STEENROD OPERATIONS, DEGREE FORMULAS AND ALGEBRAIC COBORDISM

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1. Introduction

Relying on Brosnan's theory of Steenrod operations [1], Merkurjev [4] has given a wide-ranging construction of characteristic classes with values in the Chow ring which satisfy "degree formulas". In this brief note, we give what we view as a somewhat more conceptual treatment of both Brosnan's Steenrod operations and Merkurjev's degree formulas, relying on our theory of algebraic cobordism. As algebraic cobordism requires resolution of singularities, our approach is limited to characteristic zero.

2. Oriented theories

We recall the setting of an oriented Borel-Moore theory from [3]; we fix a field k and let $\mathrm{pt} = \operatorname{Spec} k$. For a full subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$, we let $\mathcal{V}'$ denote the category with the same objects as $\mathcal{V}$, but with morphisms the projective morphisms.

Given a rank n locally free sheaf $\mathcal{E}$ on X, let $q : \mathbb{P}(\mathcal{E}) \to X$ denote the projective bundle of rank one quotients of $\mathcal{E}$, with tautological quotient invertible sheaf $q^*\mathcal{E} \to \mathcal{O}(1)_{\mathcal{E}}$. We let $O(1)_{\mathcal{E}}$ denote the line bundle on $\mathbb{P}(\mathcal{E})$ with sheaf of sections $\mathcal{O}(1)_{\mathcal{E}}$.

We call a functor $F : \mathbf{Sch}'_k \to \mathbf{Ab}_*$ *additive* if $F(\emptyset) = 0$ and the canonical map $F(X) \oplus F(Y) \to F(X \coprod Y)$ is an isomorphism for all X, Y in $\mathcal{V}$.

Definition 1. An *oriented Borel-Moore homology theory A* on $\mathbf{Sch}_k$ is given by

(D1). An additive functor

$$A_* : \mathbf{Sch}'_k \to \mathbf{Ab}_*\,,\ X \mapsto A_*(X).$$

(D2). For each l.c.i. morphism $f : Y \to X$ in $\mathbf{Sch}_k$ of relative dimension d, a homomorphism of graded groups

$$f^* : A_*(X) \to A_{*+d}(Y).$$

(D3). An element $1 \in A_0(\mathrm{pt})$ and, for each pair (X,Y) in $\mathbf{Sch}_k$, a bilinear graded pairing:

$$A_*(X) \otimes A_*(Y) \to A_*(X \times_k Y)$$
$$u \otimes v \mapsto u \times v,$$

called the external product, which is associative, commutative and admits 1 as unit element.

These satisfy

(BM1). One has $Id_X^* = Id_{A_*(X)}$ for any $X \in \mathbf{Sch}_k$. Moreover, given composable l.c.i. morphisms $f : Y \to X$ and $g : Z \to Y$ in $\mathbf{Sch}_k$ of pure relative dimension, one has $(f \circ g)^* = g^* \circ f^*$.

(BM2). Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbf{Sch}_k$. Suppose that f and g are Tor-independent, that f is projective and that g is an l.c.i. morphism, giving the cartesian square

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

Note that f' is projective and g' is an l.c.i. morphism. Then $g^* f_* = f'_* g'^*$.

(BM3). Let $f : X' \to X$ in $\mathbf{Sch}_k$ and $g : Y' \to Y$ be morphisms in $\mathbf{Sch}_k$. If f and g are projective, then for $u' \in A_*(X')$ and $v' \in A_*(Y')$ one has

$$(f \times g)_*(u' \times v') = f_*(u') \times g_*(v').$$

If f and g are l.c.i. morphisms, then for $u \in A_*(X)$ and $v \in A_*(Y)$ one has

$$(f \times g)^*(u \times v) = f^*(u) \times g^*(u')$$

(PB). For $L \to Y$ a line bundle on $Y \in \mathbf{Sch}_k$ with zero-section $s : Y \to L$, define the operator

$$\tilde{c}_1(L) : A_*(Y) \to A_{*-1}(Y)$$

by $\tilde{c}_1(L)(\eta) = s^*(s_*(\eta))$. Let $\mathcal{E}$ be a rank $n+1$ locally free coherent sheaf on $X \in \mathbf{Sch}_k$, with projective bundle $q : \mathbb{P}(\mathcal{E}) \to X$. For $i = 0, \ldots, n$, let

$$\xi^{(i)} : A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

be the composition of $q^* : A_{*+i-n}(X) \to A_{*+i}(\mathbb{P}(\mathcal{E}))$ followed by $\tilde{c}_1(O(1)_{\mathcal{E}})^i : A_{*+i}(\mathbb{P}(\mathcal{E})) \to A_*(\mathbb{P}(\mathcal{E}))$. Then the homomorphism

$$\Sigma_{i=0}^{n-1} \xi^{(i)} : \oplus_{i=0}^{n} A_{*+i-n}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

is an isomorphism.

(H). Let $E \to X$ be a vector bundle of rank r over $X \in \mathbf{Sch}_k$, and let $p : V \to X$ be an E-torsor. Then $p^* : A_*(X) \to A_{*+r}(V)$ is an isomorphism.

(CD). For integers $r, N > 0$, let $W = \mathbb{P}^N \times_S \ldots \times_S \mathbb{P}^N$ (r factors), and let $p_i : W \to \mathbb{P}^N$ be the ith projection. Let $X_0, \ldots, X_N$ be the standard homogeneous coordinations on $\mathbb{P}^N$, let $n_1, \ldots, n_r$ be non-negative integers, and let $i : E \to W$ be the subscheme defined by $\prod_{i=1}^r p_i^*(X_N)^{n_i} = 0$. Then $i_* : A_*(E) \to A_*(W)$ is injective.

Remarks 2. (1) The notion of an oriented weak Borel-Moore homology theory on an (admissible) subcategory $\mathcal{V} \subset \mathbf{Sch}_k$ is defined in [2]. We won't recall the definition here, except to note that an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ is an oriented weak Borel-Moore homology theory on $\mathbf{Sch}_k$, and that, roughly speaking, an oriented Borel-Moore homology theory is an oriented weak Borel-Moore homology theory with pull-back maps for l.c.i. morphisms.

(2) An *oriented cohomology theory* A^* *on* $\mathbf{Sm}_k$ in the sense of [2] is just an oriented Borel-Moore homology theory A_*, only with $\mathbf{Sch}_k$ replaced everywhere by $\mathbf{Sm}_k$, the fiber product W in (BM2) is required to be in $\mathbf{Sm}_k$ and the axiom (CD) omitted (*cf.* [3, Prop. 1.4]). The grading is reindexed:

$$A^*(X) := A_{\dim X - *}(X).$$

(3) The Chow group functor $X \mapsto \mathrm{CH}_*(X)$ is an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$.

(4) In case we need to emphasize the particular theory, we will write $\tilde{c}_1^A(L)$ for the first Chern class operator $\tilde{c}_1(L) : A_*(X) \to A_{*-1}(X)$.

Suppose k has characteristic zero. In [2], we construct the theory Ω_* as an oriented Borel-Moore weak homology theory on $\mathbf{Sch}_k$, and show in [3, Theorem 1.13] that Ω_* extends uniquely to to an an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$,

A main result of [2], [3] (*cf.* [3, Theorem 1.15]) is

Theorem 3. *Assume k admits resolution of singularities.*

(1) *Algebraic cobordism, $X \mapsto \Omega_*(X)$, is the universal oriented Borel-Moore homology theory on* $\mathbf{Sch}_k$.

(2) *Algebraic cobordism, considered as an oriented cohomology theory on* $\mathbf{Sm}_k$*, is the universal oriented cohomology theory on* $\mathbf{Sm}_k$*.*

3. Formal group laws

As mentioned in [2], each oriented Borel-Moore homology theory A has a formal group law $F_A(u,v) \in A_*(k)[[u,v]]$, which gives the identity of operators

$$F_A(\tilde{c}_1^A(L), \tilde{c}_1^A(M)) = \tilde{c}_1^A(L \otimes M)$$

for each pair of line bundles L, M on $X \in \mathbf{Sch}_k$.

The universal formal group law $F_{\mathbb{L}}$ has coefficient ring the Lazard ring $\mathbb{L}_*$, so for each theory A there is a canonical graded ring homomorphism

$$\phi_A : \mathbb{L}_* \to A_*(k)$$

with $\phi_A(F_{\mathbb{L}}) = F_A$. One main result of [2] is the identification of F_Ω with $F_{\mathbb{L}}$:

Theorem 4 ([2, Theorem 4])**.** *For a field k of characteristic zero, the homomorphism $\phi_\Omega : \mathbb{L}_* \to \Omega_*(k)$ is an isomorphism.*

The well-known additivity of the first Chern class in the Chow ring means that F_{CH} is the additive group

$$F_{\mathrm{CH}}(u,v) = u + v.$$

Let $\phi_+ : \mathbb{L} \to \mathbb{Z}$ be the homomorphism classifying the additive group law and let $\Omega^+_* = \Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$. By the universal property of Ω_*, we have the canonical morphism of oriented Borel-Morel theories

$$\psi_{\mathrm{CH}} : \Omega^+ \to \mathrm{CH}.$$

Theorem 5 ([2, Theorem 1, Theorem 14.1])**.** $\psi_{\mathrm{CH}} : \Omega^+ \to \mathrm{CH}$ *is an isomorphism.*

4. Twisting a theory

We recall the construction of Chern classes and twisted Chern classes.

Let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$. Using the axiom (PB), the Grothendieck construction allows us to define, for each vector bundle $E \to X$, the total Chern class operator

$$\tilde{c}_*(E) = \sum_{i=0}^{\operatorname{rnk} E} \tilde{c}_i(E),$$

with $\tilde{c}_i(E) : A_n(X) \to A_{n-i}(X)$ satisfying

(0) Given vector bundles $E \to X$ and $F \to X$ on $X \in \mathcal{V}$ one has

$$\tilde{c}_i(E) \circ \tilde{c}_j(F) = \tilde{c}_j(F) \circ \tilde{c}_i(E)$$

for any (i, j).

(1) For any line bundle L, $\tilde{c}_1(L)$ agrees with the one given in axiom (PB) of definition 1, applied to A_*.

(2) For any l.c.i. morphism $Y \to X \in \mathbf{Sch}_k$, and any vector bundle $E \to X$ over X one has

$$\tilde{c}_i(f^*E) \circ f^* = f^* \circ \tilde{c}_i(E).$$

(3) If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles over X, then for each integer $n \geq 0$ one has the following equation in $\mathrm{End}(A_*(X))$:

$$\tilde{c}_n(E) = \sum_{i=0}^{n} \tilde{c}_i(E')\, \tilde{c}_{n-i}(E'').$$

(4) For any projective morphism $Y \to X$ in $\mathbf{Sch}_k$ and any vector bundle $E \to X$ over X, one has

$$f_* \circ \tilde{c}_i(f^*E) = \tilde{c}_i(E) \circ f_*.$$

Moreover, the Chern class operators are characterized by the properties (0)-(3).

This construction can be generalized as follows:

Lemma 6. *Let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ and let $\tau = (\tau_i) \in \Pi_{i=0}^{\infty} A_i(k)$, with $\tau_0 = 1$. Then one can define in a unique way, for each $X \in \mathbf{Sch}_k$ and each vector bundle E on X, an endomorphism (of degree zero)*

$$\tilde{c}_\tau(E) : A_*(X) \to A_*(X)$$

such that the following holds:

(0) *Given vector bundles $E \to X$ and $F \to X$ one has*

$$\tilde{c}_\tau(E) \circ \tilde{c}_\tau(F) = \tilde{c}_\tau(F) \circ \tilde{c}_\tau(E).$$

(1) *For a line bundle L one has:*

$$\tilde{c}_\tau(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i\, \tau_i.$$

(2) *For any l.c.i. morphism $Y \to X$ in $\mathbf{Sch}_k$, and any vector bundle $E \to X$ over X one has*

$$\tilde{c}_\tau(f^*E) \circ f^* = f^* \circ \tilde{c}_\tau(E).$$

(3) *If* $0 \to E' \to E \to E'' \to 0$ *is an exact sequence of vector bundles over* X*, then one has:*

$$\tilde{c}_\tau(E) = \tilde{c}_\tau(E') \circ \tilde{c}_\tau(E'').$$

(4) *For any projective morphism* $Y \to X$ *in* $\mathbf{Sch}_k$ *and any vector bundle* $E \to X$ *over* X*, one has*

$$f_* \circ \tilde{c}_\tau(f^*E) = \tilde{c}_\tau(E) \circ f_*.$$

Notation 7. If X is in $\mathbf{Sm}_k$, we have the fundamental class $1_X \in A_{\dim X}(X)$, defined by $1_X := p^*(1)$, where $p : X \to \mathrm{pt}$ is the structure morphism and $1 \in A_0(k)$ is the unit. For a vector bundle $E \to X$, we write $c_i(E)$ for $\tilde{c}_i(E)(1_X)$ and $c_\tau(E)$ for $\tilde{c}_\tau(E)(1_X)$.

Now let A_* be an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ and choose elements $\tau = (\tau_i) \in \Pi_{i=0}^\infty A_i(k)$, with $\tau_0 = 1$. We form the *twisted theory* $A_*^{(\tau)}$ with $A_*^{(\tau)}(X) = A_*(X)$ for each X, with the same push-forward maps f_* and external product $\times$, but with the pull-back $f^*_{(\tau)}$ (for $f : Y \to X$ an l.c.i. morphism) defined by

$$f^*_{(\tau)}(x) := \tilde{c}_\tau(N_f)(f^*(x)).$$

Here $N_f \in K_0(Y)$ is the formal normal bundle of f: if we factor $f : Y \to X$ as a regular embedding $i : Y \to P$ followed by a smooth morphism $q : P \to X$, then

$$N_f := N_i - i^*T_q,$$

where T_q is the dual of the sheaf of relative Kähler differentials $\Omega_{P/X}$.

A direct calculation verifies:

Lemma 8. $A_*^{(\tau)}$ *is an oriented Borel-Moore homology theory on* $\mathbf{Sch}_k$*. The first Chern class operator for this theory,* $\tilde{c}_1^{A^{(\tau)}}$*, is given by*

$$\tilde{c}_1^{A^{(\tau)}}(L) = \tilde{c}_\tau^A(L) \circ \tilde{c}_1^A(L).$$

5. Steenrod operations

We construct the Steenrod operations by applying the twisting operation to a polynomial extension of $\mathrm{CH}_* \otimes \mathbb{F}_p$.

Fix a prime p and let $b_1^{(p)}, b_2^{(p)}, \ldots$ be indeterminates, with $b_n^{(p)}$ having degree $p^n - 1$; we set $b_0^{(p)} := 1$. Let $\mathbb{F}_p[b^{(p)}]$ be the polynomial ring on the $b_1^{(p)}, b_2^{(p)}, \ldots$ and set

$$\bar{\mathrm{CH}}_* := \mathrm{CH}_* \otimes_{\mathbb{Z}} \mathbb{F}_p$$
$$\bar{\mathrm{CH}}[b^{(p)}]_* := \mathrm{CH}_* \otimes_{\mathbb{Z}} \mathbb{F}_p[b^{(p)}].$$

Form the twisted theory $\bar{\mathrm{CH}}[b^{(p)}]_*^{(b^{(p)})}$, i.e., we take $\tau_{p^n-1} = b_n^{(p)}$ and $\tau_i = 0$ if i is not of the form $p^n - 1$. The universal property of Ω_* gives us the morphism of oriented Borel-Moore homology theories

$$\tilde{S}^{(p)} : \Omega_* \to \bar{\mathrm{CH}}[b^{(p)}]_*^{(b^{(p)})}.$$

Proposition 9. *The map $\tilde{S}^{(p)}$ descends to a morphism of oriented Borel-Moore homology theories*

$$\bar{S}^{(p)} : \bar{\mathrm{CH}}_* \to \bar{\mathrm{CH}}[b^{(p)}]_*^{(b^{(p)})}.$$

Proof. Since CH_* is the universal additive theory, we need only check that the formal group law for $\bar{\mathrm{CH}}[b^{(p)}]_*^{(b^{(p)})}$ is additive. But by Lemma 8, the twisted first Chern class $\tilde{c}_1^{(b^{(p)})}$ is given by

$$\tilde{c}_1^{(b^{(p)})}(L) = \sum_{n=0}^{\infty} \tilde{c}_1^{\mathrm{CH}}(L)^{p^n} b_n^{(p)} \mod p.$$

Since $\tilde{c}_1^{\mathrm{CH}}(L \otimes M) = \tilde{c}_1^{\mathrm{CH}}(L) + \tilde{c}_1^{\mathrm{CH}}(M)$, we have

$$\tilde{c}_1^{(b^{(p)})}(L \otimes M) = \tilde{c}_1^{(b^{(p)})}(L) + \tilde{c}_1^{(b^{(p)})}(M)$$

yielding the additive group law for $\bar{\mathrm{CH}}[b^{(p)}]_*^{(b^{(p)})}$. □

We omit the p from the notation for the rest of this section. Let $R := (r_1, r_2, \dots r_n)$ be a sequence of non-negative integers. Let $b^R := \prod b_i^{r_i}$, $|R| := \sum_i r_i(p^i - 1) = \deg(b^R)$. We can thus write $\bar{S} : \mathrm{CH}_* \to \bar{\mathrm{CH}}[b]_*^{(b)}$ as

$$\bar{S} = \sum_R \bar{S}_R \cdot b^R : \mathrm{CH}_* \to \bar{\mathrm{CH}}[b]_*^{(b)}.$$

Similarly, for a vector bundle $E \to X$, we write the twisted total Chern class endomorphism as

$$\tilde{c}^{(b)}(E) := \sum_R \tilde{c}_R(E) b^R;$$
$$\tilde{c}_R(E) : \mathrm{CH}_* \to \mathrm{CH}_{*-|R|}.$$

We record the principal properties of the map $\bar{S}$ and the maps $\bar{S}_R$; these properties all follow immediately from the fact that $\bar{S}$ is a morphism of oriented Borel-Moore homology theories.

(5.1)

(1) For each R, $\bar{S}_R : \bar{\mathrm{CH}}_* \to \bar{\mathrm{CH}}_{*-R}$ is an natural transformation of functors $\mathbf{Sch}'_k \to \mathbf{Ab}$, i.e., for each $X \in \mathbf{Sch}_k$, $\bar{S}_R(X) : \bar{\mathrm{CH}}_*(X) \to \bar{\mathrm{CH}}_{*-R}(X)$ is additive, and the maps $\bar{S}_R(X)$ commute with the pushforward maps f_* for $f : X \to Y$ projective.

(2) Let $f: Y \to X$ be a l.c.i. morphism. Then
$$\bar{S} \circ f^* = \tilde{c}^{(b)}(N_f) \circ f^* \circ \bar{S}$$
(3) For classes $x \in \bar{\mathrm{CH}}_*(X)$, $y \in \bar{\mathrm{CH}}_*(Y)$,
$$\bar{S}(x \times y) = \bar{S}(x) \times \bar{S}(y) \in \bar{\mathrm{CH}}[b]_*^{(b)}(X \times Y).$$

These properties yield the following formula for $\bar{S}_R$:

Proposition 10. *Let $Z \subset X$ be a subvariety of some $X \in \mathbf{Sch}_k$, let $\tilde{Z} \to Z$ be a resolution of singularities and let $f: \tilde{Z} \to X$ be the evident morphism. Then*
$$\bar{S}_R(1 \cdot Z) = f_*(c_R(-T_{\tilde{Z}})).$$

Proof. Let $p: \tilde{Z} \to \mathrm{pt}$ be the structure morphism. $1 \cdot Z$ is clearly $f_*(1_{\tilde{Z}})$. Also, $\bar{S}_R(\mathrm{pt}) = 0$ for all $R \neq \emptyset$ by reasons of degree, i.e., $\bar{S}(\mathrm{pt}) = \mathrm{id}$. Thus
$$\begin{aligned}
\bar{S}(1 \cdot Z) &= \bar{S}(f_*(p^*(1))) \\
&= f_*(\bar{S}(p^*(1))) \\
&= f_*(\tilde{c}^{(b)}(N_p)(p^*(\bar{S}(1)))) \\
&= f_*(\tilde{c}^{(b)}(N_p)(1_{\tilde{Z}})) \\
&= f_*(c^{(b)}(-T_{\tilde{Z}})).
\end{aligned}$$
Taking the coefficient of b^R finishes the proof. □

This last proposition completely describes the operations $\bar{S}_R$; by basic properties of Brosnan's Steenrod operations outlined in [1, Section 8], this also shows that they coincide with the Steenrod operations defined by Brosnan. Of course, Brosnan's operations have the advantage that they are defined in arbitrary characteristic.

6. Mod p characteristic classes

An integral lifting of the construction of the last section gives rise to interesting mod p characteristic classes in cobordism.

For $p: X \to \mathrm{pt}$ smooth and projective of dimension d, we set $[X] := p_*(1_X) \in \Omega_d(k)$. By [2, Lemma 4.15], $\Omega_d(k)$ is generated by the classes $[X]$.

Fix a prime p, let $b_n = b_n^{(p)}$, etc. Form the twisted theory $\mathrm{CH}_*[b]^{(b)}$, giving us the morphism of oriented Borel-Moore theories on $\mathbf{Sch}_k$,
$$S: \Omega_* \to \mathrm{CH}[b]_*^{(b)},$$
which we may write as $S = \sum_R S_R \cdot b^R$.

Since we are using integral coefficients rather than mod p coefficients, the map S will not descend to CH_*, however, the fact that it does modulo p has as consequence:

Lemma 11. *Let $\Omega_{>0}(k)$ be the ideal of $\Omega_*(k)$ generated by elements of degree >0. For all $R \neq \emptyset$, $S_R(\Omega_{>0}(k))$ is contained in $p\mathrm{CH}[b]_*^{(b)}(k)$.*

Proof. Since $\Omega_*(k) = \mathbb{L}$ and $\mathrm{CH}_*(k) = \Omega_*(k) \otimes_{\mathbb{L}} \mathbb{Z}$, $\Omega_{>0}(k)$ is the kernel of the canonical map $\Omega_*(k) \to \mathrm{CH}_*(k) = \mathbb{Z}$. Since $S \mod p$ factors through CH_*, the result follows. $\square$

Using $1 \in \mathrm{CH}_0(k)$ as a generator, we have the canonical identification of $\mathrm{CH}[b]_*^{(b)}(k)$ with the polynomial ring $\mathbb{Z}[b_1, b_2, \ldots]$. Thus, setting $s_R := (S_R/p) \mod p$, we have for each $R \neq \emptyset$ the well-defined homorphism

$$s_R : \Omega_{|R|}(k) \to \mathbb{F}_p.$$

Explicitly, Lemma 11 shows that, for each $R \neq \emptyset$, and each smooth projective X of dimension $|R|$ over k, $p| \deg(c_R(-T_X))$, and s_R is the unique homomorphism with

$$s_R([X]) = \frac{1}{p} \cdot \deg(c_R(-T_X)) \mod p.$$

The proof of these statements follows by a computation similar to that used in the proof of Proposition 10, and the fact that $\Omega_*(k)$ is generated by the classes $[X]$. We write $s_R(X)$ for $s_R([X])$.

The mod p characteristic numbers s_R have a nice primitivity property:

Lemma 12. *Let $X = Y \times Z$ be a product of smooth projective varieties Y and Z over k, with $\dim Y > 0$ and $\dim Z > 0$. Then for all $R \neq \emptyset$, $s_R(X) = 0$.*

Proof. Since S is a morphism of oriented Borel-Moore homology theories, S has the same formal properties (5.1) as $\bar{S}$, in particular

$$\begin{aligned} S([X]) &= S([Y \times Z]) \\ &= S([Y] \times [Z]) \\ &= S([Y]) \times S([Z]) \end{aligned}$$

Thus

$$\begin{aligned} s_R(X) &= \frac{1}{p} S_R([X]) \mod p \\ &= \sum_{\substack{R'+R''=R \\ |R'|=\dim Y,\ |R''|=\dim Z}} \frac{1}{p} S_{R'}([Y]) \cdot S_{R''}([Z]) \mod p \\ &= \sum_{\substack{R'+R''=R \\ |R'|=\dim Y,\ |R''|=\dim Z}} s_{R'}(Y) \cdot p s_{R''}(Z) \mod p \\ &= 0 \end{aligned}$$

□

7. Degree formulas

Using the generalized degree formula of [2], it is easy to show that the mod p characteristic numbers s_R satisfy a "degree formula". We fix a characteristic zero base field k and a prime number p.

For a k-scheme X of finite type over k, we let $I(X)$ denote the ideal in $\mathbb{Z}$ generated by the field extension degrees $[k(x) : k]$, as x runs over the closed points of X. Let $i(X)$ be the ideal $(p, I(X))$.

Theorem 13. *Let $f : Y \to X$ be a k-morphism of smooth projective varieties over k. Let $R = (r_1, \ldots, r_n)$ be a sequence of non-negative integers with $|R| = \dim X = \dim Y$. Then*

$$s_R(Y) \equiv \deg f \cdot s_R(X) \mod i(X).$$

Proof. It follows from the generalized degree formula [2, Theorem 8] that there are smooth projective k-schemes $\tilde{Z}_i$, morphisms $f_i : \tilde{Z}_i \to X$ and elements $\alpha_i \in \Omega_*(k)$, $i = 1, \ldots, m$, such that

(1) $\tilde{Z}_i \to f_i(\tilde{Z}_i)$ is birational.
(2) $\dim \tilde{Z}_i < \dim X$
(3) $f_*(1_Y) = \deg f \cdot 1_X + \sum_{i=1}^m \alpha_i \cdot f_{i*}(1_{\tilde{Z}_i})$.

Pushing forward the last identity to $\Omega_*(k)$ gives

$$[Y] = \deg f \cdot [X] + \sum_i \alpha_i \cdot [\tilde{Z}_i].$$

Since $\dim \tilde{Z}_i < \dim X$, it follows that α_i is in the ideal $\Omega_{*>0}(k)$ for each i. As $\Omega_d(k)$ is generated by the classes $[W]$, with W smooth and projective over k and $\dim W = d$, it follows that each α_i is a sum

$$\alpha_i = \sum_j n_{ij}[W_{ij}]$$

with W_{ij} smooth and projective over k, $\dim W_{ij} > 0$, and the n_{ij} are integers.

If $\dim \tilde{Z}_i > 0$, then it follows from Lemma 12 that $s_R(\alpha_i \cdot [\tilde{Z}_i]) = 0$. Thus

$$s_R(Y) = \deg f \cdot s_R(X) + {\sum_i}' s_R(\alpha_i \cdot [\tilde{Z}_i]),$$

where ${\sum_i}'$ means the sum over all i such that $\dim \tilde{Z}_i = 0$.

In case $\dim \tilde{Z}_i = 0$, it follows from (1) that $f_i : \tilde{Z}_i \to f_i(\tilde{Z}_i)$ is an isomorphism, and f_i thus identifies $\tilde{Z}_i$ with a closed point z_i of X. It thus follows that $[\tilde{Z}_i] = [z_i] = [k(z_i) : k] \cdot 1 \in \Omega_0(k) = \mathbb{Z}$ (*cf.* [2, Lemma 4.7]).

Thus

$$s_R(\alpha_i \cdot [\tilde{Z}_i]) = [k(z_i) : k] \cdot s_R(\alpha_i) \equiv 0 \mod i(X),$$

and we have

$$s_R(Y) \equiv \deg f \cdot s_R(X) \mod i(X)$$

as desired. □

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