let S be a regular semi-local k-scheme, essentially of finite type over k.*

1. For $Y \in \mathbf{Proj}/S$, $X \in \mathbf{Sm}/S$ and $n \geq 0$, let

$$\varphi : C_*^S(Y, r)(X) \to C^S(Y \otimes \mathbb{L}^n, r+n)(X)$$

be the map sending $W \subset X \times \square^n \times_S Y$ to $W \times (\mathbb{P}^1)^n \subset X \times (\mathbb{P}^1)^n \times \square^n \times_S Y$. Then φ is an isomorphism in $D^-(\mathbf{Ab})$.

2. For $Y, Z \in \mathbf{Proj}/S$, $X \in \mathbf{Sm}/S$, with Y of dimension d over S, there are natural isomorphisms in $D^-(\mathbf{Ab})$

$$C^S(Y \times_S Z, r)(X) \cong C^S(Z \otimes \mathbb{L}^d, r)(X \times_S Y).$$

For $Z = S \times_k Z_0$, with $Z_0 \in \mathbf{Proj}/k$, we have an isomorphism

$$C^S(Z \otimes \mathbb{L}^d, r)(X \times_S Y) \cong C^k(Z_0 \otimes \mathbb{L}^d, r)(p_*(X \times_S Y)).$$

3. For $X \in \mathbf{Sm}/S$, $Y \in \mathbf{Proj}/S$ and $r \geq n \geq 0$ integers, there are natural isomorphisms in $D^-(\mathbf{Ab})$

$$C^S(Y, r)(X \otimes \mathbb{L}^n) \cong C^S(Y, r+n)(X).$$

Proof For (1), the isomorphism

$$C^S(Y \otimes \mathbb{L}^n, r+n)(X) \cong C^S(Y, r)(X).$$

follows by induction and the projective bundle formula (theorem 5.5). Indeed, it suffices to define a natural isomorphism

$$C^S(Y \otimes \mathbb{L}^n, r+1)(X) \to C^S(Y \otimes \mathbb{L}^{n-1}, r)(X). \tag{5.1}$$

Since the projection $\mathbb{P}^1 \to \operatorname{Spec} k$ is split by the inclusion $i_\infty : \operatorname{Spec} k \to \mathbb{P}^1$, we get a direct sum decomposition

$$C^S(Y \times \mathbb{P}^1, r) \cong C^S(Y, r) \oplus C^S(Y \otimes \mathbb{L}, r)$$

By induction, we have a similar direct sum decomposition of $C^S(Y \times (\mathbb{P}^1)^n, r)$ for all n; this reduces the proof of (5.1) to showing that

$$C^S(Y \times (\mathbb{P}^1)^{n-1} \otimes \mathbb{L}, r+1)(X) \cong C^S(Y \times (\mathbb{P}^1)^{n-1}, r)(X),$$

which reduces us to the case $n = 1$. Comparing with the isomorphism

$$\alpha_0 + \alpha_1 : C^S(Y, r+1) \oplus C^S(Y, r) \to C^S(Y \times \mathbb{P}^1, r+1)$$

and noting that $p_* \circ \alpha_1 = 0$, $p_* \circ \alpha_0 = \mathrm{id}$, we see that α_1 gives an isomorphism

$$\alpha_1 : C^S(Y, r) \to C^S(Y \otimes \mathbb{L}, r+1),$$

completing the proof of the first assertion.

The first isomorphism in (2) follows from the first assertion and theorem 5.4:

$$C^S(Z\otimes\mathbb{L}^d,r)(X\times_S Y)\cong C^S(Y\times_S Z\otimes\mathbb{L}^d,d+r)(X)\cong C^S(Y\times_S Z,r)(X).$$

For the second isomorphism, $C^S(Z\otimes\mathbb{L}^d,r)(X\times_S Y)=C^k(Z_0\otimes\mathbb{L}^d,r)(X\times_S Y)$, since we have the isomorphism of schemes (over $\mathrm{Spec}\,k$)

$$T\times_S(S\times_k Z_0)\times_S\mathbb{P}^1_S\cong p_*T\times_k Z_0\times_k\mathbb{P}^1_k$$

for all S-schemes T.

For (3), it suffices to prove the case $n=1$. By theorem 5.4, we have the quasi-isomorphism

$$\int_{\mathbb{P}^1}:C^S(Y,r)(X\times\mathbb{P}^1)\to C^S(Y\times\mathbb{P}^1,r+1)(X).$$

Since $Y\times\mathbb{P}^1\cong Y\oplus Y\otimes\mathbb{L}$ in $Cor^\natural_S$, and similarly for $X\times\mathbb{P}^1$, we have the quasi-isomorphism

$$\int_{\mathbb{P}^1}:C^S(Y,r)(X)\oplus C^S(Y,r)(X\otimes\mathbb{L})\to C^S(Y\times\mathbb{P}^1,r+1)(X).$$

Since $\mathbb{L}=(\mathbb{P}^1,1-i_\infty p)$, the summand $C^S(Y,r)(X\otimes\mathbb{L})$ of $C^S(Y,r)(X\times\mathbb{P}^1)$ is the image of $\mathrm{id}-p^*i^*_\infty$, which is the same as the kernel of i^*_∞. Similarly, the map $C^S(Y,r)(X)\to C^S(Y,r)(X\times\mathbb{P}^1)$ is defined by p^*. Using the flag $0\subset\mathbb{P}^1$ to define the quasi-isomorphism of theorem 5.5,

$$\alpha_0+\alpha_1:C^S(Y,r+1)(X)\oplus C^S(Y,r)(X)\to C^S(Y\times\mathbb{P}^1,r+1)(X),$$

the inverse isomorphism (in $D^-(\mathbf{Ab})$) is given by the map of complexes

$$C^S(Y\times\mathbb{P}^1,r+1)_{Y\times\infty}(X)\xrightarrow{(p_*,i^*_\infty)}C^S(Y,r+1)(X)\oplus C^S(Y,r)(X).$$

composed with the inverse of the quasi-isomorphism

$$C^S(Y\times\mathbb{P}^1,r+1)_{Y\times\infty}(X)\hookrightarrow C^S(Y\times\mathbb{P}^1,r+1)(X).$$

We note that the image of $\int_{\mathbb{P}^1}$ lands in $C^S(Y\times\mathbb{P}^1,r+1)_{Y\times\infty}(X)$, and sends $C^S(Y,r)(X\otimes\mathbb{L})$ to $\ker i^*_\infty$ and $C^S(Y,r)(X)$ to $\ker p_*$. Thus $p_*\circ\int_{\mathbb{P}^1}$ defines a quasi-isomorphism

$$p_*\circ\int_{\mathbb{P}^1}:C^S(Y,r)(X\otimes\mathbb{L})\to C^S(Y,r+1)(X),$$

as desired. □

Remark 5.7 For later use, we extract from the proof a description of the isomorphism in corollary 5.6(2,3). In (2), the isomorphism is the composition of

$$\int_Y:C^S(Z\otimes\mathbb{L}^d,r)(X\times_S Y)\to C^S(Y\times_S Z\otimes\mathbb{L}^d,r+d)(X)$$

with the inverse of

$$-\times(\mathbb{P}^1)^n:C^S(Y\times_S Z,r)(X)\to C^S(Y\times_S Z\otimes\mathbb{L}^d,r+d)(X).$$

To describe the map in (3), let $p:Y\times\mathbb{P}^1\to Y$ be the projection. We have the composition

$$C^S(Y,r)(X\times\mathbb{P}^1)\xrightarrow{\int_{\mathbb{P}^1}}C^S(Y\times\mathbb{P}^1,r+1)(X)\xrightarrow{p_*}C^S(Y,r+1)(X).$$

Then the restriction of $p_*\circ\int_{\mathbb{P}^1}$ to the summand $C^S(Y,r)(X\otimes\mathbb{L})$ of $C^S(Y,r)(X\times\mathbb{P}^1)$,

$$p_*\circ\int_{\mathbb{P}^1}: C^S(Y,r)(X\otimes\mathbb{L})\to C^S(Y,r+1)(X),$$

is the isomorphism in (3). Iterating, we have the map

$$[p_*\circ\int_{\mathbb{P}^1}]^n: C^S(Y,r)(X\otimes\mathbb{L}^n)\to C^S(Y,r+n)(X),$$

giving the isomorphism in (3).

For $Y,Z\in\mathbf{Proj}/S$, $X\in\mathbf{Sm}/S$, we have the natural map

$$\times_S Z: z^S_{equi}(Y,r)(X)\to z^S_{equi}(Y\times_S Z,r)(X\times_S Z)$$

defined by sending a cycle W on $X\times_S Y$ to the image of W in $X\times_S Z\times_S Y\times_S Z$ via the "diagonal" embedding $X\times_S Y\to X\times_S Z\times_S Y\times_S Z$. This gives us the map

$$\times_S Z: C^S(Y,r)(X)\to C^S(Y\times_S Z,r)(X\times_S Z).$$

Taking $Z=\mathbb{P}^1$ and extending to the pseudo-abelian hull gives the map

$$\otimes\mathbb{L}: C^S(Y,r)(X)\to C^S(Y\otimes\mathbb{L},r)(X\otimes\mathbb{L}),$$

sending $W\subset X\times\square^n\times_S Y$ to $W\times\Delta_{\mathbb{P}^1}\subset X\times\mathbb{P}^1\times\square^n\times_S Y\times\mathbb{P}^1$.

Corollary 5.8 *Let S be a regular semi-local k-scheme, essentially of finite type over k. For $X,Y\in\mathbf{Proj}/S$, the map $\otimes\mathbb{L}: C^S(Y,r)(X)\to C^S(Y\otimes\mathbb{L},r)(X\otimes\mathbb{L})$ is a quasi-isomorphism.*

Proof By the projective bundle formula (theorem 5.5), the map $W\mapsto W\times\mathbb{P}^1$ gives a quasi-isomorphism

$$-\times\mathbb{P}^1: C^S(Y,r)(X)\to C^S(Y\otimes\mathbb{L},r+1)(X).$$

By corollary 5.6(3) and remark 5.7, the map

$$p_*\circ\int_{\mathbb{P}^1}: C^S(Y\otimes\mathbb{L},r)(X\otimes\mathbb{L})\to C^S(Y\otimes\mathbb{L},r+1)(X)$$

is a quasi-isomorphism. Now, for $W\in C^S(Y,r)(X)$, we have

$$p_*\circ\int_{\mathbb{P}^1}(W\otimes\mathbb{L})=p_*\circ\int_{\mathbb{P}^1}(W\times\Delta_{\mathbb{P}^1})=W\times\mathbb{P}^1,$$

hence

$$\otimes\mathbb{L}: C^S(Y,r)(X)\to C^S(Y\otimes\mathbb{L},r)(X\otimes\mathbb{L})$$

is a quasi-isomorphism. □

5.2 Duality for smooth motives. The duality results of the previous section extend to the category $SmMot_{gm}(S)$ and defines a duality on the tensor triangulated category $SmMot_{gm}(S)_{\mathbb{Q}}$.

Proposition 5.9 *Let S be a regular scheme, essentially of finite type over a field k. For $X\in\mathbf{Proj}/S$, the natural map*

$$\mathcal{H}om_{dgPrCor^\natural_S}(X,\mathbb{L}^d)^*\to\mathcal{H}om_{R\Gamma(S,\underline{dgSmMot}_S)^\natural}(X,\mathbb{L}^d)^*$$

is a quasi-isomorphism.

Proof Let $p_*X \in \mathbf{Sm}/k$ denote the scheme X, considered as a k-scheme via the structure morphism $p : S \to \operatorname{Spec} k$. We have a canonical isomorphism

$$\mathcal{H}om_{dgPrCor_S^\natural}(X, \mathbb{L}^d)^* \cong \mathcal{H}om_{dgPrCor_k^\natural}(p_*X, \mathbb{L}^d)^*,$$

induced by the isomorphisms

$$(\mathbb{P}^1)^d_S \times_S \square^n_S \times_S X \cong (\mathbb{P}^1_k)^d \times_k \square^n_k \times_k p_*X.$$

On the other hand, the complex $\mathcal{H}om_{dgPrCor_k^\natural}(p_*X, \mathbb{L}^d)^*$ is just a cubical version of the weight d Friedlander-Suslin complex of the k-scheme p_*X, hence we have isomorphisms [23]

$$H^n(\mathcal{H}om_{dgPrCor_k^\natural}(p_*X, \mathbb{L}^d)^*) \cong H^{2d+n}(X, \mathbb{Z}(d)) \cong \mathrm{CH}^d(X, -n).$$

Since the higher Chow groups of smooth k-schemes satisfy the Mayer-Vietoris property for the Zariski topology [3], the presheaf of complexes on S

$$U \mapsto \mathcal{H}om_{dgPrCor_U^\natural}(X \times_S U, \mathbb{L}^d)^*$$

satisfies the Mayer-Vietoris property on S_{Zar}. Thus, by the Brown-Gersten theorem [2] the natural map

$$\mathcal{H}om_{dgPrCor_S^\natural}(X, \mathbb{L}^d)^* \to R\Gamma(S, [U \mapsto \mathcal{H}om_{dgPrCor_U^\natural}(X \times_S U, \mathbb{L}^d)^*])$$

is a quasi-isomorphism. Since $R\Gamma(S, [U \mapsto \mathcal{H}om_{dgPrCor_U^\natural}(X \times_S U, \mathbb{L}^d)^*])$ is by definition equal to $\mathcal{H}om_{R\Gamma(S, \underline{dgSmMot}_S)^\natural}(X, \mathbb{L}^d)^*$, the result is proven. $\square$

Following remark 1.15, the tensor structure on Cor_S extends to an action of Cor_S on the presheaf of DG categories $\underline{dgSmMot}_S$, giving us an action of $Cor_S^\natural$ on the DG categories $R\Gamma(S, \underline{dgSmMot}_S)^\natural$ and $dgSmMot_S^{\mathrm{eff}\natural}$. Thus, we have an action of $Cor_S^\natural$ on the triangulated category $SmMot^{\mathrm{eff}}_{gm}(S)$:

$$\otimes : Cor_S^\natural \otimes SmMot^{\mathrm{eff}}_{gm}(S) \to SmMot^{\mathrm{eff}}_{gm}(S).$$

In particular, each object $A \in Cor_S^\natural$ gives an exact functor

$$(-) \otimes A : SmMot^{\mathrm{eff}}_{gm}(S) \to SmMot^{\mathrm{eff}}_{gm}(S)$$

sending a morphism $f : X \to Y$ in $SmMot^{\mathrm{eff}}_{gm}(S)$ to $f \otimes \mathrm{id} : X \otimes A \to Y \otimes A$.

Theorem 5.10 *Let S be a regular scheme, essentially of finite type over a field k.*

1. Let X, Y and Z be in $\mathbf{Proj}/S$, $d = dim_S Y$. Then there is a natural quasi-isomorphism

$$\mathcal{H}om_{R\Gamma(S, \underline{dgSmMot}_S)}(X, Y \times_S Z)^* \cong \mathcal{H}om_{R\Gamma(S, \underline{dgSmMot}_S)^\natural}(X \times_S Y, Z \otimes \mathbb{L}^d)^*$$

2. Let X and Y be in $\mathbf{Proj}/S$, $d = dim_S Y$, and take Z in $\mathbf{Proj}/k$. Then there is a natural quasi-isomorphism

$$\mathcal{H}om_{R\Gamma(S, \underline{dgSmMot}_S)}(X, Y \times_k Z)^* \cong \mathcal{H}om_{dgSmMot_k^\natural}(p_*(X \times_S Y), Z \otimes \mathbb{L}^d)^*$$

3. Let X and Y be in $\mathbf{Proj}/S$. Then the map

$$\otimes \mathbb{L} : \mathcal{H}om_{R\Gamma(S, \underline{dgSmMot}_S)}(X, Y)^* \to \mathcal{H}om_{R\Gamma(S, \underline{dgSmMot}_S)^\natural}(X \otimes \mathbb{L}, Y \otimes \mathbb{L})^*$$

is a quasi-isomorphism.

Proof For $X, Y \in \mathbf{Proj}/S$, let $\mathcal{H}om_{\underline{dgSmMot}_S}(X,Y)$ denote the Zariski sheaf on S associated to the presheaf

$$U \mapsto \mathcal{H}om_{dgSmMot_U}(X_U, Y_U)^*;$$

extend the notation to define the sheaf $\mathcal{H}om_{\underline{dgSmMot}^\natural_S}(X, Y\otimes\mathbb{L}^d)$, etc.

From corollary 5.6 and remark 5.7, we have isomorphisms in $D^-(Sh_S^{\mathrm{Zar}})$

$$\mathcal{H}om_{\underline{dgSmMot}_S}(X, Y\times_S Z)^* \cong \mathcal{H}om_{\underline{dgSmMot}^\natural_S}(X\times_S Y, Z\otimes\mathbb{L}^d)^*$$

and

$$\otimes\mathbb{L}: \mathcal{H}om_{\underline{dgSmMot}_S}(X,Y)^* \to \mathcal{H}om_{\underline{dgSmMot}^\natural_S}(X\otimes\mathbb{L}, Y\otimes\mathbb{L})^*$$

These induce isomorphisms (in $D(\mathbf{Ab})$) after applying $\Gamma(S, G^*(-)) \cong R\Gamma(S,-)$, proving (1) and (3). (2) follows from (1), noting that we have the isomorphism of $\mathcal{H}om_{\underline{dgSmMot}_S}(X\times_S Y, S\times_k Z\otimes\mathbb{L}^d)^*$ with the constant sheaf (on S_{Zar}) with value $\mathcal{H}om_{dgSmMot^\natural_k}(p_*(X\times_S Y), Z\otimes\mathbb{L}^d)^*$, and that for a constant sheaf of complexes $\mathcal{F}$ on S_{Zar}, the natural map

$$\Gamma(S,\mathcal{F}) \to R\Gamma(S,\mathcal{F})$$

is an isomorphism in $D(\mathbf{Ab})$. □

Definition 5.11 The DG category of motives over S, $dgSmMot_S$, is the DG category formed by inverting $\otimes\mathbb{L}$ on $dgSmMot_S^{\mathrm{eff}\natural}$. The triangulated category of motives over S, $SmMot_{gm}(S)$ is the triangulated category formed by inverting $\otimes\mathbb{L}$ on $SmMot^{\mathrm{eff}}_{gm}(S)$ (we will see in corollary 5.14 below that $SmMot_{gm}(S)$ has a natural triangulated structure).

Explicitly, $dgSmMot_S$ has objects $X\otimes\mathbb{L}^n$, $n\in\mathbb{Z}$, X in $dgSmMot_S^{\mathrm{eff}\natural}$, with

$$\mathcal{H}om_{dgSmMot_S}(X\otimes\mathbb{L}^n, Y\otimes\mathbb{L}^m)^* := \varinjlim_N \mathcal{H}om_{dgSmMot_S^{\mathrm{eff}\natural}}(X\otimes\mathbb{L}^{N+n}, Y\otimes\mathbb{L}^{N+m})^*$$

Similarly, $SmMot_{gm}(S)$ has objects $X\otimes\mathbb{L}^n$, $n\in\mathbb{Z}$, X in $SmMot^{\mathrm{eff}}_{gm}(S)$, with

$$\mathcal{H}om_{SmMot_{gm}(S)}(X\otimes\mathbb{L}^n, Y\otimes\mathbb{L}^m)^* := \varinjlim_N \mathcal{H}om_{SmMot^{\mathrm{eff}}_{gm}(S)}(X\otimes\mathbb{L}^{N+n}, Y\otimes\mathbb{L}^{N+m})^*.$$

Remarks 5.12 We can also invert $\otimes\mathbb{L}$ on $dgPrCor^\natural_U$ for all open $U\subset S$, giving us the presheaf $\underline{dgPrCor}^\natural_S[\otimes\mathbb{L}^{-1}]$ on S_{Zar}, and the associated DG category $R\Gamma(S, \underline{dgPrCor}^\natural_S[\otimes\mathbb{L}^{-1}])$. We have an isomorphism of DG categories

$$R\Gamma(S, \underline{dgPrCor}^\natural_S[\otimes\mathbb{L}^{-1}]) \cong R\Gamma(S, \underline{dgPrCor}_S)^\natural[\otimes\mathbb{L}^{-1}],$$

giving us the isomorphism of DG categories

$$dgSmMot_S \cong \mathcal{C}^b(R\Gamma(S, \underline{dgPrCor}^\natural_S[\otimes\mathbb{L}^{-1}])).$$

2. Inverting $\otimes\mathbb{L}$ on the various Lefschetz/Tate categories of definition 4.9 gives us full sub-DG categories $dgLCor_S$, $dgTMot(S)$ of $R\Gamma(S, \underline{dgPrCor}^\natural_S[\otimes\mathbb{L}^{-1}])$ and $dgSmMot_S$, resp., with

$$dgTMot(S) \cong \mathcal{C}^b(dgLCor_S),$$

and the full triangulated subcategory $DTMot(S)$ of $SmMot_{gm}(S)$.

With $\mathbb{Q}$-coefficients we have the analogous DG tensor sub-categories $dgLCor^{\rm alt}_{S\mathbb{Q}}$, $dgTMot_{S\mathbb{Q}}$ of $R\Gamma(S, \underline{dgSmMot}^{\rm alt}_S)^{\otimes\natural}[\otimes\mathbb{L}^{-1}])$, $dgSmMot_S$, resp., with

$$dgTMot_{S\mathbb{Q}} \cong \mathcal{C}^b(dgLCor^{\rm alt}_{S\mathbb{Q}}),$$

and the full tensor triangulated subcategory $DTMot(S)_{\mathbb{Q}}$ of $SmMot_{gm}(S)_{\mathbb{Q}}$.

Theorem 5.13 *For $X, Y \in dgSmMot^{\rm eff\natural}_S$, the natural map*

$$\mathcal{H}om_{dgSmMot^{\rm eff\natural}_S}(X,Y)^* \to \mathcal{H}om_{dgSmMot_S}(X,Y)^*$$

is a quasi-isomorphism, and the natural map

$$\mathrm{Hom}_{SmMot^{\rm eff}_{gm}(S)}(X, Y[n]) \to \mathrm{Hom}_{SmMot_{gm}(S)}(X, Y[n])$$

is an isomorphism for all n. Furthermore, the isomorphism

$$H^n\mathcal{H}om_{dgSmMot^{\rm eff\natural}_S}(X,Y)^* \cong \mathrm{Hom}_{SmMot^{\rm eff}_{gm}(S)^\natural}(X, Y[n])$$

induces an isomorphism

$$H^n\mathcal{H}om_{dgSmMot_S}(X\otimes\mathbb{L}^p, Y\otimes\mathbb{L}^m)^* \cong \mathrm{Hom}_{SmMot_{gm}(S)}(X\otimes\mathbb{L}^p, Y\otimes\mathbb{L}^m[n]).$$

for each $n, m, p \in \mathbb{Z}$.

Proof For $X, Y \in \mathbf{Proj}/S$, the first assertion is theorem 5.10(2). Since $dgSmMot^{\rm eff}_S$ is generated by $\mathbf{Proj}/S$ by taking translation, cone sequences and isomorphisms, the first assertion for $X, Y \in dgSmMot^{\rm eff}_S$ follows; this immediately implies the result for $dgSmMot^{\rm eff\natural}_S$.

Since cohomology commutes with filtered inductive limits, we have the isomorphism

$$H^n\mathcal{H}om_{dgSmMot_S}(X,Y)^* \cong \mathrm{Hom}_{SmMot_{gm}(S)}(X, Y[n]).$$

as claimed. Thus the first assertion implies the second. □

As immediate consequence we have

Corollary 5.14 *1. $SmMot_{gm}(S)$ is equivalent to the idempotent completion of $H^0dgSmMot_S$,*

$$SmMot_{gm}(S) \sim (H^0dgSmMot_S)^\natural \sim \mathrm{per}(R\Gamma(S, \underline{dgPrCor}^\natural_S[\otimes\mathbb{L}^{-1}])).$$

In particular, $SmMot_{gm}(S)$ has the natural structure of a triangulated category.

2. The canonical functor $SmMot^{\rm eff}_{gm}(S) \to SmMot_{gm}(S)$ is an exact fully faithful embedding.

Remark 5.15 We can also invert $\otimes\mathbb{L}$ on the DG tensor categories $dgCor^{\rm alt\natural}_S$, $R\Gamma(S, \underline{dgPrCor}^{\rm alt}_S)^{\otimes\natural}$ and $dgSmMot^{\rm eff\natural}_{S\mathbb{Q}}$, and on the tensor triangulated category $SmMot^{\rm eff}_{gm}(S)_{\mathbb{Q}}$. Setting

$$dgSmMot_{S\mathbb{Q}} := dgSmMot^{\rm eff\natural}_{S\mathbb{Q}}[\otimes\mathbb{L}^{-1}],\ SmMot_{gm}(S)_{\mathbb{Q}} := SmMot^{\rm eff}_{gm}(S)_{\mathbb{Q}}[\otimes\mathbb{L}^{-1}],$$

this gives us the DG tensor functor $dgSmMot^{\rm eff\natural}_{S\mathbb{Q}} \to dgSmMot_{S\mathbb{Q}}$, and the exact tensor functor $SmMot^{\rm eff}_{gm}(S)_{\mathbb{Q}} \to SmMot_{gm}(S)_{\mathbb{Q}}$. The analogs of theorem 5.13 and corollary 5.14 hold in this setting. Similarly, the analog of theorem 5.13 and corollary 5.14 hold for the respective subcategories of Tate motives.

5.3 Chow motives. We recall the definition of the category of Chow motives over S.

Definition 5.16 Let S be a regular scheme. Let $Ch\tilde{M}ot^{\text{eff}}(S)$ be the category with the same objects as $\mathbf{Proj}/S$, and with morphisms (for $X \to S$ of pure dimension d_X over S)

$$\mathrm{Hom}_{Ch\tilde{M}ot^{\text{eff}}(S)}(X, Y) := \mathrm{CH}_{d_X}(X \times_S Y).$$

The composition law is the usual one of composition of correspondence classes: for $W \in \mathrm{Hom}_{Ch\tilde{M}ot^{\text{eff}}(S)}(X, Y)$, $W' \in \mathrm{Hom}_{Ch\tilde{M}ot^{\text{eff}}(S)}(Y, Z)$, define

$$W' \circ W := p_{13*}(p_{12}^*(W) \cdot_{XYZ} p_{23}^*(W')),$$

where p_{ij} is the projection of $X \times_S Y \times_S Z$ on the ij factors. The operation of product over S makes $Ch\tilde{M}ot^{\text{eff}}(S)$ a tensor category. Sending $f : X \to Y$ to the graph of f defines a functor

$$\tilde{m}_S : \mathbf{Proj}/S \to Ch\tilde{M}ot^{\text{eff}}(S).$$

The category $ChMot^{\text{eff}}(S)$ of *effective Chow motives over* S is the idempotent completion of $Ch\tilde{M}ot^{\text{eff}}(S)$. We let $\mathbb{L} = (\mathbb{P}^1, 1 - i_\infty \circ p)$, and define the category of Chow motives over S as

$$ChMot^{\text{eff}}(S) := Ch\tilde{M}ot^{\text{eff}}(S)[\otimes \mathbb{L}^{-1}].$$

Take $X, Y \in \mathbf{Proj}/S$. By theorem 5.10 and [23], we have the isomorphism

$$\begin{aligned}\mathrm{Hom}_{SmMot_{gm}^{\text{eff}}(S)}(X, Y) &= H^0\mathcal{H}om_{R\Gamma(S, \underline{dgSmMot}_S)}(X, Y)^* \\ &\cong H^0\mathcal{H}om_{dgSmMot_k}(p_*(X \times_S Y), \mathbb{L}^{d_Y})^* = H^{2d_Y}(X \times_S Y, \mathbb{Z}(d_Y)) \\ &\cong \mathrm{CH}_{d_X}(X \times_S Y) = \mathrm{Hom}_{ChMot^{\text{eff}}(S)}(X, Y),\end{aligned}$$

which we denote as

$$\varphi_{X,Y} : \mathrm{Hom}_{SmMot_{gm}^{\text{eff}}(S)}(X, Y) \to Hom_{ChMot^{\text{eff}}(S)}(X, Y).$$

Lemma 5.17 *φ_{**} respects the composition: for $X, Y, Z \in \mathbf{Proj}/S$,*

$$f \in \mathrm{Hom}_{SmMot_{gm}^{\text{eff}}(S)}(X, Y),\ g \in \mathrm{Hom}_{SmMot_{gm}^{\text{eff}}(S)}(Y, Z),$$

we have

$$\varphi_{X,Z}(g \circ f) = \varphi_{Y,Z}(g) \circ \varphi_{X,Y}(f).$$

Proof If f is in the image of $z_{equi}^S(Y, 0)(X) \to \mathrm{Hom}_{SmMot_{gm}^{\text{eff}}(S)}(X, Y)$ and g is in the image of $z_{equi}^S(Z, 0)(Y) \to \mathrm{Hom}_{SmMot_{gm}^{\text{eff}}(S)}(Y, Z)$, the result is obvious, so it suffices to show that

$$z_{equi}^S(Y, 0)(X) \to \mathrm{Hom}_{SmMot_{gm}^{\text{eff}}(S)}(X, Y) = \mathrm{CH}_{d_X}(X \times_S Y)$$

is surjective for all $X, Y \in \mathbf{Proj}/S$; this is lemma 5.18 below. □

Lemma 5.18 *Take $X \in \mathbf{Sm}/S$ of dimension d_X over S. For $Y \in \mathbf{Proj}/S$, , the map*

$$z_{equi}^S(Y, r)(X) \to \mathrm{CH}_{d_X + r}(X \times_S Y),$$

sending a cycle W on $X \times_S Y$ to its cycle class, is surjective for all $r \geq 0$.

Proof Take a locally closed immersion $X \times_S Y$ in a projective space $\mathbb{P}^N_k$ and let $T \subset \mathbb{P}^N_k$ be the closure of $X \times_S Y$. Let γ be a dimension $d_X + r$ cycle on $X \times_S Y$ and let $\bar\gamma$ be the closure of γ on T. By [10, theorem 1.7], $\bar\gamma$ is rationally equivalent to a cycle $\bar\gamma'$ such that each component of $\bar\gamma'$ intersects the cycle $x \times Y$ properly on T, for each point $x \in X$ (note that $x \times Y$ is closed on $T_{k(x)}$ and is contained in the smooth locus of $T_{k(x)}$). Thus, the restriction γ' of $\bar\gamma'$ to $X \times_S Y$ is rationally equivalent to γ and is in the image of $z^S_{equi}(Y, r)(X) \to \mathrm{CH}_{d_X+r}(X \times_S Y)$, completing the proof. $\square$

Thus, we have shown

Proposition 5.19 *1. There is a fully faithful embedding* $\psi^{\mathrm{eff}} : ChMot^{\mathrm{eff}}(S) \to SmMot^{\mathrm{eff}}_{gm}(S)$*, such that the diagram*

$$\begin{array}{ccc} \mathbf{Proj}/S & \longrightarrow & ChMot^{\mathrm{eff}}(S) \\ & \searrow & \downarrow{\scriptstyle \psi^{\mathrm{eff}}} \\ & & SmMot^{\mathrm{eff}}_{gm}(S) \end{array}$$

commutes, and such that

$$\psi^{\mathrm{eff}}(X, Y) : \mathrm{Hom}_{ChMot^{\mathrm{eff}}(S)}(X, Y) \to \mathrm{Hom}_{SmMot^{\mathrm{eff}}_{gm}(S)}(X, Y)$$

is the inverse of the isomorphism $\varphi_{X,Y}$*.*

2. ψ^{eff} *extends to a fully faithful embedding* $\psi : ChMot(S) \to SmMot_{gm}(S)$*.*

3. The maps ψ^{eff} *and* ψ *are compatible with the* $\otimes$ *action of* Cor_S*. Also, the maps* $\psi^{\mathrm{eff}} : ChMot^{\mathrm{eff}}(S) \to SmMot^{\mathrm{eff}}_{gm}(S)_{\mathbb{Q}}$*,* $\psi^{\mathrm{eff}} : ChMot(S) \to SmMot_{gm}(S)_{\mathbb{Q}}$ *induced by* ψ^{eff} *and* ψ *are tensor functors.*

Lemma 5.20 *For* $X \in \mathbf{Proj}/S$*, there are maps*

$$\tilde\delta_X : \mathbb{L}^d \to X \otimes X;\ \tilde\epsilon_X : X \otimes X \otimes \mathbb{L}^d$$

in $SmMot^{\mathrm{eff}}_{gm}(S)$ *such that the composition*

$$X \otimes \mathbb{L}^d \xrightarrow{\mathrm{id}\otimes\delta_X} X \otimes X \otimes X \xrightarrow{\epsilon_X\otimes\mathrm{id}} \mathbb{L}^d \otimes X \xrightarrow{\tau} X \otimes \mathbb{L}^d$$

is the identity.

Proof By proposition 5.19, it suffices to construct $\tilde\delta_X$ and $\tilde\epsilon_X$ in $ChMot(S)$. Identifying $\mathrm{Hom}_{ChMot(S)}(\mathbb{L}^d, X \otimes X)$ with the appropriate summand of $\mathrm{CH}_d((\mathbb{P}^1)^d \otimes X \times_S X)$, $\tilde\delta_X$ is represented by the cycle $0 \times \Delta_X$. Similarly

$$\tilde\epsilon_X \in \mathrm{Hom}_{ChMot(S)}(X \otimes X, \mathbb{L}^d) \subset \mathrm{CH}_{2d}(X \times_S X \times (\mathbb{P}^1)^d)$$

is represented by the cycle $\Delta_X \times (\mathbb{P}^1)^d$; the desired identity is a straightforward computation. $\square$

Finally, we have the duality theorem:

Theorem 5.21 *Sending* $X \in \mathbf{Proj}/S$ *of dimension* d *over* S *to* $X^D := X \otimes \mathbb{L}^{-d}$ *extends to an exact duality*

$$D : SmMot_{gm}(S)^{\mathrm{op}}_{\mathbb{Q}} \to SmMot_{gm}(S)_{\mathbb{Q}}.$$

Proof For $X \in \mathbf{Proj}/S$, the maps $\tilde{\delta}_X$ and $\tilde{\epsilon}_X$ define maps

$$\delta_X : S \to X \otimes (X \otimes \mathbb{L}^{-d_X});\ \epsilon_X : (X \otimes \mathbb{L}^{-d}) \otimes X \to S.$$

The identity of lemma 5.20 tells us that $(X \otimes \mathbb{L}^{-d}, \delta_X, \epsilon_X)$ defines a dual of X in $SmMot_{gm}(S)_{\mathbb{Q}}$. Since $SmMot_{gm}(S)_{\mathbb{Q}}$ is generated by the Tate twists of objects in $\mathbf{Proj}/S$ (as an idempotently complete triangulated category), this implies that every object of $SmMot_{gm}(S)_{\mathbb{Q}}$ admits a dual (see e.g. [20, Part I, Chap. IV, theorem 1.2.5]). Since a dual of an object in a tensor category is unique up to unique isomorphism, this suffices to define the exact tensor duality involution D. □

6 Smooth motives and motives over a base

Cisinski-Déglise have defined a tensor triangulated category of effective motives over a base-scheme S, $DM^{\text{eff}}(S)$, and a tensor triangulated category of motives over S, $DM(S)$, with an exact tensor functor $DM^{\text{eff}}(S) \to DM(S)$ that inverts $\otimes\mathbb{L}$. In this section, we show how to define exact functor

$$\rho_S^{\text{eff}} : SmMot_{gm}^{\text{eff}}(S) \to DM^{\text{eff}}(S),\ \rho_S : SmMot_{gm}(S) \to DM(S)$$

which give equivalences of $SmMot_{gm}^{\text{eff}}(S)$, $SmMot_{gm}(S)$ with the full triangulated subcategories of $DM^{\text{eff}}(S)$ and $DM(S)$ generated by the motives of smooth projective S schemes, resp. the Tate twists of smooth projective S-schemes. Working with $\mathbb{Q}$ coefficients, we have the same picture, with ρ_S^{eff}, ρ_S replaced by exact tensor functors

$$\rho_{S\mathbb{Q}}^{\text{eff}} : SmMot_{gm}^{\text{eff}}(S)_{\mathbb{Q}} \to DM^{\text{eff}}(S)_{\mathbb{Q}}\ \rho_{S\mathbb{Q}} : SmMot_{gm}(S)_{\mathbb{Q}} \to DM(S)_{\mathbb{Q}},$$

giving analogous equivalences.

6.1 Cisinski-Déglise categories of motives. We summarize the construction of the category $DM^{\text{eff}}(S)$ of effective motives over S, and the category $DM(S)$ of motives over S, from [5]. Although S is allowed to be a quite general scheme in [5], we restrict ourselves to the case of a base-scheme S that is separated, smooth and essentially of finite type over a field.

Define the abelian category of *presheaves with transfer* on $\mathbf{Sm}/S$, $PST(S)$, as the category of additive presheaves of abelian groups on Cor_S, "additive" meaning taking disjoint union to direct sum. We have the representable presheaves $\mathbb{Z}_S^{tr}(Z)$ for $Z \in \mathbf{Sm}/S$ by $\mathbb{Z}_S^{tr}(Z)(X) := Cor_S(X, Z)$ and pull-back maps given by the composition of correspondences.

One gives the category of complexes $C(PST(S))$ the *Nisnevich local* model structure (which we won't need to specify). The homotopy category is equivalent to the (unbounded) derived category $D(Sh_{\text{Nis}}^{tr}(S))$, where $Sh_{\text{Nis}}^{tr}(S)$ is the full subcategory of $PST(S)$ consisting of the presheaves with transfer which restrict to Nisnevich sheaves on $\mathbf{Sm}/S$.

The operation

$$\mathbb{Z}_S^{tr}(X) \otimes_S^{tr} \mathbb{Z}_S^{tr}(X') := \mathbb{Z}_S^{tr}(X \times_S X')$$

extends to a tensor structure $\otimes_S^{tr}$ making $PST(S)$ a tensor category: one forms the *canonical left resolution* $\mathcal{L}(\mathcal{F})$ of a presheaf $\mathcal{F}$ by taking the canonical surjection

$$\mathcal{L}_0(\mathcal{F}) := \bigoplus_{X \in \mathbf{Sm}/S, s \in \mathcal{F}(X)} \mathbb{Z}_S^{tr}(X) \xrightarrow{\varphi_0} \mathcal{F}$$

setting $\mathcal{F}_1 := \ker \varphi_0$ and iterating. One then defines

$$\mathcal{F} \otimes_S^{tr} \mathcal{G} := H_0(\mathcal{L}(\mathcal{F}) \otimes_S^{tr} \mathcal{L}(\mathcal{G}))$$

noting that $\mathcal{L}(\mathcal{F}) \otimes_S^{tr} \mathcal{L}(\mathcal{G})$ is defined since both complexes are degreewise direct sums of representable presheaves.

The restriction of $\otimes_S^{tr}$ to the subcategory of cofibrant objects in $C(Sh_{\mathrm{Nis}}^{tr}(S))$ induces a tensor operation $\otimes_S^L$ on $D(Sh_{\mathrm{Nis}}^{tr}(S))$ which makes $D(Sh_{\mathrm{Nis}}^{tr}(S))$ a tensor triangulated category.

Definition 6.1 ([5, definition 10.1]) $DM^{\mathrm{eff}}(S)$ is the localization of the triangulated category $D(Sh_{\mathrm{Nis}}^{tr}(S))$ with respect to the localizing category generated by the complexes $\mathbb{Z}_S^{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_S^{tr}(X)$. Denote by $m_S(X)$ the image of $\mathbb{Z}_S^{tr}(X)$ in $DM^{\mathrm{eff}}(S)$.

Remark 6.2 1. $DM^{\mathrm{eff}}(S)$ is a tensor triangulated category with tensor product $\otimes_S$ induced from the tensor product $\otimes_S^L$ via the localization map

$$Q_S : D(Sh_{\mathrm{Nis}}^{tr}(S)) \to DM^{\mathrm{eff}}(S),$$

and satisfying $m_S(X) \otimes_S m_S(Y) = m_S(X \times_S Y)$.

2. There is a model category $C(PST_{\mathbb{A}^1}(S))$ having $C(PST(S))$ as underlying category, defined as the left Bousfield localization of $C(PST(S))$ with respect to the following complexes

1. For each *elementary Nisnevich square* with $X \in \mathbf{Sm}/S$:

$$\begin{array}{ccc} W & \hookrightarrow & X' \\ \| & & \downarrow f \\ W & \hookrightarrow & X \end{array}$$

one has the complex

$$\mathbb{Z}_S^{tr}(X' \setminus W) \to \mathbb{Z}_S^{tr}(X \setminus W) \oplus \mathbb{Z}_S^{tr}(X') \to \mathbb{Z}_S^{tr}(X)$$

Recall that the square above is an elementary Nisnevich square if f is étale, the horizontal arrows are closed immersions of reduced schemes and the square is cartesian.

2. For $X \in \mathbf{Sm}/S$, one has the complex $\mathbb{Z}_S^{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_S^{tr}(X)$.

The homotopy category of $C(PST_{\mathbb{A}^1}(S))$ is equivalent to $DM^{\mathrm{eff}}(S)$.

Definition 6.3 Let T^{tr} be the presheaf with transfers

$$T^{tr} := \mathrm{coker}(\mathbb{Z}_S^{tr}(S) \xrightarrow{i_{\infty *}} \mathbb{Z}_S^{tr}(\mathbb{P}_S^1))$$

and let $\mathbb{Z}_S(1)$ be the image in $DM^{\mathrm{eff}}(S)$ of $T^{tr}[-2]$. Let

$$\otimes T^{tr} : C(PST(S)) \to C(PST(S))$$

be the functor $C \mapsto C \otimes_S^{tr} T^{tr}$.

Let $\mathbf{Spt}_{T^{tr}}(S)$ be the model category of $\otimes T^{tr}$ spectra in $C(PST_{\mathbb{A}^1}(S))$, i.e., an object is a sequence $E := (E_0, E_1, \ldots)$, $E_n \in C(PST(S))$, with bonding maps

$$\epsilon_n : E_n \otimes_S^{tr} T^{tr} \to E_{n+1}.$$

Morphisms are given by sequences of maps in $C(PST(S))$ which strictly commute with the respective bonding maps.

The model structure on the category of T^{tr}-spectra is the *stable model structure*, defined by following the construction of Hovey [14]. One first defines the projective model structure on $\mathbf{Spt}_{T^{tr}}(S)$, with weak equivalences maps $E \to F$ for which $E_n \to F_n$ is a weak equivalence for all n; let $\mathcal{H}_{proj}\mathbf{Spt}_{T^{tr}}(S)$ be the associated homotopy category. Next, for each $E \in \mathbf{Spt}_{T^{tr}}(S)$ there is a canonical fibrant model $E \to E^f$, where $E^f := (E_0^f, E_1^f, \ldots)$ with each E_n^f fibrant in $C(PST_{\mathbb{A}^1}(S))$ and the map

$$E_n^f \to \mathcal{H}om(T^{tr}, E_{n+1}^f)$$

adjoint to the bonding map $E_n^f \otimes_S^{tr} T^{tr} \to E_{n+1}^f$ is a weak equivalence in the model category $C(PST_{\mathbb{A}^1}(S))$. A *stable weak equivalence* in $\mathbf{Spt}_{T^{tr}}(S)$ is a map $f : A \to B$ such that

$$f^* : \mathrm{Hom}_{\mathcal{H}_{proj}\mathbf{Spt}_{T^{tr}}(S)}(B, E^f) \to \mathrm{Hom}_{\mathcal{H}_{proj}\mathbf{Spt}_{T^{tr}}(S)}(A, E^f)$$

is an isomorphism for all E. Hovey's stable model structure on $\mathbf{Spt}_{T^{tr}}(S)$ has weak equivalences the stable weak equivalences; we refer the interested reader to [5] or [14] for a description of the cofibrations and fibrations.

Definition 6.4 The category of triangulated motives over S, $DM(S)$, is the homotopy category of $\mathbf{Spt}_{T^{tr}}(S)$ for the stable model structure.

We will use the following results from [5].

Theorem 6.5 ([5, section 10.3, corollary 6.12]) *Suppose that S is in $\mathbf{Sm}/k$ for a field k, take X in $\mathbf{Sm}/S$, and let $m_k(X)$, $m_S(X)$ denote the motives of X in $DM(k)$, $DM(S)$, respectively. Then there is a natural isomorphism*

$$\mathrm{Hom}_{DM(S)}(m_S(X), \mathbb{Z}(n)[m]) \cong \mathrm{Hom}_{DM(k)}(m_k(X), \mathbb{Z}(n)[m])$$

In addition, the natural map

$$\varinjlim_N \mathrm{Hom}_{DM^{\mathrm{eff}}(S)}(m_S(X) \otimes \mathbb{Z}(N), \mathbb{Z}(n+N)[m]) \to \mathrm{Hom}_{DM(S)}(m_S(X), \mathbb{Z}(n)[m])$$

is an isomorphism. Finally, the cancellation theorem holds in this setting: the natural map

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(S)}(m_S(X), \mathbb{Z}(n)[m])] \to \mathrm{Hom}_{DM^{\mathrm{eff}}(S)}(m_S(X) \otimes \mathbb{Z}(1), \mathbb{Z}(n+1)[m])$$

is an isomorphism.

Remarks 6.6 1. By [23] $\mathrm{Hom}_{DM(k)}(m_k(X), \mathbb{Z}(n)[m])$ is motivic cohomology in the sense of Voevodsky [24, chapter V], that is

$$\mathrm{Hom}_{DM(k)}(m_k(X), \mathbb{Z}(n)[m]) = H^m(X, \mathbb{Z}(n)) \cong \mathrm{CH}^n(X, 2n-m).$$

In fact, this follows immediately from [23] in the case of a perfect field; the general case follows by using the usual trick of viewing a field k as a limit of finitely generated extensions of the prime field k_0, and the fact that, for a projective systems of regular noetherian schemes S_α with affine transition maps, the functor

$$S_\alpha \mapsto \mathrm{Hom}_{DM(S)}(m_{S_\alpha}(X_{S_\alpha}), \mathbb{Z}(n)[m])$$

transforms the projective limit $\varprojlim S_\alpha$ to the inductive limit (see [7, prop 4.2.19]).

2. The isomorphisms in theorem 6.5 are proven as follows: Let $p : S \to \mathrm{Spec}\, k$

be the (smooth) structure morphism. The limit argument mentioned above reduces us to the case of k a perfect field. The restriction of base functor induces an exact functor

$$p_\sharp : DM^{\mathrm{eff}}(S) \to DM^{\mathrm{eff}}(k)$$

and the pull-back $X \mapsto X \times_k S$ induces an exact functor

$$p^* : DM^{\mathrm{eff}}(k) \to DM^{\mathrm{eff}}(S),$$

right adjoint to $p_\sharp$. In addition, one has $p^*(\mathbb{Z}(n)) \cong \mathbb{Z}(n)$, and $p_\sharp(m_S(X)\otimes\mathbb{Z}(N)) \cong m_k(X) \otimes \mathbb{Z}(N)$ for $X \in \mathbf{Sm}/S$, $N \in \mathbb{Z}$. We have similar functors on $DM(S)$, $DM(k)$. Thus, we have

$$\begin{aligned}\mathrm{Hom}_{DM^{\mathrm{eff}}(S)}&(m_S(X)\otimes\mathbb{Z}(N), \mathbb{Z}(n+N)[m])\\ &\cong \mathrm{Hom}_{DM^{\mathrm{eff}}(S)}(m_S(X)\otimes\mathbb{Z}(N), p^*\mathbb{Z}(n+N)[m])\\ &\cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(p_\sharp(m_S(X)\otimes\mathbb{Z}(N)), \mathbb{Z}(n+N)[m])\\ &\cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(m_k(X)\otimes\mathbb{Z}(N), \mathbb{Z}(n+N)[m]),\end{aligned}$$

and similarly with DM^{eff} replaced by DM. This already proves the first isomorphism. For the next two, the above isomorphisms reduce us to the case of $S = \mathrm{Spec}\, k$. Also, $DM^{\mathrm{eff}}_-(k)$ is a full subcategory of $DM^{\mathrm{eff}}(k)$. Finally, Voevodsky's identification of motivic cohomology with the higher Chow groups [23], together with the projective bundle formula, gives the limited version of the cancellation theorem we need to finish the proof.

6.2 Tensor structure. The tensor structure on $C(PST(S))$ induces a "tensor operation" on the spectrum category by the usual device of choosing a cofinal subset $\mathbb{N} \subset \mathbb{N}\times\mathbb{N}$, $i \mapsto (n_i, m_i)$, with $n_{i+1} + m_{i+1} = n_i + m_i + 1$ for each i: each pair of T^{tr} spectra $E := (E_0, E_1, \ldots)$ and $F := (F_0, F_1, \ldots)$ gives rise to a T^{tr} bispectrum

$$E \boxtimes^{tr}_S F := \begin{pmatrix} & \vdots & \\ \ldots & E_i \otimes^{tr}_S F_j & \ldots \\ & \vdots & \end{pmatrix}$$

with vertical and horizontal bonding maps induced by the bonding maps for E and F, respectively. The vertical bonding maps use in addition the symmetry isomorphism in $C(PST_{\mathbb{A}^1}(S))$. Finally, the choice of the cofinal $\mathbb{N} \subset \mathbb{N}\times\mathbb{N}$ converts a bispectrum to a spectrum.

Of course, this is not even associative, so one does not achieve a tensor operation on $\mathbf{Spt}_{T^{tr}}(S)$, but $\boxtimes^{tr}_S$ (on cofibrant objects) does pass to the localization $DM(S)$, and gives rise there to a tensor structure, making $DM(S)$ a tensor triangulated category. We write this tensor operation as $\otimes_S$, as before.

Remark 6.7 One can also define a "Spanier-Whitehead" category $DM^{\text{S-W}}(S)$ by inverting the functor $-\otimes T^{tr} = -\otimes \mathbb{Z}_S(1)[2]$ on $DM^{\mathrm{eff}}(S)$; this is clearly equivalent to inverting $-\otimes\mathbb{Z}_S(1)$. Concretely, $DM^{\text{S-W}}(S)$ has objects $X(n)$, $n \in \mathbb{Z}$, with morphisms

$$\mathrm{Hom}_{DM^{\text{S-W}}(S)}(X(n), Y(m)) := \varinjlim_N \mathrm{Hom}_{DM^{\mathrm{eff}}(S)}(X\otimes\mathbb{Z}_S(N+n), Y\otimes\mathbb{Z}_S(N+m)).$$

Sending X to $X(n)$ clearly defines an auto-equivalence of $DM^{\text{S-W}}(S)$.

$DM^{\text{S-W}}(S)$ inherits the structure of a triangulated category from $DM^{\mathrm{eff}}(S)$, by declaring a triangle $\mathcal{T}$ in $DM^{\text{S-W}}(S)$ to be distinguished if $\mathcal{T}(N)$ is the image of a distinguished triangle in $DM^{\text{eff}}(S)$ for some $N >> 0$. One shows that the symmetry isomorphism $\mathbb{Z}_S(1) \otimes \mathbb{Z}_S(1) \to \mathbb{Z}_S(1) \otimes \mathbb{Z}_S(1)$ is the identity, which implies that $DM^{\text{S-W}}(S)$ inherits a tensor structure from $DM^{\text{eff}}(S)$.

Since $- \otimes T^{tr}$ is isomorphic to the shift operator in $DM(S)$, this functor is invertible on $DM(S)$, hence $DM^{\text{eff}}(S) \to DM(S)$ factors through a canonical exact functor $\varphi_S : DM^{\text{S-W}}(S) \to DM(S)$. Giving $DM(S)$ the tensor structure described above, it is easy to see that φ_S is a tensor functor.

6.3 Motives and smooth motives. For $X \in \mathbf{Sm}/S$, we have the presheaf $U \mapsto C^S(X,0)(U)$ (definition 5.2), giving us the object $C^S(X)$ in $C^-(Sh^{tr}_{\text{Nis}}(S))$. Suppose X is in $\mathbf{Proj}/S$. Then by remark 5.3(1), $C^S(X)^0 = \mathbb{Z}^{tr}_S(X)$, giving the natural map

$$\iota_X : \mathbb{Z}^{tr}_S(X) \to C^S(X)$$

Lemma 6.8 *For $X \in \mathbf{Proj}/S$, ι_X is an isomorphism in $DM^{\text{eff}}(S)$.*

Proof Let $\mathbb{Z}^{tr}_S(X)^*$ be the complex which is $\mathbb{Z}^{tr}_S(X)$ in each degree $n \leq 0$, and with differential $d^n : \mathbb{Z}^{tr}_S(X)^n \to \mathbb{Z}^{tr}_S(X)^{n+1}$ the identity if n is even and 0 if n is odd. The projection $U \times \square^n \to U$ gives a map of complexes

$$\pi_U : \mathbb{Z}^{tr}(X)^*(U) \to C^S(X)(U)$$

functorially in U, hence a map of complexes of presheaves

$$\pi : \mathbb{Z}^{tr}(X)^* \to C^S(X)$$

On the other hand, we have

$$C^S(X)^n(U) \cong \text{Hom}_{C(PST(S))}(\mathbb{Z}^{tr}(U \times \square^n), \mathbb{Z}^{tr}(X))$$

and, in degree n π is the map induced on $\text{Hom}_{C(PST(S))}(-, \mathbb{Z}^{tr}(X))$ by

$$p_* : \mathbb{Z}^{tr}(U \times \square^n) \to \mathbb{Z}^{tr}(U).$$

As $\mathbb{Z}^{tr}(U)$ is cofibrant for all $U \in \mathbf{Sm}/S$ and p_* is a weak equivalence (both in the model category $C(PST_{\mathbb{A}^1}(S)))$ it follows that $\pi : \mathbb{Z}^{tr}(X)^* \to C^S(X)$ is an isomorphism in the homotopy category $DM^{\text{eff}}(S)$. Since ι factors through π via the homotopy equivalence

$$\iota_0 : \mathbb{Z}^{tr}(X) \to \mathbb{Z}^{tr}(X)^*$$

ι is an isomorphism as well. $\square$

Let $G^*_S : C^-(Sh^{tr}_{\text{Nis}}(S)) \to C(Sh^{tr}_{\text{Nis}}(S))$ be the Godement resolution functor, with respect to S_{Zar}. Concretely, for $\mathcal{F} \in Sh^{tr}_{\text{Nis}}(S)$, $G^0_S(\mathcal{F})$ is the sheaf

$$G^0_S(\mathcal{F})(U) := \prod_{s \in S} \mathcal{F}(U \otimes_S \mathcal{O}_{S,s}).$$

where we define $\mathcal{F}(U \otimes_S \mathcal{O}_{S,s})$ as the limit of $\mathcal{F}(U \otimes_S V)$, where V runs over the Zariski open neighborhoods of s in S. We have the natural transformation $\text{id} \to G_S$ induced by the inclusion $\mathcal{F} \to G^0_S\mathcal{F}$. Since $\mathcal{F} \to G^*_S\mathcal{F}$ is a Zariski local weak equivalence, this map induces an isomorphism in $D(Sh^{tr}_{\text{Nis}})$. Thus, lemma 6.8 gives

Lemma 6.9 *For each $X \in \mathbf{Proj}/S$, the composition*

$$\mathbb{Z}^{tr}_S(X) \xrightarrow{\iota_X} C^S(X) \to G^*_S C^S(X)$$

define an isomorphism $m_S(X) \cong G^*_S C^S(X)$ in $DM^{\text{eff}}(S)$.

Remark 6.10 Since S has finite Krull dimension, say d, Grothendieck's theorem [12] tells us that S has Zariski cohomological dimension $\leq d$. Thus, for $\mathcal{F} \in C^-(Sh^{tr}_{\mathrm{Nis}}(S))$, $G_S(\mathcal{F})$ is cohomologically bounded above. Therefore, the composition of G_S with the quotient functor $C(Sh^{tr}_{\mathrm{Nis}}(S)) \to D(Sh^{tr}_{\mathrm{Nis}}(S))$ factors (up to natural isomorphism) through a modified Godement resolution

$$G_S^- : C^-(Sh^{tr}_{\mathrm{Nis}}(S)) \to D^-(Sh^{tr}_{\mathrm{Nis}}(S)).$$

Remark 6.11 One can also define categories of motives with coefficients; for simplicity, we restrict to the case of a coefficient ring A being a localization of $\mathbb{Z}$. One simply replaces the category $PST(S)$ with the category $PST(S)_A$ of additive presheaves of A-modules on Cor_S, and then follows the same procedure as above, forming the tensor triangulated category of effective S-motives with A coefficients, $DM^{\mathrm{eff}}(S)_A$, and tensor triangulated category of S-motives with A coefficients, $DM(S)_A$. All the results described above for $DM^{\mathrm{eff}}(S)$ and $DM(S)$ hold for $DM^{\mathrm{eff}}(S)_A$ and $DM(S)_A$, suitably modified to reflect the A-module structure. We note that $DM^{\mathrm{eff}}(S)_A$ and $DM(S)_A$ are idempotently complete.

We now proceed to construct an exact functor

$$\rho_S : SmMot^{\mathrm{eff}}_{gm}(S) \to DM^{\mathrm{eff}}(S).$$

For this, consider the presheaf of DG categories $\underline{dgCor}_S$ on S_{Zar}

$$U \mapsto dgCor_U.$$

Recall that $dgCor_U$ has objects $X \in \mathbf{Sm}/U$, and the full subcategory with objects $X \in \mathbf{Proj}/U$ is our category $dgPrCor_U$. We have the "Godement extension" $R\Gamma(S, \underline{dgCor}_S)$ with objects $X \in \mathbf{Sm}/S$, containing $R\Gamma(S, \underline{dgPrCor}_S)$ as the full DG subcategory with objects $X \in \mathbf{Proj}/S$.

Take $X \in \mathbf{Sm}/S$. By construction, the presheaf on $\mathbf{Sm}/S$ of Hom-complexes

$$U \mapsto \mathcal{H}om_{R\Gamma(S,\underline{dgCor}_S)}(U, X)^*$$

is just $G_S^* C^S(X)$. Thus, for $X, Y \in \mathbf{Sm}/S$, the composition law in $R\Gamma(S, \underline{dgCor}_S)$ defines a natural map

$$\tilde{\rho}_{X,Y} : \mathcal{H}om_{R\Gamma(S,\underline{dgCor}_S)}(X, Y)^* \to \mathcal{H}om_{C(Sh^{tr}_{\mathrm{Nis}}(S))}(G_S^* C^S(X), G_S^* C^S(Y)).$$

This defines for us the functor of DG categories

$$\tilde{\rho} : R\Gamma(S, \underline{dgCor}_S) \to C(Sh^{tr}_{\mathrm{Nis}}(S)).$$

Applying the functor $\mathcal{K}^b$ and composing with the total complex functor

$$\mathrm{Tot} : \mathcal{K}^b(C(Sh^{tr}_{\mathrm{Nis}}(S))) \to K(Sh^{tr}_{\mathrm{Nis}}(S))$$

and the quotient functor

$$K(Sh^{tr}_{\mathrm{Nis}}(S)) \to DM^{\mathrm{eff}}(S)$$

gives us the exact functor

$$\mathcal{K}^b(\tilde{\rho}) : \mathcal{K}^b(R\Gamma(S, \underline{dgCor}_S)) \to DM^{\mathrm{eff}}(S)$$

If we restrict to the full subcategory $\mathcal{K}^b(R\Gamma(S, \underline{dgPrCor}_S))$ of $\mathcal{K}^b(R\Gamma(S, \underline{dgCor}_S))$ and extend canonically to the idempotent completion, we have defined an exact functor

$$\rho_S^{\mathrm{eff}} : SmMot^{\mathrm{eff}}_{gm}(S) \to DM^{\mathrm{eff}}(S).$$

with $\rho_S^{\text{eff}}(X) = G_S^* C^S(X)$ for all $X \in \mathbf{Proj}/S$. The natural isomorphism $m_S(X) \to G_S^* C^S(X)$ in $DM^{\text{eff}}(S)$ constructed in lemma 6.9 shows that the diagram

$$\begin{array}{ccc} \mathbf{Proj}/S & \longrightarrow & SmMot^{\text{eff}}_{gm}(S) \\ & \searrow^{m_S} & \downarrow{\scriptstyle \rho_S^{\text{eff}}} \\ & & DM^{\text{eff}}(S) \end{array}$$

commutes up to natural isomorphism.

We note that $\rho_S^{\text{eff}}(\mathbb{L}) = T^{tr}$, hence the composition

$$SmMot^{\text{eff}}_{gm}(S) \xrightarrow{\rho_S} DM^{\text{eff}}(S) \to DM^{\text{S-W}}(S)$$

sends $-\otimes \mathbb{L}$ to the invertible endomorphism $-\otimes T^{tr}$ on $DM^{\text{S-W}}(S)$, hence this composition factors through a canonical extension

$$\rho_S^{\text{S-W}} : SmMot_{gm}(S) \to DM^{\text{S-W}}(S).$$

Let

$$\rho_S : SmMot_{gm}(S) \to DM(S).$$

be the composition of $\rho^{\text{S-W}}$ with the canonical functor $DM^{\text{S-W}}(S) \to DM(S)$.

Remark 6.12 It is easy to check that the restriction of ρ_S^{eff} to $H^0 dgCor_S$ is a tensor functor. By the surjectivity of the cycle class map (lemma 5.18) this shows that the composition

$$\rho_S^{\text{eff}} \circ \psi^{\text{eff}} : ChMot^{\text{eff}}(S) \to DM^{\text{eff}}(S)$$

is a tensor functor. Similarly,

$$\rho_S \circ \psi : ChMot(S) \to DM(S)$$

is a tensor functor.

Theorem 6.13 *Let Y be in $\mathbf{Proj}/S$, $d = dim_S Y$, and take A, B in $DM(S)$. Then for every $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\text{Hom}_{DM(S)}(A, m_S(Y) \otimes B[n]) \cong \text{Hom}_{DM(S)}(A \otimes m_S(Y), B \otimes \mathbb{Z}(d)[2d+n])$$

Proof $Y \in ChMot(S)$ has the dual $(Y \otimes \mathbb{L}^{-d}, \delta_Y, \epsilon_Y)$ and $m_S(Y) \cong \rho_S \psi_S(Y)$, hence $m_S(Y)$ has the dual $(m_S(Y)(-d)[-2d], \rho_S\psi_S(\delta_Y), \rho_S\psi_S(\epsilon_Y))$ in $DM(S)$. This gives us the desired isomorphism. □

Corollary 6.14 *The functors*

$$\rho_S^{\text{eff}} : SmMot^{\text{eff}}_{gm}(S) \to DM^{\text{eff}}(S)$$
$$\rho_S : SmMot_{gm}(S) \to DM(S).$$

are faithful embeddings and ρ_S is fully faithful.

Proof Since

$$SmMot^{\text{eff}}_{gm}(S) \to SmMot_{gm}(S)$$

is fully faithful (corollary 5.14), it suffices to show that ρ_S is fully faithful.

Take $X, Y \in \mathbf{Proj}/S$, let $d = \dim_S Y$. By theorem 6.13, we have the duality isomorphism

$$\text{Hom}_{DM(S)}(m_S(X), m_S(Y)) \cong \text{Hom}_{DM(S)}(m_S(X) \otimes m_S(Y), \mathbb{Z}^{tr}(d)[2d+n]).$$

As the construction of the duality isomorphism arises from the duality in $ChMot(S)$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{SmMot_{gm}(S)}(X, Y[n]) & \xrightarrow{\sim} & \mathrm{Hom}_{SmMot_{gm}(S)}(X \times_S Y, \mathbb{L}^d[n]) \\ \downarrow{\scriptstyle \rho_S} & & \downarrow{\scriptstyle \rho_S} \\ \mathrm{Hom}_{DM(S)}(m_S(X), m_S(Y)[n]) & \xrightarrow[\sim]{} & \mathrm{Hom}_{DM(S)}(m_S(X \times_S Y), \mathbb{Z}^{tr}(d)[2d+n]) \end{array}$$

commutes. But by theorem 6.5 and remark 6.6, we have

$$\begin{aligned} &\mathrm{Hom}_{DM(S)}(m_S(X \times_S Y), \mathbb{Z}^{tr}(d)[2d+n]) \\ &\qquad \cong \mathrm{Hom}_{DM(k)}(m_k(p_* X \times_S Y), \mathbb{Z}^{tr}(d)[2d+n]) \\ &\qquad\qquad \cong H^{2d+n}(p_* X \times_S Y, \mathbb{Z}(d)). \end{aligned}$$

It is easy to check that the natural map

$$\begin{aligned} &H^{2d+n}(p_* X \times_S Y, \mathbb{Z}(d)) \cong H^n(C^k(\mathbb{L}^d)(p_*(X \times_S Y)) \\ &\qquad = H^n(C^S(\mathbb{L}^d)(X \times_S Y)) \to \mathrm{Hom}_{DM(S)}(m_S(X \times_S Y), \mathbb{Z}^{tr}(d)[2d+n]) \end{aligned}$$

inverts this isomorphism, from which it follows that the composition

$$\begin{aligned} &H^{2d+n}(X \times_S Y, \mathbb{Z}(d)) \cong H^n(C^k(\mathbb{L}^d))(p_*(X \times_S Y)) = H^n(C^S(\mathbb{L}^d)(X \times_S Y)) \\ &\to \mathrm{Hom}_{SmMot_{gm}(S)}(X \times_S Y, \mathbb{L}^d[n]) \xrightarrow{\rho_S} \mathrm{Hom}_{DM(S)}(m_S(X \times_S Y), \mathbb{Z}^{tr}(d)[2d+n]) \\ &\qquad \cong \mathrm{Hom}_{DM(k)}(m_k(p_* X \times_S Y), \mathbb{Z}^{tr}(d)[2d+n]) \cong H^{2d+n}(X \times_S Y, \mathbb{Z}(d)) \end{aligned}$$

is the identity. Therefore,

$$\rho_S : \mathrm{Hom}_{SmMot_{gm}(S)}(X \times_S Y, \mathbb{L}^d[n]) \to \mathrm{Hom}_{DM(S)}(m_S(X \times_S Y), \mathbb{Z}^{tr}(d)[2d+n])$$

is an isomorphism. □

Definition 6.15 Let $DTM(S)$ be the full triangulated subcategory of $DM(S)$ generated by the Tate objects $\mathbb{Z}_S^{tr}(n)$, $n \in \mathbb{Z}$, and let $DTM^{\mathrm{eff}}(S)$ be the full triangulated subcategory of $DM^{\mathrm{eff}}(S)$ generated by the Tate objects $\mathbb{Z}_S^{tr}(n)$, $n \geq 0$

Restricting to the Tate categories gives

Corollary 6.16 *The restriction of ρ_S to*

$$\rho_S^{Tate} : DTMot(S) \to DM(S)$$

is a fully faithful embedding, giving an equivalence of $DTMot(S)$ with $DTM(S)$. The version with $\mathbb{Q}$-coefficients

$$\rho_S^{Tate} : DTMot(S)_{\mathbb{Q}} \to DM(S)_{\mathbb{Q}}$$

defines an equivalence of $DTMot(S)_{\mathbb{Q}}$ with $DTM(S)_{\mathbb{Q}}$ as tensor triangulated categories.

Similarly, we have the equivalences

$$\rho_S^{\mathrm{eff}\,Tate} : DTMot^{\mathrm{eff}}(S) \to DTM^{\mathrm{eff}}(S)$$

and

$$\rho_{S\mathbb{Q}}^{\mathrm{eff}\,Tate} : DTMot^{\mathrm{eff}}(S)_{\mathbb{Q}} \to DTM^{\mathrm{eff}}(S)_{\mathbb{Q}}$$

Proof This is immediate consequence of corollary 6.14, except for the assertion that $\rho_S^{\text{eff Tate}}$ and $\rho_{S\mathbb{Q}}^{\text{eff Tate}}$ are full. But it follows from the weak cancellation theorem 6.5 that $DTM^{\text{eff}}(S) \to DTM(S)$ is fully faithful, whence the result. □

7 Moving lemmas

We proceed to extend the Friedlander-Lawson-Voevodsky moving lemmas to the case of a semi-local regular base scheme, and use these results to prove theorem 5.4 and theorem 5.5. We refer the reader to §5.1 for the notation involving the various presheaves of equi-dimensional cycles.

7.1 Friedlander-Lawson-Voevodsky moving lemmas. The Friedlander-Lawson moving lemmas [10] are applied in [11] to prove the main moving lemmas and duality statements for the cycle complexes of equi-dimensional cycles. This involves a two-step process: one moves cycles on $\mathbb{P}^n$ by one process (which we will not need to recall explicitly) and then one uses a variation of the projecting cone argument to extend the moving process to a smooth projective variety. The main result following from the first step is

Theorem 7.1 ([11, theorem 6.1]) *Let k be a field, and n, r, d, e integers. Then there are maps of presheaves of abelian monoids on* $\mathbf{Sm}/k$

$$H_U^{\pm} : z_{equi}^k(\mathbb{P}^n, r)^{\text{eff}}(U) \to z_{equi}^k(\mathbb{P}^n, r)^{\text{eff}}(U \times \mathbb{A}^1)$$

such that

1. *There is an integer $m \geq 0$ such that*
$$i_0^* \circ H_U^+ = (m+1) \cdot \text{id}_{z_{equi}^k(\mathbb{P}^n, r)^{\text{eff}}(U)}$$
$$i_0^* \circ H_U^- = m \cdot \text{id}_{z_{equi}^k(\mathbb{P}^n, r)^{\text{eff}}(U)}$$
where $i_0 : U \to U \times \mathbb{A}^1$ is the zero-section.
2. *Let $i_x : \operatorname{Spec} k' \to U \times \mathbb{A}^1 \setminus \{0\}$ be a k'-point of $U \times \mathbb{A}^1 \setminus \{0\}$, where $k' \supset k$ is some extension field of k. Then for each $W \in z_{equi}^k(\mathbb{P}^n, s)_{\leq d}^{\text{eff}}(k')$, $Z \in z_{equi}^k(\mathbb{P}^n, r)_{\leq e}^{\text{eff}}(U)$, with $0 \leq r, s \leq n$, $r + s \geq n$, the cycles $x^*(H_U^{\pm}(Z))$ and W intersect properly on $\mathbb{P}_{k'}^n$.*

The second step involves the projecting cone construction, which enables one to go from moving cycles on $\mathbb{P}^n$ to moving cycles on a smooth projective $X \subset \mathbb{P}^N$ of dimension n. We first recall the situation over a base-field k.

Let $D_0, \ldots, D_n \subset \mathbb{P}^N$ be degree f hypersurfaces, and let $F_0, \ldots, F_n$ be the corresponding defining equations. If $X \cap D_0 \cap \ldots \cap D_n = \emptyset$, then

$$F := (F_0 : \ldots : F_n) : \mathbb{P}^N \setminus \cap_{i=0}^n D_i \to \mathbb{P}^n$$

restricts to X to define a finite surjective morphism

$$F_X : X \to \mathbb{P}^n$$

For Z an effective cycle of dimension r on X, we thus have the effective cycle $F_X^*(F_{X*}(Z))$, which can be written as

$$F_X^*(F_{X*}(Z)) = Z + R_F(Z)$$

for a uniquely determined effective cycle $R_F(Z)$ of dimension r. Sending Z to $R_F(Z)$ thus gives a map

$$R_F : z_{equi}^k(X, r)^{\text{eff}}(k) \to z_{equi}^k(X, r)^{\text{eff}}(k).$$

Let $\mathcal{U}_X(d)$ be the open subscheme of $H^0(\mathbb{P}^N, \mathcal{O}(d))^n$ consisting of $D_0, \ldots, D_n$ such that $\cap_{i=0}^n D_i \cap X = \emptyset$. It is easy to see that the complement of $\mathcal{U}_X(d)$ in $H^0(\mathbb{P}^N, \mathcal{O}(d))^n$ is a hypersurface (in fact, the defining equation of this complement is exactly the classical Chow form of X, whose coefficients give the Chow point of X in the Chow variety of dimension n, degree deg$X$ effective cycles on $\mathbb{P}^N$).

For $F \in \mathcal{U}_X(f)$, let $\mathrm{ram}_X(F) \subset X$ be the ramification locus of F_X. Choose integers $f_0, \ldots, f_n \geq 1$ and let $\mathcal{R}_X(d_*) \subset \prod_{i=0}^n \mathcal{U}_X(d_i)$ be the open subscheme consisting of those tuples $(F^0, \ldots, F^n)$, $F^i := (F^i_0 : \ldots : F^i_n) \in \mathcal{U}_X(d_i)$ for which

$$\cap_{i=0}^n \mathrm{ram}_X(F^i) = \emptyset.$$

Remark 7.2 It follows from Bertini's theorem that, if $d_i \geq 2$ for all i, then $\mathcal{R}_X(d_*)$ is a dense open subscheme of $\prod_{i=0}^n \mathcal{U}_X(d_i)$ (even in positive characteristic).

The next moving result is a translation of [10, proposition 1.3 and theorem 1.7].

Proposition 7.3 *Let $X \subset \mathbb{P}^N_k$ be a smooth closed subscheme of dimension n. Fix dimensions $0 \leq r, s \leq n$ with $r + s \geq n$ and a degree $e \geq 1$. Then there is an increasing function*

$$G := G_{r,s,N,e,X} : \mathbb{N} \to \mathbb{N},$$

depending on X and the integers r, s, N and e, and, for each $d_* := (d_0, \ldots, d_n)$, there is a closed subset $\mathcal{B}_X(d_*, e, r, s)$ of $\mathcal{R}_X(d_*)$, such that the following holds: Let k' be a field extension of k, and take $F^0, \ldots, F^n \in \mathcal{R}_X(d_*)(k') \setminus \mathcal{B}_X(d_*, e, r, s)(k')$. Suppose that $d_0 \geq G(e)$ and $d_i \geq G(d_{i-1}^n \cdot e)$ for $i = 1, \ldots, n$. Then

1. *For each pair of effective cycles Z, W on $X_{k'}$ with Z of dimension r, W of dimension s and both having degree $\leq e$, the cycles W and $R_X(F^n) \circ \ldots \circ R_X(F^0)(Z)$ intersect properly on $X_{k'}$.*
2. *Let Z, W be a pair of effective cycles on $X_{k'}$ with Z of dimension r, W of dimension s and both having degree $\leq e$. Suppose that Z and W intersect properly on $X_{k'}$. Then for every $j = 0, \ldots, n$, the cycles W and $(F_X^{j*} F_{X*}^j) \circ \ldots \circ (F_X^{0*} F_{X*}^0)(Z)$ intersect properly on $X_{k'}$.*

Remark 7.4 If we have two closed subsets $\mathcal{B}^1_X(d_*, e, r, s), \mathcal{B}^2_X(d_*, e, r, s)$ satisfying the conditions of proposition 7.3, then the intersection $\mathcal{B}^1_X(d_*, e, r, s) \cap \mathcal{B}^2_X(d_*, e, r, s)$ also satisfies proposition 7.3. Thus, without loss of generality, we may assume that $\mathcal{B}_X(d_*, e, r)$ is the *minimal* closed subset of $\mathcal{R}_X(d_*)$ satisfying the conditions of proposition 7.3.

Assuming this to be the case, we have the following description of $\mathcal{B}_X(d_*, e, r, s)$:

$\mathcal{B}_X(d_*, e, r, s) = \{F^* := (F^0, \ldots, F^n) \in \mathcal{R}_X(d_*) \mid \exists$ an extension field $k' \supset k(F^*)$,

$Z \in z_{equi}(X, r)^{\mathrm{eff}}_{\leq e}(k'), W \in z_{equi}(X, s)^{\mathrm{eff}}_{\leq e}(k')$ such that either

1. W and $R_X(F^n) \circ \ldots \circ R_X(F^0)(Z)$ do not intersect properly on $X_{k'}$

or

2. W and Z intersect properly on $X_{k'}$, but this is not the case for W and $(F_X^{j*} F_{X*}^j) \circ \ldots \circ (F_X^{0*} F_{X*}^0)(Z)$ for some $j, 0 \leq j \leq n.\}$

To see this, it suffices to see that the set described is closed; for this it suffices to see that this set is closed under specialization. This follows from the fact the the Chow varieties parametrizing effective cycles of fixed degree and dimension on X are proper over k.

7.2 Extending the moving lemma. We now consider the two moving lemmas in the setting of a smooth projective scheme over a regular base-scheme B. The extension of the first moving lemma is a direct corollary of theorem 7.1.

Corollary 7.5 *Let k be a field, and n, r, d, e integers, B a regular k-scheme, essentially of finite type over k. Then there are maps of presheaves of abelian monoids on $\mathbf{Sm}/B$*

$$H_U^{\pm} : z_{equi}^B(\mathbb{P}_B^n, r)^{\text{eff}}(U) \to z_{equi}^B(\mathbb{P}_B^n, r)^{\text{eff}}(U \times \mathbb{A}^1)$$

such that

1. *Let $i_0 : U \to U \times \mathbb{A}^1$ be the zero-section.There is an integer $m \geq 0$ such that*
$$i_0^* \circ H_U^+ = (m+1) \cdot \mathrm{id}_{z_{equi}^B(\mathbb{P}^n, r)^{\text{eff}}(U)}$$
$$i_0^* \circ H_U^- = m \cdot \mathrm{id}_{z_{equi}^B(\mathbb{P}^n, r)^{\text{eff}}(U)}.$$
2. *Let $i : T \to U \times \mathbb{A}^1 \setminus \{0\}$ be a morphism of B-schemes. Then for each $W \in z_{equi}^B(\mathbb{P}_B^n, s)_{\leq d}^{\text{eff}}(T)$, $Z \in z_{equi}^B(\mathbb{P}_B^n, r)_{\leq e}^{\text{eff}}(U)$, with $0 \leq r, s \leq n$, $r+s \geq n$, the cycles $i^*(H_U^{\pm}(Z))$ and W intersect properly on $\mathbb{P}_T^n$.*

Proof First of all, it suffices to prove the result if B is of finite type over k. Let $p : B \to \operatorname{Spec} k$ be the structure morphism and $p_* : \mathbf{Sm}/B \to \mathbf{Sm}/k$ the base-restriction functor, $p^* : \mathbf{Sm}/k \to \mathbf{Sm}/B$ the pull-back functor $p^*(Y) := Y \times_k B$. Then we have the evident identifications

$$z_{equi}^B(\mathbb{P}_B^n, r)(U) = z_{equi}^k(\mathbb{P}_k^n, r)(p_*U),$$

induced by the canonical isomorphism $U \times_B \mathbb{P}_B^n \cong p_*U \times_k \mathbb{P}_k^n$. Thus the maps $H_{p_*U}^{\pm}$ given by theorem 7.1 give rise to maps

$$H_U^{\pm} : z_{equi}^B(\mathbb{P}_B^n, r)(U) \to z_{equi}^B(\mathbb{P}_B^n, r)(U \times \mathbb{A}^1)$$

which clearly satisfy our condition (1).

For (2), if we have two cycles $A \in z_{equi}^B(\mathbb{P}_B^n, r)^{\text{eff}}(T)$ and $A' \in z_{equi}^B(\mathbb{P}_B^n, s)^{\text{eff}}(T)$ such that, for each point $t \in T$, the fibers $t^*(A), t^*(A')$ intersect properly on $\mathbb{P}_t^n$, then A and A' intersect properly on $T \times_B \mathbb{P}_B^n$ and $A \cdot A'$ is in $z_{equi}^B(\mathbb{P}_B^n, r+s-n)^{\text{eff}}(T)$. This, together with theorem 7.1(2), proves (2). □

We now show how to extend the second step of the moving lemma. Take $X \in \mathbf{Proj}/B$ with a fixed embedding $X \hookrightarrow \mathbb{P}_B^N$ over B. For each $b \in B$, let $\mathcal{C}_{X_b}(d) = \mathbb{P}(H^0(\mathbb{P}_s^N, \mathcal{O}(d)))^{n+1} \setminus \mathcal{U}_{X_b}(d)$. Let $V \subset \operatorname{Proj}_B(p_*\mathcal{O}(d))^{n+1} \times \mathbb{P}^N$ be the incidence correspondence with points $\{((f_0 : \ldots : f_n), x) \mid x \in \cap_{i=}^n (f_i = 0)\}$. Let $\mathcal{C}_X(d) = p_1(V \cap p_2^{-1}(X))$, and let $\mathcal{U}_X(d) = \mathbb{P}(p_*\mathcal{O}(d)))^{n+1} \setminus \mathcal{C}_X(d)$. Then clearly $\mathcal{C}_X(d) \subset \mathbb{P}(H^0(\mathbb{P}_s^N, \mathcal{O}(d)))^{n+1}$ is a closed subset with fiber $\mathcal{C}_{X_b}(d)$ over $b \in B$, hence $\mathcal{U}_X(d)$ has fiber $\mathcal{U}_{X_b}(d)$ over $b \in B$.

Let $F_d : \mathcal{U}_X(d) \times_B X \to \mathcal{U}_X(d) \times_B \mathbb{P}^n$ be the $\mathcal{U}_X(d)$-morphism parametrized by $\mathcal{U}_X(d)$, i.e., $F_d((f_0 : \ldots : f_n), x) := ((f_0 : \ldots : f_n), (f_0(x) : \ldots : f_n(x)))$. We have the ramification locus $\mathrm{ram}_X(F)_{\mathcal{U}} \subset \mathcal{U}_X(d) \times_B X$.

For a sequence $d_* := (d_0, \ldots, d_n)$, let

$$\mathcal{U}_X(d_*) := \mathcal{U}_X(d_0) \times_B \ldots \times_B \mathcal{U}_X(d_n)$$

let $p_i : \mathcal{U}_X(d_*) \to \mathcal{U}_X(d_i)$ be the projection, and let $\mathcal{F}_X(d_*) \subset \mathcal{U}_X(d_*) \times_B X$ be the intersection $\cap_{i=0}^n (p_i \times \mathrm{id}_X)^{-1}(\mathrm{ram}_X(F_{d_i})_{\mathcal{U}})$. Finally, let $q : \mathcal{U}_X(d_*) \times_B X \to \mathcal{U}_X(d_*)$

be the projection and set

$$\mathcal{R}_X(d_*) := \mathcal{U}_X(d_*) \setminus q(\mathcal{F}_X(d_*))$$

Clearly $\mathcal{R}_X(d_*)$ is an open subscheme of $\mathcal{U}_X(d_*)$ with fiber $\mathcal{R}_{X_b}(d_*)$ over $b \in B$.

Each $F \in \mathcal{U}_X(d_*)(B)$ thus determines a finite B-morphism

$$F_X : X \to \mathbb{P}^n_B$$

and gives rise to the map

$$R_F : z^B(X,r)^{\mathrm{eff}} \to z^B(X,r)^{\mathrm{eff}}$$

with

$$F_X^*(F_{X*})(Z) = Z + R_F(Z)$$

for each $Z \in z^B(X,r)^{\mathrm{eff}}$.

Lemma 7.6 *Fix integers e, r, s and a sequence of integers $d_0, \ldots, d_n$, with $d_i \geq 2$, $0 \leq r, s \leq n$, $r + s \geq n$ and $e \geq 1$. For each $b \in B$, we have the closed subset $\mathcal{B}_{X_b}(d_*, e, r, s)$ of $\mathcal{R}_{X_b}(d_*) \subset \mathcal{R}_X(d_*)$ given by proposition 7.3; we assume following remark 7.4 that $\mathcal{B}_{X_b}(d_*, e, r, s)$ is minimal for each b. Let*

$$\mathcal{B}_X(d_*, e, r, s) := \cup_{b \in B} \mathcal{B}_{X_b}(d_*, e, r, s).$$

Then $\mathcal{B}_X(d_*, e, r, s)$ is closed in $\mathcal{R}_X(d_*)$.

Proof Let $b \to b'$ be a specialization of points of B, x a point of $\mathcal{B}_{X_b}(d_*, e, r, s)$, and $x \to x'$ an extension of $b \to b'$ to a specialization of x to a point of $\mathcal{R}_{X_{b'}}(d_*)$. Using the properness of Chow varieties, it follows from the description of $\mathcal{B}_X(d_*, e, r, s)$ in remark 7.4 that x' is in $\mathcal{B}_{X_{b'}}(d_*, e, r, s)$, proving the result. □

We specialize to the case of a semi-local B, still regular and essentially of finite type over k. Recall the function $G_{r,s,N,e,X} : \mathbb{N} \to \mathbb{N}$ from proposition 7.3, defined for $X \in \mathbf{Proj}/k$, with a given embedding $X \hookrightarrow \mathbb{P}^N_k$, and integers r, s, e. Given $X \in \mathbf{Proj}/B$ with a given embedding over B, $X \hookrightarrow \mathbb{P}^N_B$, let $b_1, \ldots, b_m$ be the closed points of B and define

$$G_{r,s,N,e,X}(m) := \max_i \{G_{r,s,N,e,X_{b_i}}(m)\}.$$

Proposition 7.7 *Let B be a semi-local regular k-scheme, with k an infinite field. Take X in $\mathbf{Proj}/B$ of relative dimension n over B, with a fixed closed immersion $X \hookrightarrow \mathbb{P}^N_B$ over B. Fix dimensions $0 \leq r, s \leq n$ and a degree $e \geq 1$ with $r + s \geq n$ and let $G := G_{r,s,N,e,X}$. Then for all tuple of integers $d_0, \ldots, d_n$ with $d_0 \geq G(e)$, $d_i \geq G(d_{i-1}^n \cdot e)$ for $i = 1, \ldots, n$, there is a point $(F^0, \ldots, F^n) \in \mathcal{R}_X(d_*)(B) \setminus \mathcal{B}_X(d_*, e, r, s)(B)$.*

Proof We note that the fiber of $\mathcal{R}_X(d_*)$ over $b \in B$ is $\mathcal{R}_{X_b}(d_*)$ and similarly for $\mathcal{B}_X(d_*, e, r, s)$. Then $\mathcal{R}_X(d_*) \setminus \mathcal{B}_X(d_*, e, r, s) \to B$ is an open subscheme of the product of projective spaces $\prod_{i=0}^n \mathrm{Proj}\,_B(p_*\mathcal{O}_{\mathbb{P}^N}(d_i))$ over B; by proposition 7.3, the fiber $\mathcal{R}_X(d_*) \setminus \mathcal{B}_X(d_*, e, r, s)$ is non-empty for each closed point b of B. As k is assumed infinite, B has infinite residue fields over each of its closed points. Thus, for each closed point b of B, there is a $k(b)$-point F_b^* in $\mathcal{R}_X(d_*)_b \setminus \mathcal{B}_X(d_*, e, r, s)_b$. Since B is semi-local, there is a B-point F^* of $\prod_{i=0}^n \mathrm{Proj}\,_B(p_*\mathcal{O}_{\mathbb{P}^N}(d_i))$ with $F^*(b) = F_b^*$ for each closed point b; F^* is automatically a B point of $\mathcal{R}_X(d_*) \setminus \mathcal{B}_X(d_*, e, r, s)$, using again the fact that B is semi-local. □

Remark 7.8 Take $(F^0, \ldots, F^n) \in \mathcal{R}_X(d_*)(B) \setminus \mathcal{B}_X(d_*, e, r, s)(B)$ as given by proposition 7.7. Then for each B-scheme $T \to B$, and each pair of cycles

$$Z \in z^B(X, r)^{\mathrm{eff}}_{\leq e}(T),\ W \in z^B(X, s)^{\mathrm{eff}}_{\leq e}(T),$$

the cycles W and $R_X(F^n) \circ \ldots \circ R_X(F^0)(Z)$ intersect properly on X_T and the intersection $W \cdot_{X_T} R_X(F^n) \circ \ldots \circ R_X(F^0)(Z)$ is in $z^B(X, r+s-n)^{\mathrm{eff}}(T)$.

Indeed, this follows from the fact that the operation $R_X(F)$ is compatible with taking fibers, hence, for all $t \in T$, the cycles W_t and $R_{X_t}(F^n_t) \circ \ldots \circ R_{X_t}(F^0_t)(Z_t)$ intersect properly on X_t. Thus W and $R_X(F^n) \circ \ldots \circ R_X(F^0)(Z)$ intersect properly on X_T and the intersection $W \cdot_{X_T} R_X(F^n) \circ \ldots \circ R_X(F^0)(Z)$ is equi-dimensional (of dimension $r+s-n$) over T.

Similarly, if W and Z intersect properly on X_T and $W \cdot_{X_T} Z$ is in $z^B(X, r+s-n)^{\mathrm{eff}}(T)$, then for each $j = 0, \ldots, n$, W and $Z_j := (F^{j*}_X F^j_{X*}) \circ (F^{0*}_X F^0_{X*})(Z)$ intersect properly on X_T and $W \cdot Z_j$ is in $z^B(X, r+s-n)^{\mathrm{eff}}(T)$.

We can now prove our extension of [11, theorem 6.3].

Theorem 7.9 *Let k be a field, B a regular k-scheme, essentially of finite type over k, $X \in \mathbf{Proj}/B$ of dimension n over B with a given embedding $X \hookrightarrow \mathbb{P}^N_B$. Let r, s, e, d be given integers with $e \geq 1$, $0 \leq r, s \leq n$, $r+s \geq n$. Then is a map of presheaves*

$$H_{X,U} : z^B_{equi}(X, r)(U) \to z^B_{equi}(X, r)(U \times \mathbb{A}^1);\ U \in \mathbf{Sm}/B,$$

such that

1. *Let $i_0 : U \to U \times \mathbb{A}^1$ be the zero-section. Then $i_0^* \circ H_{X,U} = \mathrm{id}_{z^B_{equi}(X,r)(U)}$.*
2. *Let $i : T \to U \times \mathbb{A}^1 \setminus \{0\}$ be a morphism of B-schemes. Then for each $W \in z^B_{equi}(X, s)^{\mathrm{eff}}_{\leq e}(T)$, $Z \in z^B_{equi}(X, r)^{\mathrm{eff}}_{\leq e}(U)$, with $0 \leq r, s \leq n$, $r+s \geq n$, the cycles $i^*(H_{X,U}(Z))$ and W intersect properly on X_T and $i^*(H_{X,U}(Z)) \cdot_{X_T} W$ is in $z^B_{equi}(X, r+s-n)^{\mathrm{eff}}(T)$.*
3. *Let $i : T \to U \times \mathbb{A}^1$ be a morphism of B-schemes and take*

$$W \in z^B_{equi}(X, s)^{\mathrm{eff}}_{\leq e}(T),$$
$$Z \in z^B_{equi}(X, r)^{\mathrm{eff}}_{\leq e}(U),$$

with $0 \leq r, s \leq n$, $r+s \geq n$. Suppose that $(p_U \circ i)^*(Z)$ and W intersect properly on X_T and $(p_U \circ i)^*(Z) \cdot_{X_T} W$ is in $z^B_{equi}(X, r+s-n)^{\mathrm{eff}}(T)$. Then the cycles $i^*(H_{X,U}(Z))$ and W intersect properly on X_T and $i^*(H_{X,U}(Z)) \cdot_{X_T} W$ is in $z^B_{equi}(X, r+s-n)^{\mathrm{eff}}(T)$.

Proof The proof follows that in [11] and [10]. If k is a finite field, we use the fact that k admits a infinite pro-l extension k_l for every prime l prime to the characteristic, plus the usual norm argument, to reduce to the case of an infinite field. We may assume that B is irreducible. Thus for each $b \in B$, the embedded subscheme $X_b \subset \mathbb{P}^N_b$ has degree d_X independent of b.

We denote the maps

$$H^{\pm}_U : z^B_{equi}(\mathbb{P}^n_B, r)^{\mathrm{eff}}(U) \to z^B_{equi}(\mathbb{P}^n_B, r)^{\mathrm{eff}}(U \times \mathbb{A}^1)$$

given by corollary 7.5 by $H^{\pm}_U(d)$, noting explicitly the dependence on the degree d. Similarly,we write $m(d)$ for the integer m that appears in corollary 7.5(1).

Choose some $F^* \in \mathcal{R}_X(d_*)(B) \setminus \mathcal{B}_X(d_*)(B)$, as given by proposition 7.7. Let $p_U : U \times \mathbb{A}^1 \to U$ be the projection. We have the maps of presheaves

$$F^i_{X*} : z^B_{equi}(X, r)^{\text{eff}}_d \to z^B_{equi}(\mathbb{P}^n, r)^{\text{eff}}_d$$
$$F^{i*}_X : z^B_{equi}(\mathbb{P}^n, r)^{\text{eff}}_d \to z^B_{equi}(X, r)^{\text{eff}}_{d^n_i ed}$$

and similarly without the degree restrictions. Also, for $W \in z^B_{equi}(X, s)^{\text{eff}}(U)$, $Z \in z^B_{equi}(\mathbb{P}^n_B, r)^{\text{eff}}(U)$, $W \cdot_{U\times_B X} F^{i*}_X(Z)$ is defined and is in $z^B_{equi}(X, s+r-n)^{\text{eff}}(U)$ if and only if $F^i_{X*}(W) \cdot_{U\times_B \mathbb{P}^n_B} Z$ is defined and is in $z^B_{equi}(\mathbb{P}^n_B, s+r-n)^{\text{eff}}(U)$. Finally, we note that, for $Z \in z^B_{equi}(X, r)^{\text{eff}}_d(U)$, $R_{F^i}(Z)$ is in $z^B_{equi}(X, r)^{\text{eff}}_{d(d^n_i e-1)}(U)$.

We now define a sequence of maps of presheaves on $\mathbf{Sm}/B$,

$$H^{\pm}_{X,U,j} : z^B_{equi}(X, r)^{\text{eff}}(U) \to z^B_{equi}(X, r)^{\text{eff}}(U \times \mathbb{A}^1);\ U \in \mathbf{Sm}/B.$$

Define the integers δ_j inductively by $\delta_0 = e$, and

$$\delta_j = (m(\delta_{j-1}) + 1) d^n_i e \delta_{j-1}$$

for $j \geq 1$. These numbers have the property that, if Z is in $z^B_{equi}(X, r)^{\text{eff}}_{\leq\delta_j}(U)$, then $(m(\delta_j) + 1) \cdot R_{F^j}(Z)$ is in $z^B_{equi}(X, r)^{\text{eff}}_{\leq\delta_{j+1}}(U)$. For $Z \in z^B_{equi}(X, r)^{\text{eff}}(U)$, define

$$H^{\pm}_{X,U,0}(Z) := F^{0*}_X \circ H^{\pm}_U(\delta_0) \circ F^0_{X*}(Z).$$

For $j = 1, \ldots, n$, let

$$H^{\pm}_{X,U,j}(Z) := F^{j*}_X \circ H^{\pm}_U(\delta_j) \circ F^j_{X*}(R_{F^{j-1}} \circ \ldots \circ R_{F^0}(Z)).$$

Finally, we let

$$H^{+}_{X,U,n+1}(Z) = p^*_U(R_{F^n} \circ \ldots \circ R_{F^0}(Z))$$

and $H^-_{X,U,n+1}(Z) = 0$. Set

$$H_{X,U}(Z) := \sum_{j=0}^{n+1} (-1)^j (H^+_{X,U,j}(Z) - H^-_{X,U,j}(Z)).$$

Thus, $U \mapsto H_{X,U}$ defines a map of presheaves on $\mathbf{Sm}/B$

$$H_X : z^B_{equi}(X, r) \to z^B_{equi}(X, r)((-) \times \mathbb{A}^1).$$

We have

$$i^*_0(H^+_{X,U,j}(Z) - H^-_{X,U,j}(Z)) = \begin{cases} F^{*0}_X \circ F^0_{X*}(Z) & \text{for } j = 0 \\ F^{j*}_X \circ F^j_{X*}(R_{F^{j-1}} \circ \ldots \circ R_{F^0}(Z)) & \text{for } j = 1, \ldots, n \\ R_{F^n} \circ \ldots \circ R_{F^0}(Z) & \text{for } j = n+1. \end{cases}$$

Since $F^{j*}_X \circ F^j_{X*} = \text{id} + R_{F^j}$, and $i^*_0 \circ (H^+_U - H^-_U) = \text{id}$, this implies that

$$i^*_0 \circ H_{X,U}(Z) = Z$$

for all $Z \in z^B_{equi}(X, r)(U)$. This verifies the property (1).

For (2), let $i : T \to U \times \mathbb{A}^1 \setminus \{0\}$ be a morphism of B-schemes, take $W \in z^B_{equi}(X, s)^{\text{eff}}_{\leq e}(T)$ and $Z \in z^k_{equi}(X, r)^{\text{eff}}_{\leq e}(U)$, with $0 \leq r, s \leq n$, $r + s \geq n$. Take $j = 0, \ldots, n$. By corollary 7.5 and our remarks above, the cycles $i^*(H^{\pm}_{X,U,j}(Z))$ and W intersect properly on X_T and $i^*(H^{\pm}_{X,U,j}(Z)) \cdot_{X_T} W$ is in $z^B_{equi}(X, r+s-n)^{\text{eff}}(T)$. By remark 7.8, the same holds for $j = n+1$. This proves (2).

For (3), under the given assumptions for Z and W, it follows from remark 7.8 that W and

$$Z_j := i^* \circ p_U^*(F_X^{j*} F_{X*}^j) \circ \ldots \circ (F_X^{0*} F_{X*}^0)(Z)$$

intersect properly on X_T. Since $H^{\pm}_{X,U,j}(Z)$ is effective and $i_0^* \circ H^{\pm}_{X,U,j}(Z)$ plus some effective cycle is a multiple of $(F_X^{j*} F_{X*}^j) \circ \ldots \circ (F_X^{0*} F_{X*}^0)(Z)$, (3) follows from (2). □

Fix a field extension k' of k, and let $Y \subset \mathbb{P}^N_{k'}$ be a closed subset. Let $\mathcal{C}_Y(s,e)$ be the Chow scheme parametrizing effective cycles of dimension s and degree $\leq e$ on Y. For $L \supset k'$ an extension field and W an effective dimension s cycle on $\mathbb{P}^N_L$ of degree $\leq e$, and with support in Y_L, we let $\mathrm{chow}(W) \in \mathcal{C}_(s,e)(\bar{L})$ denote the Chow point of W.

For $Y \subset \mathbb{P}^N_B$, we have the relative Chow scheme $\mathcal{C}_{Y/B}(s,e) \to B$, with (reduced) fiber over $b \in B$ the Chow scheme of $Y_b \subset \mathbb{P}^N_b$.

Definition 7.10 For a fixed regular noetherian base-scheme B, take $Y \in$ **Proj**$/B$ and fix an embedding $Y \hookrightarrow \mathbb{P}^N_B$ over B. Let $\mathcal{C} \subset \mathcal{C}_{Y/B}(s,e)$ be any collection of locally closed subsets of $\mathcal{C}_{Y/B}(s,e)$.

Let

$$z^B_{equi}(Y,r)_{\mathcal{C}}(X) \subset z^B_{equi}(Y,r)(X)$$

be the subgroup generated by integral closed subschemes $Z \subset X \times_B Y$ such that,

1. Z is in $z^B_{equi}(Y,r)(X)$.
2. Take $W \in z^B_{equi}(Y,s)^{\mathrm{eff}}_{\leq e}(X)$ and suppose that, for each $x \in X$, the cycle $i_x^*(W)$ on $Y_{k(x)}$ has Chow point $\mathrm{chow}(W)$ in $\mathcal{C}(\bar{k(x)})$. Then the intersection $W \cdot_{X \times_B Y} Z$ is defined and is in $z^B_{equi}(Y, r+s-n)(X)$.

This defines the subpresheaf $z^B_{equi}(Y,r)_{\mathcal{C}}$ of $z^B_{equi}(Y,r)$. The subpresheaves

$$z^B_{equi}(Y,r)^{\mathrm{eff}}_{\mathcal{C},\leq e} \subset z^B_{equi}(Y,r)^{\mathrm{eff}}_{\leq e};\ z^B_{equi}(Y,r)_{\mathcal{C},\leq e} \subset z^B_{equi}(Y,r)_{\leq e},$$

etc., are defined similarly.

We let $C^B(Y,r)_{\mathcal{C}}(X) \subset C^B(Y,r)(X)$ be the subcomplex associated to the cubical object

$$n \mapsto z^B_{equi}(Y,r)_{\mathcal{C}}(X \times \square^n)$$

This gives us the presheaf of subcomplexes $C^B(Y,r)_{\mathcal{C}} \subset C^B(Y,r)$. The subcomplex $C^B(Y,r)_{\mathcal{C},\leq e} \subset C^B(Y,r)_{\leq e}$ (see definition 5.2) is defined similarly.

Theorem 7.11 *Let B be a semi-local regular scheme, essentially of finite type over some field k. Take $X \in$ **Proj**$/B$ of relative dimension n over B, and integers r,s,e with $e \geq 1$, $0 \leq r,s \leq n$, $r+s \geq n$. Fix an embedding $X \hookrightarrow \mathbb{P}^N_B$. Let $\mathcal{C} \subset \mathcal{C}_X(s,e)$ be a collection of locally closed subsets. Then the inclusion $C^B(X,r)_{\mathcal{C}} \subset C^B(X,r)$ is a quasi-isomorphism.*

Proof Since

$$C^B(X,r)_{\mathcal{C}} = \cup_{e\geq 1} C^B(X,r)_{\mathcal{C},\leq e};\ C^B(X,r) = \cup_{e \geq 1} C^B(X,r)_{\leq e},$$

it suffices to show that the map

$$\iota : \frac{C^B(X,r)_{\leq e}}{C^B(X,r)_{\mathcal{C},\leq e}} \to \frac{C^B(X,r)}{C^B(X,r)_{\mathcal{C}}}$$

induced by the inclusions $C^B(X,r)_{\mathcal{C},\leq e} \subset C^B(X,r)_{\mathcal{C}}$ and $C^B(X,r)_{\leq e} \subset C^B(X,r)$ gives the zero-map on homology.

Let

$$H_X : z^B_{equi}(X, r) \to z^B_{equi}(X, r)((-) \times \mathbb{A}^1)$$

be the map given by theorem 7.9 for the given values of r, s, e. By theorem 7.9(3), H_X restricts to a map

$$H_X : z^B_{equi}(X, r)_{\mathcal{C}} \to z^B_{equi}(X, r)((-) \times \mathbb{A}^1)_{\mathcal{C}}$$

By theorem 7.9(2), $i_1^* \circ H_X$ defines a map

$$G_X : z^B_{equi}(X, r) \to z^B_{equi}(X, r)_{\mathcal{C}}$$

and by theorem 7.9(1), $i_0^* \circ H_X = \mathrm{id}$.

Identifying $U \times \mathbb{A}^1 \times_B X \times \square^n$ with $U \times_B X \times \square^{n+1}$ via the exchange of factors

$$U \times \mathbb{A}^1 \times_B X \times \square^n \to U \times_B X \times \square^n \times \mathbb{A}^1 = U \times_B X \times \square^{n+1},$$

$(-1)^n H_X$ gives us maps

$$h_X^{-n} : C^B(X, r)^{-n}_{\leq e} \to C^B(X, r)^{-n-1},\ h_{X\mathcal{C}}^{-n} : C^B(X, r)^{-n}_{\leq e, \mathcal{C}} \to C^B(X, r)^{-n-1}_{\mathcal{C}}$$

and thus induces a degree -1 map

$$\bar{h}_X : \frac{C^B(X, r)_{\leq e}}{C^B(X, r)_{\leq e, \mathcal{C}}} \to \frac{C^B(X, r)}{C^B(X, r)_{\mathcal{C}}}$$

which gives a homotopy between the map ι and the zero-map, completing the proof. □

We use theorem 7.11 to prove theorem 5.4:

Corollary 7.12 *Let B be a semi-local regular scheme, essentially of finite type over some field k. Take $X, Y \in$ **Proj**$/B$, $U \in$ **Sm**$/B$ and let $p = dim_B X$. Then for $0 \leq r \leq dim_B Y$, the map*

$$\int_X : C^B(Y, r)(U \times_B X) \to C^B(X \times_B Y, r + p)(U)$$

is a quasi-isomorphism.

Proof Let $s = \dim_B Y$. Fix embeddings of Y, X into some $\mathbb{P}^N_B$, with Y of degree e. The Segre embedding $\mathbb{P}^N \times \mathbb{P}^N \to \mathbb{P}^M$ gives us an embedding of $X \times_B Y$ in $\mathbb{P}^M_B$ such that $x \times Y$ has degree e for each $x \in X$. Take $\mathcal{C} \subset \mathcal{C}_{X \times_B Y/B}(s, e)$ to be the family of cycles $x \times Y$, $x \in X$. Note that the map $\int_X$ identifies $z_{equi}(Y, r)(U \times_B X)$ with $z_{equi}(X \times_B Y, r + p)_{\mathcal{C}}(U)$, and thus gives an isomorphism

$$\int_X : C^B(Y, r)(U \times_B X) \to C^B(X \times_B Y, r + p)_{\mathcal{C}}(U).$$

Since $r + p + s \geq p + s = \dim_B X \times_B Y$, we may apply theorem 7.11 to conclude that the inclusion

$$C^B(X \times_B Y, r + p)_{\mathcal{C}}(U) \subset C^B(X \times_B Y, r + p)(U)$$

is a quasi-isomorphism, completing the proof. □

To complete this section, we prove the projective bundle formula.

Proof of theorem 5.5 We proceed by induction on n. Let $\pi : Y \times \mathbb{P}^n \to Y$ be the projection. By theorem 7.11, the inclusion

$$C^B(Y \times \mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}} \subset C^B(Y \times \mathbb{P}^n, r)$$

is a quasi-isomorphism for $r \geq 1$ Here $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ is the hyperplane $X_n = 0$. We have the intersection map

$$i^*_{\mathbb{P}^{n-1}} : z_{equi}(Y \times \mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}} \to z_{equi}(Y \times \mathbb{P}^{n-1}, r-1)$$

and the cone-map

$$C_{p_0}(-) : z_{equi}(Y \times \mathbb{P}^{n-1}, r-1) \to z_{equi}(Y \times \mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}}$$

where $p_0 := (0 : \ldots, 0 : 1)$. Let $\iota_0 : \mathrm{Spec}\, k \to \mathbb{P}^n$ be the inclusion of the point p_0 and let $\pi : \mathbb{P}^n \to \mathrm{Spec}\, k$ be the projection. Let

$$\mu : \mathbb{P}^n \times (\mathbb{A}^1 \setminus \{0\}) \to \mathbb{P}^n$$

be the multiplication map

$$\mu((x_0 :, \ldots : x_n), t) := (x_0 : \ldots : x_{n-1} : tx_n).$$

We have as well the natural transformation

$$H_U : z_{equi}(Y \times \mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}}(U) \to z_{equi}(Y \times \mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}}(U \times \mathbb{A}^1)$$

which sends a cycle $Z \in z_{equi}(Y \times \mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}}(U)$ to the closure of $\mu^*(Z)$. One checks that this is well-defined and satisfies

$$i_1^* \circ H_U = \mathrm{id}_{z_{equi}(Y\times\mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}}(U)};\ i_0^* \circ H_U = C_{p_0} \circ i^*_{\mathbb{P}^{n-1}} + \alpha_0 \circ \pi_*,$$

where $\alpha_0 := \iota_{0*}$. As in the proof of theorem 7.11, the maps

$$h_U := (-1)^n H_U : C^B_n(Y \times \mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}}(U) \to C^B_{n+1}(Y \times \mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}}(U)$$

define a homotopy between the identity and $C_{p_0} \circ i^*_{\mathbb{P}^{n-1}} + \alpha_0 \circ \pi_*$.

On the other hand, since $p_0 \cap \mathbb{P}^{n-1} = \emptyset$, $i^*_{\mathbb{P}^{n-1}}$ is the zero map on the image of α_{0*}. Also, since $\pi(C_{p_0}(Z))$ has dimension $< \dim_B Z$ for any equi-dimensional closed subset Z of $Y \times_B \mathbb{P}^{n-1}$, π_* is zero on the image of C_{p_0}. Finally,

$$\pi_* \circ \alpha_0 = \mathrm{id};\ i^*_{\mathbb{P}^{n-1}} \circ C_{p_0} = \mathrm{id}$$

hence, for $r \geq n \geq 1$,

$$\alpha_0 + C_{p_0} : C^B(Y, r) \oplus C^B(Y \times \mathbb{P}^{n-1}, r-1) \to C^B(Y \times \mathbb{P}^n, r)_{\{Y\times\mathbb{P}^{n-1}\}}$$

is a homotopy equivalence. By induction

$$\sum_{j=0}^{n-1} \alpha_j : C^B(Y, r-j-1) \to C^B(Y \times \mathbb{P}^{n-1}, r-1)$$

is a quasi-isomorphism; since $C_{p_0} \circ \alpha_j = \alpha_{j+1}$ (where we use p_0 and the subspaces $C_{p_0}(\mathbb{P}^j \subset \mathbb{P}^{n-1})$ for the flag of linear subspaces of $\mathbb{P}^n_k$ needed to define the maps α_j), the induction goes through. □

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