A SURVEY OF ALGEBRAIC COBORDISM

MARC LEVINE

Abstract. This paper is based on the author’s lecture at the ICM Satellite Conference in Algebra at Suzhou University, August 30-September 2, 2002, describing a joint work with Fabien Morel.

1. Introduction

Together with Fabien Morel, we have constructed a theory of algebraic cobordism, which lifts the theory of complex cobordism to algebraic varieties over a field of characteristic zero, as the theory of the Chow ring lifts singular cohomology, or the theory of algebraic $K_0$ lifts the topological $K_0$. In this paper, we give an introduction to this theory for the non-expert. For those interested in more details, we refer the reader to [5, 6, 7, 8].

We would like to thank the organizers of the Satellite Conference in Algebra at Suzhou University for providing a warm and stimulating atmosphere at the conference and for promoting the continued exchange of ideas among mathematicians.

2. Complex cobordism

We recall that the Thom space $\text{Th}(E)$ of a vector bundle $E \to X$ is the quotient space $D(E)/S(E)$, where $D(E)$ and $S(E)$ are the disk bundle and sphere bundle

$$D(E) := \{ v \in E \mid \|v\| \leq 1 \},$$
$$S(E) := \{ v \in E \mid \|v\| = 1 \},$$

with respect to a chosen metric on $E$. It is easy to see that $\text{Th}(E)$ is independent of choice of metric; in fact, one can define $\text{Th}(E)$ without reference to a metric as

$$\text{Th}(E) := \mathbb{P}(E \oplus e_{\mathbb{C}})/\mathbb{P}(E),$$

where $\mathbb{P}$ is the associated bundle of projective spaces, and $e_{\mathbb{C}}$ is the trivial complex line bundle.

The author gratefully acknowledges support from the N.S.F., grant DMS 0140445.
Taking the example of the universal rank $n$ complex vector bundle $\mathcal{U}_n \rightarrow \text{Gr}(n, \infty)$, we have the $2n$th universal Thom space

$$MU_{2n} := \text{Th}(\mathcal{U}_n).$$

The bundle $\mathcal{U}_n \oplus e_C \rightarrow \text{Gr}(n, \infty)$ is classified by an inclusion $i_n : \text{Gr}(n, \infty) \rightarrow \text{Gr}(n+1, \infty)$, giving the isomorphism

$$\mathcal{U}_n \oplus e_C \cong i_n^* \mathcal{U}_{n+1}.$$

This in turn yields the map of Thom spaces $\text{Th}(\mathcal{U}_n \oplus e_C) \rightarrow \text{Th}(\mathcal{U}_{n+1})$. In addition, one has the homeomorphism $\text{Th}(\mathcal{U}_n \oplus e) \cong S^2 \wedge \text{Th}(\mathcal{U}_n)$, which yields the connecting maps

$$S^2 \wedge MU_{2n} \xrightarrow{\epsilon_n} MU_{2n+2}.$$

We set $MU_{2n+1} := S^1 \wedge MU_{2n}$. The sequence of spaces

$$MU_0 = \text{pt}., MU_1, MU_2, \ldots, MU_{2n}, MU_{2n+1}, \ldots$$

with attaching maps

$$S^1 \wedge MU_{2n} = M_{2n+1} \xrightarrow{id} M_{2n+1}$$

$$S^1 \wedge MU_{2n+1} = S^2 \wedge MU_{2n} \xrightarrow{\epsilon_n} MU_{2n+2}$$

defines the Thom spectrum $MU$; for a topological space $X$, the complex cobordism of $X$ is defined as the set of stable homotopy classes of pointed maps

$$MU^n(X) := \lim_{N \rightarrow \infty} [\Sigma^N X_+, MU_{N+n}].$$

Sending $X$ to the graded group $MU^*(X)$ evidently defines a contravariant functor. In fact, this satisfies the axioms of a cohomology theory on topological spaces.

2.1. Quillen’s construction. Restricting to differentiable manifolds, the cohomology theory $MU^*$ was given a more geometric flavor by Quillen [9], following work of Thom. In [9] Quillen describes $MU^n(X)$ as generated (for $n$ even) by the set of complex oriented proper maps $f : Y \rightarrow X$ of codimension $n$. Here a complex orientation is given by factoring $f$ through a closed embedding $i : Y \rightarrow E$, where $E \rightarrow X$ is a complex vector bundle, together with a complex structure on the normal bundle $N_i$ of $Y$ in $E$ (for $n$ even). For $n$ odd, one puts a complex structure on $N_i \oplus e_R$. One then imposes the cobordism relation by identifying $f^{-1}(X \times 0)$ and $f^{-1}(X \times 1)$, if $f : Y \rightarrow X \times \mathbb{R}$ is a proper complex oriented map, transverse to $X \times \{0,1\}$.

From this definition, it becomes apparent that $MU^*(X)$ has natural push-forward maps $f_* : MU^n(X) \rightarrow MU^{n-d}(X')$ for a proper complex-oriented map $f : X \rightarrow X'$ of relative dimension $d$. Pull-back is defined
by noting that, given a differentiable map \( g : X' \to X \), and a complex-oriented map \( f : Y \to X \), one can change \( f \) by a homotopy to make \( f \) transverse to \( g \). One then defines \( g^*(f) \) as the projection \( Y \times_X X' \to X' \).

One also has the compatibility \( g^* f_* = f'_* g^* \) for cartesian squares

\[
\begin{array}{ccc}
X' \times_X Y & \xrightarrow{g'} & Y \\
\downarrow f' & & \downarrow f \\
X' & \xrightarrow{g} & X
\end{array}
\]

with \( f \) proper and complex-oriented, and \( g \) transverse to \( f \).

Disjoint union defines the addition in \( MU^*(X) \), and “reversing” the orientation defines the minus. Taking products of maps defines external products

\[
MU^n(X) \otimes MU^m(Y) \to MU^{n+m}(X \times Y).
\]

Taking \( X = Y \) and pulling back by the diagonal defines cup products on \( MU^*(X) \), making \( MU^*(X) \) a graded commutative ring with identity \( 1_X = \text{id}_X \).

2.2. Chern classes and the projective bundle formula. Let \( L \to X \) be a complex line bundle on a differentiable manifold \( X \), and let \( s : X \to L \) be the zero section. Define \( c_1(L) \in MU^2(X) \) by

\[
c_1(L) = s^* s_*(1_X).
\]

One has the projective bundle formula: Let \( E \to X \) be a rank \( n + 1 \) vector bundle on \( X \), \( L \to \mathbb{P}(E) \) the tautological line bundle on \( \mathbb{P}(E) \), and let \( \xi = c_1(L) \). Then \( MU^*(\mathbb{P}(E)) \) is a free \( MU^*(X) \)-module, with basis \( 1, \xi, \ldots, \xi^n \). In fact, \( MU^* \) is the universal cohomology theory with Chern classes and a projective bundle formula.

2.3. The formal group law. It is not the case that \( c_1(L \otimes M) = c_1(L) + c_1(M) \) ! To make this failure precise, one considers the universal case of the tautological complex line bundle \( L_n \) on \( \mathbb{P}^n \) and the limit bundle \( L_\infty \) on \( \mathbb{P}^\infty \). Letting \( \xi_n = c_1(L_n) \), sending \( u \) to \( \xi_n \) defines an isomorphism

\[
MU^*(\mathbb{P}^n) \cong MU^*(pt.)/u^{n+1}.
\]

Taking limits gives \( MU^*(\mathbb{P}^\infty) \cong MU^*(pt.)[[u]] \), with \( u \) mapping to \( \xi_\infty \). Similarly, we have \( MU^*(\mathbb{P}^\infty \times \mathbb{P}^\infty) \cong MU^*(pt.)[[u, v]] \), with \( u \) going to \( c_1(p_1^* L_\infty) \), and \( v \) to \( c_1(p_2^* L_\infty) \). We have the power series \( F(u, v) \in MU^*(pt.)[[u, v]] \) defined as the element corresponding to \( c_1(p_1^* L_\infty \otimes p_2^* L_\infty) \). Thus, for any two line bundles \( L, M \), we have

\[
c_1(L \otimes M) = F(c_1(L), c_1(M)).
\]
From the elementary properties of tensor product, we see that $F$ defines a commutative formal group law on $MU^*(pt.)$, that is
\[
F(u, 0) = F(0, u) = u, \\
F(u, v) = F(v, u), \\
F(u, F(v, w)) = F(F(u, v), w).
\]
In fact, Quillen [9] has shown this is the universal formal group law, so the failure of $c_1$ to be additive is as complete as it can possibly be.

2.4. The Lazard ring. The coefficient ring of the universal formal group was first studied by Lazard [4], and is thus known as the Lazard ring $\mathbb{L}$. The Lazard ring is known to be a polynomial ring over $\mathbb{Z}$ in infinitely many variables
\[
\mathbb{L} = \mathbb{Z}[x_1, x_2, \ldots].
\]
$\mathbb{L}$ is naturally a graded ring with $\deg(x_i) = -i$.

Explicitly, one constructs $\mathbb{L}$ and the universal group law $F_L$ as follows: Let $\mathbb{L} = \mathbb{Z}[\{A_{ij} \mid i, j \geq 1\}]$, where we give $A_{ij}$ degree $-i - j + 1$, and let $F \in \mathbb{L}[[u, v]]$ be the power series $F = u + v + \sum ij A_{ij}u^iv^j$. Let
\[
\mathbb{L} = \mathbb{L}/F(u, v) = F(v, u), \quad F(F(u, v), w) = F(u, F(v, w)).
\]
and let $F_L$ be the image of $F$ in $\mathbb{L}[[u, v]]$. Then $(F_L, \mathbb{L})$ is evidently the universal commutative dimension 1 formal group; $\mathbb{L}$ is thus the Lazard ring.

3. Oriented cohomology theories

We abstract the properties of $MU^*$ in an algebraic setting. Fix a base field $k$ and let $\text{Sm}_k$ denote the category of smooth quasi-projective $k$-schemes.

Definition 3.1. An oriented cohomology theory on $\text{Sm}_k$ is given by
\begin{itemize}
  \item[D1.] A contravariant functor $A^*$ from $\text{Sm}_k$ to graded rings.
  \item[D2.] For each projective morphism $f : X \to Y$ in $\text{Sm}_k$ of relative codimension $d$ an $A^*(Y)$-linear push-forward homomorphism $f_* : A^*(X) \to A^{*+d}(Y)$.
\end{itemize}
These satisfy:
\begin{itemize}
  \item[A1.] $(f \circ g)_* = f_* \circ g_*$. \quad $\text{id}_* = \text{id}$.
\end{itemize}
A2. Let 

\[
\begin{array}{ccc}
W & \xrightarrow{g'} & X \\
m & \downarrow & f' \\
Y & \xrightarrow{g} & Z
\end{array}
\]

be a cartesian square, with \(X, Y, Z\) and \(W\) in \(\text{Sm}_k\), and with \(f\) projective. Then

\[g^* f_* = f'_* g'^*.\]

A3. **Projective bundle formula.** For a line bundle \(L\) on \(X \in \text{Sm}_k\) with zero-section \(s : X \to L\), define

\[c_1(L) := s^* s_* (1_X) \in A^1(X).\]

Let \(E \to X\) be a rank \(n + 1\) vector bundle over \(X \in \text{Sm}_k\), and \(\mathbb{P}(E) \to X\) the associated projective bundle. Let \(\xi = c_1(O(1))\). Then \(A^*(\mathbb{P}(E))\) is a free module over \(A^*(X)\) with basis \(1, \xi, \ldots, \xi^n\).

A4. **Homotopy.** Let \(p : V \to X\) be an \(\mathbb{A}^n\) bundle over \(X \in \text{Sm}_k\). Then \(p^* : A^*(X) \to A^*(V)\) is an isomorphism.

**Remark 3.2.** The reader should note that an oriented cohomology theory as defined above is not a cohomology theory in the usual sense, as there is no requirement of a Mayer-Vietoris property. One should perhaps call the above data an oriented pre-cohomology theory, but we have chosen not to use this terminology.

4. **The formal group law**

Let \(A^*\) be an oriented cohomology theory. We have

\[\lim_{n,m} A^*(\mathbb{P}^n \times \mathbb{P}^m) \cong A^*(k)[[u, v]],\]

the isomorphism sending \(u\) to \(c_1(p_1^*O(1))\) and \(v\) to \(c_1(p_2^*O(1))\). The class of \(c_1(p_1^*O(1) \otimes p_2^*O(1))\) thus gives a power series \(F_A(u, v) \in A^*(k)[[u, v]]\) with

\[c_1(p_1^*O(1) \otimes p_2^*O(1)) = F_A(c_1(p_1^*O(1)), c_1(p_2^*O(1))).\]

By naturality, we have, for \(X \in \text{Sm}_k\) with line bundles \(L, M\), the identity

\[c_1(L \otimes M) = F_A(c_1(L), c_1(M)).\]

In addition, \(F_A(u, v) = u + v \mod uv\), \(F_A(u, v) = F_A(v, u)\), and \(F_A(F_A(u, v), w) = F_A(u, F_A(v, w))\). Thus, \(F_A\) gives a formal group law with coefficients in \(A^*(k)\). In particular, for each oriented cohomology
theory $A$, there is a canonical ring homomorphism $\phi_A : \mathbb{L} \to A^*(k)$ classifying the group law $F_A$.

Note that $c_1 : \text{Pic}(X) \to A^1(X)$ is a group homomorphism if and only if $F_A(u, v) = u + v$; we call such a theory ordinary.

**Examples 4.1.** (1) The Chow ring of algebraic cycles modulo rational equivalence, $\text{CH}^*$, and étale cohomology $H_{\text{ét}}^2(-, \mathbb{Z}/n(*))$ (also with $\mathbb{Z}_l(*)$ or $\mathbb{Q}_l(*)$ coefficients). These are all ordinary theories. Similarly, if $\sigma : k \to \mathbb{C}$ is an embedding, and $X$ is in $\text{Sm}_k$, let $X^\sigma(\mathbb{C})$ be the complex manifold of $\mathbb{C}$-points on $X \times_k \mathbb{C}$. We have the ordinary theory

$$X \mapsto H^*(X^\sigma(\mathbb{C}), \mathbb{Z})$$

where $H^*(-, \mathbb{Z})$ is singular cohomology.

(2) For $X \in \text{Sm}_k$, we have the Grothendieck group of algebraic vector bundles on $X$, $K_0^{\text{alg}}(X)$. For a projective morphism $f : Y \to X$, we have the pushforward $f_* : K_0^{\text{alg}}(Y) \to K_0^{\text{alg}}(X)$, defined by sending $E$ to the alternating sum $\sum_i (-1)^i [R^i f_*(E)]$. Here, we need to identify $K_0^{\text{alg}}(X)$ with the Grothendieck group of coherent sheaves on $X$, for which we require $X$ to be regular (e.g. smooth over $k$).

This does not define an oriented cohomology theory, since there is no natural grading on $K_0^{\text{alg}}$ which respects the pushforward maps in the proper manner. To correct this, we adjoin a variable $\beta$ (of degree -1), and its inverse $\beta^{-1}$, and define

$$f_*([E]\beta^n) := f_*([E])\beta^{n-d}$$

for $f : Y \to X$ projective, $d = \dim_k X - \dim_k Y$. This defines the oriented cohomology theory theory $K_0[\beta, \beta^{-1}]$.

$K_0[\beta, \beta^{-1}]$ is not an ordinary theory, in fact, its formal group law is the multiplicative group

$$F_{K_0}(u, v) = u + v - \beta uv.$$ 

To see this, it follows from the definition of $c_1$ that $c_1(L) = \beta^{-1}(1 - L^\vee)$, where $L^\vee$ is the dual line bundle. If $L = \mathcal{O}_X(D)$, $M = \mathcal{O}_X(E)$ for smooth transverse divisors $D$ and $E$, we have the exact sequences

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$$

and

$$0 \to \mathcal{O}_X(-D - E) \to \mathcal{O}_X(-E) \oplus \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \otimes_{\mathcal{O}_X} \mathcal{O}_E \to 0.$$
From these, one finds the relation in $K_0^{alg}$

$$[1] - [(L \otimes M)^\vee] = ([1] - [L^\vee]) + ([1] - [M^\vee]) - ([1] - [L^\vee]) \cdot ([1] - [M^\vee]),$$

which yields the stated group law.

(3) As in (1), let $\sigma : k \to \mathbb{C}$ be an embedding. We have the oriented cohomology theories

$$X \mapsto K^0_{top}(X^\sigma(\mathbb{C}))[\beta, \beta^{-1}]$$

$$X \mapsto MU^*(X^\sigma(\mathbb{C}))$$

These are both extraordinary theories (i.e., not ordinary). The group law for $K^0_{top}$ is the multiplicative group, and for $MU^*$ the universal group law.

5. Algebraic cobordism

Let $P_{Sch_k}$ be the category with objects finite type $k$-schemes, and with morphisms the projective maps $Y \to X$; let $P_{Sm_k}$ be the full subcategory of $P_{Sch_k}$ with objects the smooth quasi-projective schemes over $k$. We can now state our main result on algebraic cobordism:

**Theorem 5.1** ([5, 6, 7]). Let $k$ be a field of characteristic zero.

1. There is a universal oriented cohomology theory $\Omega^*$ on $Sm_k$.
2. The homomorphism $\phi_\Omega : \mathbb{L} \to \Omega^*(k)$ is an isomorphism.
3. For $X$ of dimension $d$, write $\Omega_n(X) := \Omega^{d-n}(X)$. Then the covariant functor $\Omega_*$ on $P_{Sm_k}$ extends to a covariant functor on $P_{Sch_k}$ satisfying
   (a) $\Omega_*$ has pull-back homomorphisms for smooth quasi-projective morphisms, compatible in cartesian squares.
   (b) Let $i : Z \to X$ be a closed embedding with open complement $j : U \to X$. Then the sequence

   $$\Omega_*(Z) \xrightarrow{j_*} \Omega_*(X) \xrightarrow{i_*} \Omega_*(U) \to 0.$$ 

is exact.

*Idea of construction:* For a finite type $k$-scheme $X$, let $\mathcal{M}(X)$ be the set of isomorphism classes of morphisms $f : Y \to X$, with $Y \in Sm_k$ and $f$ projective. $\mathcal{M}(X)$ is a graded monoid under disjoint union, with $f : Y \to X$ in degree $\text{dim}_k(Y)$, let $\mathcal{M}(X)^+$ be the group completion. Composition with a projective morphism $g : X \to X'$ makes $\mathcal{M}^+$ a functor on $P_{Sm_k}$.

We construct $\Omega_*(X)$ as a quotient of $\mathcal{M}(X)^+$ in three steps:
(1) Impose the relation of “classical cobordism”: $f^{-1}(0) = f^{-1}(1)$ for $f : W \to X \times \mathbb{A}^1$, $W \in \text{Sm}_k$, $f$ projective, and $f$ transverse with respect to the inclusion $X \times \{0,1\} \to X \times \mathbb{A}^1$.

(2) For $L \to X$ a globally generated line bundle and $f : Y \to X$ in $\mathcal{M}(X)$, let $i : D \to Y$ be the zero locus of a general section of $f^* L$. Set $c_1(L)(f) := f \circ i : D \to X$. One checks that this is well-defined modulo the relation of classical cobordism.

Impose the universal formal group law:

$$c_1(L \otimes M)(f) = F_{L}(c_1(L), c_1(M))(f)$$

for globally generated line bundles $L$ and $M$ on $X$ and $f: Y \to X$ in $\mathcal{M}(X)$.

(3) Impose the “Gysin relation”, by identifying $c_1(\mathcal{O}_W(D))(\text{id}_W)$ with the class of $i : D \to W$ for $D$ a smooth divisor on $W$.

Remarks 5.2. (1) The above gives the rough outline of a somewhat simplified version of the actual construction. We refer the reader to [7, 8] for more details.

(2) The restriction to characteristic zero in Theorem 5.1 arises from a heavy use of resolution of singularities [3]. In addition, the weak factorization theorem of [1] is used in an essential way in the proof of Theorem 5.1(2).

6. Degree formulas

In the paper [10], Rost made a number of conjectures based on the theory of algebraic cobordism in the Morel-Voevodsky stable homotopy category; assuming these conjecture, Rost is able to construct the so-called splitting varieties which play a crucial role in Voevodsky’s approach to proving the Bloch-Kato conjecture. Many of Rost’s conjectures have been proved by homotopy-theoretic means (cf. [2]); our construction of an algebro-geometric cobordism gives an alternate proof of these results, and settles many remaining open questions as well. We give a sampling of some of these results.

6.1. The generalized degree formula. All the degree formulas follow from the “generalized degree formula”. Before stating this result, we first define the degree homomorphism

$$\text{deg} : \Omega_*(X) \to \Omega_*(k).$$

We assume the base-field $k$ has characteristic zero.

Let $X$ be an irreducible finite type $k$-scheme and let $i_x : x \to X$ be the generic point of $X$, with structure map $p_x : x \to \text{Spec} k$. By
Theorem 5.1, we have the commutative diagram

\[
\begin{array}{ccc}
\Omega_* (k) & \xrightarrow{p^*_x} & \Omega_* (k(x)) \\
\downarrow{\phi_{\Omega/k}} & & \downarrow{\phi_{\Omega/k(x)}} \\
\phi_{\Omega/k} \rightarrow & & \phi_{\Omega/k(x)} \\
\end{array}
\]

with \(\phi_{\Omega/k}\) and \(\phi_{\Omega/k(x)}\) isomorphisms. Thus the base-change homomorphism \(p^*_x: \Omega_* (k) \to \Omega_* (k(x))\) is also an isomorphism.

Let \(f: Y \to X\) be in \(\mathcal{M}(X)\), with \(X\) irreducible. Define \(\deg f \in \Omega_* (k)\) to be the element mapping to \(f_x: Y \times_X x \to x\) in \(\Omega_* (k(x))\) under the isomorphism \(p^*_x: \Omega_* (k) \to \Omega_* (k(x))\). More generally, if \(\eta\) in any element of \(\Omega_* (X)\), let \(\deg(\eta) \in \Omega_* (k)\) be the element with \(p^*_x(\deg(\eta)) = i^*_x \eta \in \Omega_* (k(x))\).

Theorem 6.2 (Generalized degree formula). Let \(X\) be an irreducible finite type \(k\)-scheme, and let \(\eta\) be in \(\Omega_* (X)\). Let \(f_0: B_0 \to X\) be a resolution of singularities of \(X\), with \(B_0\) quasi-projective over \(k\). Then there are \(a_i \in \Omega_* (k), f_i: B_i \to X\) in \(\mathcal{M}(X)\), \(i = 1, \ldots, s\), such that

1. \(f_i: B_i \to f(B_i)\) is birational and \(f(B_i)\) is a proper closed subset of \(X\), \(i = 1, \ldots, s\).
2. \(\eta - (\deg \eta) [f_0] = \sum_i a_i [f_i]\) in \(\Omega_* (X)\).

Proof. It follows from the definitions of \(\Omega_*\) that, for \(X\) an irreducible finite type \(k\)-scheme, we have

\[
\Omega_* (k(X)) = \lim_{\text{U}} \Omega_* (U),
\]

where the limit is over dense open subschemes \(U\) of \(X\), and \(\Omega_* (k(X))\) is the value at \(\text{Spec} k(X)\) of the functor \(\Omega_*\) on finite type \(k(X)\)-schemes. Thus, there is a smooth open subscheme \(j: U \to X\) of \(X\) such that \(j^* \eta = (\deg \eta) [U]\) in \(\Omega_* (U)\). Shrinking \(U\) if necessary, we may assume that \(B_0 \to X\) is an isomorphism over \(U\). Thus, \(j^*(\eta - (\deg \eta) [f_0]) = 0\) in \(\Omega_* (U)\).

Let \(W = X \setminus U\). From the localization sequence

\[
\Omega_* (W) \xrightarrow{i_*} \Omega_* (X) \xrightarrow{j^*} \Omega_* (U)
\]

we find an element \(\eta_1 \in \Omega_* (W)\) with \(i_* (\eta_1) = \eta - (\deg \eta) [f_0]\), and noetherian induction completes the proof. \(\square\)

Remarks 6.3. (1) If \(X\) is in \(\text{Sm}_k\), we can take \(f_0 = \text{id}_X\), giving the formula

\[
\eta - (\deg \eta) [\text{id}_X] = \sum_{i=1}^m a_i [f_i]
\]
in $\Omega_*(X)$.

(2) If $X$ is in $\text{Sm}_k$, and $\eta = [f]$ for some $f : Y \to X$ in $\mathcal{M}(X)$, we have

$$[f] - (\deg f)[\text{id}_X] = \sum_{i=1}^m a_i[f]_i$$

in $\Omega_*(X)$.

(3) If $f : Y \to X$ is in $\mathcal{M}(X)$, and $\dim Y = \dim X$, then $\deg f$ is an integer, namely, the usual degree of $f$ if $f$ is dominant, and zero if $f$ is not dominant. Indeed, the map $\Omega_*(k(X)) \to \Omega_*(\overline{k(X)})$ is an isomorphism ($\overline{k(X)}$ the algebraic closure of $k(X)$), and it is clear that the image of $[f]$ in $\Omega_*(\overline{k(X)})$ is $[k(Y) : k(X)] \cdot [\text{Spec } k(X)]$ if $f$ is dominant, and zero if not.

6.4. Classical cobordism and algebraic cobordism. From the universal property of $\Omega^*$, one sees that a homomorphism of fields $\sigma : k \to \mathbb{C}$ yields a natural homomorphism $\mathcal{R}_\sigma : \Omega^*(X) \to MU^{2*}(X^\sigma(\mathbb{C}))$, with $f : Y \to X$ going to the class of the map of complex manifolds $f^\sigma : Y^\sigma(\mathbb{C}) \to X^\sigma(\mathbb{C})$.

Let $P = P(c_1, \ldots, c_d)$ be a degree $d$ (weighted) homogeneous polynomial. If $X$ is smooth and projective of dimension $d$ over $k$, we have the Chern number

$$P(X) := \deg(P(c_1(\Theta^{X^\sigma(\mathbb{C})}), \ldots, c_d(\Theta^{X^\sigma(\mathbb{C})}))).$$

$P(X)$ is in fact independent of the choice of $\sigma$.

Let $s_d$ be the polynomial which corresponds to $\sum_i \xi_i^d$, where $\xi_1, \ldots$ are the Chern roots. The following divisibility is known: If $d = p^n - 1$ for some prime $p$, and $\dim X = d$, then $s_d(X)$ is divisible by $p$.

In addition, for integers $d = p^n - 1$ and $r \geq 1$, there are mod $p$ characteristic classes $t_{d,r}$, with $t_{d,1} = s_d/p \mod p$. The $s_d$ and the $t_{d,r}$ have the following properties:

(6.1)

1. $s_d(X) \in p\mathbb{Z}$ is defined for $X$ smooth and projective of dimension $d = p^n - 1$. $t_{d,r}(X) \in \mathbb{Z}/p$ is defined for $X$ smooth and projective of dimension $rd = r(p^n - 1)$.
2. $s_d$ and $t_{d,r}$ extend to homomorphisms $s_d : \Omega^{-d}(k) \to p\mathbb{Z}$, $t_{d,r} : \Omega^{-rd}(k) \to \mathbb{Z}/p$.
3. If $X$ and $Y$ are smooth projective varieties with $\dim X, \dim Y > 0$, $\dim X + \dim Y = d$, then $s_d(X \times Y) = 0$.
4. If $X_1, \ldots, X_s$ are smooth projective varieties with $\sum_i \dim X_i = rd$, then $t_{d,r}(\prod_i X_i) = 0$ unless $d \mid \dim X_i$ for each $i$. 

Theorem 6.5. Let $f : Y \to X$ be a morphism of smooth projective $k$-schemes of dimension $d$, $d = p^n - 1$ for some prime $p$. Then there is a zero-cycle $\eta$ on $X$ such that

$$s_d(Y) - (\deg f)s_d(X) = p \cdot \deg(\eta).$$

Theorem 6.6. Let $f : Y \to X$ be a morphism of smooth projective $k$-schemes of dimension $rd$, $d = p^n - 1$ for some prime $p$. Suppose that $X$ admits a sequence of surjective morphisms

$$X = X_0 \to X_1 \to \ldots \to X_{r-1} \to X_r = \text{Spec } k$$

such that

1. $\dim X_i = d(r - i)$.
2. Let $\eta$ be a zero-cycle on $X_i \times_{X_{i+1}} \text{Spec } k(X_{i+1})$. Then $p | \deg(\eta)$.

Then

$$t_{d,r}(Y) = \deg(f)t_{d,r}(X).$$

Proof. These two theorems follow easily from the generalized degree formula Theorem 6.2 and the amplifications of Remark 6.3. Indeed, for Theorem 6.5, we have the identity

$$[Y \to X] - (\deg f)[id_X] = \sum_{i=1}^{m} a_i[B_i \to X]$$

in $\Omega_*(X)$ with the $a_i$ in $\Omega_*(k)$ and with $\dim(B_i) < d$ for all $i$. Pushing forward to $\Omega_*(k)$ gives the identity

$$[Y] - (\deg f)[X] = \sum_{i=1}^{m} a_i[B_i]$$

in $\Omega_d(k)$. We can express each $a_i$ as a sum

$$a_i = \sum_l n_{il}[Y_{il}],$$

where the $Y_{il}$ are smooth projective varieties over $k$. Applying $s_d$ gives

$$s_d(Y) - \deg(f)s_d(X) = \sum_{i,l} n_{il} s_d(Y_{il} \times B_i).$$

As $\dim(B_i) < d$ for all $i$, we have $\dim(Y_{il}) > 0$ for all $i, l$.

Since $s_d$ vanishes on non-trivial products, only the terms with $B_i$ a point $z_i$ of $X$ survive in this last sum. Rewriting the sum, this gives

$$s_d(Y) - \deg(f)s_d(X) = \sum_j m_j s_d(Y_j) \deg(z_j)$$
for smooth projective dimension $d$ $k$-schemes $Y_j$, integers $m_j$ and points $z_j$ of $X$. Since $s_d(Y_j) = pn_j$ for suitable integers $n_j$, we have

$$s_d(Y) - \deg(f)s_d(X) = p \deg(\sum_j m_jn_jz_j),$$

proving Theorem 6.5.

For Theorem 6.6, we have as before

$$[Y \to X] - (\deg f)[\text{id}_X] = \sum_{i=1}^{m} a_i[B_i \to X]$$

in $\Omega_*(X)$. We then decompose each $B_j \to X_2$ using Theorem 6.2, giving

$$[B_j] = a_{0j}[X_2] + \sum_i n_{ij}a_{ij}[B_{ij}].$$

in $\Omega_*(X_2)$. Iterating, we have the identity in $\Omega_*(k)$

$$[Y] - (\deg f)[X] = \sum_{I=(i_0,...,i_r)} n_I[\prod_{j=0}^{r} Y_{ij}],$$

where the $Y_{ij}$ are smooth projective $k$-schemes. In addition, the conditions on the tower imply that for each product $\prod_{j=0}^{r} Y_{ij}$ such that $d \mid \dim Y_{ij}$ for all $j$ has $p \mid n_I$. Thus, arguing as above, we see that $t_{d,r}(Y) = \deg(f)t_{d,r}(X)$. 

\[\square \]

7. Comparison results

In this last section, we explain how one can recover both the Chow ring $\text{CH}^*(X)$ and $K_0^\text{alg}(X)$ from $\Omega^*(X)$.

Suppose we have a formal group law $(f, R)$, giving the canonical homomorphism $\phi_f : \mathbb{L} \to R$. Let $\Omega^*_{(f,R)}$ be the functor

$$\Omega^*_{(f,R)}(X) = \Omega^*(X) \otimes_{\mathbb{L}} R,$$

where $\Omega^*(X)$ is an $\mathbb{L}$-algebra via the isomorphism $\phi_\Omega : \mathbb{L} \to \Omega^*(X)$. For $X$ a finite type $k$-scheme, define $\Omega^*_{(f,R)}(X)$ similarly.

Since $\otimes$ is right exact, the theories $\Omega^*_{(f,R)}$ and $\Omega^*_{(f,R)}$ have the same formal properties as $\Omega^*$ and $\Omega_*$. In particular, $\Omega^*_{(f,R)}$ is an oriented cohomology theory, and $\Omega^*_{(f,R)}$ satisfies localization. The universal property of $\Omega^*$ gives the analogous universal property for $\Omega^*_{(f,R)}$.

In particular, let $\Omega^*_{+}$ be the theory with $(f(u, v), R) = (u + v, \mathbb{Z})$, and let $\Omega^*_{\times}$ be the theory with $(f(u, v), R) = (u + v - \beta uv, \mathbb{Z}[\beta, \beta^{-1}])$. We thus have the canonical natural transformations of oriented theories

$$(7.1) \quad \Omega^*_{+} \to \text{CH}^*; \quad \Omega^*_{\times} \to K_0^\text{alg}[\beta, \beta^{-1}].$$
Theorem 7.1. The natural transformations (7.1) are isomorphisms.

Proof. We define maps backwards:

\[ CH^* \to \Omega^*_{+}; \quad K^\text{alg}_0[\beta, \beta^{-1}] \to \Omega^*_x. \]

For \( CH^* \), we first note that \( \Omega^n(k)_+ = 0 \) for \( n \neq 0 \), since \( \mathbb{L} \cong \Omega^*(k) \), and \( \mathbb{L} \) is generated by the coefficients of the universal group law. To map \( CH^* \) to \( \Omega^*_{+} \), send a subvariety \( Z \subset X \) to the map \( \tilde{Z} \to X \), where \( \tilde{Z} \to Z \) is a resolution of singularities of \( Z \). It follows from localization that the class of \( \tilde{Z} \) in \( \Omega^*_+(X) \) is independent of the choice of the resolution. A similar argument shows that the relations defining \( CH^* \) go to zero.

It is evident that the composition \( CH^* \to \Omega^*_{+} \to CH^* \) is the identity. Finally, the generalized degree formula Theorem 6.2 shows that the map \( CH^* \to \Omega^*_{+} \) is surjective, proving the result.

For \( K^\text{alg}_0[\beta, \beta^{-1}] \), we use a “Chern character” to define the backwards map. In fact, sending a line bundle \( L \) to \( \text{ch}(L) := \sum_i c_i \Omega^i(L)^i \) is easily seen to satisfy

\[ \text{ch}(L \otimes M) = \text{ch}(L)\text{ch}(M). \]

Defining \( \text{ch}(\oplus_i L_i) = \sum_i \text{ch}(L_i) \) and using the splitting principle defines the ring homomorphism \( \text{ch} : K^\text{alg}_0[\beta, \beta^{-1}] \to \Omega^*_x \). One calculates the associated Todd genus as follows: The respective projective bundle formulas give isomorphisms

\[ \lim_{\to} K^\text{alg}_0[\beta, \beta^{-1}](\mathbb{P}^n) \cong \mathbb{Z}[\beta, \beta^{-1}][[u]]; \quad \lim_{\to} \Omega^*_x(\mathbb{P}^n) \cong \mathbb{Z}[\beta, \beta^{-1}][[v]], \]

with \( u \) going to \( 1 - O(-1) \), and \( v \) going to the class of a hyperplane \( H \). Thus, one can write \( \text{ch}(1 - O(-1)) \) as \( \phi([H]) \) in \( \Omega^*_x(\mathbb{P}^n) \) for a unique power series \( \phi(v) = av + \ldots \). Then \( \text{Todd}(v)^{-1} = \phi(v)/v \). But one easily computes \( \phi(v) = v \), giving \( \text{Todd}(v) = 1 \). The Riemann-Roch formula then gives

\[ \text{ch}(\mathcal{O}_Z) = [Z] \in \Omega^*_x(X) \]

for \( Z \to X \) a smooth closed subscheme of \( X \in \text{Sm}_k \). This and localization implies that \( \text{ch} : K^\text{alg}_0[\beta, \beta^{-1}] \to \Omega^*_x \) is surjective; one easily computes that the composition

\[ K^\text{alg}_0[\beta, \beta^{-1}] \to \Omega^*_x \to K^\text{alg}_0[\beta, \beta^{-1}] \]

is the identity, completing the proof. \( \square \)

References


Department of Mathematics, Northeastern University, Boston, MA 02115, U.S.A.

E-mail address: marc@neu.edu