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Abstract

Let k be a number field, and let $A \subset \mathbb{P}^1(k)$ be a finite set of rational points. Deligne and Goncharov have defined the motivic fundamental group $\pi_1^{mot}(X, x)$ of $X := \mathbb{P}^1 \setminus A$ with base-point x being either a k-point of X or a tangential base-point. We extend the construction of the motivic fundamental group to the setting of a smooth S-scheme $p : X \to S$ with section $x : S \to X$, in case S is itself smooth over a field, X satisfies the Beilinson-Soulé vanishing conjectures and the motive of X in $DM(S)_{\mathbb{Q}}$ is a mixed Tate motive. Finally, letting $\mathrm{Gal}(\mathrm{MT}(X))$ be the Tannaka group of the Tannakian category of mixed Tate motives over X, we identify $\pi_1^{mot}(X, x)$ with the kernel of the map $p_* : \mathrm{Gal}(\mathrm{MT}(X)) \to \mathrm{Gal}(\mathrm{MT}(S))$.

Tate motives and the fundamental group

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Introduction

In [11], P. Deligne defined the motivic fundamental group of $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ over a number field k as an object in the category of systems of realizations. This is a Tannakian category over $\mathbb{Q}$, which he constructed as tuples (Betti, de Rham, ℓ-adic, crystalline), with compatibilities between them, a definition close to the one given by U. Jannsen [24]. The Betti-de Rham component is the mixed Hodge structure, defined by J. Morgan [32], on the nilpotent completion $\varprojlim_N \mathbb{Q}[\pi^{\text{top}}(X, x)]/I^N$ of the topological fundamental group $\pi_1^{\text{top}}(X(\mathbb{C}), x)$, for all complex embeddings $k \subset \mathbb{C}$, where the base-point x is either a point in $X(k)$ or a non-trivial tangent vector at $\bar{x} \in (\mathbb{P}^1 \setminus X)(k)$.

A. Beilinson [12, proposition 3.4] showed that for any smooth complex variety X, and for base-point $x \in X(\mathbb{C})$, the ind-system

$$\varinjlim_N \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[\pi^{\mathrm{top}}(X,x)]/I^N, \mathbb{Q}),$$

which is a Hopf algebra over $\mathbb{Q}$, arises from the cohomology of a cosimplicial scheme $\mathcal{P}_x(X)$. As pointed out by Z. Wojtkoviak [39], the Hopf algebra structure on $\varinjlim_N(\mathbb{Q}[\pi^{\mathrm{top}}(X,x)]/I^N)^\vee$ similarly arises from operations on $\mathcal{P}_x(X)$. These key results have many consequences. For instance, one can use $\mathcal{P}_x(X)$ to define the mixed Hodge structure on $\varprojlim_N \mathbb{Q}[\pi^{\mathrm{top}}(X,x)]/I^N$, *cf.* [18]. Even more, the cosimplicial scheme $\mathcal{P}_x(X)$, regardless of the geometry of X, defines an ind-Hopf algebra object $H_{gm}(\mathcal{P}_x(X))$ in Voevodsky's triangulated category of motives $DM_{gm}(k)_{\mathbb{Q}}$ [15, chapter V]; here

$$H_{gm} : \mathbf{Sm}/k^{\mathrm{op}} \to DM_{gm}(k)$$

is the "cohomological motive" functor, dual to the canonical functor $M_{gm} : \mathbf{Sm}/k \to DM_{gm}(k)$. If in addition X is the complement in $\mathbb{P}^1_k$ of a finite set of k-rational points, then $H_{gm}(\mathcal{P}_x(X))_{\mathbb{Q}}$ is actually an ind-Hopf algebra in the full triangulated subcategory $\mathrm{DMT}_{gm}(k)$ of $DM_{gm}(k)_{\mathbb{Q}}$ spanned by the Tate objects $\mathbb{Q}(n)$.

As explained in [31], if a field k satisfies the Beilinson-Soulé vanishing conjecture, that is, if the motivic cohomology $H^p(k, \mathbb{Q}(q))$ vanishes for $p \le 0$, $q > 0$, there is a t-structure defined on $\mathrm{DMT}_{gm}(k)$, the heart of which is the abelian category $\mathrm{MT}(k)$ of mixed Tate motives over k. $\mathrm{MT}(k)$ is a $\mathbb{Q}$-linear, abelian rigid tensor category with the structure of a functorial exact weight filtration W_*. Taking the associated graded object with respect to W_* defines a neutral fiber functor gr^W_*, endowing $\mathrm{MT}(k)$ with the structure of a Tannakian category over $\mathbb{Q}$.

By the work of Borel [6], we know that if k is a number field, then k does satisfy the Beilinson-Soulé conjecture. Thus Beilinson's construction allows one to define the ind-Hopf algebra object $H^0(H_{gm}(\mathcal{P}_x(X))$ in $\mathrm{MT}(k)$, if k is a number field. In [12, théorème 4.4] P. Deligne and A. Goncharov show that the dual $\pi_1^{\mathrm{mot}}(X,x)$ of $H^0(H_{gm}(\mathcal{P}_x(X))_{\mathbb{Q}})$, which is a pro-group scheme object in $\mathrm{MT}(k)$, yields Deligne's original motivic fundamental group upon applying the appropriate realization functors, in case $x \in X(k)$ and $X \subset \mathbb{P}^1_k$ is the complement of a finite set of k-points of $\mathbb{P}^1$. In addition, they show that, even for a tangential base-point x, there is a pro-group scheme object $\pi_1^{\mathrm{mot}}(X,x)$ in $\mathrm{MT}(k)$ which maps to Deligne's motivic fundamental group under realization, without, however, making an explicit construction of $\pi_1^{\mathrm{mot}}(X,x)$ in this case. Using this construction as starting point, they go on to construct a motivic fundamental group for any unirational variety over the number field k, as a pro-group scheme over the larger Tannakian category of Artin-Tate motives $\mathrm{MAT}(k)$ (see [12] for details).

Using recent work of Cisinski-Déglise [10], one now has available a reasonable candidate for the category of motives over a base X, at least if X is a smooth variety over a perfect field k. The resulting triangulated category $DM(X)$ has Tate objects $\mathbb{Z}_X(n)$ which properly compute the motivic cohomology of X (defined using Voevodsky's category $DM_{gm}(k)$). In addition, if $X \subset \mathbb{P}^1_k$ is an open defined over a number field k, then the observation made in [31] carries over to the full triangulated subcategory $\mathrm{DMT}(X)$ of the category $DM(X)_{\mathbb{Q}}$ generated by the Tate objects $\mathbb{Q}_X(n)$. Thus, assuming k is a number field, there is a heart

$\mathrm{MT}(X) \subset \mathrm{DMT}(X)$ which is a $\mathbb{Q}$-linear abelian rigid tensor category, and which receives $\mathrm{MT}(k)$ by pull back via the structure morphism $p : X \to \operatorname{Spec} k$.

By Tannaka duality, we therefore have the Tannaka group schemes $G(\mathrm{MT}(X), \mathrm{gr}^W_*)$ and $G(\mathrm{MT}(k), \mathrm{gr}^W_*)$ over $\mathbb{Q}$, and the functors $p^* : \mathrm{MT}(k) \to \mathrm{MT}(X)$, $x^* : \mathrm{MT}(X) \to \mathrm{MT}(k)$ give a canonical split short exact sequence

$$1 \longrightarrow K \longrightarrow G(\mathrm{MT}(X), \mathrm{gr}^W_*) \underset{x_*}{\overset{p_*}{\rightleftarrows}} G(\mathrm{MT}(k), \mathrm{gr}^W_*) \longrightarrow 1,$$

where K is defined as the kernel of p_* . The splitting x_* also defines an action of the Tannaka group $G(\mathrm{MT}(k), \mathrm{gr}^W_*)$ on K, which lifts the $\mathbb{Q}$ group-scheme K to a group-scheme object K_x in $\mathrm{MT}(k)$.

In [12, section 4.19], Deligne and Goncharov use the group-scheme $\pi_1^{\mathrm{mot}}(X, x)$ over $\mathrm{MT}(k)$ to *define* $\mathrm{MT}(X)$ as the category of $\mathrm{MT}(k)$ representations of $\pi_1^{\mathrm{mot}}(X, x)$. In [12, section 4.22] they ask about the relationship between $\mathrm{MT}(k(\mathbb{P}^1))$, defined as above as a subcategory of Voevodsky's category $DM(k(\mathbb{P}^1))_{\mathbb{Q}}$, and $\varinjlim_{X \subset \mathbb{P}^1} \mathrm{MT}(X)$ (this is the formulation for $k = \bar{\mathbb{Q}}$, in general, one needs to use the Artin-Tate motives MAT). The purpose of this article is to give an answer to this question in the following form: the intrinsic definition of $\mathrm{MT}(X)$ mentioned above is equivalent to the category of K_x-representations in $\mathrm{MT}(k)$, assuming $\mathbb{P}^1 \setminus X$ consists of k-rational points.

We now describe our main result for X as above.

Theorem 1 *Let k be a number field, $A \subset X(k)$ a finite (possibly empty) set of k-points of $\mathbb{P}^1$, let $X := \mathbb{P}^1 \setminus A$ and take $x \in X(k)$. Then the pro-group scheme objects K_x and $\pi_1^{\mathrm{mot}}(X, x)$ are isomorphic as pro-group-schemes in* $\mathrm{MT}(k)$*.*

The equivalence of $\mathrm{MT}(X)$ with the category of K_x-representations in $\mathrm{MT}(k)$ follows directly from this.

In fact, we have a more general result. Let S be a smooth k-scheme, and let $X \to S$ be a smooth S-scheme with a section $x : S \to X$. One can easily extend the construction of $\pi_1^{\mathrm{mot}}(X, x)$ to this setting, if we assume that X satisfies the Beilinson-Soulé vanishing conjectures, and, in addition, that the motive of X in $DM(S)_{\mathbb{Q}}$ is in the Tate subcategory $\mathrm{DMT}(S)$ (see definition 6.3.1 for details). Note that S also satisfies the Beilinson-Soulé vanishing conjectures, as the section x identifies $H^*(S, \mathbb{Q}(*))$ with a summand of $H^*(X, \mathbb{Q}(*))$. Replacing $\mathrm{MT}(k)$ with $\mathrm{MT}(S)$, we have the split exact sequence as above

$$1 \longrightarrow K \longrightarrow G(\mathrm{MT}(X), \mathrm{gr}^W_*) \underset{x_*}{\overset{p_*}{\rightleftarrows}} G(\mathrm{MT}(S), \mathrm{gr}^W_*) \longrightarrow 1.$$

defining the pro-group scheme object K_x of $\mathrm{MT}(S)$. Our main result in this more general setting is

Theorem 2 *Suppose X satisfies the Beilinson-Soulé vanishing conjectures, and suppose that the motive of X in $DM(S)_{\mathbb{Q}}$ is in the Tate subcategory* $\mathrm{DMT}(S)$ *of $DM(S)_{\mathbb{Q}}$. Then the pro-group scheme objects K_x and $\pi_1^{\mathrm{mot}}(X, x)$ are isomorphic as pro-group-schemes in* $\mathrm{MT}(S)$*.*

We now explain the ideas that go into the proof. In [2] S. Bloch and I. Kriz construct a group-scheme $G_{BK}(k)$ over $\mathbb{Q}$, by applying the bar construction to the *cycle algebra* $\mathcal{N}_k :=$

$\mathbb{Q} \oplus \oplus_{r\geq 1}\mathcal{N}_k(r)$. The rth component $\mathcal{N}_k(r)$ of $\mathcal{N}_k$ is a shifted, alternating version of Bloch cycle complex,

$$\mathcal{N}_k^m(r) = z^r(k, 2r-m)^{\mathrm{Alt}};$$

the alternation makes the product on $\mathcal{N}_k$ strictly graded-commutative. The additional grading r is the *Adams grading*. The reduced bar construction gives us the Adams graded Hopf algebra $H^0(\bar{B}(\mathcal{N}_k))$ and $G_{BK}(k)$ is the pro group scheme $\mathrm{Spec}\, H^0(\bar{B}(\mathcal{N}_k))$. Bloch-Kriz define the category of "Bloch-Kriz" mixed Tate motives over k, $\mathrm{MT}_{BK}(k)$, as the finite dimensional graded representations of $G_{BK}(k)$ in $\mathbb{Q}$-vector spaces.

In [26], I. Kriz and P. May consider, for an Adams graded commutative differential graded algebra (cdga) $\mathcal{A} = \mathbb{Q}\cdot\mathrm{id}\oplus\oplus_{r\geq 1}\mathcal{A}(r)$ over $\mathbb{Q}$, the "bounded" derived category $\mathcal{D}^f_{\mathcal{A}}$ of Adams graded dg $\mathcal{A}$ modules. $\mathcal{D}^f_{\mathcal{A}}$ admits a functorial exact weight filtration, arising from the Adams grading; in case $\mathcal{A}$ is cohomologically connected, $\mathcal{D}^f_{\mathcal{A}}$ has a t-structure, defined by pulling back the usual t-structure on $\mathcal{D}^f_{\mathbb{Q}} \cong \oplus_n D^b(\mathbb{Q})$ via the functor $M \mapsto M \otimes^L_{\mathcal{A}} \mathbb{Q}$ from $\mathcal{D}^f_{\mathcal{A}}$ to $\mathcal{D}^f_{\mathbb{Q}}$. In particular, they define the heart $\mathcal{H}^f_{\mathcal{A}}$. Next, assuming $\mathcal{A}$ cohomologically connected, they construct an exact functor

$$\rho : D^b\big(\text{co-rep}^f_{\mathbb{Q}}(H^0(\bar{B}(\mathcal{A})))\big) \to \mathcal{D}^f_{\mathcal{A}}$$

where $\text{co-rep}^f_{\mathbb{Q}}(H^0(\bar{B}(\mathcal{A})))$ is the category of graded co-representations of $H^0(\bar{B}(\mathcal{A}))$ in finite-dimension $\mathbb{Q}$-vector spaces. Furthermore, they show that ρ identifies the categories $\mathcal{H}^f_{\mathcal{A}}$ and $\text{co-rep}^f_{\mathbb{Q}}(H^0(\bar{B}(\mathcal{A})))$ (although ρ is not in general an equivalence). For those who prefer group-schemes to Hopf algebras, let $G_{\mathcal{A}} := \mathrm{Spec}\, H^0(\bar{B}(\mathcal{A}))$. Then $G_{\mathcal{A}}$ is a pro-affine group scheme over $\mathbb{Q}$ with $\mathbb{G}_m$ action, and $\text{co-rep}^f_{\mathbb{Q}}(H^0(\bar{B}(\mathcal{A})))$ is equivalent to the category of graded represention s of $G_{\mathcal{A}}$ in finite dimensional $\mathbb{Q}$-vector spaces.

Taking $\mathcal{A} = \mathcal{N}_k$, and noting that the Beilinson-Soulé vanishing conjectures for k are equivalent to the cohomological connectedness of $\mathcal{A}$, this gives an equivalence of the heart $\mathcal{H}^f_{\mathcal{N}_k}$ with the Bloch-Kriz mixed Tate motives $\mathrm{MT}_{BK}(k)$.

M. Spitzweck [37] (see [29, section 5] for a detailed account) defines an equivalence

$$\theta_k : \mathcal{D}^f_{\mathcal{N}_k} \to \mathrm{DMT}(k) \subset DM_{gm}(k)_{\mathbb{Q}}$$

for k an arbitrary field. In addition, under the assumption that k satisfies the Beilinson-Soulé conjectures, or equivalently, that $\mathcal{N}_k$ is cohomologically connected, θ_k restricts to an equivalence

$$\theta_k : \mathcal{H}^f_{\mathcal{N}_k} \to \mathrm{MT}(k).$$

From the discussion above, this gives an equivalence of $\text{co-rep}^f_{\mathbb{Q}}(H^0(\bar{B}(\mathcal{N}_k)))$ with $\mathrm{MT}(k)$, and in fact identifies $G_{BK}(k) \ltimes \mathbb{G}_m$ as the Tannaka group of $(\mathrm{MT}(k), \mathrm{gr}^*_W)$.

Our first task is to extend this picture from k to X. To this aim, one defines the cycle algebra $\mathcal{N}(X)$ by replacing k with X in the definition of $\mathcal{N}_k$ and modifying the construction further by using complexes of cycles which are equi-dimensional over X. This yields an Adams graded cdga over $\mathbb{Q}$ together with a map of Adams graded cdgas $p^* : \mathcal{N}(k) \to \mathcal{N}(X)$ arising from the structure morphism $p : X \to \mathrm{Spec}\, k$. Essentially the same construction as for k gives an equivalence

$$\theta_X : \mathcal{D}^f_{\mathcal{N}(X)} \to \mathrm{DMT}(X) \subset DM(X)_{\mathbb{Q}} \qquad (*)$$

and if X satisfies the Beilinson-Soulé vanishing conjectures, θ_X restricts to an equivalence $\mathcal{H}^f_{\mathcal{N}(X)} \sim \mathrm{MT}(X)$. Defining the $\mathbb{Q}$ pro-group scheme $G_{BK}(X)$ as above,

$$G_{BK}(X) := G_{\mathcal{N}(X)} = \mathrm{Spec}\,(H^0(\bar{B}(\mathcal{N}_X))),$$

we also have the equivalence of $\mathrm{MT}(X)$ with the graded representations of $G_{BK}(X)$ in finite dimensional $\mathbb{Q}$-vector spaces, giving the identification of $G_{BK}(X) \ltimes \mathbb{G}_m$ with the Tannaka group of $(\mathrm{MT}(X), \mathrm{gr}^W_*)$, and identifying $p_* : G(\mathrm{MT}(X), \mathrm{gr}^W_*) \to G(\mathrm{MT}(k), \mathrm{gr}^W_*)$ with the map

$$\tilde{p} \times \mathrm{id} : G_{BK}(X) \ltimes \mathbb{G}_m \to G_{BK}(k) \ltimes \mathbb{G}_m$$

induced from $p^* : \mathcal{N}(k) \to \mathcal{N}(X)$.

A k-point x of X gives an augmentation $\epsilon_x : \mathcal{N}(X) \to \mathcal{N}(k)$. We discuss the general theory of augmented cdgas in section 2, leading to the *relative bar construction* $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$ of a cdg $\mathcal{N}$ algebra $\mathcal{A}$ with augmentation $\epsilon : \mathcal{A} \to \mathcal{N}$, as an ind-Hopf algebra in $\mathcal{H}^f_{\mathcal{N}}$. Let $G_{\mathcal{A}/\mathcal{N}}(\epsilon) = \mathrm{Spec}\, H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$ and let $G_{\mathcal{A}/\mathcal{N}}(\epsilon)_{\mathbb{Q}}$ be the pro-group scheme over $\mathbb{Q}$ gotten from $G_{\mathcal{A}/\mathcal{N}}(\epsilon)$ by applying the fiber functor $\mathrm{gr}^W_* : \mathcal{H}^f_{\mathcal{N}} \to \mathrm{Vec}_{\mathbb{Q}}$. Note that Tannaka duality gives a canonical action of $G_{\mathcal{N}}$ on $G_{\mathcal{A}/\mathcal{N}}(\epsilon)_{\mathbb{Q}}$.

Of course, in order to make a reasonable relative bar construction, one needs to use a good model for $\mathcal{A}$ as an $\mathcal{N}$-algebra. This is provided by using the *relative minimal model* $\mathcal{A}\{\infty\}_{\mathcal{N}}$ of $\mathcal{A}$ over $\mathcal{N}$, for which the derived tensor product is just the usual tensor product.

In section 2.8, especially theorem 2.8.3, we show that

1. $G_{\mathcal{A}/\mathcal{N}}(\epsilon)_{\mathbb{Q}} = \mathrm{Spec}\, H^0(\bar{B}(\mathcal{A}\{\infty\}_{\mathcal{N}} \otimes_{\mathcal{N}} \mathbb{Q}))$.

2. There is an exact sequence of pro-group schemes over $\mathbb{Q}$:

$$1 \to G_{\mathcal{A}/\mathcal{N}}(\epsilon)_{\mathbb{Q}} \to G_{\mathcal{A}} \xrightarrow{p_*} G_{\mathcal{N}} \to 1$$

   The splitting ϵ^* to p^* defines a splitting $\epsilon_* : G_{\mathcal{N}} \to G_{\mathcal{A}}$ to p_*.

3. The conjugation action of $G_{\mathcal{N}}$ on $G_{\mathcal{A}/\mathcal{N}}(\epsilon)_{\mathbb{Q}}$ given by the splitting ϵ_* is the same as the canonical action.

To do this, we use an alternate description of dg modules over an Adams graded cdga $\mathcal{N}$, that of *flat dg connections*. Kriz and May describe dg modules M over $\mathcal{N}$ as $\mathcal{N}^+ := \bigoplus_{r>0} \mathcal{N}(r)$-valued connections over $M \otimes_{\mathcal{N}} \mathbb{Q}$ (for the canonical augmentation $\mathcal{N} \to \mathbb{Q}$). Writing $\mathcal{A}\{\infty\}^+_{\mathcal{N}}$ as $\mathcal{N}^+ \oplus \mathcal{I}$, with this decomposition coming from the augmentation $\mathcal{A}\{\infty\}_{\mathcal{N}} \to \mathcal{N}$, the absolute (i.e. $\mathcal{A}\{\infty\}^+_{\mathcal{N}}$-valued) connection on $H^0(\bar{B}(\mathcal{A})) = H^0(\bar{B}(\mathcal{A}\{\infty\}_{\mathcal{N}}))$ induces a $\mathcal{N}^+$-valued connection on $H^0(\bar{B}(\mathcal{A}\{\infty\}_{\mathcal{N}} \otimes_{\mathcal{N}} \mathbb{Q}))$. Similarly, the structure of $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$ as an ind-Hopf algebra in $\mathcal{H}^f_{\mathcal{N}}$ gives an $\mathcal{N}^+$-valued connection on

$$H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon)) \otimes_{\mathcal{N}} \mathbb{Q} = H^0(\bar{B}(\mathcal{A}\{\infty\}_{\mathcal{N}} \otimes_{\mathcal{N}} \mathbb{Q})).$$

Using this description, it is easy to make the identifications necessary for proving (1)-(3) above. H. Esnault has interpreted this argument as saying that $G_{\mathcal{A}/\mathcal{N}}(\epsilon)$ is the Gauß-Manin connection of $G_{\mathcal{A}}$ associated to $\mathcal{A}/\mathcal{N}$.

Applying this theory to the splitting $\epsilon_x : \mathcal{N}(X) \to \mathcal{N}(k)$, the $\mathbb{Q}$ pro-group scheme K, and the lifting K_x to a $\mathrm{MT}(k)$ pro-group scheme, gives us the isomorphism of pro-group schemes

$$K \cong \mathrm{Spec}\, H^0(\bar{B}(\mathcal{N}(X)\{\infty\}_{\mathcal{N}(k)} \otimes_{\mathcal{N}(k)} \mathbb{Q}))$$

and the isomorphism of pro-group scheme objects in $\mathcal{H}^f_{\mathcal{N}(k)}$

$$K_x \cong \operatorname{Spec} H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}(k)}(\mathcal{N}(X), \epsilon_x)). \qquad (**)$$

One can make the dg $\mathcal{N}(k)$-module $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}(k)}(\mathcal{N}(X), \epsilon_x))$ explicit as an object in MT(k) via Spitzweck's theorem. This relies on a crucial property of the transformation from dg $\mathcal{N}(k)$ modules to motives (see theorem 5.6.2 for a more general statement):

Take $X \in \mathbf{Sm}/k$. If the motive of X in $DM(k)_{\mathbb{Q}}$ is in DMT(k) and X satisfies the Beilinson-Soulé vanishing conjectures, then the motive of $\mathcal{N}_X\{\infty\}_{\mathcal{N}(k)}$ is canonically isomorphic to $H_{gm}(X)_{\mathbb{Q}}$.

The explicit decription of the Beilinson simplicial scheme underlying the Deligne-Goncharov construction, together with this essential fact, allows one to conclude that K_x with its MT(k) structure induced by the Gauß-Manin connection is precisely $\pi_1^{\rm mot}(X, x)$, when x comes from a rational point $x \in X(k)$ (see sections 6.5 and 6.6). In other words, we have the isomorphism of pro-group schemes over MT(k):

$$\pi_1^{\rm mot}(X, x) \cong \operatorname{Spec} H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}_k}(\mathcal{N}_X, \epsilon_x)).$$

Combining this with our identification ($**$) proves theorem 1. Replacing k with a more general base-scheme $S \in \mathbf{Sm}/k$, the program outlined above proves theorem 2.

In this article, we do not consider the case of the base-point x being a non-trivial tangent vector at some point $\bar{x} \in \mathbb{P}^1 \setminus X$. As mentioned above, Deligne-Goncharov [12] show in this case as well that the motivic π_1, defined by Deligne [11] as a system of realizations, comes from MT(k). This defines $\pi_1^{\rm mot}(X, x)$ as an object in MT(k), but does not give a direct construction in MT(k). However, the results of [28] give a section ϵ_x to $p_* : \mathcal{N}(k) \to \mathcal{N}(X)$ (in the homotopy category of cdgas) for tangential base-points x as well as for k-points, so we do have a relative bar construction available even for tangential base-points. In order to extend our main theorem 6.6.1 to this case, one should define realization functors on the categories of Tate motives, described as dg modules over the cycle algebra, and check that the realization of $\operatorname{Spec} H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}_k}(\mathcal{N}_X, \epsilon_x))$ agrees with Deligne's motivic π_1.

Outline: The paper is organized as follows: We begin in section 1 with a review of the theory of dg modules over an Adams-graded cdga, following for the most part the discussion of Kriz-May [26], but adding some new material dealing with the category of "weight-bounded" modules. In section 2 we describe an extension of the classical model structure on cdgas over a field of characteristic zero (*cf.* [7]) to the category of cdgas over a cdga. This enables us to extend the theory of minimal models and the bar construction to the relative case. We conclude this section with our main result on the relative bar construction, theorem 2.8.3. In fact, the reader who is moderately familiar with the Kriz-May theory of dg modules over a cdga could simply skim the first two sections to absorb our notation, and accept theorem 2.8.3 on faith for the first reading.

We then proceed to a review of the recently available theory of motives over a base-scheme, due to Cisinski-Déglise [9, 10], in section 3. Next, in section 4, we take a look at generalizations of the Bloch-Kriz cycle algebra to a functorial construction for smooth schemes over k, modifying a construction of Joshua [25]. In section 5, we describe the "cohomological motive" functor to the Cisinski-Déglise category and show how a $\mathbb{Q}$-version of this functor can be described using the cycle algebra. This section is the technical heart of the paper. In it, we prove our main results relating motives and cycle algebras: our generalization of Spitzweck's representation theorem, theorem 5.3.2, identifying the derived category of dg modules over the cycle algebra $\mathcal{N}(S)$ to the triangulated category of Tate motives over S, and our two main results relating the cycle complex of a smooth S-scheme X to the geometric motive of X, theorem 5.5.3 and theorem 5.6.2. We put everything together in section 6, giving our generalization of the Deligne-Goncharov motivic π_1 and proving our main results, theorem 1 and theorem 2 (these are corollary 6.6.2 and theorem 6.6.1, respectively).

Acknowledgements: Together with H. Esnault, we gave a seminar in the winter 2006-7 at the university of Duisburg-Essen on [12], to try to understand the constructions and results of Deligne-Goncharov, as well as the various constructions of mixed Tate motives and the relationships between them, as developed in the works of Bloch, Bloch-Kriz, Kriz-May and Spitzweck, and summarized in [29]; this paper is to a large extent a product of that seminar. We thank all the seminar participants for their willingness to give talks. In particular we thank Phùng Hô Hai for various discussions on Tannakian categories.

Most importantly, this paper is a revision of a joint work with Hélène Esnault [14]. This joint work also contained a proof of theorem 1, with proof along the same lines as the one given here. It was Esnault who had originally suggested relating the Deligne-Goncharov motivic π_1 to the Bloch-Kriz cycle Hopf algebras as a way of answering the question of Deligne and Goncharov on the relation of $\mathrm{MT}(k(t))$ to $\pi_1(X, x)$-representations in $\mathrm{MT}(k)$. This paper would never have existed had it not been for the many fruitful discussions and numerous insights Esnault has shared with us; we take this opportunity to thank her for her crucial contribution to this work. Finally, we would like to thank the referee for making a number of useful suggestions.

1 Differential graded algebras

We fix notation and recall some basic facts on commutative differential graded algebras (cdgas) over $\mathbb{Q}$. This material is taken mainly from [26], with some refinements and additions.

In what follows a cdga will always mean a cdga over $\mathbb{Q}$.

1.1 Adams graded cdgas

Definition 1.1.1 (1) A *cdga* $(A^*, d, \cdot)$ (over $\mathbb{Q}$) consists of a unital, graded-commutative $\mathbb{Q}$-algebra $(A^* := \oplus_{n\in\mathbb{Z}} A^n, \cdot)$ together with a graded homomorphism $d = \oplus_n d^n$, $d^n : A^n \to A^{n+1}$, such that

1. $d^{n+1} \circ d^n = 0$.
2. $d^{n+m}(a \cdot b) = d^n a \cdot b + (-1)^n a \cdot d^m b$; $a \in A^n$, $b \in A^m$.

A^* is called *connected* if $A^n = 0$ for $n < 0$ and $A^0 = \mathbb{Q} \cdot 1$, *cohomologically connected* if $H^n(A^*) = 0$ for $n < 0$ and $H^0(A^*) = \mathbb{Q} \cdot 1$.

(2) An *Adams graded* cdga is a cdga A together with a direct sum decomposition into subcomplexes $A^* := \oplus_{r\geq 0} A^*(r)$ such that $A^*(r)\cdot A^*(s) \subset A^*(r+s)$. In addition, we require that $A^*(0) = \mathbb{Q}\cdot \mathrm{id}$. An Adams graded cdga is said to be (cohomologically) connected if the underlying cdga is (cohomologically) connected.

For $x \in A^n(r)$, we call n the *cohomological degree* of x, $n := \deg x$, and r the *Adams degree* of x, $r := |x|$.

Note that an Adams graded cdga A has a canonical augmentation $A \to \mathbb{Q}$ with augmentation ideal $A^+ := \oplus_{r>0} A^*(r)$.

1.2 The bar construction

We let **Ord** denote the category with objects the sets $[n] := \{0, \ldots, n\}$, $n = 0, 1, \ldots$, and morphisms the non-decreasing maps of sets. The morphisms in **Ord** are generated by the *coface maps* $\delta_i^n : [n] \to [n+1]$ and the *codegeneracy maps* $\sigma_i^n : [n] \to [n-1]$, where δ_i^n is the strictly increasing map omitting i from its image and σ_i^n is the non-decreasing surjective map sending i and $i+1$ to i. For a category $\mathcal{C}$, we have the categories of *cosimplicial objects* in $\mathcal{C}$ and *simplicial objects* in $\mathcal{C}$, namely, the categories of functors $\mathbf{Ord} \to \mathcal{C}$ and $\mathbf{Ord}^{\mathrm{op}} \to \mathcal{C}$, respectively. For a cosimplicial object $X : \mathbf{Ord} \to \mathcal{C}$, we often write δ_i^n and σ_i^n for the coface maps $X(\delta_i^n)$ and $X(\sigma_i^n)$, and for a simplicial object $S : \mathbf{Ord}^{\mathrm{op}} \to \mathcal{C}$, we often write d_i^n and s_i^n for the *face* and *degeneracy* maps $S(\delta_i^n)$ and $S(\sigma_i^n)$.

Let A be a cdga. We begin by defining the simplicial cdga $B_\bullet(A)$ as follows: Tensor product (over $\mathbb{Q}$) is the coproduct in the category of cdgas, so for a finite set S, we have $A^{\otimes S}$, giving the functor $A^{\otimes ?}$ from finite sets to cdgas. Thus, if we have a simplicial set S such that $S[n]$ is a finite set for all n, we may form the simplicial cdga $A^{\otimes S}$, $n \mapsto A^{\otimes S[n]}$. We have the representable simplicial sets $\Delta[n] := \mathrm{Hom}_{\mathbf{Ord}}(-, [n])$; setting $[0,1] := \Delta[1]$ gives us the simplicial cdga

$$B_\bullet(A) := A^{\otimes [0,1]}.$$

The two inclusions $[0] \to [1]$ define the maps $i_0, i_1 : \Delta[0] \to \Delta[1]$. Letting $\{0,1\}$ denote the constant simplicial set with two elements, the maps i_0, i_1 give rise to the map of simplicial sets $i_0 \amalg i_1 : \{0,1\} \to [0,1]$, which makes $B_\bullet(A)$ into a simplicial $A \otimes A = A^{\otimes\{0,1\}}$ algebra.

Suppose we have augmentations $\epsilon_1, \epsilon_2 : A \to \mathbb{Q}$. Define $\bar{B}_\bullet(A, \epsilon_1, \epsilon_2)$ by

$$\bar{B}_\bullet(A, \epsilon_1, \epsilon_2) := B_\bullet(A) \otimes_{A\otimes A} \mathbb{Q}$$

using $\epsilon_1 \otimes \epsilon_2 : A \otimes A \to \mathbb{Q}$ as structure map. Since $\bar{B}_n(A, \epsilon_1, \epsilon_2)$ is a complex for each n, we can form a double complex by using the usual alternating sum of the face maps $d_i^n : \bar{B}_{n+1}(A, \epsilon_1, \epsilon_2) \to \bar{B}_n(A, \epsilon_1, \epsilon_2)$ as the second differential, and let $\bar{B}(A, \epsilon_1, \epsilon_2)$ denote the total complex of this double complex. We use cohomological grading throughout, so $\bar{B}_n(A, \epsilon_1, \epsilon_2)^m$ has total degree $m - n$. For $\epsilon_1 = \epsilon_2 = \epsilon$, we write $\bar{B}(A, \epsilon)$ or simply $\bar{B}(A)$; this is the *reduced bar construction* for (A, ϵ). As is usual, we denote a decomposable element $x_1 \otimes \ldots \otimes x_n$ of $\bar{B}(A)$ by $[x_1|, \ldots |x_n]$. Note that

$$\deg([x_1|\ldots|x_m]) = -m + \sum_i \deg(x_i).$$

The bar construction $\bar{B} := \bar{B}(A)$ has the following structures: a differential $d : \bar{B} \to \bar{B}$ of degree $+1$ coming from the differential in A, a product

$$\cup : \bar{B} \otimes \bar{B} \to \bar{B}$$
$$[x_1|\ldots|x_p] \cup [x_{p+1}|\ldots|x_{p+q}] = \sum_\sigma \mathrm{sgn}(\sigma)[x_{\sigma(1)}|\ldots|x_{\sigma(p+q)}]$$

where the sum is over all (p, q) shuffles $\sigma \in S_{p+q}$ (and the sign is the *graded* sign of σ, taking into account the degrees of the x_i) , a co-product

$$\delta : \bar{B} \to \bar{B} \otimes \bar{B}$$
$$\delta([x_1|\ldots|x_n]) := \sum_{i=0}^{n} (-1)^{i\deg([x_{i+1}|\ldots|x_n])}[x_1|\ldots x_i] \otimes [x_{i+1}|\ldots|x_n]$$

and an involution

$$\iota : \bar{B} \to \bar{B},$$
$$\iota([x_1|x_2|\ldots|x_{n-1}|x_n]) := (-1)^m[x_n|x_{n-1}|\ldots|x_2|x_1];\ m = \sum_{1\le i<j\le n} \deg(x_i)\cdot\deg(x_j),$$

making $(\bar{B}(A), d, \cup, \delta, \iota)$ a differential graded Hopf algebra over $\mathbb{Q}$, which is graded-commutative with respect to the product $\cup$. The cohomology $H^*(\bar{B}(A))$ is thus a graded Hopf algebra over $\mathbb{Q}$, in particular, $H^0(\bar{B}(A))$ is a commutative Hopf algebra over $\mathbb{Q}$.

Let $\mathcal{I}(A)$ be the kernel of the augmentation $H^0(\bar{B}(A)) \to \mathbb{Q}$ induced by ϵ. The coproduct δ on $H^0(\bar{B}(A))$ induces the structure of a co-Lie algebra on $\gamma_A := \mathcal{I}(A)/\mathcal{I}(A)^2$.

From the formula for the coproduct, we see that, modulo tensors of degree $< m$, we have

$$\delta([x_1|\ldots|x_m]) = 1 \otimes [x_1|\ldots|x_m] + [x_1|\ldots|x_m] \otimes 1$$

This implies that the pro-affine $\mathbb{Q}$-algebraic group $G := \mathrm{Spec}\, H^0(\bar{B}(A))$ is pro-unipotent. In addition, in case A is cohomologically connected, $H^0(\bar{B}(A))$ is, as a $\mathbb{Q}$-algebra, a polynomial algebra with indecomposables γ_A (see, e.g., [2, theorem 2.30, corollary 2.31]).

Suppose $A = \oplus_{r\ge0} A^*(r)$ is an Adams graded cdga, with canonical augmentation $\epsilon : A \to \mathbb{Q}$. The Adams grading on A induces an Adams grading on $B_\bullet(A)$ and thus on $\bar{B}(A)$; explicitly $\bar{B}(A)$ has the Adams grading $\bar{B}(A) = \oplus_{r\ge0}\bar{B}(A)(r)$ where the Adams degree of $[x_1|\ldots|x_m]$ is

$$|[x_1|\ldots|x_m]| := \sum_j |x_j|.$$

Thus $H^0(\bar{B}(A)) = \oplus_{r\ge0} H^0(\bar{B}(A)(r))$ becomes an Adams graded Hopf algebra over $\mathbb{Q}$, commutative as a $\mathbb{Q}$-algebra. We also have the Adams graded co-Lie algebra $\gamma_A = \oplus_{r>0}\gamma_A(r)$.

Remark 1.2.1 Let A be a cohomologically connected Adams graded cdga. The Adams grading equips the pro-unipotent affine $\mathbb{Q}$ group scheme $G := \mathrm{Spec}\, H^0(\bar{B}(A))$ with a grading, or, equivalently, with a $\mathbb{G}_m$-action. Thus γ_A is a positively graded nilpotent co-Lie algebra, and there is an equivalence of categories between the continuous graded co-representations of $H^0(\bar{B}(A))$ in finite dimensional graded $\mathbb{Q}$-vector spaces, $\text{co-rep}^f_{\mathbb{Q}}(H^0(\bar{B}(A)))$, and the continuous graded co-representations of γ_A in finite dimensional graded $\mathbb{Q}$-vector spaces, $\text{co-rep}^f_{\mathbb{Q}}(\gamma_A)$.

1.3 The category of cell modules

Kriz and May [26] define a triangulated category directly from an Adams graded cdga A without passing to the bar construction or using a co-Lie algebra. We recall some of their work here, with some extensions.

Let A^* be a graded algebra over $\mathbb{Q}$. We let $A[n]$ be the left A^*-module which is A^{m+n} in degree m, with the A^*-action given by left multiplication. If $A^*(*) = \oplus_{n\geq 0, r\geq 0} A^n(r)$ is a bi-graded $\mathbb{Q}$-algebra, we let $A{<}r{>}[n]$ be the left $A^*(*)$-module which is $A^{m+n}(r+s)$ in bi-degree (m, s), with action given by left multiplication.

Definition 1.3.1 Let A be a cdga.
(1) A *dg A-module* (M^*, d) consists of a complex $M^* = \oplus_n M^n$ of $\mathbb{Q}$-vector spaces with differential d, together with a graded, degree zero map $A^* \otimes_{\mathbb{Q}} M^* \to M^*,\ a \otimes m \mapsto a \cdot m$, which makes M^* into a graded A^*-module, and satisfies the Leibniz rule

$$d(a \cdot m) = da \cdot m + (-1)^{\deg a} a \cdot dm;\ a \in A^*, m \in M^*.$$

(2) If $A = \oplus_{r\geq 0} A^*(r)$ is an Adams graded cdga, an *Adams graded dg A-module* is a dg A-module M^* together with a decomposition into subcomplexes $M^* = \oplus_s M^*(s)$ such that $A^*(r) \cdot M^*(s) \subset M^*(r+s)$. We say $x \in M^*$ has *Adams degree s* if $x \in M^*(s)$, and write this as $|x| = s$.

(3) An Adams graded dg A-module M is a *cell module* if

(a) M is free as a bi-graded A-module, where we forget the differential structure. That is, there is a set J and elements $b_j \in M^{n_j}(r_j)$, $j \in J$, such that the maps $a \mapsto a \cdot b_j$ induces an isomorphism of bi-graded A-modules

$$\oplus_{j\in J} A{<}-r_j{>}[-n_j] \to M.$$

(b) There is a filtration on the index set J:

$$J_{-1} = \emptyset \subset J_0 \subset J_1 \subset \ldots J_n \subset \ldots \subset J$$

such that $J = \cup_{n=0}^{\infty} J_n$ and for $j \in J_n$,

$$db_j = \sum_{i\in J_{n-1}} a_{ij} b_i.$$

A *finite cell module* is a cell module with finite index set J.

We denote the category of dg A-modules by $\mathcal{M}_A$, the A-cell modules by $\mathcal{CM}_A$ and the finite cell modules by $\mathcal{CM}_A^f$.

1.4 The derived category

Let A be an Adams graded cdga and let M and N be Adams graded dg A-modules. Let $\mathcal{H}om_A(M,N)$ be the Adams graded dg A-module with $\mathcal{H}om_A(M,N)^n(r)$ the A-module consisting of maps $f : M \to N$ with $f(M^a(s)) \subset N^{a+n}(s+r)$, $f(am) = (-1)^{np}af(m)$ for $a \in A^p$ and $m \in M$, and with differential d defined by $df(m) = d(f(m))(-1)^n f(dm)$ for $f \in \mathcal{H}om(M,N)^n(r)$. Similarly, let $M \otimes_A N$ be the Adams graded dg A-module with underlying module $M \otimes_A N$ and with differential $d(m \otimes n) = dm \otimes n + (-1)^{\deg m} m \otimes dn$.

For $f : M \to N$ a morphism of Adams graded dg A-modules, we let $\mathrm{Cone}(f)$ be the Adams graded dg A-module with

$$\mathrm{Cone}(f)^n(r) := N^n(r) \oplus M^{n+1}(r)$$

and differential $d(n,m) = (dn + f(m), -dm)$. Let $M[1]$ be the Adams graded dg A-module with $M[1]^n(r) := M^{n+1}(r)$ and differential $-d$, where d is the differential of M. A sequence of the form

$$M \xrightarrow{f} N \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{j} M[1]$$

where i and j are the evident inclusion and projection, is called a *cone sequence.*

Definition 1.4.1 Let A be an Adams graded cdga over $\mathbb{Q}$. We let $\mathcal{M}_A$ denote the category of Adams graded dg A-modules, $\mathcal{K}_A$ the *homotopy category*, i.e. the objects of $\mathcal{K}_A$ are the objects of $\mathcal{M}_A$ and

$$\mathrm{Hom}_{\mathcal{K}_A}(M,N) = H^0(\mathcal{H}om_A(M,N)(0)).$$

The category $\mathcal{K}_A$ is a triangulated category, with distinguished triangles those triangles which are isomorphic in $\mathcal{K}_A$ to a cone sequence.

Definition 1.4.2 The *derived category* $\mathcal{D}_A$ of dg A-modules is the localization of $\mathcal{K}_A$ with respect to morphisms $M \to N$ which are quasi-isomorphisms on the underlying complexes of $\mathbb{Q}$-vector spaces. For M in $\mathcal{D}_A$, we denote the nth cohomology of M, as a complex of $\mathbb{Q}$-vector spaces, by $H^n(M)$.

We define the homotopy category of A-cell modules, resp. finite cell modules, as the full subcategory of $\mathcal{K}_A$ with objects in $\mathcal{CM}_A$, resp. in $\mathcal{CM}_A^f$,

$$\mathcal{KCM}_A^f \subset \mathcal{KCM}_A \subset \mathcal{K}_A.$$

Note that for $A = \mathbb{Q}$, $\mathcal{M}_{\mathbb{Q}}$ is just the category of complexes of graded $\mathbb{Q}$-vector spaces, and $\mathcal{D}_{\mathbb{Q}}$ is the unbounded derived category of graded $\mathbb{Q}$-vector spaces.

Proposition 1.4.3 ([26, construction 2.7]) *Let A be an Adams graded cdga. Then the evident functor*

$$\mathcal{KCM}_A \to \mathcal{D}_A$$

is an equivalence of triangulated categories. Explicitly, let $f : M' \to M$ be a quasi-isomorphism in $\mathcal{M}_A$, $N \in \mathcal{CM}_A$. Then the induced map

$$f : \mathrm{Hom}_{\mathcal{K}_A}(N, M') \to \mathrm{Hom}_{\mathcal{K}_A}(N, M)$$

is an isomorphism.

We let $\mathcal{D}_A^f \subset \mathcal{D}_A$ be the full subcategory with objects those M isomorphic in $\mathcal{D}_A$ to a finite cell module. As an immediate consequence of proposition 1.4.3, we have

Proposition 1.4.4 *$\mathcal{KCM}_A^f \to \mathcal{D}_A^f$ is an equivalence of triangulated categories.*

Example 1.4.5 (Tate objects) For $n \in \mathbb{Z}$, let $\mathbb{Q}(n)$ be the object of $\mathcal{CM}_A^f$ which is the free rank one A-module with generator b_n having Adams degree $-n$, cohomological degree 0 and $db_n = 0$, i.e., $\mathbb{Q}(n) = A<n>$. We sometimes write $\mathbb{Q}_A(n)$ for $\mathbb{Q}(n)$; $\mathbb{Q}(n)$ is called a *Tate object.*

1.5 Weight filtration

Let M be an Adams graded dg A-module which is free as a bi-graded A-module. Choose a basis $\mathcal{B} := \{b_j \mid j \in J\}$, $M = \oplus_j A \cdot b_j$. Write

$$db_j = \sum_i a_{ij} b_i;\ \ a_{ij} \in A.$$

Since $|a_{ij}| \geq 0$ and d has Adams degree 0, it follows that

$$|b_i| \leq |b_j| \text{ if } a_{ij} \neq 0.$$

Thus, we have the subcomplex

$$W_n^{\mathcal{B}} M = \oplus_{\{j,\ |b_j| \leq n\}} A \cdot b_j$$

of M.

The subcomplex $W_n^{\mathcal{B}} M$ is independent of the choice of basis: if $\mathcal{B}' = \{b'_j\}$ is another basis and if $|b'_j| = n$, then as $b'_j = \sum_i e_{ij} b_i$ and $|e_{ij}| \geq 0$, it follows that $b'_j \in W_n^{\mathcal{B}} M$ and hence $W_n^{\mathcal{B}'} M \subset W_n^{\mathcal{B}} M$. By symmetry, $W_n^{\mathcal{B}} M \subset W_n^{\mathcal{B}'} M$. We may thus write $W_n M$ for $W_n^{\mathcal{B}} M$.

This gives us the increasing filtration as an Adams graded dg A-module

$$W_* M : \ \ \ldots \subset W_n M \subset W_{n+1} M \subset \ldots \subset M$$

with $M = \cup_n W_n M$.

Similarly, for $n \geq n'$, define $W_{n/n'} M$ as the cokernel of the inclusion $W_{n'} M \to W_n M$, i.e., $W_{n/n'} M$ is the Adams graded dg A-module with basis the b_j having $n' < |b_j| \leq n$ and with differential induced by the differential in $W_n M$. We write gr_n^W for $W_{n/n-1}$ and $W^{>n}$ for $W_{\infty/n}$.

It is not hard to see that $W_n M$ is functorial in M. In particular, if $f : M \to M'$ is a homotopy equivalence of cell modules with homotopy inverse $g : M' \to M$, then f and g restricted to $W_n M$ and $W_n M'$ give inverse homotopy equivalences $W_n f : W_n M \to W_n M'$, $W_n g : W_n M' \to W_n M$. Thus the W filtration in $\mathcal{CM}_A$ defines a functorial tower of endo-functors on $\mathcal{KCM}_A$:

$$\ldots \to W_n \to W_{n+1} \to \ldots \to \mathrm{id} \tag{1.5.1}$$

Lemma 1.5.1 *1. The endo-functor W_n is exact for each n.*

2. For $n' \leq n \leq \infty$, the sequence of endo-functors $W_{n'} \to W_n \to W_{n/n'}$ canonically extends to a distinguished triangle of endo-functors.

Proof For (1), it follows directly from the definition that W_n transforms a cone sequence into a cone sequence. For (2), take $M \in \mathcal{CM}_A$. The sequence

$$0 \to W_{n'}M \to W_n M \to W_{n/n'}M \to 0$$

is split exact as a sequence of bi-graded A-modules. Thus (2) follows from the general fact that a sequence in $\mathcal{CM}_A$

$$0 \to N' \xrightarrow{i} N \xrightarrow{p} N'' \to 0$$

that is split exact as a sequence of bi-graded A-modules extends canonically to a distinguished triangle in $\mathcal{KCM}_A$. To see this, choose a splitting s to p (as bi-graded A-modules), and define $t : N'' \to N'[1]$ by $i \circ t = s \circ d_{N''} - d_N \circ s$. It is then easy to check that t is a map of complexes and $(s,t) : N'' \to N \oplus N'[1]$ defines the map of complexes

$$(s,t) : N' \to \mathrm{Cone}(i)$$

making the diagram

$$\begin{array}{ccccccc} N' & \xrightarrow{i} & N & \xrightarrow{p} & N'' & \xrightarrow{t} & N'[1] \\ \| & & \| & & \downarrow{\scriptstyle (s,t)} & & \| \\ N' & \xrightarrow{i} & N & \longrightarrow & \mathrm{Cone}(i) & \longrightarrow & N'[1] \end{array}$$

commute. In particular, (s,t) is an isomorphism in $\mathcal{KCM}_A$. One sees similarly that another choice s' of splitting leads to a homotopic map (s',t').

Note that it is not necessary for M to be a cell module to define W_nM; being free as a bi-graded A-module suffices. However, it is not clear that W_nM is a quasi-isomorphism invariant in general. To side-step this issue, we use instead

Definition 1.5.2 Define the tower of exact endo-functors on $\mathcal{D}_A$

$$\ldots \to W_n \to W_{n+1} \to \ldots \to \mathrm{id}$$

using (1.5.1) and the equivalence of categories in proposition 1.4.3. We define $W_{n/n'}$, gr^W_n and $W^{>n}$ on $\mathcal{D}_A$ similarly.

Remark 1.5.3 Since $\mathcal{KCM}_A \to \mathcal{D}_A$ is an equivalence of triangulated categories, the natural distinguished triangles

$$W_{n'} \to W_n \to W_{n/n'} \to W_{n'}[1]$$

in $\mathcal{KCM}_A$ give us natural distinguished triangles

$$W_{n'} \to W_n \to W_{n/n'} \to W_{n'}[1]$$

in $\mathcal{D}_A$.

One uses the weight filtration for inductive arguments, for example:

Lemma 1.5.4 *Let M be a finite A-cell module. Suppose N is a summand of M in $\mathcal{D}_A$. Then there is a finite A-cell module M' with $N \cong M'$ in $\mathcal{D}_A$.*

Proof By proposition 1.4.3 there is an isomorphism $N' \cong N$ in $\mathcal{D}_A$ with with N' an object in $\mathcal{CM}_A$. Thus we may assume that N is a cell module. Since $\mathcal{KCM}_A \to \mathcal{D}_A$ is an equivalence, N is a summand of M in $\mathcal{KCM}_A$. Write $M = N \oplus N'$ in $\mathcal{KCM}_A$ and let $p : M \to M$ be the projection $M \to N$ followed by the inclusion $N \to M$.

Since M is finite, there is a minimal n with $W_nM \neq 0$. Thus $W_{n-1}N$ is homotopy equivalent to zero and $N \cong W_{\infty/n-1}N$ in $\mathcal{KCM}_A$. Hence, we may assume that $W_{n-1}N = 0$ in $\mathcal{CM}_A$. Similarly, we may assume that $M = W_{n+r}M$ and $N = W_{n+r}N$ in $\mathcal{CM}_A$ for some $r \geq 0$. We proceed by induction on r.

As $A^*(0) = \mathbb{Q}{\cdot}\mathrm{id}$, it follows that $W_nM = A \otimes_{\mathbb{Q}} M_0$ for a finite complex of finite dimensional graded $\mathbb{Q}$-vector spaces M_0. Indeed, choose a finite bi-graded A-basis $\{b_j\}$ for W_nM and let M_0 be the finite dimensional $\mathbb{Q}$-vector space spanned by the b_j. Since $W_{n-1}M = 0$, all the b_j have Adams degree n. Writing $db_j = \sum_i a_{ij}b_i$ and noting that the differential has Adams degree 0, it follows that $|a_{ij}| = 0$ for all i, j, i.e., $a_{ij} \in \mathbb{Q} \cdot \mathrm{id}$. Consequently M_0 is a subcomplex of M and $W_nM = A \otimes_{\mathbb{Q}} M_0$ as an Adams graded dg module.

But such an M_0 is homotopy equivalent to the direct sum of its cohomologies; replacing M_0 with $\oplus_n H^n(M_0)[-n]$ and changing notation, we may assume that $d_{M_0} = 0$. Thus $W_nM = A \otimes_{\mathbb{Q}} M_0$ for M_0 a finite dimensional bi-graded $\mathbb{Q}$-vector space; using again the fact that $A(r) = 0$ for $r < 0$ and $A(0) = \mathbb{Q} \cdot \mathrm{id}$, we see that $W_np = \mathrm{id} \otimes q$ with $q : M_0 \to M_0$ an idempotent endomorphism of the bi-graded $\mathbb{Q}$-vector space M_0. Thus $W_nN \cong A \otimes \mathrm{im}(q)$, hence W_nN is homotopy equivalent to a finite A-cell module. This also takes care of the case $r = 0$.

Using the distinguished triangle

$$W_nN \to N \to W_{n+r/n}N \to W_nN[1]$$

we may replace N with the shifted cone of the map $W_{n+r/n}N \to A \otimes \mathrm{im}(q)[1]$. Since $W_{n+r/n}N$ is a summand of $W_{n+r/n}M$, it follows by induction on r that $W_{n+r/n}N$ is homotopy equivalent to a finite cell module, hence the cone of $W_{n+r/n}N \to A \otimes \mathrm{im}(q)$ is homotopy equivalent to a finite cell module as well.

Definition 1.5.5 Let $\mathcal{D}_A^{+w} \subset \mathcal{D}_A$ be the full subcategory of $\mathcal{D}_A$ with objects M such that $W_nM \cong 0$ for some n. Similarly, let $\mathcal{CM}_A^{+w} \subset \mathcal{CM}_A$ be the full subcategory with objects M such that $W_nM = 0$ for some n and let $\mathcal{KCM}_A^{+w}$ be the homotopy category of $\mathcal{CM}_A^{+w}$.

Lemma 1.5.6 *1. The natural map $\mathcal{KCM}_A^{+w} \to \mathcal{KCM}_A$ is an equivalence of $\mathcal{KCM}_A^{+w}$ with the full subcategory of $\mathcal{KCM}_A$ with objects the M such that $W_nM \cong 0$ in $\mathcal{KCM}_A$ for $n << 0$.*

2. The equivalence $\mathcal{KCM}_A \to \mathcal{D}_A$ induces an equivalence $\mathcal{KCM}_A^{+w} \to \mathcal{D}_A^{+w}$.

Proof Since $\mathcal{KCM}_A^{+w}$ is the homotopy category of the full subcategory $\mathcal{CM}_A^{+w}$ of $\mathcal{CM}_A$, the functor $\mathcal{KCM}_A^{+w} \to \mathcal{KCM}_A$ is a full embedding. Suppose that $W_nM \cong 0$ in $\mathcal{KCM}_A$. We have the cell module $W^{>n}M$ and the distinguished triangle

$$W_nM \to M \to W^{>n}M \to W_nM[1]$$

in $\mathcal{KCM}_A$. Thus the map $M \to W^{>n}M$ is an isomorphism in $\mathcal{KCM}_A$; since $W^{>n}M$ is in $\mathcal{CM}_A^{+w}$, the essential image of $\mathcal{KCM}_A^{+w}$ in $\mathcal{KCM}_A$ is as described.

For (2), following definition 1.5.2, W_nM is defined by choosing an isomorphism $P \to M$ in $\mathcal{D}_A$ with $P \in \mathcal{CM}_A$ and taking $W_nM := W_nP$. Since $W_nP = W_nM \cong 0$ in $\mathcal{D}_A$, it follows that $W_nP \cong 0$ in $\mathcal{KCM}_A$, so P is isomorphic to an object in $\mathcal{KCM}_A^{+w}$. Thus $\mathcal{D}_A^{+w}$ is the essential image of $\mathcal{KCM}_A^{+w}$ in $\mathcal{D}_A$. Since $\mathcal{KCM}_A \to \mathcal{D}_A$ is an equivalence, this proves (2).

Remark 1.5.7 Take $M \in \mathcal{D}_A^{+w}$. Then there is an n_0 such that $W_nM \cong 0$ for all $n \leq n_0$. Indeed, by definition, $W_{n_0}M \cong 0$ for some n_0. Thus $M \to W^{>n_0}M$ is an isomorphism in $\mathcal{D}_A$. If $n < n_0$, then $W_nM \to W_nW^{>n_0}M \cong 0$ is an isomorphism in $\mathcal{D}_A$.

Another result using induction on the weight filtration is

Lemma 1.5.8 *Let M be an Adams graded dg A-module.*
1. M is a finite A-cell module if and only if M is free and finitely generated as a bi-graded A-module.

2. M is in $\mathcal{CM}_A^{+w}$ if and only if M is free as a bi-graded A-module and there is an integer r_0 such that $|m| \geq r_0$ for all $m \in M$.

Proof We first prove (1). Clearly a finite A-cell module is free and finitely generated as a bi-graded A-module. Conversely, suppose M is free and finitely generated over A; choose a basis $\mathcal{B}$ for M.

Clearly $W_n^{\mathcal{B}}M = 0$ for $n << 0$; let N be the minimum integer n such that $W_n^{\mathcal{B}}M \neq 0$ and let $\mathcal{B}_N$ be the set of basis elements b of Adams degree N. Since $A(0) = \mathbb{Q} \cdot \mathrm{id}$, it follows that $\mathcal{B}_N$ forms a $\mathbb{Q}$ basis for W_NM and the differential on $\mathcal{B}_N$ is given by

$$de_\alpha = \sum_\beta a_{\alpha\beta} e_\beta$$

with $a_{\alpha\beta} \in \mathbb{Q}$ and $e_\beta \in \mathcal{B}_N$. Changing the $\mathbb{Q}$ basis for $W_N^{\mathcal{B}}M$, we may assume that the subset $\mathcal{B}_N^0$ of $\mathcal{B}_N$ of e_α such that $de_\alpha = 0$ forms an $\mathbb{Q}$ basis for the kernel of d on the $\mathbb{Q}$-span of $\mathcal{B}_N$. Since $d^2 = 0$, the two-step filtration

$$\mathcal{B}_N^0 \subset \mathcal{B}_N$$

exhibits W_NM as a finite cell module.

The result follows by induction on the length of the weight filtration: By induction $W_{\mathcal{B}}^{>N}M := M/W_n^{\mathcal{B}}M$ is a finite cell module with basis say $\{b'_j \mid j \in J\}$ for some filtration on J. Since $M = W_N^{\mathcal{B}}M \oplus W_{\mathcal{B}}^{>N}M$ as an A-module, we just take the union of the two bases, and the concatenation of the filtrations, to present M as a finite cell module.

The proof of (2) is similar. In fact, the same proof as for (1) shows that the sub-dg A-module $W_n^{\mathcal{B}}M$ of M is in $\mathcal{CM}_A^{+w}$ for all n and that we may find an A basis $\mathcal{B}_n$ for $W_n^{\mathcal{B}}M$ and a filtration

$$\emptyset = \mathcal{B}_n^{r_0-1} \subset \mathcal{B}_n^{r_0} \subset \ldots \subset \mathcal{B}_n^{2n-1} \subset \mathcal{B}_n^{2n} = \mathcal{B}_n$$

that exhibits $W_n^{\mathcal{B}}M$ as a cell module. In addition, we may assume that $\mathcal{B}_i$ with its filtration is just $\mathcal{B}_n^{2i}$ with the induced filtration, for all $i \leq n$. Thus, taking the union of the $\mathcal{B}_n$ gives the desired basis for M, showing that M is in $\mathcal{CM}_A^{+w}$.

1.6 Bounded below modules

Definition 1.6.1 Let $\mathcal{D}_A^+ \subset \mathcal{D}_A$ be the full subcategory with objects the Adams graded dg A-modules M having $H^n(M) = 0$ for $n << 0$, as usual, we call such an M *bounded below.*

Lemma 1.6.2 *Suppose that A is cohomologically connected, and M is an Adams graded dg A-module with $H^n(M) = 0$ for $n < n_0$. Then there is a quasi-isomorphism $P \to M$ with P an A-cell module having basis $\{e_\alpha\}$ with $\deg(e_\alpha) \geq n_0$ for all α. If in addition there is an r_0 such that $H^n(M)(r) = 0$ for all (r, n) with $r < r_0$, we may find $P \to M$ as above satisfying the additional condition $|e_\alpha| \geq r_0$ for all α.*

Proof We first note the following elementary facts: Let $V = \oplus_{n,r} V^n(r)$ be a bi-graded $\mathbb{Q}$-vector space, which we consider as a complex with zero differential. Then the complex $A \otimes_{\mathbb{Q}} V$ is a cell-module, since a bi-graded $\mathbb{Q}$ basis for V gives a bi-graded A basis with 0 differential. In addition, the map $v \mapsto 1 \otimes v$ gives a map

$$V^n := \oplus_r V^n(r) \to H^n(A \otimes V).$$

Finally, suppose there is an n_0 such that $V^{n_0} \neq 0$ but $V^n = 0$ for all $n < n_0$. Then as $H^n(A) = 0$ for $n < 0$ and $H^0(A) = \mathbb{Q}$, the map

$$V^n \to H^n(A \otimes_{\mathbb{Q}} V)$$

is an isomorphism for all $n \leq n_0$.

We begin the construction of $P \to M$ by taking V to be a bi-graded $\mathbb{Q}$ subspace of $\oplus_{n \geq n_0} M^n$ representing $\oplus_n H^n(M)$, giving the map of Adams graded dg A modules

$$\phi_{n_0} : P_0 := \oplus_{n \geq n_0} A \otimes H^n(M)[-n] \to M.$$

From the discussion above, we see that ϕ_{n_0} is an isomorphism on H^n for $n \leq n_0$ and a surjection on H^n for $n > n_0$. If in addition there is an r_0 such that $H^n(M)(r) = 0$ for $r < r_0$ and all n, then P_0 has a bi-graded A-basis $\mathcal{S}_0$ with $|v| \geq r_0$ for each $v \in \mathcal{S}_0$.

Suppose by induction we have constructed a sequence of inclusions of A-cell modules

$$P_0 \to P_1 \to \ldots \to P_{r-1}$$

and maps of Adams graded dg A-modules

$$\phi_{n_0+i} : P_i \to M$$

with the following properties:

1. The P_i have A-bases $\mathcal{S}(i) := \mathcal{S}_0 \amalg \ldots \amalg \mathcal{S}_i$. In addition, for all $i \geq 1$, all the elements in $\mathcal{S}_i$ are of cohomological degree $n_0 + i - 1$, and for $v \in \mathcal{S}_i$, dv is in P_{i-1}.

2. The map $P_i \to P_{i+1}$ is the one induced by the inclusion $\mathcal{S}(i) \subset \mathcal{S}(i+1)$.

3. $\phi_{n_0+i} : P_i \to M$ induces an isomorphism on H^n for $n \leq n_0 + i$ and a surjection for all n.

4. If $H^n(M)(r) = 0$ for $r < r_0$ and all n, then $v \in \mathcal{S}(i)$ has Adams degree $|v| \geq r_0$.

We now show how to continue the induction. For this, let $n_r = n_0 + r$ and let $V \subset P_{r-1}^{n_r}$ be a bi-graded $\mathbb{Q}$-subspace of representatives for the kernel of the surjection $H^{n_r}(P_{r-1}) \to H^{n_r}(M)$. Let

$$P_r := P_{r-1} \oplus A \otimes_{\mathbb{Q}} V$$

as bi-graded A-module, where the differential is given by using the differential on P_{r-1}, setting

$$d((0, 1 \otimes v)) = (v, 0) \in P_{r-1}^{n_r}$$

for $v \in V$ and extending by the Leibniz rule. Note that, for $v \in V \subset P_{r-1}^{n_r}$, there is an $m_v \in M^{n_r - 1}$ with $dm_v = \phi_{r-1}(v)$; chosing a bi-graded $\mathbb{Q}$-basis $\mathcal{S}_r$ for V and extending the assignment $s \mapsto m_s$ from $\mathcal{S}_r$ to all of V by $\mathbb{Q}$-linearity, we have a $\mathbb{Q}$-linear map

$$f : V \to M^{n_r - 1}$$

with $d(f(v)) = \phi_{r-1}(v)$ for all $v \in V$. Thus, we may define the map of dg A-modules

$$\phi_r : P_r \to M$$

by using ϕ_{r-1} on P_r, f on $1 \otimes V$ and extending by A-linearity. Clearly P_r is an A-cell module with A-basis $\mathcal{S}(r) := \mathcal{S}(r-1) \amalg \mathcal{S}_r$.

In case $H^n(M)(r) = 0$ for $r < r_0$ and all n, clearly all bi-homogeneous elements of V have Adams degree $\geq r_0$, so $|v| \geq r_0$ for all $v \in \mathcal{S}_r$.

We can compute the cohomology of P_r by using the sequence of A-cell modules

$$0 \to P_{r-1} \to P_r \to A \otimes_{\mathbb{Q}} V \to 0,$$

where we consider V as a complex with zero differential, which is split exact as a sequence of bi-graded A-modules. The resulting long exact cohomology sequence shows that $P_{r-1} \to P_r$ induces an isomorphism in cohomology H^n for $n < n_r - 1$ and we have the exact sequence

$$0 \to H^{n_r - 1}(P_{r-1}) \to H^{n_r - 1}(P_r) \to V \xrightarrow{\partial} H^{n_r}(P_{r-1}) \to H^{n_r}(P_r) \to 0.$$

In addition, one can compute the coboundary ∂ by lifting the element $1 \otimes v \in (A \otimes_{\mathbb{Q}} V)^{n_r - 1}$ to the element $(0, 1 \otimes v) \in P_r^{n_r - 1}$ and applying the differential d_{P_r}. From this, we see that the sequence

$$0 \to V \xrightarrow{\partial} H^{n_r}(P_{r-1}) \to H^{n_r}(P_r) \to 0$$

is exact, hence $H^{n_r - 1}(P_{r-1}) \to H^{n_r - 1}(P_r)$ is an isomorphism. This also shows that $\phi_r : P_r \to M$ induces an isomorphism on H^n for $n \leq n_r$ and the induction continues.

If we now take P to be the direct limit of the P_r, it follows that P is an A-cell module with basis elements all in cohomological degree $\geq n_0$, and that the map $\phi : P \to M$ induced from the ϕ_r is a quasi-isomorphism. If there is an r_0 such that $H^*(M)(r) = 0$ for $r < r_0$, then by (4) above, the basis $\mathcal{S} := \cup_r \mathcal{S}(r)$ clearly has $|e| \geq r_0$ for all $e \in \mathcal{S}$. This completes the proof.

1.7 Tor and Ext

The Hom functor $\mathcal{H}om_A(M,N)$ and tensor product functor $M\otimes_A N$ define bi-exact bi-functors

$$\mathcal{H}om_A : \mathcal{KCM}_A^{\mathrm{op}} \otimes \mathcal{KCM}_A \to \mathcal{D}_A$$
$$\otimes_A : \mathcal{KCM}_A \otimes \mathcal{KCM}_A \to \mathcal{KCM}_A.$$

Via proposition 1.4.3, these give well-defined derived functors of $\mathcal{H}om_A$ and $\otimes_A$:

$$R\mathcal{H}om_A : \mathcal{D}_A^{\mathrm{op}} \otimes \mathcal{D}_A \to \mathcal{D}_A$$
$$\otimes_A^L : \mathcal{D}_A \otimes \mathcal{D}_A \to \mathcal{D}_A.$$

Restricting to $\mathcal{KCM}_A^f$, we have the derived functors for the finite categories

$$R\mathcal{H}om_A : \mathcal{D}_A^{f\mathrm{op}} \otimes \mathcal{D}_A^f \to \mathcal{D}_A^f$$
$$\otimes_A^L : \mathcal{D}_A^f \otimes \mathcal{D}_A^f \to \mathcal{D}_A^f.$$

In both settings, these bi-functors are adjoint:

$$R\mathcal{H}om_A(M \otimes^L N, K) \cong R\mathcal{H}om_A(M, R\mathcal{H}om_A(N,K)).$$

We have as well the restriction of $\otimes^L$ to $\mathcal{D}_A^{+w}$:

$$\otimes_A^L : \mathcal{D}_A^{+w} \otimes \mathcal{D}_A^{+w} \to \mathcal{D}_A^{+w}.$$

The derived tensor product makes $\mathcal{D}_A$ into a triangulated tensor category with unit $\mathbb{1} := A$ and $\mathcal{D}_A^{+w}$, $\mathcal{D}_A^+$ and $\mathcal{D}_A^f$ are triangulated tensor subcategories. By lemma 1.5.4, $\mathcal{D}^f$ is closed under taking summands in $\mathcal{D}_A$; this property is obvious for $\mathcal{D}_A^{+w}$.

Define $M^\vee := R\mathcal{H}om_A(M,A)$ and call M *strongly dualizable* if the canonical map $M \to M^{\vee\vee}$ is an isomorphism in $\mathcal{D}_A$. Note that M is strongly dualizable if M is *rigid*, i.e., there exists an $N \in \mathcal{D}_A$ and morphisms $\delta : A \to M \otimes_A^L N$ and $\epsilon : N \otimes_A^L M \to A$ such that

$$(\mathrm{id}_M \otimes \epsilon) \circ (\delta \otimes \mathrm{id}_M) = \mathrm{id}_M$$
$$(\mathrm{id}_N \otimes \delta) \circ (\epsilon \otimes \mathrm{id}_N) = \mathrm{id}_N$$

We have

Proposition 1.7.1 ([26, theorem 5.7]) *$M \in \mathcal{D}_A$ is rigid if and only if M is in $\mathcal{D}_A^f$, i.e., $M \cong N$ in $\mathcal{D}_A$ for some finite A-cell module N.*

The precise statement found in [26, theorem 5.7] is that M is rigid if and only if M is a *summand* in $\mathcal{D}_A$ of some finite cell module, so the proposition follows from this and lemma 1.5.4; Kriz and May are working in a more general setting in which lemma 1.5.4 does not hold.

Example 1.7.2 For $n \geq 0$, $\mathbb{Q}(\pm n) \cong (\mathbb{Q}(\pm 1))^{\otimes n}$ and for all n, $\mathbb{Q}(n)^\vee \cong \mathbb{Q}(-n)$.

1.8 Change of ring

If $\phi : A \to A'$ is a homomorphism of Adams graded cdgas, we have the functor

$$- \otimes_A A' : \mathcal{M}_A \to \mathcal{M}_{A'}$$

which induces a functor on cell modules and the homotopy category

$$\phi_* : \mathcal{KCM}_A \to \mathcal{KCM}_{A'}.$$

Via proposition 1.4.3, we have the change of rings functor

$$\phi_* : \mathcal{D}_A \to \mathcal{D}_{A'}$$

on the derived category. By proposition 1.4.3 and lemma 1.5.6, the respective restrictions of ϕ_* define exact tensor functors

$$\phi_* : \mathcal{D}_A^{+w} \to \mathcal{D}_{A'}^{+w}$$
$$\phi_* : \mathcal{D}_A^{f} \to \mathcal{D}_{A'}^{f}.$$

From [26] we have

Theorem 1.8.1 ([26, proposition 4.2]) *If ϕ is a quasi-isomorphism, then*

$$\phi_* : \mathcal{D}_A \to \mathcal{D}_{A'}$$

is an equivalence of triangulated tensor categories.

Noting the ϕ_* is compatible with the weight filtrations, the theorem immediately yields

Corollary 1.8.2 *If ϕ is a quasi-isomorphism, then*

$$\phi_* : \mathcal{D}_A^{+w} \to \mathcal{D}_{A'}^{+w}$$

is an equivalence of triangulated tensor categories.

In addition, we have

Corollary 1.8.3 *If ϕ is a quasi-isomorphism, then*

$$\phi_* : \mathcal{D}_A^{f} \to \mathcal{D}_{A'}^{f}$$

is an equivalence of triangulated tensor categories.

Proof Since an equivalence of tensor triangulated categories induces an equivalence on the subcategories of rigid objects, the result follows from theorem 1.8.1 and proposition 1.7.1.

Proposition 1.8.4 *Let $\phi : A \to B$ be a map of cdgas. Then $\phi_* : \mathcal{D}_A^{+w} \to \mathcal{D}_B^{+w}$ is conservative, i.e., $\phi_*(M) \cong 0$ implies $M \cong 0$, or equivalently, if $\phi_*(f)$ is an isomorphism then f is an isomorphism.*

Proof Take $M \in \mathcal{D}^{+w}$, and let

$$\mathcal{S} := \{n \mid M \cong W^{>n}M\}.$$

Then $\mathcal{S} \neq \emptyset$; we claim that either $M \cong 0$ or $\mathcal{S}$ has a maximal element. Indeed, if $\mathcal{S}$ has no maximum then $W_nM \cong 0$ for all n. But since

$$\varinjlim_n W_nM \to M$$

is an isomorphism, this implies that M is acyclic, hence $M \cong 0$ in $\mathcal{D}_A$.

Thus, we may find a cell module P and quasi-isomorphism $P \to M$ such that $W_{n-1}P = 0$, but W_nP is not acyclic. In particular P has a basis $\{e_\alpha\}$ with $|e_\alpha| \geq n$ for all α. If $|e_\alpha| = n$ then since there are no basis elements with Adams grading $< n$, we have

$$de_\alpha = \sum_j a_{\alpha j}e_j$$

with $|a_{\alpha j}| = 0$, $|e_j| = n$, i.e., $a_{\alpha j} \in \mathbb{Q} = A(0)$. Since W_nP is not acyclic, it thus follows that $(W_nP) \otimes_A \mathbb{Q}$ is also not acyclic: if $(W_nP) \otimes_A \mathbb{Q}$ were acyclic, this complex would be zero in the homotopy category $\mathcal{KCM}_\mathbb{Q}$, which would make W_nP 0 in $\mathcal{KCM}_A$. As $W_n(P \otimes_A B) = (W_nP) \otimes_A B$ and

$$(W_nP) \otimes_A \mathbb{Q} = (W_nP \otimes_A B) \otimes_B \mathbb{Q}$$

it follows that $P \otimes_A B$ is not isomorphic to zero in $\mathcal{KCM}_B$, and thus $\phi_*(M)$ is non-zero in $\mathcal{D}_B^{+w}$.

Example 1.8.5 Each Adams graded cdga A has a canonical augmentation $\epsilon : A \to \mathbb{Q}$, given by projection on $A^0(0) = \mathbb{Q} \cdot \mathrm{id}$.

In particular, we have the functor

$$q := \epsilon_* : \mathcal{CM}_A \to \mathcal{M}_\mathbb{Q},\ qM := M \otimes_A \mathbb{Q}$$

and the exact tensor functors

$$\begin{aligned} q &: \mathcal{D}_A \to \mathcal{D}_\mathbb{Q}, \\ q^{+w} &: \mathcal{D}_A^{+w} \to \mathcal{D}_\mathbb{Q}^{+w}, \\ q^f &: \mathcal{D}_A^f \to \mathcal{D}_\mathbb{Q}^f. \end{aligned}$$

Explicitly, q is given on the derived level by $qM := M \otimes_A^L \mathbb{Q}$.

1.9 Finiteness conditions

$\mathcal{M}_\mathbb{Q}$ is just the category of graded $\mathbb{Q}$-vector spaces, so $\mathcal{D}_\mathbb{Q}$ is equivalent to the product of the unbounded derived categories

$$\mathcal{D}_\mathbb{Q} \cong \prod_{n\in\mathbb{Z}} D(\mathbb{Q}).$$

Similarly

$$\mathcal{D}_\mathbb{Q}^f \cong \oplus_{n\in\mathbb{Z}} D^b(\mathbb{Q}),$$

where $D^b(\mathbb{Q})$ is the bounded derived category of finite dimensional $\mathbb{Q}$-vector spaces. Finally,

$$\mathcal{D}_{\mathbb{Q}}^{+w} \cong \bigcup_N \prod_{n\geq N} D(\mathbb{Q}) \ \subset \ \prod_{n\in\mathbb{Z}} D(\mathbb{Q}).$$

Remark 1.9.1 The inclusion $\mathbb{Q} \to A$ splits ϵ, identifying $\mathcal{D}_{\mathbb{Q}}$, $\mathcal{D}_{\mathbb{Q}}^{+w}$, etc., with full subcategories of $\mathcal{D}_A$, $\mathcal{D}_A^{+w}$, etc. Under this identification, and the decomposition of $\mathcal{D}_{\mathbb{Q}}$ into its Adams graded pieces described above, the functor q is identified with the functor $\mathrm{gr}_*^W := \prod_{n\in\mathbb{Z}} \mathrm{gr}_n^W$. Indeed, if P is an A-cell module with basis $\{e_\alpha\}$, then as $A(r) = 0$ for $r < 0$ and $A(0) = \mathbb{Q}\cdot\mathrm{id}$, the differential d decomposes as $d = d^0 + d^+$ with

$$d^0 e_\alpha = \sum_\beta a^0_{\alpha\beta} e_\beta,\ d^+ e_\alpha = \sum_\beta a^+_{\alpha\beta} e_\beta$$

where $|a^0_{\alpha\beta}| = 0$, $|a^+_{\alpha\beta}| > 0$. Since d has Adams degree 0, it follows that $|e_\beta| < |e_\alpha|$ if $a^+_{\alpha\beta} \neq 0$, and $|e_\beta| = |e_\alpha|$ if $a^0_{\alpha\beta} \neq 0$. Thus $\mathrm{gr}_*^W P$ is the complex of graded $\mathbb{Q}$-vector spaces with $\mathbb{Q}$ basis $\{e_\alpha\}$ and with $d_{\mathrm{gr}_*^W P} e_\alpha = d^0 e_\alpha$. As qP has exactly the same description, we have the identification of gr_*^W and q as described.

Lemma 1.9.2 *Let M be in $\mathcal{D}_A^{+w}$. Then M is in $\mathcal{D}_A^f$ if and only if*

1. $\mathrm{gr}_n^W M$ *is in* $D^b(\mathbb{Q}) \subset D(\mathbb{Q})$ *for all* n.
2. $\mathrm{gr}_n^W M \cong 0$ *for all but finitely many* n.

Proof It is clear that $M \in \mathcal{D}_A^f$ satisfies the conditions (1) and (2). Conversely, suppose $M \in \mathcal{D}_A^{+w}$ satisfies (1) and (2). If $M \cong 0$, there is nothing to prove, so assume M is not isomorphic to 0. By proposition 1.8.4, $qM = \prod_n \mathrm{gr}_n^W M$ is not isomorphic to zero. Take N minimal such that $\mathrm{gr}_N^W M$ is not isomorphic to zero. By (2), there is a maximal N' such that $\mathrm{gr}_{N'}^W M$ is not isomorphic to zero.

If $N = N'$, then $M \cong \mathrm{gr}_N^W M$ is in $D^b(\mathbb{Q})$ by (1), hence $M \cong \oplus_{i=1}^s A{<}{-N}{>}[m_i]$, and thus M is in $\mathcal{D}_A^f$. In general, we apply remark 1.5.3, giving the distinguished triangle

$$\mathrm{gr}_N^W M \to M \to M^{>N} \to \mathrm{gr}_N^W M[1];$$

note that $\mathrm{gr}_n^W M^{>N} \cong 0$ for $n > N'$. By induction on $N' - N$, $M^{>N}$ is in $\mathcal{D}_A^f$; since $\mathcal{D}_A^f$ is a full triangulated subcategory of $\mathcal{D}_A$, closed under isomorphism, it follows that M is in $\mathcal{D}_A^f$.

1.10 Model structure

Let **cdga** denote the category of Adams graded commutative differential graded algebras over $\mathbb{Q}$. In the non-Adams graded case, Bousfield and Guggenheim [7] have defined a model structure on cdgas with weak equivalences the quasi-isomorphisms. As we are interested in possibly non-connected Adams graded cdgas, we modify their definitions slightly.

Definition 1.10.1 1. A morphism $\phi : A \to B$ in **cdga** is a *weak equivalence* if ϕ induces an isomorphism

$$\phi_* : H^n(A(r)) \to H^n(B(r))$$

for all $n, r \geq 1$.

2. A morphism $\phi : A \to B$ in **cdga** is a *fibration* if $\phi(r) : A(r)^n \to B(r)^n$ is surjective for all $n, r \geq 1$

3. A morphism $\phi : A \to B$ in **cdga** is a *cofibration* if ϕ has the left lifting property with respect to acyclic fibrations.

The proof that this defines a model structure on **cdga** is word for word the same as the proof in [7, chapter 4]; we will give details of the proof in §2, where we discuss a more general situation. We denote the homotopy category of **cdga** by $\mathcal{H}(\mathbf{cdga})$.

1.11 Minimal models

Definition 1.11.1 A cdga A is said to be *generalized nilpotent* if

1. As a graded $\mathbb{Q}$-algebra, $A = \mathrm{Sym}^* E$ for some $\mathbb{Z}$-graded $\mathbb{Q}$-vector space E, i.e., $A = \Lambda^* E_{\mathrm{odd}} \otimes \mathrm{Sym}^* E_{\mathrm{ev}}$. In addition, $E_n = 0$ for $n \leq 0$.

2. For $n \geq 0$, let $A_{(n)} \subset A$ be the subalgebra generated by the elements of degree $\leq n$. Set $A_{(n+1,0)} = A_{(n)}$ and for $q \geq 0$ define $A_{(n+1,q+1)}$ inductively as the subalgebra generated by $A_{(n)}$ and
$$A^{n+1}_{(n+1,q+1)} := \{x \in A^{n+1}_{(n+1)} \, | dx \in A_{(n+1,q)}\}.$$
Then for all $n \geq 0$,
$$A_{(n+1)} = \cup_{q \geq 0} A_{(n+1,q)}.$$

Note that a generalized nilpotent cdga is automatically connected.

Definition 1.11.2 Let A be a cdga. An *n-minimal model* of A is a map of cdgas
$$s : A\{n\} \to A,$$
with $A\{n\}$ generalized nilpotent and generated (as an algebra) in degrees $\leq n$, such that s induces an isomorphism on H^m for $1 \leq m \leq n$ and an injection on H^{n+1}.

Remark 1.11.3 Let $s : A\{n\} \to A$ be an n-minimal model of A. Then $A\{n\}_{(n-1)} \subset A\{n\}$ is clearly generalized nilpotent and the inclusion in $A\{n\}$ is an isomorphism in degrees $\leq n-1$. Thus $H^p(A\{n\}_{(n-1)}) \to H^p(A\{n\})$ is an isomorphism for $p \leq n-1$ and injective for $p = n$, and hence $s : A\{n\}_{(n-1)} \to A$ is an $n-1$-minimal model.

Define the above notions for Adams graded cdgas by giving everything an Adams grading.

By lemma 2.4.4, a generalized nilpotent cdga is cofibrant, so the minimal model $s : A\{\infty\} \to A$ is a cofibrant replacement, that is, s is a weak equivalence and $A\{\infty\}$ is cofibrant.

Theorem 1.11.4 *Let A be an Adams graded cdga.*
1. For each $n = 1, 2, \ldots, \infty$, there is an n-minimal model of A: $A\{n\} \to A$.

2. If $\psi : A \to B$ is a quasi-isomorphism of Adams graded cdgas, and $s : A\{n\} \to A$, $t : B\{n\} \to B$ are n-minimal models, then there is an isomorphism of Adams graded cdgas, $\phi : A\{n\} \to B\{n\}$ such that $\psi \circ s$ is homotopic to $t \circ \phi$.

See [7, chapter 4] for a proof in the non-Adams graded case; the Adams graded case is exactly the same, where one proceeds by a double induction, first with respect to the Adams degree and then with respect to the cohomological degree. For details, theorem 1.11.4(1) is a special case of proposition 2.4.9 and theorem 1.11.4(2) is a special case of proposition 2.4.14.

Corollary 1.11.5 *If A is cohomologically connected, there is a quasi-isomorphism of Adams graded cdgas $A' \to A$ with A' connected. Similarly, if $\phi : A \to B$ is a map of cohomologically connected Adams graded cdgas, there is a diagram of Adams graded cdgas*

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow[\phi]{} & B \end{array}$$

that commutes up to homotopy, with the vertical maps being quasi-isomorphisms, such that A' and B' are connected.

Proof For the first assertion, just take $A' = A\{\infty\}$. For the second, let $B' = B\{\infty\}$. Since $\phi : A\{\infty\} \to A$ is a quasi-isomorphism of A-cell modules, ϕ is a homotopy equivalence of A-cell modules (proposition 1.4.3), so taking the tensor product yields a quasi-isomorphism

$$A\{\infty\} \otimes_A B \to B.$$

Clearly $A\{\infty\} \otimes_A B$ is a generalized nilpotent cdga, so we need only apply theorem 1.11.4(2).

This result, together with theorem 1.8.1, corollary 1.8.2 and corollary 1.8.3, allows us to replace "cohomologically connected" with "connected" in statements involving $\mathcal{D}_A$, $\mathcal{D}_A^{+w}$ or $\mathcal{D}_A^f$.

1.12 t-structure

To define a t-structure on $\mathcal{D}_A^{+w}$ or $\mathcal{D}_A^f$, one needs to assume that A is cohomologically connected; by corollaries 1.8.2 or 1.8.3, we may assume that A is connected. Recall from example 1.8.5 the functor

$$q := \epsilon_* : \mathcal{CM}_A \to \mathcal{M}_{\mathbb{Q}}$$

associated to the augmentation $\epsilon : A \to \mathbb{Q}$, and its extension to exact tensor functors on the various derived categories.

Define full subcategories $\mathcal{D}_A^{\le 0}$, $\mathcal{D}_A^{\ge 0}$ and $\mathcal{H}_A$ of $\mathcal{D}_A^{+w}$ by

$$\begin{aligned} \mathcal{D}_A^{\le 0} &:= \{M \in \mathcal{D}_A^{+w} \mid H^n(qM) = 0 \text{ for } n > 0\} \\ \mathcal{D}_A^{\ge 0} &:= \{M \in \mathcal{D}_A^{+w} \mid H^n(qM) = 0 \text{ for } n < 0\} \\ \mathcal{H}_A &:= \{M \in \mathcal{D}_A^{+w} \mid H^n(qM) = 0 \text{ for } n \neq 0\}. \end{aligned}$$

The arguments of Kriz-May [26] show that this defines a t-structure $(\mathcal{D}_A^{\le 0}, \mathcal{D}_A^{\ge 0})$ on $\mathcal{D}_A^{+w}$ with heart $\mathcal{H}_A$. Since Kriz-May use $\mathcal{D}_A^+$ instead of $\mathcal{D}_A^{+w}$, we give a sketch of the argument here, with the necessary modifications.

Remark 1.12.1 As we have identified the functor q with $\prod_n \mathrm{gr}^W_n$ (remark 1.9.1) we can describe the category $\mathcal{D}^{\leq 0}_A$ as the $M \in \mathcal{D}^{+w}_A$ such that $H^m(\mathrm{gr}^W_n M) = 0$ for all $m > 0$ and all n. We have a similar description of $\mathcal{D}^{\geq 0}_A$ and $\mathcal{H}_A$.

Recall that an *essentially full* subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is a full subcategory such that, if $b \to a$ is an isomorphism in $\mathcal{A}$ with b in $\mathcal{B}$, then a is in $\mathcal{B}$.

Definition 1.12.2 We recall from [5] that a *t-structure* on a triangulated category $\mathcal{D}$ consists of essentially full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $\mathcal{D}$ such that

1. $\mathcal{D}^{\leq 0}[1] \subset \mathcal{D}^{\leq 0}$, $\mathcal{D}^{\geq 0}[-1] \subset \mathcal{D}^{\geq 0}$
2. $\mathrm{Hom}_{\mathcal{D}}(M, N[-1]) = 0$ for M in $\mathcal{D}^{\leq 0}$, N in $\mathcal{D}^{\geq 0}$
3. Each $M \in \mathcal{D}$ admits a distinguished triangle
$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1]$$
with $M^{\leq 0}$ in $\mathcal{D}^{\leq 0}$, $M^{>0}$ in $\mathcal{D}^{\geq 0}[-1]$.

Write $\mathcal{D}^{\leq n}$ for $\mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n}$ for $\mathcal{D}^{\geq 0}[-n]$.

A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is *non-degenerate* if $A \in \cap_{n\leq 0}\mathcal{D}^{\leq n}$, $B \in \cap_{n\geq 0}\mathcal{D}^{\geq n}$ imply $A \cong 0 \cong B$.

Lemma 1.12.3 *Suppose that A is connected.*

1. Take M in $\mathcal{D}^{\leq 0}_A$. Then there is an A-cell module $P \in \mathcal{CM}^{+w}_A$ with basis $\{e_\alpha\}$ such that $\deg(e_\alpha) \leq 0$ *for all α, and a quasi-isomorphism $P \to M$.*

2. For $N \in \mathcal{D}^{\geq 0}_A$, there is an A-cell module $P \in \mathcal{CM}^{+w}_A$ with basis $\{e_\alpha\}$ such that $\deg(e_\alpha) \geq 0$ *for all α, and a quasi-isomorphism $P \to N$.*

Proof For (1) choose a quasi-isomorphism $Q \to M$ with Q in $\mathcal{CM}^{+w}_A$. Let $\{e_\alpha\}$ be a basis for Q. Decompose the differential d_Q as $d_Q = d^0_Q + d^+_Q$ as in remark 1.9.1. Making a $\mathbb{Q}$-linear change of basis if necessary, we may assume that the collection $\mathcal{S}_0$ of e_α with $\deg e_\alpha = 0$ and $d^0_Q e_\alpha = 0$ forms a basis of

$$\ker[d^0 : \oplus_{\deg e_\alpha = 0}\mathbb{Q}e_\alpha \to \oplus_{\deg e_\beta = 1}\mathbb{Q}e_\beta].$$

Let $\tau^{\leq 0}Q$ be the A submodule of Q with basis $\{e_\alpha \mid \deg e_\alpha < 0\} \cup \mathcal{S}_0$. We claim that $\tau^{\leq 0}Q$ is a subcomplex of Q. Indeed, we have

$$\begin{aligned} d_Q e_\alpha &= d^0_Q e_\alpha + d^+_Q e_\alpha \\ &= \sum_\beta a^0_{\alpha\beta} e_\beta + \sum_\beta a^+_{\alpha\beta} e_\beta \end{aligned}$$

with $|a^0_{\alpha\beta}| = 0 = \deg a^0_{\alpha\beta}$, $|a^+_{\alpha\beta}| > 0$. Since A is connected, $\deg a^+_{\alpha\beta} \geq 1$. As d_Q has cohomological degree $+1$, it follows that $\deg e_\beta \leq \deg e_\alpha$ if $a^+_{\alpha\beta} \neq 0$. Similarly, $\deg e_\beta = \deg e_\alpha + 1$ if $a^0_{\alpha\beta} \neq 0$.

Take e_α with $\deg e_\alpha = -1$. Since $d_Q^2 = 0$, it follows that $(d_Q^0)^2 = 0$, from which it follows that e_β is in $\mathcal{S}_0$ if $a^0_{\alpha\beta} \neq 0$. Now take $e_\alpha \in \mathcal{S}_0$. Write

$$de_\alpha = \sum_{\deg b^+_{\alpha\beta}=1} b^+_{\alpha\beta} f^0_\beta + \sum_{\deg b^+_{\alpha\beta}>1} b^+_{\alpha\beta} f_\beta$$

with the $\{b^+_{\alpha\beta}\}$ being chosen $\mathbb{Q}$ independent in $A^{*\geq 1}$, the f_β in the $\mathbb{Q}$ span of the degree ≤ -1 part of the basis $\{e_\alpha\}$ and the f^0_β in $\mathbb{Q}$ span of the degree 0 part of $\{e_\alpha\}$. We have

$$0 = d^2 e_\alpha = \sum_{\deg b_{\alpha\beta}=1} b^+_{\alpha\beta} d^0(f^0_\beta) + \ldots$$

with the $\ldots$ involving only the degree ≤ 0 part of the basis (and coefficients from A). Since the $b^+_{\alpha\beta}$ are $\mathbb{Q}$ independent, we have $d^0 f^0_\beta = 0$ for all β in the first sum, hence the f^0_β are in the $\mathbb{Q}$-span of $\mathcal{S}_0$. Thus $\tau^{\leq 0}Q$ is a subcomplex of Q, as claimed.

So far we have only needed that Q is a cell module. We will now use that Q lies in $\mathcal{CM}_A^{+w}$. We claim that $\tau^{\leq 0}Q \to Q$ is a quasi-isomorphism. By proposition 1.8.4 applied to the augmentation $A \to \mathbb{Q}$, the functor

$$q : \mathcal{D}_A^{+w} \to \mathcal{D}_\mathbb{Q}^{+w}$$

is conservative, thus it suffices to see that $q\tau^{\leq 0}Q \to qQ$ is a quasi-isomorphism. Now, qQ represents $qM \in \mathcal{D}_\mathbb{Q}$, and by assumption qM is in $\mathcal{D}_\mathbb{Q}^{\leq 0}$, hence qQ is in $\mathcal{D}_\mathbb{Q}^{\leq 0}$. But by construction $q\tau^{\leq 0}Q \to qQ$ is an isomorphism on H^n for all $n \leq 0$. Since $H^n(q\tau^{\leq 0}Q) = 0$ for $n > 0$, it follows that $q\tau^{\leq 0}Q \to qQ$ is a quasi-isomorphism, as desired.

For (2), we may assume that N is an object in $\mathcal{CM}_A^{+w}$ and thus $W_{r_0-1}N = 0$ for some r_0. The result then follows from lemma 1.6.2.

Lemma 1.12.4 *Suppose that A is connected. Then* $\mathrm{Hom}_{\mathcal{D}_A^{+w}}(M, N[-1]) = 0$ *for M in $\mathcal{D}_A^{\leq 0}$, N in $\mathcal{D}_A^{\geq 0}$.*

Proof By lemma 1.12.3 we may assume that M and $N[-1]$ are A-cell modules with bases $\{e_\alpha\}$ for M and $\{f_\beta\}$ for $N[-1]$ satisfying $\deg e_\alpha \leq 0$ and $\deg f_\beta \geq 1$ for all α, β. By lemma 1.5.6, we also have

$$\mathrm{Hom}_{\mathcal{D}_A^{+w}}(M, N[-1]) = \mathrm{Hom}_{\mathcal{KCM}_A^{+w}}(M, N[-1]).$$

But if $\phi : M \to N[-1]$ is a map in $\mathcal{KCM}_A^{+w}$, then ϕ is given by a degree 0 map of complexes, so

$$\phi(e_\alpha) = \sum_\beta a_{\alpha\beta} f_\beta$$

for $a_{\alpha\beta} \in A$ with $\deg(a_{\alpha\beta}) + \deg(f_\beta) = \deg(e_\alpha)$ Since $A^i = 0$ for $i < 0$, this is impossible.

Lemma 1.12.5 *Suppose that A is connected. For $M \in \mathcal{D}_A^{+w}$, there is a distinguished triangle*

$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1]$$

with $M^{\leq 0}$ in $\mathcal{D}_A^{\leq 0}$, $M^{>0}$ in $\mathcal{D}_A^{\geq 1}$.

Proof We may assume that M is in $\mathcal{CM}_A^{+w}$. We perform exactly the same construction as in the proof of lemma 1.12.3, giving us a sub A-cell module $\tau^{\leq 0}M$ of M such that

(a) $\tau^{\leq 0}M$ has a basis $\{e_\alpha\}$ with $\deg e_\alpha \leq 0$ for all α

(b) The map $q\tau^{\leq 0}M \to qM$ induced by applying q to $\tau^{\leq 0}M \to M$ gives an isomorphism on H^n for $n \leq 0$.

Let $M^{\leq 0} = \tau^{\leq 0}M$ and let $M^{>0}$ be the cone of $\tau^{\leq 0}M \to M$. This gives us the distinguished triangle

$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1]$$

in $\mathcal{D}_A^{+w}$. By construction, $M^{\leq 0}$ is in $\mathcal{D}_A^{+w}$. Applying q to the distinguished triangle gives the distinguished triangle in $\mathcal{D}_{\mathbb{Q}}^{+w}$

$$qM^{\leq 0} \to qM \to qM^{>0} \to qM^{\leq 0}[1];$$

by (b) and the fact that $H^1(qM^{\leq 0}) = 0$, it follows that $H^n(qM^{>0}) = 0$ for $n \leq 0$. Thus $M^{>0}$ is in $\mathcal{D}_A^{\geq 1}$, as desired.

Theorem 1.12.6 *Suppose A is cohomologically connected. Then $(\mathcal{D}_A^{\leq 0}, \mathcal{D}_A^{\geq 0})$ is a non-degenerate t-structure on $\mathcal{D}_A^{+w}$.*

Proof Replacing A with its minimal model, we may assume that A is connected. The property (1) of definition 1.12.2 is obvious; properties (2) and (3) follow from lemmata 1.12.4 and 1.12.5, respectively.

For $A \in \cap_{n\leq 0}\mathcal{D}_A^{\leq n}$, it follows that $H^n(qA) = 0$ for all n, i.e., $qA \cong 0$ in $\mathcal{D}_{\mathbb{Q}}^{+w}$. Since q is conservative, $A \cong 0$ in $\mathcal{D}_A^{+w}$. The case of $B \in \cap_{n\geq 0}\mathcal{D}_A^{\geq n}$ is similar, hence the t-structure is non-degenerate.

Definition 1.12.7 Let $\mathcal{D}_A^{f,\leq 0} := \mathcal{D}_A^f \cap \mathcal{D}_A^{\leq 0}$, $\mathcal{D}_A^{f,\geq 0} := \mathcal{D}_A^f \cap \mathcal{D}_A^{\geq 0}$, $\mathcal{H}_A^f := \mathcal{H}_A \cap \mathcal{D}_A^f = \mathcal{D}_A^{f,\leq 0} \cap \mathcal{D}_A^{f,\geq 0}$.

Corollary 1.12.8 *If A is cohomologically connected, then $(\mathcal{D}_A^{f,\leq 0}, \mathcal{D}_A^{f,\geq 0})$ is a non-degenerate t-structure on $\mathcal{D}_A^f$ with heart $\mathcal{H}_A^f$.*

Proof Since $\mathcal{D}_A^f$ is a full triangulated subcategory of $\mathcal{D}_A^{+w}$, closed under isomorphisms in $\mathcal{D}_A^{+w}$, all the properties of a non-degenerate t-structure are inherited from the non-degenerate t-structure on $(\mathcal{D}_A^{\leq 0}, \mathcal{D}_A^{\geq 0})$ on $\mathcal{D}_A^{+w}$ given by theorem 1.12.6, except perhaps for the condition (3) of definition 1.12.2. So, take $M \in \mathcal{D}_A^f$. Since $(\mathcal{D}_A^{\leq 0}, \mathcal{D}_A^{\geq 0})$ is a t-structure on $\mathcal{D}_A^{+w}$, we have a distinguished triangle

$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1]$$

with $M^{\leq 0}$ in $\mathcal{D}_A^{\leq 0}$, $M^{>0}$ in $\mathcal{D}_A^{\geq 0}[-1]$. Applying the exact functor gr_n^W (see remark 1.5.3) gives the distinguished triangle

$$\mathrm{gr}_n^W M^{\leq 0} \to \mathrm{gr}_n^W M \to \mathrm{gr}_n^W M^{>0} \to \mathrm{gr}_n^W M^{\leq 0}[1]$$

in the derived category of $\mathbb{Q}$-vector spaces $D(\mathbb{Q})$, such that $\mathrm{gr}^W_n M^{\le 0}$ is in $D(\mathbb{Q})^{\le 0}$ and $\mathrm{gr}^W_n M^{>0}$ is in $D(\mathbb{Q})^{\ge 1}$, i.e., $H^n(\mathrm{gr}^W_n M^{\le 0}) = 0$ for $n > 0$, $H^n(\mathrm{gr}^W_n M^{>0}) = 0$ for $n \le 0$. However, since M is in $\mathcal{D}^f_A$, it follows that $\mathrm{gr}^W_n M$ is in $D^b(\mathbb{Q})$ for all n and is isomorphic to 0 for all but finitely many n (lemma 1.9.2). The long exact cohomology sequence for a distinguished triangle in $D(\mathbb{Q})$ thus shows that $\mathrm{gr}^W_n M^{\le 0}$ and $\mathrm{gr}^W_n M^{>0}$ are in $D^b(\mathbb{Q})$ for all n and are isomorphic to zero for all but finitely many n. Applying lemma 1.9.2 again shows $M^{\le 0}$ and $M^{>0}$ are in $\mathcal{D}^f_A$.

Lemma 1.12.9 *(1) The restriction of $\otimes^L$ to $\mathcal{H}_A$ and $\mathcal{H}^f_A$ makes these into abelian tensor categories.*

(2) The weight filtrations on $\mathcal{D}^{+w}_A$ and $\mathcal{D}^f_A$ restrict to define exact functorial filtrations on $\mathcal{H}_A$ and $\mathcal{H}^f_A$.

(3) $\mathcal{H}^f_A$ is the smallest abelian subcategory of $\mathcal{H}^f_A$ containing the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$ and closed under extensions in $\mathcal{H}^f_A$.

Proof (1) is more or less obvious: for cell modules M and N, we have $q(M \otimes_A N) \cong qM \otimes_{\mathbb{Q}} qN$; the Künneth formula for $H^n(qM \otimes_{\mathbb{Q}} qN)$ thus shows that $\mathcal{D}^{\le 0}_A$ and $\mathcal{D}^{\ge 0}_A$ are closed under $\otimes^L_A$.

For (2), note that the augmentation $\epsilon : A \to \mathbb{Q}$ is a homomorphism of Adams graded cdgas, and that $q = \epsilon_*$. Thus q is compatible with the weight filtrations on $\mathcal{D}_A$ and $\mathcal{D}_{\mathbb{Q}}$ (and also on the finite categories). In particular, we have

$$q(\mathrm{gr}^W_n M) \cong \mathrm{gr}^W_n qM.$$

On the other hand, for C in $\mathcal{D}^{+w}_{\mathbb{Q}}$ we have

$$C \cong \oplus_m H^m(C)[-m]$$

Furthermore $H^m(C)$ is isomorphic to its associated weight graded $\oplus_n \mathrm{gr}^W_n H^m(C)$. All this implies that

$$M \text{ is in } \mathcal{D}^{\le 0}_A \iff \mathrm{gr}^W_n M \text{ is in } \mathcal{D}^{\le 0}_A \text{ for all } n$$

and similarly for $\mathcal{D}^{\ge 0}_A$. Thus, the t-structure $(\mathcal{D}^{\le 0}_A, \mathcal{D}^{\ge 0}_A)$ on $\mathcal{D}^{+w}_A$ induces a t-structure $(W_n\mathcal{D}^{\le 0}_A, W_n\mathcal{D}^{\ge 0}_A)$ on the full subcategory $W_n\mathcal{D}^{+w}_A$ with objects the W_nM, $M \in \mathcal{D}^{+w}_A$. The same holds for $\mathcal{D}^{\ge 0}_A$, from which it follows that the truncation functors $\tau_{\le 0}$, $\tau_{\ge 0}$ associated with the t-structure $(\mathcal{D}^{\le 0}_A, \mathcal{D}^{\ge 0}_A)$ commute with the functors W_n. This proves (2).

For (3), we argue by induction on the weight filtration. Let $\mathcal{H}^T_A \subset \mathcal{H}^f_A$ be any full abelian subcategory containing all the $\mathbb{Q}(n)$ and closed under extension in $\mathcal{H}^f_A$. Since $A(0) = \mathbb{Q} \cdot \mathrm{id}$, the full subcategory $\mathcal{D}^f_A(-n)$ of $\mathcal{D}^f_A$ consisting of M with $M \cong \mathrm{gr}^W_n M$ is equivalent to the bounded derived category of (ungraded) finite dimensional $\mathbb{Q}$-vector spaces, $D^b(\mathbb{Q})$, with the equivalence sending a complex C to $\mathbb{Q}(-n) \otimes_{\mathbb{Q}} C$. The t-structure on $\mathcal{D}^f_A$ restricts to a t-structure on $\mathcal{D}^f_A(-n)$ which is equivalent to the standard t-structure on $D^b(\mathbb{Q})$.

Thus, if we have $M \in \mathcal{H}^f_A$, then $\mathrm{gr}^W_n M \cong \mathbb{Q}(-n)^{r_n}$ for some $r_n \ge 0$. If N is the minimal n such that $W_nM \ne 0$, then we have the exact sequence

$$0 \to \mathrm{gr}^W_N M \to M \to W^{>N} M \to 0$$

By induction on the length of the weight filtration, $W^{>N}M$ is in $\mathcal{H}_A^T$, hence M is in $\mathcal{H}_A^T$ and thus $\mathcal{H}_A^T = \mathcal{H}_A^f$.

Lemma 1.12.10 *For $N, M \in \mathcal{H}_A^f$, $n \le m \in \mathbb{Z}$, we have*

$$\mathrm{Hom}_{\mathcal{H}_A^f}(W^{>m}M, W_nN) = 0$$

Proof If $M = \mathbb{Q}(-a)$, $N = \mathbb{Q}(-b)$ with $a > b$, then

$$\mathrm{Hom}_{\mathcal{H}_A^f}(M, N) = H^0(A(a-b)) = 0$$

since A is connected. The result in general follows by induction on the weight filtration.

Proposition 1.12.11 $\mathcal{H}_A^f$ *is a neutral Tannakian category over* $\mathbb{Q}$*.*

Proof Since $\mathbb{Q}(n)^\vee = \mathbb{Q}(-n)$, it follows from lemma 1.12.9 that $M \mapsto M^\vee$ restricts from $\mathcal{D}_A^f$ to an exact involution on $\mathcal{H}_A^f$. Since $\mathcal{D}_A^f$ is rigid, it follows that $\mathcal{H}_A^f$ is rigid as well. Also

$$\mathrm{Hom}_{\mathcal{H}_A^f}(\mathbb{Q}(-a), \mathbb{Q}(-b)) = \begin{cases} H^0(A(a-b)) = 0 & \text{if } a \neq b \\ H^0(A(0)) = \mathbb{Q} \cdot \mathrm{id} & \text{if } a = b. \end{cases}$$

By induction on the weight filtration, this implies that $\mathrm{Hom}_{\mathcal{H}_A^f}(M, N)$ is a finite dimensional $\mathbb{Q}$-vector space for all M, N in $\mathcal{H}_A^f$. Since the identity for the tensor product is $\mathbb{Q}(0)$, it follows that $\mathcal{H}_A^f$ is $\mathbb{Q}$ linear.

We have the rigid tensor functor $q : \mathcal{H}_A^f \to \mathcal{H}_\mathbb{Q}^f$. Noting that $\mathcal{H}_\mathbb{Q}^f$ is equivalent to the category of finite dimensional graded $\mathbb{Q}$-vector spaces, composing q with the functor "forget the grading" from $\mathcal{H}_\mathbb{Q}^f$ to $\mathrm{Vec}_\mathbb{Q}$ defines the rigid tensor functor

$$\omega : \mathcal{H}_A^f \to \mathrm{Vec}_\mathbb{Q}.$$

The forgetful functor $\mathcal{H}_\mathbb{Q}^f \to \mathrm{Vec}_\mathbb{Q}$ is faithful, so we need only see that $q : \mathcal{H}_A^f \to \mathcal{H}_\mathbb{Q}^f$ is faithful. Sending $M \in \mathrm{Vec}_\mathbb{Q}$ to $\mathbb{Q}(-n) \otimes M$ defines an equivalence of $\mathrm{Vec}_\mathbb{Q}$ with the full subcategory $\mathrm{gr}_n^W \mathcal{H}^f$ of $\mathcal{H}_A^f$ consisting of M which are isomorphic to $\mathrm{gr}_n^W M$. Via this identification, we can further identify q with the functor

$$M \mapsto \mathrm{gr}_*^W M := \oplus_n \mathrm{gr}_n^W M.$$

Let $f : M \to N$ be a map in $\mathcal{H}_A^f$ such that $\mathrm{gr}_n^W f = 0$ for all n; we claim that $f = 0$. By induction on the length of the weight filtration, it follows that $W^{>n} f = 0$, where n is the mininal integer such that $W_nM \oplus W_nN \neq 0$. Thus f is given by a map

$$\tilde{f} : W^{>n}M \to \mathrm{gr}_n^W N.$$

But $\tilde{f} = 0$ by lemma 1.12.10, hence $f = 0$ as desired.

Notation 1.12.12 We denote the truncation to the heart,

$$\tau_{\le 0}\tau^{\ge 0} : \mathcal{D}_A^{+w} \to \mathcal{H}_A,$$

by H_A^0.

1.13 Connection matrices

A convenient way to define an A-cell module is by a connection matrix (called a *twisting matrix* in [26]).

Let (M, d_M) be a complex of Adams graded $\mathbb{Q}$-vector spaces. An *A-connection* for M is a map (of bi-graded $\mathbb{Q}$-vector spaces)

$$\Gamma : M \to A^+ \otimes_{\mathbb{Q}} M$$

of Adams degree 0 and cohomological degree 1. One says that Γ is *flat* if

$$d\Gamma + \Gamma^2 = 0.$$

This means the following: $A \otimes_{\mathbb{Q}} M$ has the standard tensor product differential, so $d\Gamma := d_{A^+\otimes_{\mathbb{Q}}M} \circ \Gamma + \Gamma \circ d_M$ using the usual differential in the complex of maps M to $A^+ \otimes_{\mathbb{Q}} M$. Also, we extend Γ to

$$\Gamma : A^+ \otimes M \to A^+ \otimes M$$

using the Leibniz rule, so that Γ^2 is defined.

Remark 1.13.1 Given a connection $\Gamma : M \to A^+ \otimes_{\mathbb{Q}} M$, define

$$d_0 : M \to A \otimes_{\mathbb{Q}} M = M \oplus A^+ \otimes_{\mathbb{Q}} M,\ m \mapsto d_M m \oplus \Gamma m$$

and extend d_0 to $d_\Gamma : A \otimes_{\mathbb{Q}} M \to A \otimes_{\mathbb{Q}} M$ by the Leibniz rule. Then Γ is flat if and only if d_Γ endows $A \otimes_{\mathbb{Q}} M$ with the structure of a dg A-module, i.e. $d_\Gamma^2 = 0$.

If $\Gamma : M \to A^+ \otimes_{\mathbb{Q}} M$ is a connection, call Γ *nilpotent* if M admits a filtration by bi-graded $\mathbb{Q}$ subspaces

$$0 = M_{-1} \subset M_0 \subset \ldots \subset M_n \subset \ldots \subset M$$

such that $M = \cup_n M_n$ and such that

$$d_M(M_n) \subset M_{n-1};\ \Gamma(M_n) \subset A^+ \otimes M_{n-1}$$

for every $n \geq 0$.

The following result is obvious:

Lemma 1.13.2 *Let $\Gamma : M \to A^+ \otimes_{\mathbb{Q}} M$ be a flat nilpotent connection. Then the dg A-module $(A \otimes_{\mathbb{Q}} M, d_\Gamma)$ is a cell module.*

Indeed, choosing a $\mathbb{Q}$ basis $\mathcal{B}$ for M such that $\mathcal{B}_n := M_n \cap \mathcal{B}$ is a $\mathbb{Q}$ basis for M_n for each n gives the necessary filtered A basis for $A \otimes_{\mathbb{Q}} M$. In addition, we have

Lemma 1.13.3 *Let $\Gamma : M \to A^+ \otimes_{\mathbb{Q}} M$ be a flat connection. Suppose there is an integer r_0 such that $|m| \geq r_0$ for all $m \in M$. Then Γ is nilpotent.*

Proof The proof is essentially the same as that of lemma 1.5.8(2): If M is concentrated in a single Adams degree r_0, then Γ is forced to be the zero-map. Thus, taking $M_0 = \ker(d_M) \subset M$ and $M_1 = M$ shows that Γ is nilpotent. In general, one shows by induction on the length of the weight filtration that the restriction of Γ to $W_n M := \oplus_{r \leq n} M(r)$ is nilpotent for every n, and then a limit argument completes the proof.

A morphism $f : (M, d_M, \Gamma) \to (M', d_{M'}, \Gamma')$ is a map of bi-graded vector spaces

$$f := f_0 + f^+ : M \to A \otimes M' = M' \oplus A^+ \otimes M'$$

such that

$$d_{\Gamma'} f = f d_\Gamma.$$

In particular, we may identify the category of complexes of $\mathbb{Q}$-vector spaces with the sub-category consisting of complexes with flat connection 0 and morphisms $f = f^0 + f^+$ with $f^+ = 0$.

Definition 1.13.4 We denote the category of flat nilpotent connections over A by $Conn_A$. We let $Conn_A^{+w}$ be the full subcategory consisting of flat nilpotent connections on M with $M(r) = 0$ for $r << 0$, and $Conn_A^f$ the full subcategory of flat nilpotent connections on M with M finite dimensional over $\mathbb{Q}$.

It follows from lemma 1.13.3 that a flat connection on M with $M(r) = 0$ for $r << 0$ (or with M finite dimensional over $\mathbb{Q}$) is automatically nilpotent.

1.14 The homotopy category of connections

Define a tensor operation on $Conn_A$ by

$$(M, \Gamma) \otimes (M', \Gamma') := (M \otimes M', \Gamma \otimes \mathrm{id} + \mathrm{id} \otimes \Gamma')$$

with $\Gamma \otimes \mathrm{id} + \mathrm{id} \otimes \Gamma'$ suitably interpreted as a connection by using the necessary symmetry isomorphisms. Complexes of $\mathbb{Q}$ vector spaces act on $Conn_A$ by

$$(M, \Gamma) \otimes K := (M, \Gamma) \otimes (K, 0).$$

Let I be the complex

$$\mathbb{Q} \xrightarrow{\delta} \mathbb{Q} \oplus \mathbb{Q}$$

with $\mathbb{Q}$ in degree -1, and with connection 0. We have the two inclusions $i_0, i_1 : \mathbb{Q} \to I$. Two maps $f, g : (M, \Gamma) \to (M', \Gamma')$ are said to be homotopic if there is a map $h : (M, \Gamma) \otimes I \to (M', \Gamma')$ with $f = h \circ (\mathrm{id} \otimes i_0)$, $g = h \circ (\mathrm{id} \otimes i_1)$.

Definition 1.14.1 Let $\mathcal{H}Conn_A$ denote the homotopy category of $Conn_A$, i.e., the objects are the same as $Conn_A$ and morphisms are homotopy classes of maps in $Conn_A$. Similarly, we have the full subcategories

$$\mathcal{H}(Conn_A^f) \subset \mathcal{H}(Conn_A^{+w}) \subset \mathcal{H}Conn_A$$

with objects $Conn_A^f$, resp. $Conn_A^{+w}$.

If M is an A-cell module, then let M_0 be the complex of $\mathbb{Q}$-vector spaces $M \otimes_A \mathbb{Q}$. Using the identity splitting $\mathbb{Q} \to A$ to the augmentation $A \to \mathbb{Q}$, we have the canonical isomorphism of A-modules

$$A \otimes_{\mathbb{Q}} M_0 \cong M.$$

Using the decomposition $A = \mathbb{Q} \oplus A^+$, we can decompose the differential on $A \otimes_{\mathbb{Q}} M_0$ induced by the above isomorphism as

$$d = d^0 + d^+$$

where d^0 maps $\mathbb{Q} \otimes M_0$ to $\mathbb{Q} \otimes M_0$ and d^+ maps $\mathbb{Q} \otimes M_0$ to $A^+ \otimes M_0$.

We can thus make M_0 into a complex of Adams graded $\mathbb{Q}$-vector spaces by using the differential d^0. The map

$$d^+ : M_0 \to A^+ \otimes M_0$$

gives a connection and the flatness condition follows from $d^2 = 0$. Nilpotence follows from the filtration condition (definition 1.3.1(3b)) for an A-basis of M.

Conversely, if (M_0, d^0) is a complex of Adams graded $\mathbb{Q}$-vector spaces, and

$$\Gamma : M_0 \to A^+ \otimes M_0$$

is a flat nilpotent connection, make the free Adams graded A-module $A \otimes_{\mathbb{Q}} M_0$ a cell module by taking d_Γ to be the differential (see remark 1.13.1 and lemma 1.13.2).

Lemma 1.14.2 *The correspondences*

$$(M, d_M = d^0 + d^+) \mapsto ((M_0, d_0)d^+), \ ((M_0, d_0)d^+) \mapsto (M = A \otimes_{\mathbb{Q}} M_0, d^0 + d^+)$$

define an equivalence of the category of A-cell modules with the category of flat nilpotent A-connections. This equivalence respectives the homotopy relations and tensor products.

Proof Indeed the functor which assigns to a flat nilpotent connection (M_0, d_{M_0}, Γ) the cell module $(A \otimes_{\mathbb{Q}} M_0, d_\Gamma)$ is essentially surjective by the previous discussion, and the map on Hom groups is an isomorphism.

Define the shift operator by $(M, \Gamma)[1] := (M[1], -\Gamma[1])$. Given a morphism $f : (M, \Gamma) \to (M', \Gamma')$ of flat nilpotent connections, decompose $f : M \to A \otimes M'$ as $f := f^0 + f^+$. Define the *cone* of f as having underlying complex $\mathrm{Cone}(f^0)$, with connection $(-\Gamma[1] \oplus \Gamma') + f^+$. This gives us the cone sequence

$$(M, \Gamma) \to (M', \Gamma') \to \mathrm{Cone}(f) \to (M, \Gamma)[1].$$

The next result is immediate:

Lemma 1.14.3 *Using the cone sequences as distinguished triangles makes $\mathcal{H}Conn_A$ into a triangulated tensor category. The equivalence of lemma 1.14.2 passes to an equivalence of $\mathcal{H}Conn_A$ with the homotopy category $\mathcal{KCM}_A$ of A-cell modules, as triangulated tensor categories.*

Thus, via proposition 1.4.3 we have defined an equivalence of $\mathcal{H}Conn_A$ with $\mathcal{D}_A$ as triangulated tensor categories. This restricts to equivalences of $\mathcal{H}(Conn_A^{+w})$ with $\mathcal{D}_A^{+w}$ and $\mathcal{H}(Conn_A^f)$ with $\mathcal{D}_A^f$.

The weight filtration in $\mathcal{D}_A$ can be described in the language of flat connections: Let M be an Adams graded complex of $\mathbb{Q}$-vector spaces, which we decompose into Adams graded pieces as $M = \oplus_r M(r)$. Set

$$W_n M := \oplus_{r \leq n} M(r)$$

giving us the subcomplex W_nM of M. If $\Gamma : M \to A^+ \otimes M$ is a flat connection, then as Γ has Adams degree 0, it follows that Γ restricts to a flat nilpotent connection

$$W_n\Gamma : W_nM \to A^+ \otimes W_nM.$$

It is easy to see that this filtration corresponds to the weight filtration on $\mathcal{D}_A$ via the equivalence of lemma 1.14.3 and proposition 1.4.3.

Let $\mathcal{H}Conn_A^{+w} \subset \mathcal{H}Conn_A$ be the full subcategory of objects M such that $W_nM \cong 0$ for some n, and let $\mathcal{H}Conn_A^f \subset \mathcal{H}Conn_A^{+w}$ be the full subcategory of objects M such that $\oplus_n H^n(M)$ is finite dimensional. It is easy to see that the inclusions $\mathcal{H}(Conn_A^f) \subset \mathcal{H}Conn_A^f$ and $\mathcal{H}(Conn_A^{+w}) \subset \mathcal{H}Conn_A^{+w}$ are equivalences, giving us the equivalences

$$\mathcal{H}Conn_A^f \sim \mathcal{D}_A^f,\ \mathcal{H}Conn_A^{+w} \sim \mathcal{D}_A^{+w}.$$

Now suppose that A is connected. It is easy to see that the standard t-structure on the derived category $D(\mathbb{Q})$ of complexes over $\mathbb{Q}$ induces a t-structure on the homotopy category $\mathcal{H}Conn_A^{+w}$. Under the equivalence $\mathcal{H}Conn_A^{+w} \sim \mathcal{D}_A^{+w}$, the t-structure on $\mathcal{D}_A^{+w}$ defined in section 1.12 corresponds to the pair of subcategories $(\mathcal{H}Conn_A^{\leq 0}, \mathcal{H}Conn_A^{\geq 0})$, hence these give the corresponding t-structure on $\mathcal{H}Conn_A^{+w}$.

Definition 1.14.4 Suppose that A is connected. Let $Conn_A^0$ denote the filtered abelian tensor category of flat A-connections on Adams graded $\mathbb{Q}$-vector spaces V with $V(r) = 0$ for $r << 0$. Let $Conn_A^{0f} \subset Conn_A^0$ be the full sub-category of flat connections on finite dimensional Adams graded $\mathbb{Q}$-vector spaces.

Lemma 1.14.5 *Suppose that A is connected. Then the equivalence of lemma 1.14.3 defines an equivalence of filtered abelian tensor categories*

$$\mathcal{H}_A \sim Conn_A^0.$$

and this restricts to an equivalence of filtered Tannakian categories

$$\mathcal{H}_A^f \sim Conn_A^{0f}.$$

Proof The first equivalence follows from the discussion above. Also, $\mathcal{D}_A^f$ is equivalent to the full subcategory $\mathcal{H}Conn_A^f$ of $\mathcal{H}Conn_A$ with objects the flat nilpotent connections on complexes M such that $\oplus_n H^n(M)$ is finite dimensional, compatible with the restrictions of the respective t-structures, giving the second equivalence.

Remarks 1.14.6
1. By lemma 1.13.3, the flat connection Γ for an object (M, Γ) in $Conn_A^0$ is automatically nilpotent.

2. $Conn_A^{0f}$ may also be defined as the full subcategory of $\mathcal{H}Conn_A^f$ consisting of complexes M with $H^*(M) = H^0(M)$.

We can also give an explicit description of the truncation functors for this t-structure in the language of flat nilpotent connections. Let (M, d) is a complex of Adams graded $\mathbb{Q}$-vector spaces with a flat nilpotent connection

$$\Gamma : M \to A^+ \otimes M$$

such that (M, d, Γ) is in $Conn_A^{+w}$. Then we can decompose Γ as

$$\Gamma := \sum_{i \geq 1} \Gamma^{(i)}$$

by writing

$$[A^+ \otimes M]^{n+1} = \oplus_{i \geq 1} A^i \otimes M^{n-i+1}$$

and letting $\Gamma^{(i),n} : M^n \to A^i \otimes M^{n-i+1}$ be the composition

$$M^n \xrightarrow{\Gamma^n} [A^+ \otimes M]^{n+1} \to A^i \otimes M^{n-i+1}.$$

The flatness condition for Γ when restricted to the component which maps M^n to $A^1 \otimes M^n$ yields the commutative diagram

$$\begin{array}{ccc} M^n & \xrightarrow{\Gamma^{(1),n}} & A^1 \otimes_{\mathbb{Q}} M^n \\ {\scriptstyle d^n}\downarrow & & \downarrow{\scriptstyle 1 \otimes d^{n+1}} \\ M^{n+1} & \xrightarrow[\Gamma^{(1),n+1}]{} & A^1 \otimes_{\mathbb{Q}} M^{n+1}. \end{array}$$

This implies that Γ restricts to a flat connection $\tau_{\leq n}\Gamma$ on the subcomplex $\tau_{\leq n}M$:

$$\tau_{\leq n}\Gamma : \tau_{\leq n}M \to A^+ \otimes \tau_{\leq n}M;$$

$\tau_{\leq n}\Gamma$ is nilpotent by lemma 1.13.3.

This in turn implies that Γ descends to a connection on the quotient complex $\tau^{>n}M := M/\tau_{\leq n}M$:

$$\tau^{>n}\Gamma : \tau^{>n}M \to A^+ \otimes \tau^{>n}M$$

which is in fact a flat nilpotent connection. Indeed, the only question for flatness is for the terms in $\Gamma^2 + d\Gamma$ which factor via Γ or d through $A^+ \otimes M^{*\leq n}$, but which have non-zero image in $A^+ \otimes \tau^{>n}M$. There are three such terms:

$$\Gamma^{(1),n} \circ \Gamma^{(i+1-n),i}, (1 \otimes d^n) \circ \Gamma^{(i+1-n),i}, (1 \otimes d^{n-1}) \circ \Gamma^{(i+2-n),i}$$

where we use the convention that $\Gamma^{(0),i} = d^i$. For a term of the first type, the fact that $\Gamma^{(1)}$ commutes with d implies that the composition factors through $A^{i+1-n} \otimes (M^n / \ker d^n)$. The second term similarly factors through $A^{i+1-n} \otimes (M^n / \ker d^n)$, while the third term goes to zero in $A^{i+2-n} \otimes (M^n / \ker d^n)$.

As before, the nilpotence of $\tau^{>n}\Gamma$ follows from lemma 1.13.3.

Thus for each (M, d, Γ) in $Conn_A^{+w}$ we have the sequence of complexes with flat nilpotent connection

$$0 \to (\tau_{\leq n}M, d, \tau_{\leq n}\Gamma) \to (M, d, \Gamma) \to (\tau^{>n}M, d, \tau^{>n}\Gamma) \to 0$$

which is exact as a sequence of bi-graded $\mathbb{Q}$-vector spaces. When we take the associated cell modules, this gives us the canonical distinguished triangle for the t-structure we have described for $\mathcal{D}_A^{+w}$.

In particular, the truncation functor $H_A^n := \tau^{\geq n}\tau_{\leq n}$ can be explicitly described in the language of flat nilpotent connections. Namely, the restricted connection

$$\Gamma^{(1),n} : M^n \to A^1 \otimes M^n$$

defines a connection (not necessarily flat) on the Adams graded $\mathbb{Q}$-vector space M^n for each n, and the differential d gives a map in the category of connections

$$d^n : (M^n, \Gamma^{(1),n}) \to (M^{n+1}, \Gamma^{(1),n+1}).$$

In short, $(M, d, \Gamma^{(1)})$ is a complex in the category of connections. Thus $\Gamma^{(1)}$ induces a connection on $H^n(M)$:

$$H^n(\Gamma) := H^n(\Gamma^{(1)}) : H^n(M) \to A^1 \otimes H^n(M).$$

On M^n, the flatness condition for Γ, when restricted to the component which maps M^n to $A^2 \otimes M^n$, gives the identity:

$$(\mathrm{id} \otimes d^{n+1}) \circ \Gamma^{(2),n} - \Gamma^{(1),n+1} \circ \Gamma^{(1),n} + \Gamma^{(2),n+1} \circ d^n = 0$$

and thus $H^n(\Gamma^{(1)})$ is flat. $H^n(\Gamma^{(1)})$ is nilpotent by lemma 1.13.3.

The canonical quasi-isomorphism of complexes

$$\tau^{\geq n}\tau_{\leq n}(M, d_M) \to H^n(M, d_M)$$

thus gives rise to a quasi-isomorphism of complexes with flat nilpotent connection

$$\tau^{\geq n}\tau_{\leq n}(M, d_M, \Gamma) \to (H^n(M, d_M), H^n(\Gamma^{(1)})).$$

Definition 1.14.7 Let A be a cohomologically connected cdga with 1-minimal model $A\{1\}$. We let $QA := A\{1\}^1$ and let $\partial : QA \to \Lambda^2 QA$ denote the differential $d : A\{1\}^1 \to \Lambda^2 A\{1\}^1 = A\{1\}^2$. Then (QA, ∂) is co-Lie algebra over $\mathbb{Q}$. If A is an Adams graded cdga, then QA becomes an Adams graded co-Lie algebra.

In the Adams graded case, we let co-rep(QA) denote the category of co-modules M over QA, where M is a bi-graded $\mathbb{Q}$-vector space such that the Adams degrees in M are bounded below.

Remark 1.14.8 Let us suppose that A is a generalized nilpotent Adams graded cdga. Then the co-Lie algebra QA is given by the restriction of d to A^1, noting that d factors as

$$d : A^1 \to \Lambda^2 A^1 \subset A^2.$$

If now M is an Adams graded $\mathbb{Q}$-vector space (concentrated in cohomological degree 0) and $\Gamma : M \to A^+ \otimes M$ is a flat connection, then Γ is actually a map

$$\Gamma : M \to A^1 \otimes M$$

and the flatness condition is just saying the Γ makes M into an Adams graded co-module for the co-Lie algebra QA. If in addition the Adams degrees occuring in M have a lower bound, then Γ is automatically nilpotent (lemma 1.13.3).

Thus, we have an equivalence of the category $Conn^0_A$ with co-rep(QA), which restricts to an equivalence of $Conn^{0f}_A$ with the category co-rep$^f(QA)$ of finite dimensional co-modules over QA.

Putting this together with the above discussion, we have equivalences

$$\mathcal{H}_A \sim Conn^0_A \sim \text{co-rep}(QA)$$

which restrict to equivalences

$$\mathcal{H}^f_A \sim Conn^{0f}_A \sim \text{co-rep}^f(QA).$$

1.15 Summary

In [26] the relations between the various constructions we have presented above are discussed. We summarize the main points here.

Definition 1.15.1 1. Let $H = \mathbb{Q}\cdot \mathrm{id} \oplus \oplus_{r\geq 1} H(r)$ be an Adams Hopf algebra over $\mathbb{Q}$. We let co-rep(H) denote the abelian tensor category of co-modules M over H, where M is a bi-graded $\mathbb{Q}$ vector space such that the Adams degrees in M are bounded below. Let co-rep$^f(H) \subset$ co-rep(H) be the full subcategory of co-modules M such that M is finite dimensional over $\mathbb{Q}$.

2. Let $\gamma = \oplus_{r\geq 1}\gamma(r)$ be an Adams graded co-Lie algebra over $\mathbb{Q}$. We let co-rep(γ) denote the abelian tensor category of co-modules M over γ, where M is a bi-graded $\mathbb{Q}$ vector space such that the Adams degrees in M are bounded below. Let co-rep$^f(\gamma) \subset$ co-rep(γ) be the full subcategory of co-modules M such that M is finite dimensional over $\mathbb{Q}$.

The Adams grading induces a functorial exact weight filtration on the abelian categories co-rep(H) and co-rep(γ) by setting

$$W_nM := \oplus_{r\leq n} M(r).$$

The subcategories co-rep$^f(H)$ and co-rep$^f(\gamma)$ are Tannakian categories over $\mathbb{Q}$, with neutral fiber functor the associated graded for the weight filtration gr^W_*.

Let $H_+ = \oplus_{r\geq 1}H(r) \subset H$ be the augmentation ideal, $\gamma_H := H_+/H_+^2$ the co-Lie algebra of H. For an H co-module $\delta : M \to H\otimes M$ we have the associated γ_H co-module $\bar{M}$ with the same underlying bi-graded $\mathbb{Q}$ vector space, and with co-action $\bar\delta : \bar{M} \to \bar{M}\otimes\gamma_H$ given by the composition

$$M \xrightarrow{\delta} M\otimes H = M\oplus M\otimes H_+ \to M\otimes H_+ \to M\otimes\gamma_H.$$

Then the association $M \mapsto \bar{M}$ induces equvalences of filtered abelian tensor categories

$$\text{co-rep}(H) \sim \text{co-rep}(\gamma_H),\ \text{co-rep}^f(H) \sim \text{co-rep}^f(\gamma_H).$$

For an Adams graded cdga A, we have the Adams graded Hopf algebra $\chi_A := H^0(\bar{B}(A))$ and the Adams graded co-Lie algebra $\gamma_A := \gamma_{\chi_A}$. We have as well the co-Lie algebra QA defined using the 1-minimal model of A (definition 1.14.7).

Theorem 1.15.2 *Let A be an Adams graded cdga. Suppose that A is cohomologically connected.*

(1) There is a functor $\rho : D^b(\text{co-rep}^f(\chi_A)) \to \mathcal{D}^f_A$. ρ respects the weight filtrations and sends Tate objects to Tate objects. ρ induces a functor on the hearts

$$\mathcal{H}(\rho) : \text{co-rep}^f(\chi_A) \to \mathcal{H}^f_A$$

which is an equivalence of filtered Tannakian categories, respecting the fiber functors gr^W_*.

(2) Let $A\{1\}$ be the 1-minimal model of A. Then $A\{1\} \to A$ induces an isomorphism of graded Hopf algebras $\chi_{A\{1\}} \to \chi_A$ and graded co-Lie algebras

$$QA \cong \gamma_{A\{1\}} \cong \gamma_A.$$

(3) The functor ρ is an equivalence of triangulated categories if and only if A is 1-minimal.

(4) Sending a co-module $M \in \text{co-rep}(\chi_A)$ to the γ_A co-module $\bar{M}$ defines equivalences of neutral Tannakian categories

$$\text{co-rep}(\chi_A) \sim \text{co-rep}(\gamma_A);\ \text{co-rep}^f(\chi_A) \sim \text{co-rep}^f(\gamma_A).$$

Putting this together with our discussion on connections in section 1.13 gives

Corollary 1.15.3 *Let A be a cohomologically connected Adams graded cdga. We have equivalences of filtered abelian tensor categories*

$$\text{co-rep}(\chi_A) \sim \text{co-rep}(\gamma_A) \sim \text{co-rep}(QA) \sim Conn^0_A$$

and equivalences of filtered neutral Tannakian categories

$$\text{co-rep}^f(\chi_A) \sim \text{co-rep}^f(\gamma_A) \sim \text{co-rep}^f(QA) \sim Conn^0_A \cap Conn^f_A.$$

2 Relative theory of cdgas

The theory of cdgas over $\mathbb{Q}$ generalizes to a large extent to cdgas over a cdga $\mathcal{N}$. In this section, we give the main constructions in this direction that we will need. As in section 1, all cdgas are cdgas over $\mathbb{Q}$.

2.1 Definitions and model structure

We fix a base cdga $\mathcal{N}$. A *cdga over* $\mathcal{N}$ is a cdga $\mathcal{A}$ together with a homomorphism of cdgas $\phi : \mathcal{N} \to \mathcal{A}$. An *augmented cdga over* $\mathcal{N}$ has in addition a splitting $\pi : \mathcal{A} \to \mathcal{N}$ to ϕ. The same notions apply for an Adams graded cdga $\mathcal{A}$ over an Adams graded cdga $\mathcal{N}$. Let $\mathbf{cdga}_{\mathcal{N}}$ denote the category of Adams graded augmented cdgas over $\mathcal{N}$, where a map $\mathcal{A} \to \mathcal{B}$ is a dg $\mathcal{N}$-algebra morphism compatible with the augmentations.

Definition 2.1.1 1. A morphism $\phi : \mathcal{A} \to \mathcal{B}$ in $\mathbf{cdga}_{\mathcal{N}}$ is a *weak equivalence* if ϕ induces an isomorphism

$$\phi_* : H^n(\mathcal{A}(r)) \to H^n(\mathcal{B}(r))$$

for all $n, r \geq 1$.

2. A morphism $\phi : \mathcal{A} \to \mathcal{B}$ in $\mathbf{cdga}_{\mathcal{N}}$ is a *fibration* if $\phi(r) : \mathcal{A}(r)^n \to \mathcal{B}(r)^n$ is surjective for all $n, r \geq 1$

3. A morphism $\phi : \mathcal{A} \to \mathcal{B}$ in $\mathbf{cdga}$ is a *cofibration* if ϕ has the left lifting property with respect to acyclic fibrations.

As usual, we call $\phi : \mathcal{A} \to \mathcal{B}$ a quasi-isomorphism if ϕ induces an isomorphism on H^n for all n.

The category $\mathbf{cdga}_{\mathcal{N}}$ has push-outs and pull-backs: the push-out in the diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{B} \\ \downarrow & & \\ \mathcal{A} & & \end{array}$$

is $\mathcal{A} \otimes_{\mathcal{C}} \mathcal{B}$, with the morphisms $\mathcal{A}, \mathcal{B} \to \mathcal{A} \otimes_{\mathcal{C}} \mathcal{B}$ given by $a \mapsto a \otimes 1$, $b \mapsto 1 \otimes b$; the augmentation is induced from that of $\mathcal{A}$ and $\mathcal{B}$.The pull-back in the diagram

$$\begin{array}{ccc} & & \mathcal{B} \\ & & \downarrow{\scriptstyle f} \\ \mathcal{A} & \xrightarrow[g]{} & \mathcal{C} \end{array}$$

is $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$, i.e., the sub-complex of $\mathcal{A} \oplus \mathcal{B}$ of elements (a, b) with $f(a) = g(b)$. The product is $(a, b) \cdot (a', b') := (aa', bb')$ and the augmentation is induced from that of $\mathcal{A}$ and $\mathcal{B}$. Additionally, small filtered colimits and limits exist in $\mathbf{cdga}_{\mathcal{N}}$.

We proceed to show that definition 2.1.1 makes $\mathbf{cdga}_{\mathcal{N}}$ a model category, closely following [7, chapter 4]. We call a map which is a (co)fibration and a weak equivalence an *acyclic* (co)fibration.

The proof of the following lemma is easy and is left to the reader.

Lemma 2.1.2 *The cofibrations in* $\mathbf{cdga}_{\mathcal{N}}$ *satisfy*

1. *Every isomorphism in* $\mathbf{cdga}_{\mathcal{N}}$ *is a cofibration*

2. *Cofibrations are closed under push-out by an arbitrary morphism*

3. *If* $\mathcal{A}_1 \to \mathcal{A}_2 \to \ldots$ *is a sequence of cofibrations, then* $\mathcal{A}_1 \to \varinjlim_n \mathcal{A}_n$ *is a cofibration.*

4. *If* $\{i_j : \mathcal{A}_j \to \mathcal{B}_j\}_{j \in J}$ *is a set of cofibrations, then*

$$\otimes_{j \in J} \mathcal{A}_j \to \otimes_{j \in J} \mathcal{B}_j$$

is a cofibration (where $\otimes$ *means* $\otimes_{\mathcal{N}}$*).*

5. *Recall that a map* $f : \mathcal{A} \to \mathcal{B}$ *is a* retract *of a map* $g : \mathcal{C} \to \mathcal{D}$ *if there is a commutative diagram*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{C} & \xrightarrow{p} & \mathcal{A} \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{B} & \xrightarrow[j]{} & \mathcal{D} & \xrightarrow[q]{} & \mathcal{B} \end{array}$$

with $pi = \mathrm{id}_{\mathcal{A}}$, $qj = \mathrm{id}_{\mathcal{B}}$. *Then any retract of a cofibration is a cofibration.*

Let **cdga** denote the category of Adams graded cdgas over $\mathbb{Q}$. There is a bi-functor

$$\otimes : \mathbf{cdga}_{\mathcal{N}} \times \mathbf{cdga} \to \mathbf{cdga}_{\mathcal{N}}$$

defined by letting $\mathcal{A} \otimes B$ be the Adams graded cdga over $\mathcal{N}$ with $(\mathcal{A} \otimes B)(r) := \oplus_s \mathcal{A}(r) \otimes_{\mathbb{Q}} B(r-s)$, product

$$(a \otimes b)(a' \otimes b') := (-1)^{\deg b \deg a'} aa' \otimes bb'$$

and augmentation $n \mapsto \epsilon(n) \otimes 1$.

Following [7, §4.4], we have the *elementary cofibrations* in **cdga** [7, §4.4]. As preparation for the definition, for $n \geq 0$, $r \geq 1$, let $S(n,r)$ be the cdga over $\mathbb{Q}$ freely generated as a graded-commutative algebra over $\mathbb{Q}$ by a single element $e \in S(n,r)(r)^n$ with $de = 0$. Similarly, let $T(n,r)$ be the cdga over $\mathbb{Q}$ freely generated by elements $a \in T(n,r)(r)^n$, $b \in T(n,r)(r)^{n+1}$ with $b = da$. Set $T(-1,0) = \mathbb{Q}$ (in degree 0).

Lemma 2.1.3 *1. Let $i : A \to B$ be a cofibration in* **cdga**. *Then $\mathcal{N} \otimes A \to \mathcal{N} \otimes B$ is a cofibration in* $\mathbf{cdga}_{\mathcal{N}}$.

2. The following maps are cofibrations in **cdga** *(the* elementary cofibrations*):*

a. the map $\theta : S(n,r) \to T(n-1,r)$ with $\theta(e) = b$

b. the map $\sigma : \mathbb{Q} \to S(n,r)$ with $\sigma(1) = 1$

c. the map $\tau : \mathbb{Q} \to T(n,r)$ with $\tau(1) = 1$.

Proof We have the restriction of scalars functor $\mathcal{U} : \mathbf{cdga}_{\mathcal{N}} \to \mathbf{cdga}$ with respect to the identify map $\mathbb{Q} \to \mathcal{N}$; the functor $\mathcal{N} \otimes - : \mathbf{cdga} \to \mathbf{cdga}_{\mathcal{N}}$ is left adjoint to $\mathcal{U}$. As $\mathcal{U}$ maps weak equivalences to weak equivalences and fibrations to fibrations, $\mathcal{N} \otimes -$ sends cofibrations to cofibrations, proving (1).

(2) is an exercise, left to the reader.

Write $T_{\mathcal{N}}(n,r) := \mathcal{N} \otimes T(n,r)$, $S_{\mathcal{N}}(n,r) := \mathcal{N} \otimes S(n,r)$. Given $\mathcal{A} \in \mathbf{cdga}_{\mathcal{N}}$, we have bijections of sets (for $n, r \geq 1$):

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{cdga}_{\mathcal{N}}}(T_{\mathcal{N}}(n,r), \mathcal{A}) \leftrightarrow \mathcal{A}(r)^n \\
&\mathrm{Hom}_{\mathbf{cdga}_{\mathcal{N}}}(S_{\mathcal{N}}(n,r), \mathcal{A}) \leftrightarrow Z^n(\mathcal{A}(r)) := \{y \in \mathcal{A}(r)^n \mid dy = 0\}
\end{aligned}$$

We let $\phi_x : T_{\mathcal{N}}(n,r) \to \mathcal{A}$ be the morphism corresponding to $x \in \mathcal{A}(r)^n$ and $\phi'_y : S_{\mathcal{N}}(n,r) \to \mathcal{A}$ be the morphism corresponding to $y \in Z^n(\mathcal{A}(r))$.

It follows from lemma 2.1.3 that the maps θ, σ, τ induce cofibrations

$$\begin{aligned}
\theta &: S_{\mathcal{N}}(n,r) \to T_{\mathcal{N}}(n-1,r) \\
\sigma &: \mathcal{N} \to S_{\mathcal{N}}(n,r) \\
\tau &: \mathcal{N} \to T_{\mathcal{N}}(n,r)
\end{aligned}$$

in $\mathbf{cdga}_{\mathcal{N}}$.

We refer the reader to [7, definition 4.1] for the list of axioms defining a (closed) model category. The axioms CM1, CM2, CM3 are easy to verify and are left to the reader; CM4(b) is satisfied by the definition of a cofibration. We need to verify the axioms CM4(a) and CM5.

For CM5, we need to show that every morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathbf{cdga}_{\mathcal{N}}$ can be factored as pi, with i a cofibration and p a fibration and either

(a) i is a weak equivalence, or

(b) p is a weak equivalence.

To check (a), the maps ϕ_x induce the map

$$\phi_{\mathcal{B}} : \otimes_{x\in\mathcal{B}(r)^n,r,n\geq 1}\, T_{\mathcal{N}}(n,r) \to \mathcal{B};$$

clearly $\phi_{\mathcal{B}}$ is a fibration. We have the cofibration $\tau : \mathcal{N} \to T_{\mathcal{N}}(n,r)$, giving us the cofibration

$$\psi_{\mathcal{B}} : \mathcal{N} = \otimes_{x\in\mathcal{B}(r)^n,r,n\geq 1}\mathcal{N} \to \otimes_{x\in\mathcal{B}(r)^n,r,n\geq 1} T_{\mathcal{N}}(n,r).$$

Since each $T(n,r)$ is acyclic, $\psi_{\mathcal{B}}$ is also a weak equivalence. Taking the push-out of $\psi_{\mathcal{B}}$ by the augmentation $\mathcal{N} \to \mathcal{A}$ gives us the acyclic cofibration

$$i : \mathcal{A} \to K_f := \mathcal{A} \otimes_{\mathcal{N}} \otimes_{x\in\mathcal{B}(r)^n,r,n\geq 1} T_{\mathcal{N}}(n,r).$$

The maps $f : \mathcal{A} \to \mathcal{B}$ and $\phi_{\mathcal{B}}$ give the map

$$p : K_f \to \mathcal{B}$$

which is a fibration. As i is a weak equivalence and $f = pi$, CM5(a) is proved.

To check (b), we form a sequence of maps

$$\begin{array}{ccccccc} \mathcal{A} & \xrightarrow{\beta_1} & L_f(1) & \xrightarrow{\beta_2} & L_f(2) & \xrightarrow{\beta_3} & \cdots \\ {\scriptstyle f}\downarrow & \swarrow_{\psi_1} & & \swarrow_{\psi_2} & & & \\ \mathcal{B} & & & & & & \end{array} \tag{2.1.1}$$

To define $L_f(1)$, set

$$L_f(1) := \mathcal{A} \otimes_{\mathcal{N}} [\otimes_{r,n\geq 1,x\in\mathcal{B}(r)^n} T_{\mathcal{N}}(n,r)] \otimes_{\mathcal{N}} [\otimes_{r,n\geq 1,y\in Z(\mathcal{B}(r)^n)} S_{\mathcal{N}}(n,r)].$$

Define $\psi_1 : L_f(1) \to \mathcal{B}$ by ϕ_x in the factor indexed by $x \in \mathcal{B}(r)^n$ to x, and ϕ'_y in the factor indexed by $y \in Z(\mathcal{B}(r)^n)$. Clearly ψ_1 is a fibration. We have the evident map $\beta_1 : \mathcal{A} \to L_f(1)$; as in the proof of CM5(a), β_1 is a cofibration. Furthermore $H^n(\psi_1) : H^n(L_f(1)) \to H^n(\mathcal{B})$ is surjective for all $n \geq 1$.

Let

$$R(r)^n := \{(w,y) \in L_f(1)(r)^{n+1} \times \mathcal{B}(r)^n \mid dw = 0 \text{ and } dy = \psi_1(w)\}$$

Define $\beta_2 : L_f(1) \to L_f(2)$ via the push-out diagram

$$\begin{array}{ccc} \otimes_{(w,y)\in R(r)^n,r,n\geq 1} S_{\mathcal{N}}(n+1,r) & \longrightarrow & L_f(1) \\ \downarrow{\scriptstyle \otimes\theta_w} & & \downarrow{\scriptstyle \beta_2} \\ \otimes_{(w,y)\in R(r)^n,r,n\geq 1} T_{\mathcal{N}}(n,r) & \longrightarrow & L_f(2) \end{array}$$

We let $\psi_2 : L_f(2) \to \mathcal{B}$ be the map induced by $\psi_1 : L_f(1) \to \mathcal{B}$ and the maps $T_{\mathcal{N}}(n.r) \to L_f(2)$. As each θ_w is a cofibration, β_2 is a cofibration and since ψ_1 is a fibration, so is ψ_2. Note that, for $n \geq 1$, $H^n(\psi_2) : H^n(L_f(2)) \to H^n(\mathcal{B})$ restricts to an isomorphism on the image of $H^n(\beta_2)$ since $\ker H^n(\psi_2) = \ker H^n(\beta_2)$ by construction.

Iterating this procedure gives the diagram (2.1.1) with the following properties:

(i) each map β_m is a cofibration

(ii) each map ψ_m is a fibration

(iii) for $n \geq 1$, $H^n(\psi_m) : H^n(L_f(m)) \to H^n(\mathcal{B})$ restricts to an isomorphism on the image of $H^n(\beta_m)$.

Let $L_f := \varinjlim_n L_f(n)$, $i : \mathcal{A} \to L_f$, $p : L_f \to \mathcal{B}$ the maps given on the limit by the diagram (2.1.1). Then i is a cofibration, f is a fibration, and by (iii), f is also a weak equivalence. This proves CM5(b).

To prove CM4(a), we need to show: given a commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{X} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{B} & \longrightarrow & \mathcal{Y} \end{array}$$

with i a cofibration and a weak equivalence, and f a fibration, there exist a lifting

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{X} \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \mathcal{B} & \longrightarrow & \mathcal{Y} \end{array}$$

For this, factor $i : \mathcal{A} \to \mathcal{B}$ as we did in the proof of CM5(a): $\mathcal{A} \xrightarrow{i_0} K_i \xrightarrow{p} \mathcal{B}$. In particular, i_0 is a cofibration and weak equivalence, and p is a fibration. Since i is a weak equivalence, so is p. This gives us the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i_0} & K_i \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{B} & = & \mathcal{B} \end{array}$$

which admits a lifting $\mathcal{B} \to K_i$, since i is a cofibration. Thus, it suffices to prove CM4(a) for the cofibration $i_0 : \mathcal{A} \to K_i$. But $K_i = \mathcal{A} \otimes_{\mathcal{N}} [\otimes_{j \in J} T_{\mathcal{N}}(r, n)]$, with every $T_{\mathcal{N}}(n, r)$ appearing in the tensor product having $n, r \geq 1$. It is clear that there is a lifting $T_{\mathcal{N}}(n, r) \to \mathcal{X}$ for every diagram

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow{\scriptstyle f} \\ T_{\mathcal{N}}(n, r) & \longrightarrow & \mathcal{Y} \end{array}$$

with f a fibration and $n, r \geq 1$, giving us the desired lifting $K_i \to \mathcal{X}$. This completes the proof of CM4(a), giving

Proposition 2.1.4 *With cofibrations, fibrations and weak equivalences defined as in definition 2.1.1,* $\mathbf{cdga}_{\mathcal{N}}$ *is a closed model category.*

We denote the homotopy category of $\mathbf{cdga}_{\mathcal{N}}$ by $\mathcal{H}(\mathbf{cdga}_{\mathcal{N}})$.

2.2 Path objects and the homotopy relation

Let $\mathcal{M}$ be a model category. As usual, we call an object A of $\mathcal{M}$ *cofibrant* if the from the initial object $\emptyset \to \mathcal{A}$ is a cofibration, and *fibrant* if the map $A \to *$ to the final object is a fibration.

Recall that for an object B in a model category $\mathcal{M}$, a *path object* for B is a factorization of the diagonal map $B \to B \times B$ as pi, with $i : B \to B^I$ a weak equivalence and $p : B^I \to B \times B$ a fibration. Let $p_i : B^I \to B$, $i = 1, 2$ be $\pi_i \circ p$, where $\pi_1, \pi_2 : B \times B \to B$ are the two projections.

Two morphisms $f, g : A \to B$ in $\mathcal{M}$ are *right homotopic* if there is a path object B^I, p_1, p_2 and a morphism $h : A \to B^I$ with $f = p_1 h$, $g = p_2 h$.

The main results on model categories state that right homotopy with respect to a fixed path object defines an equivalence relation $\sim$ on $\mathrm{Hom}_{\mathcal{M}}(A, B)$, if A is cofibrant and B is fibrant (see [36, chap. 1, §1, lemmas 4, 5(i) and their duals]. In addition, the category with objects the fibrant and cofibrant (*bifibrant*) objects of $\mathcal{M}$, and with morphisms the right homotopy classes of morphisms in $\mathcal{M}$, is equivalent to the homotopy category of $\mathcal{M}$ [36, chap. 1, §1, theorem 1].

Passing to $\mathbf{cdga}_{\mathcal{N}}$, we give a construction of a path object for each $\mathcal{B} \in \mathcal{N}$.

Let *cdga* denote the category of commutative differential graded algebras over $\mathbb{Q}$ (without Adams grading and without augmentation). We have the bi-functor

$$\otimes : \mathbf{cdga}_{\mathcal{N}} \times cdga \to \mathbf{cdga}_{\mathcal{N}}$$

where $\mathcal{A} \otimes B$ has Adams degree r summand $(\mathcal{A} \otimes B)(r) := \mathcal{A}(r) \otimes_{\mathbb{Q}} B$ for $r \geq 1$. The product is $(a \otimes b)(a' \otimes b') := (-1)^{\deg b \deg a'} aa' \otimes bb'$, and the augmentation is induced by that of A.

Let Ω^* be the cdga of polynomial differential forms on $\mathbb{A}^1$, that is, $\Omega^0 := \mathbb{Q}[t]$, $\Omega^1 := \Omega^1_{\mathbb{Q}[t]/\mathbb{Q}} = \mathbb{Q}[t]dt$ and the differential is the usual one. We have the unit map $\eta : \mathbb{Q} \to \Omega^*$ and two restriction maps

$$i_0^*, i_1^* : \Omega^* \to \mathbb{Q}$$

with $i_\epsilon^*(f) := f(\epsilon)$, $\epsilon = 0, 1$. Clearly η is a quasi-isomorphism and $(i_0^*, i_1 *) : \Omega^* \to \mathbb{Q} \times \mathbb{Q}$ is surjective.

For $\mathcal{B} \in \mathbf{cdga}_{\mathcal{N}}$, let $\mathcal{B}^I := \mathcal{B} \otimes \Omega^*$, $i_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^I$ the map $\mathrm{id} \otimes \eta$ and $p_{\mathcal{B}} : \mathcal{B}^I \to \mathcal{B} \times_{\mathcal{N}} \mathcal{B}$ the map $(\mathrm{id} \otimes i_0^*, \mathrm{id} \otimes i_1^*)$. Clearly p is a fibration and i is a weak equivalence, giving us the desired path object. For $\mathcal{A}, \mathcal{B}$ in $\mathbf{cdga}_{\mathcal{N}}$, we write $\sim_\Omega$ for the relation on $\mathrm{Hom}_{\mathbf{cdga}_{\mathcal{N}}}(\mathcal{A}, \mathcal{B})$ given by right homotopy with respect to the path object $\mathcal{B} \otimes_{\mathbb{Q}} \Omega^*$.

Note that for $\mathcal{B} \in \mathbf{cdga}_{\mathcal{N}}$, the augmentation $\mathcal{B} \to \mathcal{N}$ is always a fibration, hence all objects in $\mathbf{cdga}_{\mathcal{N}}$ are fibrant. Thus for $\mathcal{A}$ cofibrant, and $f, g : \mathcal{A} \to \mathcal{B}$, we have $f \sim g$ if and only if $f \sim_\Omega g$.

The results from the theory of model categories, as recalled above, thus gives us

Proposition 2.2.1 *The category $\mathcal{H}(\mathbf{cdga}_{\mathcal{N}})$ is equivalent to the category with objects the cofibrant objects of $\mathbf{cdga}_{\mathcal{N}}$ and with morphisms (for $\mathcal{A}, \mathcal{B}$ cofibrant)* $\mathrm{Hom}_{\mathbf{cdga}_{\mathcal{N}}}(\mathcal{A}, \mathcal{B}) / \sim_\Omega$.

2.3 Indecomposables

For $\mathcal{A} \in \mathbf{cdga}_{\mathcal{N}}$, let $\mathcal{A}^+$ denote the kernel of the augmentation $\mathcal{A} \to \mathcal{N}$. Let

$$Q\mathcal{A} := \mathcal{A}^+ / (\mathcal{A}^+ \cdot \mathcal{A}^+).$$

The Leibniz rule for $d_{\mathcal{A}}$ implies that $d_{\mathcal{A}}$ induces a differential on $Q\mathcal{A}$, making $(Q\mathcal{A}, d)$ an Adams graded $\mathcal{N}$-module. Sending $\mathcal{A}$ to $Q\mathcal{A}$ thus gives a functor $Q : \mathbf{cdga}_{\mathcal{N}} \to \mathcal{M}_{\mathcal{N}}$.

Lemma 2.3.1 *1. Let $f, g : \mathcal{A} \to \mathcal{B}$ be morphisms in $\mathbf{cdga}_{\mathcal{N}}$. If $f \sim_\Omega g$ then $H^*(Qf) = H^*(Qg)$.*

2. Let $f : \mathcal{A} \to \mathcal{B}$ be a weak equivalence between cofibrant objects of $\mathbf{cdga}_{\mathcal{N}}$. Then $Qf : Q\mathcal{A} \to Q\mathcal{B}$ is a quasi-isomorphism.

Proof For (1), it clearly suffices to show that $Qi : Q\mathcal{B} \to Q(\mathcal{B}\otimes_{\mathbb{Q}}\Omega^*)$ is a quasi-isomorphism. Since $\mathbb{Q}[t]$ has a unit, and $\Omega^1 \cdot \Omega^1 = 0$, the evident map

$$Q(\mathcal{B} \otimes_{\mathbb{Q}} \Omega^*) \to Q\mathcal{B} \otimes_{\mathbb{Q}} \Omega^*$$

is an isomorphism; via this isomorphism Qi is transformed to

$$-\otimes 1 : Q\mathcal{B} \to Q\mathcal{B} \otimes_{\mathbb{Q}} \Omega^*.$$

As the unit $\mathbb{Q} \to \Omega^*$ is a quasi-isomorphism, so is $-\otimes 1$, proving (1).

For (2), since $\mathcal{A}$ and $\mathcal{B}$ are cofibrant, it follows from proposition 2.2.1 that there is a morphism $g : \mathcal{B} \to \mathcal{A}$ with $gf \sim_\Omega \mathrm{id}_{\mathcal{A}}$ and $fg \sim_\Omega \mathrm{id}_{\mathcal{B}}$. By (1), $H^*(Qf)$ has inverse $H^*(Qg)$, completing the proof.

2.4 Relative minimal models

The notions of generalized nilpotent algebras and minimal models (over $\mathbb{Q}$) extend without difficulty to augmented cdgas over $\mathcal{N}$. Specifically:

Definition 2.4.1 An Adams graded cdga $\mathcal{A}$ over $\mathcal{N}$ is said to be *generalized nilpotent over $\mathcal{N}$* if

1. As a bi-graded $\mathcal{N}$-algebra, $\mathcal{A} = \mathrm{Sym}^* E \otimes \mathcal{N}$ for some Adams graded $\mathbb{Z}$-graded $\mathbb{Q}$-vector space E, i.e., $\mathcal{A} = \Lambda^* E^{\mathrm{odd}} \otimes \mathrm{Sym}^* E^{\mathrm{ev}} \otimes \mathcal{N}$, where the parity refers to the cohomological degree. In addition, $E(r)^n = 0$ if $n \le 0$ or if $r \le 0$.

2. For $n \ge 0$, let $\mathcal{A}_{(n)} \subset \mathcal{A}$ be the $\mathcal{N}$-subalgebra generated by the subspace $E^{\le n}$ of E consisting of elements of cohomological degree $\le n$. Set $\mathcal{A}_{(n+1,0)} = \mathcal{A}_{(n)}$ and for $q \ge 0$ define $\mathcal{A}_{(n+1,q+1)}$ inductively as the $\mathcal{N}$-subalgebra generated by $\mathcal{A}_{(n)}$ and

$$\mathcal{A}^{n+1}_{(n+1,q+1)} := \{x \in \mathcal{A}^{n+1}_{(n+1)} \mid dx \in \mathcal{A}_{(n+1,q)}.\}$$

   Then for all $n \ge 0$,

$$\mathcal{A}_{(n+1)} = \cup_{q\ge 0} \mathcal{A}_{(n+1,q)}.$$

3. If $\mathcal{A} = (\mathrm{Sym}^* E \otimes \mathcal{N}, d)$, satisfying (1) and (2), and there is an integer n such that $\deg e \le n$ for all homogeneous $e \in E$, we say that $\mathcal{A}$ is *generated in degree $\le n$*.

Remark 2.4.2 We can phrase the condition (2) above differently: For each $n \geq 0$, $E^{\leq n+1}$ has an increasing exhaustive bi-graded filtration

$$E^{\leq n} = F_0 E^{\leq n+1} \subset F_1 E^{\leq n+1} \subset \ldots \subset F_m E^{\leq n+1} \subset \ldots \subset E^{\leq n+1}$$

such that

$$d(F_m E^{\leq n+1} \otimes \mathcal{N}) \subset \mathrm{Sym}^*(F_{m-1} E^{\leq n+1}) \otimes \mathcal{N}$$

Indeed, if $\mathcal{A} = \mathrm{Sym}^* E \otimes \mathcal{N}$ satisfies (2), define $F_m E^{\leq n+1}$ by

$$F_m E^{\leq n+1} \otimes 1 = (E^{\leq n+1} \otimes 1) \cap \mathcal{A}^*_{(n+1,m)}.$$

Conversely, it is easy to see that the existence of such a filtration $F_* E^{\leq n+1}$ for all n implies (2).

Remark 2.4.3 Suppose that $\mathcal{N}$ is connected, that is, that $\mathcal{N}(r)^n = 0$ for $r \geq 1$, $n \leq 0$. Then the subalgebra $\mathcal{A}_{(n)}$ can be defined directly from $\mathcal{A}$, independent of the choice of bi-graded $\mathbb{Q}$-vector space E with $\mathcal{A} = \mathrm{Sym}^* E \otimes_{\mathbb{Q}} \mathcal{N}$. In fact, $\mathcal{A}_{(n)}$ is just the $\mathcal{N}$-subalgebra of $\mathcal{A}$ generated by the elements $x \in \mathcal{A}$ with $\deg x \leq n$. The inductive definition of $\mathcal{A}_{(n,q)}$ thus shows that these subalgebras are also independent of the choice of E.

Lemma 2.4.4 *Let $\mathcal{A}$ be a generalized nilpotent cdga over $\mathcal{N}$. Then $\mathcal{A}$ is cofibrant in* $\mathbf{cdga}_{\mathcal{N}}$.

Proof We write $\mathcal{A}$ as a colimit of elementary cofibrations, with $\mathcal{N}$ as the initial source. Indeed, let E be a generating bi-graded $\mathbb{Q}$-vector space for $\mathcal{A}$ with filtration $F^* E$ satisfying the properties given remark 2.4.2.

In particular, for each $y \in F^0 E$, $dy = 0$. Choose for each $r, n \geq 1$ a $\mathbb{Q}$-basis $y_\alpha^{n,r}$ of $F^0 E(r)^n$. Let $L_{\mathcal{A}}(0) = \otimes_{y_\alpha^{n,r}} S_{\mathcal{N}}(n,r)$ and let $\beta_0 : L_{\mathcal{A}}(0) \to \mathcal{A}$ be the tensor product of maps $\phi'_{y_\alpha^{n,r}}$. By definition, β_0 identifies $L_{\mathcal{A}}(0)$ with the $\mathcal{N}$-subalgebra of $\mathcal{A}$ generated by $F^0 E$. Let $i_0 : \mathcal{N} \to L_{\mathcal{A}}(0)$ be the coproduct of the cofibrations $\sigma : \mathcal{N} \to S_{\mathcal{N}}(n,r)$.

Next, for each $r, n \geq 1$, choose a subset $\{x_\alpha^{n,r}\}$ of $F^1 E(r)^n$ that maps bijectively to a basis of $\mathrm{gr}^1_F E(r)^n$. For each $x = x_\alpha^{n,r}$, dx is in the sub-algebra $\beta_0(L_{\mathcal{A}}(0))$ of $\mathcal{A}$, giving us the diagram

$$\begin{array}{ccc} S_{\mathcal{N}}(n+1,r) & \xrightarrow{\phi'_{dx}} & L_{\mathcal{A}}(0) \\ \theta \big\downarrow & & \\ T_{\mathcal{N}}(n,r) & & \end{array}$$

We let $L_{\mathcal{A}}(1)$ be defined as the push-out in the diagram

$$\begin{array}{ccc} \otimes_{x_\alpha^{n,r}} S_{\mathcal{N}}(n+1,r) & \xrightarrow{\phi'_{dx_\alpha^{n,r}}} & L_{\mathcal{A}}(0) \\ \otimes_{x_\alpha^{n,r}} \theta \big\downarrow & & \\ \otimes_{x_\alpha^{n,r}} T_{\mathcal{N}}(n,r) & & \end{array}$$

The maps $\phi_{x_\alpha^{n,r}}$ together with β_0 give the map

$$\beta_1 : L_{\mathcal{A}}(1) \to \mathcal{A}$$

identifying $L_{\mathcal{A}}(1)$ with the $\mathcal{N}$-subalgebra generated by F^1E. We have as well the cofibration $i_1 : L_{\mathcal{A}}(0) \to L_{\mathcal{A}}(1)$, defined as the push-out of $\otimes_{x_\alpha^{n,r}}\theta$, giving the commutative diagram

$$\begin{array}{ccccc} \mathcal{N} & \xrightarrow{i_0} & L_{\mathcal{A}}(0) & \xrightarrow{i_1} & L_{\mathcal{A}}(1) \\ {\scriptstyle\epsilon}\downarrow & \swarrow{\scriptstyle\beta_0} & & {\scriptstyle\beta_1}\swarrow & \\ \mathcal{A} & & & & \end{array}$$

Continuing in the way, we have cofibrations $i_n : L_{\mathcal{A}}(n-1) \to L_{\mathcal{A}}(n)$, injections $\beta_n : L_{\mathcal{A}}(n) \to \mathcal{A}$ identifying $L_{\mathcal{A}}(n)$ with the subalgebra of $\mathcal{A}$ generated by F^nE, giving a commutative diagram

$$\begin{array}{ccccccccc} \mathcal{N} & \xrightarrow{i_0} & L_{\mathcal{A}}(0) & \xrightarrow{i_1} & L_{\mathcal{A}}(1) & \xrightarrow{i_2} \cdots \xrightarrow{i_n} & L_{\mathcal{A}}(n) & \xrightarrow{i_{n+1}} & \cdots \\ {\scriptstyle\epsilon}\downarrow & \swarrow{\scriptstyle\beta_0} & {\scriptstyle\beta_1}\swarrow & & & {\scriptstyle\beta_n}\swarrow & & & \\ \mathcal{A} & & & & & & & & \end{array}$$

As the map $\varinjlim_n L_{\mathcal{A}}(n) \to \mathcal{A}$ is thus an isomorphism and $\varinjlim_n L_{\mathcal{A}}(n)$ is cofibrant, the proof is complete.

Lemma 2.4.5 *Let $\mathcal{A}$ be a generalized nilpotent cdga over a cdga $\mathcal{N}$. If $\mathcal{N}$ is cohomologically connected, then so is $\mathcal{A}$.*

Proof We have just seen that $\mathcal{A}$ is isomorphic to $\varinjlim_n L_{\mathcal{A}}(n)$, with each map $i_n : L_{\mathcal{A}}(n-1) \to L_{\mathcal{A}}(n)$ being the push-out in a diagram of the form

$$\begin{array}{ccc} \otimes_{x_\alpha^{n,r}} S_{\mathcal{N}}(n+1,r) & \xrightarrow{\phi'_{dx_\alpha^{n,r}}} & L_{\mathcal{A}}(0) \\ {\scriptstyle \otimes_{x_\alpha^{n,r}}\theta}\downarrow & & \\ \otimes_{x_\alpha^{n,r}} T_{\mathcal{N}}(n,r) & & \end{array}$$

with $n, r \geq 1$. In particular, i_n is injective and we have the exact sequence

$$0 \to L_{\mathcal{A}}(n-1) \xrightarrow{i_n} L_{\mathcal{A}}(n) \xrightarrow{p_n} [\otimes_{x_\alpha^{n,r}} S_{\mathcal{N}}(n,r)]^+ \to 0$$

where the last term is kernel of the augmentation $\otimes_{x_\alpha^{n,r}} S_{\mathcal{N}}(n,r) \to \mathcal{N}$.

$S_{\mathcal{N}}(n,r)$ is the free $\mathcal{N}$ algebra on a generator e with $\deg e = n \geq 1$, $|e| = r \geq 1$ and $de = 0$. Thus, each finite tensor product $\otimes_{i=1}^p S_{\mathcal{N}}(n_i, r_i)$ is the free $\mathcal{N}$ algebra on generators $e_1, \ldots, e_p$ with $\deg e_i = n_i \geq 1$, $|e_i| = r_i \geq 1$ and $de_i = 0$. As a dg $\mathcal{N}$-module, we thus have

$$[\otimes_{i=1}^p S_{\mathcal{N}}(n_i, r_i)]^+ \cong \oplus_\alpha \mathcal{N}{<}{-r_\alpha}{>}[-n_\alpha]$$

with $r_\alpha, n_\alpha \geq 1$. As $\mathcal{N}$ is cohomologically connected, it follows that $H^m([\otimes_{i=1}^p S_{\mathcal{N}}(n_i, r_i)]^+) = 0$ for $m \leq 0$.

Since $\otimes_{x_\alpha^{n,r}} S_{\mathcal{N}}(n,r)$ is by definition the colimit of the tensor products over finite subsets of $\{x_\alpha^{n,r}\}$, we have

$$H^m([\otimes_{x_\alpha^{n,r}} S_{\mathcal{N}}(n,r)]^+) = 0$$

for $m \leq 0$. By induction (starting with $L_{\mathcal{A}}(-1) := \mathcal{N}$), it follows that each $L_{\mathcal{A}}(n)$ is cohomologically connected, and hence so is $\mathcal{A}$.

Proposition 2.4.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be generalized nilpotent cdgas over $\mathcal{N}$ and let $f : \mathcal{A} \to \mathcal{B}$ be a weak equivalence. Then f is an isomorphism.*

Before we give the proof, we note the following version of Nakayama's lemma

Lemma 2.4.7 *Let $N = \mathbb{Q}\cdot\mathrm{id}\oplus\oplus_{r\geq 1}N_r$ be a graded $\mathbb{Q}$-algebra. Let $E = \oplus_{r\geq 1}E_r$, $F = \oplus_{r\geq 1}F_r$ be graded $\mathbb{Q}$ vector spaces, and let $A = \mathrm{Sym}^*E \otimes_{\mathbb{Q}} N$, $B = \mathrm{Sym}^*F \otimes_{\mathbb{Q}} N$, and let $\phi : A \to B$ be an N-algebra morphism, respecting the gradings induced by the grading of E, F and N. Let $\bar{\phi} : \mathrm{Sym}^*E \to \mathrm{Sym}^*F$ be the map $\phi \otimes_N \mathrm{id}_{\mathbb{Q}}$, with respect to the augmentation $N \to \mathbb{Q}$, and let $Q\bar{\phi}E \to F$ be the map on indecomposables induced by $\bar{\phi}$. Then*

$$\phi \text{ is an isomorphism } \Leftrightarrow \bar{\phi} \text{ is an isomorphism } \Leftrightarrow Q\bar{\phi} \text{ is an isomorphism.}$$

Proof The implications $\Rightarrow$ are obvious. Suppose that $Q\bar{\phi}$ is an isomorphism. Let $F^n\mathrm{Sym}^*E$ be the ideal $\oplus_{m\geq n}\mathrm{Sym}^mE$ and define $F^n\mathrm{Sym}^*F$ similarly. Then $\bar{\phi}$ induces the map $\mathrm{gr}^*_F\bar{\phi}$ on the associated graded, and we clearly have

$$\mathrm{gr}^*_F\bar{\phi} = \mathrm{Sym}^*(Q\bar{\phi}).$$

Thus $\mathrm{gr}^*_F\bar{\phi}$ is an isomorphism.

Let $(\mathrm{Sym}^*E)(r), (\mathrm{Sym}^*F)(r)$ denote the respective degree r summands, where we use the grading induced from that of E, F. Since E and F are positively graded, the filtration on $(\mathrm{Sym}^*E)(r), (\mathrm{Sym}^*F)(r)$ induced by $F^*\mathrm{Sym}^*E, F^*\mathrm{Sym}^*F$ is finite, and thus the fact that $\mathrm{gr}^*_F\bar{\phi}$ is an isomorphism implies that

$$\bar{\phi} : (\mathrm{Sym}^*E)(r) \to (\mathrm{Sym}^*F)(r)$$

is an isomorphism for each r.

Now suppose that $\bar{\phi}$ is an isomorphism. Let F^nA be the (two-sided) ideal $(\oplus_{r\geq n}N_r)A$ of A, and define F^nB similarly. As ϕ is an N-algebra map, ϕ respects the filtrations. As $\mathrm{gr}^*_F\phi = \bar{\phi}\otimes\mathrm{id}_N$, $\mathrm{gr}^*_F\phi$ is an isomorphism. Letting $A(r), B(r)$ be the degree r summand with respect to the grading induced from that of E, F, the fact that N is positively graded implies that the filtrations induced by F on $A(r), B(r)$ are finite, and thus ϕ is an isomorphism.

Proof (Proof of proposition 2.4.6) Write

$$\mathcal{A} = \mathrm{Sym}^*E \otimes_{\mathbb{Q}} \mathcal{N},\ \mathcal{B} = \mathrm{Sym}^*F \otimes_{\mathbb{Q}} \mathcal{N},$$

as $\mathcal{N}$-algebras, where E and F are bi-graded $\mathbb{Q}$-vector spaces, with filtrations satisfying the conditions of remark 2.4.2. In particular, we have

$$d(E) \subset [(\mathrm{Sym}^*E)^+ \cdot (\mathrm{Sym}^*E)^+] \otimes_{\mathbb{Q}} \mathcal{N}$$

hence $d_{Q\mathcal{A}} = \mathrm{id}\otimes d_{\mathcal{N}}$; similarly $d_{Q\mathcal{B}} = \mathrm{id}\otimes d_{\mathcal{N}}$. Thus

$$H^*(Q\mathcal{A}) \cong E \otimes_{\mathbb{Q}} H^*(\mathcal{N}),\ H^*(Q\mathcal{B}) \cong F \otimes_{\mathbb{Q}} H^*(\mathcal{N})$$

as bi-graded $H^*(\mathcal{N})$-modules, and the map $H^*(Qf)$ gives an isomorphism of bi-graded $H^*(\mathcal{N})$-modules

$$H^*(Qf) : E \otimes_{\mathbb{Q}} H^*(\mathcal{N}) \to F \otimes_{\mathbb{Q}} H^*(\mathcal{N}).$$

Using the augmentation $H^*(\mathcal{N}) \to \mathbb{Q}$, we thus have the isomorphism

$$\overline{H^*(Qf)} : E \to F$$

of bi-graded $\mathbb{Q}$-vector spaces. But clearly $\overline{H^*(Qf)}$ is just the map $Q\bar{f}$ induced by f by first applying $- \otimes_{\mathcal{N}} \mathbb{Q}$ (via the augmentation $\mathcal{N} \to \mathbb{Q}$) and then taking the indecomposables.

Using just the Adams grading, and ignoring the cohomological grading, we consider E and F as positively graded $\mathbb{Q}$ vector spaces, and $\mathcal{N}$ as a positively graded $\mathbb{Q}$-algebra. By lemma 2.4.7, it follows that $f : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{N}$-algebra isomorphism and hence an isomorphism in $\mathbf{cdga}_{\mathcal{N}}$.

Definition 2.4.8 Let $\mathcal{A}$ be an augmented Adams graded cdga over $\mathcal{N}$, $n \geq 1$ an integer, or $n = \infty$. An *n-minimal model* over $\mathcal{N}$ of $\mathcal{A}$ is a map of augmented Adams graded cdgas over $\mathcal{N}$

$$s : \mathcal{A}\{n\}_{\mathcal{N}} \to \mathcal{A},$$

with $\mathcal{A}\{n\}_{\mathcal{N}}$ generalized nilpotent over $\mathcal{N}$, generated in degree $\leq n$, and such that s induces an isomorphism on H^m for $1 \leq m \leq n$ and an injection on H^{n+1}. A *minimal model over* $\mathcal{N}$, $\mathcal{A}\{\infty\}_{\mathcal{N}} \to \mathcal{A}$, is a relative n-minimal model for all n.

If the base-cdga $\mathcal{N}$ is understood, we call an n-minimal model over $\mathcal{N}$ a *relative n-minimal model.*

Proposition 2.4.9 *Let $\mathcal{N}$ be a cohomologically connected Adams graded cdga, $\mathcal{A}$ an augmented Adams graded cdga over $\mathcal{N}$. Then for each $n = 1, 2, \ldots, \infty$, there is an n-minimal model over $\mathcal{N}$: $\mathcal{A}\{n\}_{\mathcal{N}} \to \mathcal{A}$.*

Proof This result is the relative analog of theorem 1.11.4 and the proof is essentially the same (see [7, chapter 7] for the details in the absolute case). The construction of the n-minimal model over $\mathcal{N}$ is essentially the same as for cdgas over $\mathbb{Q}$ except that we use both the cohomological degree and the Adams degree for induction.

In detail: The augmentation gives a canonical decomposition of $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{N} \oplus \mathcal{I}$$

with $\mathcal{I}$ an Adams graded dg $\mathcal{N}$-ideal in $\mathcal{A}$. Let $E_{10}(1) \subset \mathcal{I}^1(1)$ be a $\mathbb{Q}$-subspace of representatives for $H^1(\mathcal{I}(1))$, where we give $E_{10}(1)$ cohomological degree 1 and Adams degree 1. Using the $\mathcal{N}$-module structure of $\mathcal{A}$, we have the evident mapping

$$E_{10}(1) \otimes_{\mathbb{Q}} \mathcal{N} \to \mathcal{A},$$

which extends to

$$\mathrm{Sym}^* E_{10}(1) \otimes_{\mathbb{Q}} \mathcal{N} \to \mathcal{A}$$

using the algebra structure. Clearly this is a map of augmented cdgas over $\mathcal{N}$, and induces an isomorphism on $H^1(-)(1)$, because $\mathcal{N}(r) = 0$ for $r < 0$ and $\mathcal{N}(0) = \mathbb{Q} \cdot \mathrm{id}$.

One then proceeds as in the case $\mathcal{N} = \mathbb{Q}$ to adjoin elements in degree 1 and Adams degree 1 to successively kill elements in the kernel of the map on $H^2(-)(1)$. Since $\mathcal{N}(r) = 0$ for $r < 0$ and $\mathcal{N}(0) = \mathbb{Q} \cdot \mathrm{id}$, this does not affect H^1 in Adams degree ≤ 1. Thus we have constructed a bi-graded $\mathbb{Q}$-vector space $E_1(1)$, of Adams degree 1 and cohomological degree

1, a generalized nilpotent cdga over $\mathcal{N}$, $\mathcal{A}_{1,1} := \mathrm{Sym}^* E_1(1) \otimes \mathcal{N}$ and a map of cdgas over $\mathcal{N}$, $\mathcal{A}_{1,1} \to \mathcal{A}$, that induces an isomorphism on $H^1(-)(1)$ and an injection on $H^2(-)(1)$.

This completes the Adams degree ≤ 1 part for the construction of the 1-minimal model. So far, we have not used the cohomological connectivity of $\mathcal{N}$, this comes in now: Use the canonical augmentation of $\mathcal{A}_{1,1}$ to write $\mathcal{A}_{1,1} = \mathcal{N} \oplus \mathcal{I}_{1,1}$.

Claim 2.4.10 *$H^p(\mathcal{I}_{1,1}(r)) = 0$ for $r > 1$, $p \leq 1$.*

To prove the claim, we use the same filtration that we used in the proof of lemma 2.4.5. The same induction argument as in lemma 2.4.5, using of course the cohomological connnectedness of $\mathcal{N}$, shows that the lowest degree cohomology of $\mathcal{I}_{1,1}(r)$ comes from $\oplus_{i=1}^{r-1} \mathrm{Sym}^i E_1(1) \otimes H^1(\mathcal{N}(r-i))$ plus $\mathrm{Sym}^r E_1(1) \otimes H^0(\mathcal{N}(0))$. Since all the elements of $E_1(1)$ have cohomological degree 1, this proves the claim.

To construct the Adams degree ≤ 1 part of the n-minimal model in case $n > 1$, we continue the construction, first adjoining elements of Adams degree 1 and cohomological degree 2 to generate all of $H^2(\mathcal{A})(1)$, and then adjoining elements of Adams degree 1 and cohomological degree 2 to kill the kernel on $H^3(-)(1)$. Continuing in this manner gives the generalized nilpotent cdga over $\mathcal{N}$,

$$\mathcal{A}_{1,n} := \mathrm{Sym}^* E_n(1) \otimes \mathcal{N},$$

with $E_n(1)$ in Adams degree 1 and cohomological degree $1, \ldots, n$, together with a map over $\mathcal{N}$, $\mathcal{A}_{1,n} \to \mathcal{A}$, that induces an isomorphism on $H^i(-)(1)$ for $1 \leq i \leq n$ and an injection for $i = n+1$. If we are in the case $n = \infty$, we just take the colimit of the $\mathcal{A}_{1,n}$. In addition, writing $\mathcal{A}_{1,n} = \mathcal{N} \oplus \mathcal{I}_{1,n}$, we prove as above

$$H^p(\mathcal{I}_{1,n}(r)) = 0 \text{ for } r > 1, p \leq 1.$$

Now suppose we have constructed bi-graded $\mathbb{Q}$-vector spaces

$$E_n(1) \subset E_n(2) \subset \ldots \subset E_n(m)$$

(for fixed n with $1 \leq n \leq \infty$) with $E_n(j)$ having Adams degrees $1, \ldots, j$ and cohomological degrees $1, \ldots, n$, a differential on $\mathcal{A}_{m,n} := \mathrm{Sym}^* E_n(m) \otimes \mathcal{N}$ making $\mathcal{A}_{m,n}$ a generalized nilpotent cdga over $\mathcal{N}$, and a map $\mathcal{A}_{m,n} \to \mathcal{A}$ of cdgas over $\mathcal{N}$ that is an isomorphism on $H^i(-)(j)$ for $1 \leq i \leq n$, $j \leq m$, and an injection for $i = n+1$, $j \leq m$. In addition, writing $\mathcal{A}_{n,m} = \mathcal{N} \oplus \mathcal{I}_{n,m}$, we have

$$H^p(\mathcal{I}_{m,n}(r)) = 0 \text{ for } r > m, p \leq 1. \tag{2.4.1}$$

We extend $E_n(m)$ to $E_n(m+1)$ by simply repeating the construction for $E_n(1)$ described above, but working in Adams degree $m+1$ rather than 1; using (2.4.1) allows us to start the construction by adjoining generators for $H^1(\mathcal{I}(m+1))$, just as in the case of Adams weight 1. Again, as $\mathcal{N}(r) = 0$ for $r < 0$ and $\mathcal{N}(0) = \mathbb{Q} \cdot \mathrm{id}$, the inclusion $\mathcal{A}_{m,n} \to \mathcal{A}_{m+1,n}$ is an isomorphism in Adams degree $\leq m$. In addition, the argument used to prove the claim shows that (2.4.1) extends from m to $m+1$ and the induction goes through.

Taking $E_n := \cup_m E_n(m)$, we thus have a differential on $\mathcal{A}\{n\}_\mathcal{N} := \mathrm{Sym}^* E_n \otimes \mathcal{N}$ making $\mathcal{A}\{n\}_\mathcal{N}$ a generalized nilpotent cdga over $\mathcal{N}$, and a map $\mathcal{A}\{n\}_\mathcal{N} \to \mathcal{A}$ of cdgas over $\mathcal{N}$ that is an isomorphism on $H^i(-)$ for $1 \leq i \leq n$ and an injection for $i = n+1$, completing the proof.

Remark 2.4.11 Suppose that both $\mathcal{N}$ and $\mathcal{A}$ are cohomologically connected. Then

$$\mathcal{A}\{n\}_{\mathcal{N}} \to \mathcal{A}$$

induces an isomorphism on H^i for all $i \leq n$. In particular, the map $\mathcal{A}\{\infty\}_{\mathcal{N}} \to \mathcal{A}$ is a quasi-isomorphism.

Proposition 2.4.12 *Suppose that $\mathcal{N}$ is cohomologically connected, $\mathcal{A} \in$ **cdga**$_{\mathcal{N}}$. Let $s : \mathcal{A}_{\mathcal{N}} \to \mathcal{A}$, $s' : \mathcal{A}'_{\mathcal{N}} \to \mathcal{A}$ be relative minimal models. Then there is an isomorphism $\phi : \mathcal{A}_{\mathcal{N}} \to \mathcal{A}'_{\mathcal{N}}$ in* **cdga**$_{\mathcal{N}}$ *such that $s' \circ \phi \sim_\Omega s$.*

Proof By definition, the maps s, s' are weak equivalences in **cdga**$_{\mathcal{N}}$, and thus we have the isomorphism in $\mathcal{H}(\mathbf{cdga}_{\mathcal{N}})$

$$s'^{-1}s : \mathcal{A}_{\mathcal{N}} \to \mathcal{A}'_{\mathcal{N}}.$$

Since $\mathcal{A}_{\mathcal{N}}$ and $\mathcal{A}_{\mathcal{N}}$ are both generalized nilpotent $\mathcal{N}$-algebras, $\mathcal{A}_{\mathcal{N}}$ and $\mathcal{A}_{\mathcal{N}}$ are both cofibrant (see lemma 2.4.4), and thus there is a morphism $\phi : \mathcal{A}_{\mathcal{N}} \to \mathcal{A}'_{\mathcal{N}}$ in **cdga**$_{\mathcal{N}}$ representing the isomorphism $s'^{-1}s$ in $\mathcal{H}(\mathbf{cdga}_{\mathcal{N}})$. Thus ϕ is a weak equivalence and $s' \circ \phi \sim_\Omega s$. By proposition 2.4.6, ϕ is an isomorphism in **cdga**$_{\mathcal{N}}$.

Thus, the relative minimal model is unique up to (non-canonical) isomorphism in **cdga**$_{\mathcal{N}}$. In fact, in case $\mathcal{N}$ is connected, the same holds for the relative n-minimal models. For this, we first note the following simple extension of proposition 2.4.9.

Lemma 2.4.13 *Suppose that $\mathcal{N}$ is cohomologically connected. Let $s_n : \mathcal{A}\{n\}_{\mathcal{N}} \to \mathcal{A}$ be an n-minimal model for some n, $1 \leq n < \infty$. Then there is a monomorphism of generalized nilpotent cdgas over $\mathcal{N}$, $i : \mathcal{A}\{n\}_{\mathcal{N}} \to \mathcal{A}_{\mathcal{N}}$, such that*

1. *The morphism $s_n : \mathcal{A}\{n\}_{\mathcal{N}} \to \mathcal{A}$ extends to a morphism $s : \mathcal{A}_{\mathcal{N}} \to \mathcal{A}$ in* **cdga**$_{\mathcal{N}}$.
2. *$s : \mathcal{A}_{\mathcal{N}} \to \mathcal{A}$ is a relative minimal model of $\mathcal{A}$.*

If in addition $\mathcal{N}$ is connected, then $\mathcal{A}\{n\}_{\mathcal{N}}$ is equal to the $\mathcal{N}$-subalgebra $\mathcal{A}_{\mathcal{N}(n)}$ of $\mathcal{A}_{\mathcal{N}}$ generated by elements $x \in \mathcal{A}_{\mathcal{N}}$ with $\deg x \leq n$.

Proof Write $\mathcal{A}\{n\}_{\mathcal{N}} = \mathrm{Sym}^* E_n \otimes \mathcal{N}$ as an $\mathcal{N}$-algebra, where E_n is a bi-graded $\mathbb{Q}$-vector space with filtration satisfying the conditions of remark 2.4.2 and such that each $e \in E_n$ has $\deg e \leq n$. We now just apply the inductive construction of the relative minimal model of $\mathcal{A}$ as given in the proof of proposition 2.4.9, starting with the generating vector space E_n, to construct a relative minimal model $\mathcal{A}_{\mathcal{N}} \to \mathcal{A}$ as an augmented $\mathcal{N}$-algebra containing $\mathcal{A}\{n\}_{\mathcal{N}}$. This proves (1).

Suppose $\mathcal{N}$ is connected. Let $E \subset E_n$ be the bi-graded $\mathbb{Q}$ vector space of $\mathcal{N}$-algebra generators for $\mathcal{A}_{\mathcal{N}}$ constructed by the inductive procedure of proposition 2.4.9. Then $E^{\leq n} = E_n^{\leq n}$; as $\mathcal{N}$ is connected, this immediately implies $\mathcal{A}\{n\}_{\mathcal{N}} = \mathcal{A}_{\mathcal{N}(n)}$.

Proposition 2.4.14 *Suppose that $\mathcal{N}$ is connected and take $\mathcal{A} \in$ **cdga**$_{\mathcal{N}}$. Suppose we have relative n-minimal models*

$$s_n : \mathcal{A}\{n\}_{\mathcal{N}} \to \mathcal{A};\ s'_n : \mathcal{A}\{n\}'_{\mathcal{N}} \to \mathcal{A}.$$

Then there is an isomorphism $\phi_n : \mathcal{A}\{n\}_{\mathcal{N}} \to \mathcal{A}\{n\}'_{\mathcal{N}}$ in **cdga**$_{\mathcal{N}}$ *such that $s'_n \circ \phi_n \sim_\Omega s_n$.*

Proof By lemma 2.4.13, we may extend s_n and s'_n to relative minimal models $s : \mathcal{A}_\mathcal{N} \to \mathcal{A}$, $s' : \mathcal{A}'_\mathcal{N} \to \mathcal{A}$, giving us commutative diagrams

$$\begin{array}{ccc} \mathcal{A}\{n\}_\mathcal{N} & \xrightarrow{i} & \mathcal{A}_\mathcal{N} \\ & \searrow^{s_n} & \downarrow s \\ & & \mathcal{A}; \end{array} \qquad \begin{array}{ccc} \mathcal{A}\{n\}'_\mathcal{N} & \xrightarrow{i'} & \mathcal{A}'_\mathcal{N} \\ & \searrow^{s'_n} & \downarrow s' \\ & & \mathcal{A} \end{array}$$

in $\mathbf{cdga}_\mathcal{N}$, such that i and i' are monomorphisms, giving identifications

$$\mathcal{A}\{n\}_\mathcal{N} = \mathcal{A}_{\mathcal{N}(n)};\ \mathcal{A}\{n\}'_\mathcal{N} = \mathcal{A}'_{\mathcal{N}(n)}.$$

By proposition 2.4.12 there is an isomorphism $\phi : \mathcal{A}_\mathcal{N} \to \mathcal{A}'_\mathcal{N}$ in $\mathbf{cdga}_\mathcal{N}$ with $s' \circ \phi \sim_\Omega s$. Restricting ϕ to $\mathcal{A}_{\mathcal{N}(n)}$ gives the isomorphism

$$\phi_n : \mathcal{A}\{n\}_\mathcal{N} = \mathcal{A}_{\mathcal{N}(n)} \to \mathcal{A}\{n\}'_\mathcal{N} = \mathcal{A}'_{\mathcal{N}(n)}$$

and a choice of a right homotopy $h : \mathcal{A}_\mathcal{N} \to \mathcal{A} \otimes_\mathbb{Q} \Omega$ between $s' \circ \phi$ and s restricts to a right homotopy between $s'_n \circ \phi$ and s_n.

Remark 2.4.15 A generalized nilpotent cdga over $\mathcal{N}$ is automatically a cell-module over $\mathcal{N}$. Indeed, for $\mathcal{A} = \mathrm{Sym}^* E \otimes \mathcal{N}$ satisfying the conditions of definition 2.4.1, one has the filtration on $E^{\leq n}$ given by remark 2.4.2. Combining this filtration with the filtration by degree on $\mathrm{Sym}^* E$ gives a filtration on $\mathrm{Sym}^* E$ which exhibits $\mathcal{A}$ as an $\mathcal{N}$-cell module.

2.5 Relative bar construction

One forms the bar construction for a cdga $\mathcal{A}$ over $\mathcal{N}$ just as for cdgas over $\mathbb{Q}$, replacing $\otimes_\mathbb{Q}$ with $\otimes_\mathcal{N}$. However, for this construction to have good cohomological properties, one should replace $\mathcal{A}$ with a quasi-isomorphic cdga $\mathcal{A}'$ which is a cell module over $\mathcal{N}$, so that $\otimes_\mathcal{N} = \otimes^L_\mathcal{N}$. This is accomplished by using the minimal model $\mathcal{A}\{\infty\}$. In any case, we give the "pre-derived" definition for an arbitrary cdga $\mathcal{A}$ over $\mathcal{N}$.

Definition 2.5.1 Let $\mathcal{A}$ be an augmented Adams graded cdga over $\mathcal{N}$. Define the simplicial cdga $B^{pd}_\bullet(\mathcal{A}/\mathcal{N})$ by

$$B^{pd}_\bullet(\mathcal{A}/\mathcal{N}) := \mathcal{A}^{\otimes_\mathcal{N}[0,1]}$$

The inclusion $\{0,1\} \to [0,1]$ makes $B^{pd}_\bullet(\mathcal{A}/\mathcal{N})$ a simplicial cdga over $A \otimes A$. Given two (possibly equal) augmentations $\epsilon_1, \epsilon_2 : \mathcal{A} \to \mathcal{N}$, set

$$B^{pd}_\bullet(\mathcal{A}/\mathcal{N}, \epsilon_1, \epsilon_2) := B^{pd}_\bullet(\mathcal{A}/\mathcal{N}) \otimes_{\mathcal{A}\otimes\mathcal{A}} \mathcal{N}.$$

and let $\bar{B}^{pd}_\mathcal{N}(\mathcal{A}, \epsilon_1, \epsilon_2)$ be the total complex associated to $B^{pd}_\bullet(\mathcal{A}/\mathcal{N}, \epsilon_1, \epsilon_2)$.

Remark 2.5.2 Let $\mathcal{A}$ be a generalized nilpotent algebra over $\mathcal{N}$, and write $\mathcal{A}$, as a bi-graded $\mathcal{N}$-algebra, as

$$\mathcal{A} = \mathrm{Sym}^* E \otimes_\mathbb{Q} \mathcal{N},$$

where E is a bi-graded $\mathbb{Q}$-vector space satisfying the conditions of definition 1.11.1. Since $\mathcal{A}(r) = 0$ for $r < 0$ and $\mathcal{A}(0) = \mathbb{Q}\cdot\mathrm{id}$, each bi-homogeneous element $e \in E$ has Adams degree $|e| \geq 1$. Thus, $W_{-1}\mathcal{A} = 0$. Since

$$\mathcal{A}^{\otimes_{\mathcal{N}} n} \cong \mathrm{Sym}^* E^{\otimes_{\mathbb{Q}} n} \otimes_{\mathbb{Q}} \mathcal{N}$$

the same holds for $\mathcal{A}^{\otimes_{\mathcal{N}} n}$. In particular, $\mathcal{A}^{\otimes_{\mathcal{N}} n}$ is in $\mathcal{CM}_{\mathcal{N}}^{+w}$ for each $n \geq 0$.

As $\bar{B}^{pd}_{\mathcal{N}}(\mathcal{A}, \epsilon_1, \epsilon_2)$ is the total complex of a double complex built out of the $\mathcal{A}^{\otimes_{\mathcal{N}} n}$, we see that the dg $\mathcal{N}$-module $\bar{B}^{pd}_{\mathcal{N}}(\mathcal{A}, \epsilon_1, \epsilon_2)$ is in $\mathcal{CM}_{\mathcal{N}}^{+w}$. Finally, if $\epsilon_1 = \epsilon_2 = \epsilon$, then, using the same formulas as in §1.2, $\bar{B}^{pd}_{\mathcal{N}}(\mathcal{A}, \epsilon)$ has the natural structure of a dg Hopf algebra in $\mathcal{CM}_{\mathcal{N}}^{+w}$, and thus a Hopf algebra in $\mathcal{D}_{\mathcal{N}}^{+w}$.

Definition 2.5.3 Let $\mathcal{A}$ be an augmented Adams graded cdga over $\mathcal{N}$ with augmentation ϵ. Suppose that $\mathcal{N}$ is cohomologically connnected and let $\mathcal{A}\{\infty\}_{\mathcal{N}} \to \mathcal{A}$ be the relative minimal model of $\mathcal{A}$ over $\mathcal{N}$. Define

$$B_\bullet(\mathcal{A}/\mathcal{N}) := B^{pd}_\bullet(\mathcal{A}\{\infty\}_{\mathcal{N}}/\mathcal{N}),\ \bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon) := \bar{B}^{pd}_{\mathcal{N}}(\mathcal{A}\{\infty\}_{\mathcal{N}}, \epsilon\{\infty\}).$$

Remarks 2.5.4
1. Still supposing $\mathcal{N}$ to be cohomologically connected, we may apply the truncation functor

$$H^0_{\mathcal{N}} : \mathcal{D}_{\mathcal{N}}^{+w} \to \mathcal{H}_{\mathcal{N}}$$

to the dg Hopf algebra $\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon)$ in $\mathcal{D}_{\mathcal{N}}^{+w}$, giving us the Hopf algebra $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$ in $\mathcal{H}_{\mathcal{N}}$. We may also form the co-Lie algebra object $\gamma_{\mathcal{A}/\mathcal{N}}$ in $\mathcal{H}_{\mathcal{N}} = Conn^0_{\mathcal{N}}$:

$$\gamma_{\mathcal{A}/\mathcal{N}} := H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))_+ / H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))_+^2$$

with $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))_+ \subset H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$ the augmentation ideal.

We let $\bar{B}_{\bullet\leq m}(\mathcal{A}/\mathcal{N}, \epsilon)$ denote the restriction of the simplicial object $\bar{B}_\bullet(\mathcal{A}/\mathcal{N}, \epsilon)$ to the full subcategory $\{[0], \ldots, [m]\}$ of **Ord**, and $\bar{B}^{\leq m}_{\mathcal{N}}(\mathcal{A}, \epsilon) \subset \bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon)$ the associated total complex of $\bar{B}_{\bullet\leq m}(\mathcal{A}/\mathcal{N}, \epsilon)$.

If we suppose that $\mathcal{A}$ is in $\mathcal{D}^f_{\mathcal{N}}$, then $H^0_{\mathcal{N}}(\bar{B}^{\leq m}_{\mathcal{N}}(\mathcal{A}, \epsilon))$ is in $\mathcal{H}^f_{\mathcal{N}}$ for each m, hence $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$ has the structure of an ind-Hopf algebra in $\mathcal{H}^f_{\mathcal{N}}$ with

$$H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon)) = \varinjlim_{m\to\infty} H^0_{\mathcal{N}}(\bar{B}^{\leq m}_{\mathcal{N}}(\mathcal{A}, \epsilon))$$

in $\mathcal{H}_{\mathcal{N}}$.

2. In our definition of $B_\bullet(\mathcal{A}/\mathcal{N})$, we made a choice of a relative minimal model of $\mathcal{A}$; by proposition 2.4.12, this choice is unique up to (non-unique) isomorphism, and thus the same is true for $B_\bullet(\mathcal{A}/\mathcal{N})$. Furthermore, two different relative minimal models are canonically isomorphic in $\mathcal{H}(\mathbf{cdga}_{\mathcal{N}})$, and therefore the Hopf algebra object $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$ is independent of the choice of relative minimal model, up to unique isomorphism. In case $\mathcal{A}$ is in $\mathcal{D}^f_{\mathcal{N}}$, the same holds for $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$ as an ind-Hopf algebra in $\mathcal{H}^f_{\mathcal{N}}$.

2.6 Base-change

We consider a quasi-isomorphism $\phi : \mathcal{N}' \to \mathcal{N}$ of cohomologically connected cdgas. Given an augmented cdga $\mathcal{A}$ over $\mathcal{N}$ with augmentation $\epsilon : \mathcal{A} \to \mathcal{N}$, we have $\mathcal{A} = \mathcal{I} \oplus \mathcal{N}$, with $\mathcal{I}$ the kernel of ϵ. In particular, $\mathcal{I}$ is a (non-unital) $\mathcal{N}$-algebra. Via ϕ, we make $\mathcal{I}$ a (non-unital) $\mathcal{N}'$-algebra, and thus give $\mathcal{A}' := \mathcal{I} \oplus \mathcal{N}'$ the structure of a cdga over $\mathcal{N}'$, with augmentation $\epsilon' : \mathcal{A}' \to \mathcal{N}'$ the projection on $\mathcal{N}'$ with kernel $\mathcal{I}$.

This construction yields the commutative diagram of cdgas

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{\phi'} & \mathcal{A} \\ {\scriptstyle p'}\uparrow\downarrow{\scriptstyle \epsilon'} & & {\scriptstyle p}\uparrow\downarrow{\scriptstyle \epsilon} \\ \mathcal{N}' & \xrightarrow[\phi]{} & \mathcal{N} \end{array} \tag{2.6.1}$$

with ϕ and ϕ' quasi-isomorphisms.

Now let $f' : \mathcal{A}'\{n\}_{\mathcal{N}'} \to \mathcal{A}'$ be a relative n-minimal model over $\mathcal{A}'$ over $\mathcal{N}'$. Since the composition $\phi' f' : \mathcal{A}'\{n\}_{\mathcal{N}'} \to \mathcal{A}$ is an $\mathcal{N}'$-module map, $\phi' f'$ factors through a unique map

$$f : \mathcal{A}'\{n\}_{\mathcal{N}'} \otimes_{\mathcal{N}'} \mathcal{N} \to \mathcal{A}$$

of cdgas over $\mathcal{N}$. Similarly, the $\mathcal{N}'$-augmentation of $\mathcal{A}'\{n\}_{\mathcal{N}'}$ induces an $\mathcal{N}$-augmentation of $\mathcal{A}'\{n\}_{\mathcal{N}'} \otimes_{\mathcal{N}'} \mathcal{N}$, making f a map of augmented cdgas over $\mathcal{N}$.

Lemma 2.6.1 $f : \mathcal{A}'\{n\}_{\mathcal{N}'} \otimes_{\mathcal{N}'} \mathcal{N} \to \mathcal{A}$ *is a relative* n*-minimal model of* $\mathcal{A}$ *over* $\mathcal{N}$*.*

Proof As $\mathcal{A}'\{n\}_{\mathcal{N}'}$ is a generalized nilpotent algebra over $\mathcal{N}'$, with generators in degree $\leq n$, the same follows for $\mathcal{A}'\{n\}_{\mathcal{N}'} \otimes_{\mathcal{N}'} \mathcal{N}$ as an algebra over $\mathcal{N}$. ϕ is a quasi-isomorphism, so $\phi_* : \mathcal{D}_{\mathcal{N}'} \to \mathcal{D}_{\mathcal{N}}$ is an equivalence of triangulated categories. We can compute cohomology of a dg module via maps in the derived category; as $\mathcal{A}'\{n\}_{\mathcal{N}'}$ is an $\mathcal{N}'$-cell module, we have $\phi_*(\mathcal{A}'\{n\}_{\mathcal{N}'}) = \mathcal{A}'\{n\}_{\mathcal{N}'} \otimes_{\mathcal{N}'} \mathcal{N}$, hence the canonical map

$$\mathcal{A}'\{n\}_{\mathcal{N}'} \to \mathcal{A}'\{n\}_{\mathcal{N}'} \otimes_{\mathcal{N}'} \mathcal{N}$$

is a quasi-isomorphism of cdgas. Since $\phi' : \mathcal{A}' \to \mathcal{A}$ is a quasi-isomorphism, and $\mathcal{A}'\{n\}_{\mathcal{N}'} \to \mathcal{A}'$ is a relative n-minimal model, the map on H^i induced by f is an isomorphism for $1 \leq i \leq n$ and an injection for $i = n+1$, i.e., $f : \mathcal{A}'\{n\}_{\mathcal{N}'} \otimes_{\mathcal{N}'} \mathcal{N} \to \mathcal{A}$ is a relative n-minimal model.

Remark 2.6.2 Still assuming $\mathcal{N}$ and $\mathcal{N}'$ cohomologically connected, write $\mathcal{A}\{n\}_{\mathcal{N}}$ for the n-minimal model $\mathcal{A}'\{n\}_{\mathcal{N}'} \otimes_{\mathcal{N}'} \mathcal{N}$. We have the change of rings isomorphism

$$\mathcal{A}'\{n\}_{\mathcal{N}'}^{\otimes_{\mathcal{N}'} m} \otimes_{\mathcal{N}'} \mathcal{N} \to \mathcal{A}\{n\}_{\mathcal{N}}^{\otimes_{\mathcal{N}} m}$$

and the quasi-isomorphism

$$\mathcal{A}'\{n\}_{\mathcal{N}'}^{\otimes_{\mathcal{N}'} m} \to \mathcal{A}'\{n\}_{\mathcal{N}'}^{\otimes_{\mathcal{N}'} m} \otimes_{\mathcal{N}'} \mathcal{N}$$

Thus on the bar construction

$$\bar{B}^{pd}_{\mathcal{N}'}(\mathcal{A}'\{n\}_{\mathcal{N}'}, \epsilon') \xrightarrow{\alpha} \bar{B}^{pd}_{\mathcal{N}'}(\mathcal{A}'\{n\}_{\mathcal{N}'}, \epsilon') \otimes_{\mathcal{N}'} \mathcal{N} \xrightarrow{\beta} \bar{B}^{pd}_{\mathcal{N}}(\mathcal{A}\{n\}_{\mathcal{N}}, \epsilon)$$

the map α is a quasi-isomorphism and the map β is an isomorphism.

In particular, taking $n = \infty$, we have the canonical isomorphism

$$\phi_*(H^0_{\mathcal{N}'}(\bar{B}_{\mathcal{N}'}(\mathcal{A}', \epsilon'))) \cong H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$$

of Hopf algebra objects in $\mathcal{H}_{\mathcal{N}}$. Since $\phi_* : \mathcal{H}_{\mathcal{N}'} \to \mathcal{H}_{\mathcal{N}}$ is an equivalence, we are thus free to replace $\mathcal{N}$ with a quasi-isomorphic $\mathcal{N}'$ in a study of $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}, \epsilon))$. For instance, we may use the minimal model $\mathcal{N}\{\infty\} \to \mathcal{N}$ as a replacement for $\mathcal{N}$.

2.7 Connection matrices

Generalized nilpotent algebras over $\mathcal{N}$ fit well into the connection matrix point of view described in section 1.13. Indeed, suppose that $\mathcal{A} = \mathrm{Sym}^*E \otimes \mathcal{N}$ is generalized nilpotent over $\mathcal{N}$, with augmentation $\epsilon : \mathcal{A} \to \mathcal{N}$ induced by writing $\mathrm{Sym}^*E = \mathbb{Q} \oplus \mathrm{Sym}^{*\geq 1}E$.

Using the augmentation of $\mathcal{N}$, we write $\mathcal{N} = \mathbb{Q} \cdot \mathrm{id} \oplus \mathcal{N}^+$, which writes $\mathcal{A}$ as

$$\mathcal{A} = \mathrm{Sym}^*E \otimes \mathrm{id} \oplus \mathrm{Sym}^*E \otimes \mathcal{N}^+.$$

Thus, the differential on $\mathcal{A}$ is completely determined by its restriction to $\mathrm{Sym}^*E \otimes \mathrm{id}$, giving the decomposition

$$d = d^0 + \Gamma$$

with d^0 a differential on Sym^*E and $\Gamma : \mathrm{Sym}^*E \to \mathrm{Sym}^*E \otimes \mathcal{N}^+$ a flat connection. In addition, (Sym^*E, d^0) an Adams graded cdga over $\mathbb{Q}$ with augmentation ϵ^0 induced by the projection to $\mathrm{Sym}^0E = \mathbb{Q}$. Finally, the connection Γ is nilpotent since Sym^*E has all Adams degrees ≥ 0 (lemma 1.13.3).

Using the tensor structure in the category of flat nilpotent connections, the flat nilpotent connection $\Gamma : \mathrm{Sym}^*E \to \mathrm{Sym}^*E \otimes \mathcal{N}^+$ gives rise to a flat nilpotent connection on $(\mathrm{Sym}^*E)^{\otimes n}$ for all n. These fit together to give a flat nilpotent connection on the bar construction:

$$\bar{B}(\Gamma) : \bar{B}((\mathrm{Sym}^*E, d^0), \epsilon^0) \to \bar{B}((\mathrm{Sym}^*E, d^0), \epsilon^0) \otimes \mathcal{N}^+.$$

This defines a Hopf algebra object in $Conn_{\mathcal{N}}$.

Proposition 2.7.1 *Let $\mathcal{N}$ be cohomologically connected. The $\mathcal{N}$-cell module corresponding to $\bar{B}((\mathrm{Sym}^*E, d^0), \epsilon^0)$ with flat nilpotent connection $\bar{B}(\Gamma)$ is isomorphic to $\bar{B}^{pd}_{\mathcal{N}}(\mathcal{A}, \epsilon)$, as dg Hopf algebra objects in $\mathcal{CM}^{+w}_{\mathcal{N}}$.*

Proof We check instead the equivalent statement that the dg Hopf algebra in $Conn_{\mathcal{N}}$ corresponding to $\bar{B}^{pd}_{\mathcal{N}}(\mathcal{A}, \epsilon)$ is $(\bar{B}((\mathrm{Sym}^*E, d^0), \epsilon^0), \bar{B}(\Gamma))$.

We note that we have canonical isomorphisms

$$\mathcal{A}^{\otimes_{\mathcal{N}} n} \cong (\mathrm{Sym}^*E)^{\otimes_{\mathbb{Q}} n} \otimes_{\mathbb{Q}} \mathcal{N} = (\mathrm{Sym}^*E)^{\otimes_{\mathbb{Q}} n} \otimes \mathrm{id} \oplus (\mathrm{Sym}^*E)^{\otimes_{\mathbb{Q}} n} \otimes_{\mathbb{Q}} \mathcal{N}^+$$

respecting differentials and multiplications. Tracing this isomorphism through the definition we have given of the flat nilpotent connection on $\bar{B}((\mathrm{Sym}^*E, d^0), \epsilon^0)$ completes the proof.

2.8 Semi-direct products

Let $\epsilon : \mathcal{A} \to \mathcal{N}$ be an augmented Adams graded cdga over $\mathcal{N}$. We suppose that $\mathcal{N}$ is generalized nilpotent and that $\mathcal{A}$ is generalized nilpotent over $\mathcal{N}$. We let $G_{\mathcal{A}} := \operatorname{Spec} H^0(\bar{B}(\mathcal{A}))$, $G_{\mathcal{N}} := \operatorname{Spec} H^0(\bar{B}(\mathcal{N}))$ be the $\mathbb{Q}$-algebraic group schemes defined with respect to the canonical augmentations $\mathcal{A} \to \mathbb{Q}$, $\mathcal{N} \to \mathbb{Q}$. The $\mathcal{N}$-algebra structure $\pi^* : \mathcal{N} \to \mathcal{A}$ induces the map of algebraic groups $\pi : G_{\mathcal{A}} \to G_{\mathcal{N}}$; the augmentation ϵ gives a splitting $s : G_{\mathcal{N}} \to G_{\mathcal{A}}$ to π.

Lemma 2.8.1 *The map π is flat.*

Proof Following our remarks in §1.2, $H^0(\bar{B}(\mathcal{A}))$ and $H^0(\bar{B}(\mathcal{N}))$ are polynomial algebras over $\mathbb{Q}$ on $\mathcal{A}^1$, $\mathcal{N}^1$ respectively, and the map

$$H^0(\bar{B}(\pi^*)) : H^0(\bar{B}(\mathcal{N})) \to H^0(\bar{B}(\mathcal{A}))$$

is just the polynomial extension of the linear injection

$$\pi^* : \mathcal{N}^1 \to \mathcal{A}^1.$$

That is, $H^0(\bar{B}(\pi^*))$ identifies $H^0(\bar{B}(\mathcal{A}))$ with a polynomial extension of $H^0(\bar{B}(\mathcal{N}))$.

Lemma 2.8.2 *Let e denote the identity in $G_{\mathcal{N}}$. The fiber $\pi^{-1}(e)$ is canonically isomorphic to* $\operatorname{Spec} H^0(\bar{B}(\mathcal{A} \otimes_{\mathcal{N}} \mathbb{Q}))$ *as group schemes over $\mathbb{Q}$.*

Proof We have the natural map of Hopf algebras

$$H^0(\bar{B}(\mathcal{A})) \otimes_{H^0(\bar{B}(\mathcal{N}))} \mathbb{Q} \to H^0(\bar{B}(\mathcal{A} \otimes_{\mathcal{N}} \mathbb{Q})).$$

Writing $\mathcal{A} = \operatorname{Sym}^* E \otimes \mathcal{N}$ as an $\mathcal{N}$-algebra, $H^0(\bar{B}(\mathcal{A} \otimes_{\mathcal{N}} \mathbb{Q}))$ is a polynomial algebra on $(\operatorname{Sym}^* E)^1$, while $H^0(\bar{B}(\mathcal{A}))$ is the polynomial algebra on $\mathcal{A}^1 = (\operatorname{Sym}^* E)^1 \oplus \mathcal{N}^1$, and $H^0(\bar{B}(\mathcal{N}))$ is the polynomial algebra on $\mathcal{N}^1$. This shows that the above map is an algebra isomorphism.

Set $K := \operatorname{Spec} H^0(\bar{B}(\mathcal{A} \otimes_{\mathcal{N}} \mathbb{Q})) = \operatorname{Spec} H^0(\bar{B}(\operatorname{Sym}^* E))$. The splitting s gives an action of $G_{\mathcal{N}}$ on K and an isomorphism of $G_{\mathcal{A}}$ with the semi-direct product

$$G_{\mathcal{A}} \cong K \ltimes G_{\mathcal{N}}.$$

Let K_s denote the $\mathbb{Q}$-group scheme K with this $G_{\mathcal{N}}$-action.

On the other hand, we have seen (proposition 2.7.1) that writing $\mathcal{A} = \operatorname{Sym}^* E \otimes \mathcal{N}$ gives $\operatorname{Sym}^* E$ a flat nilpotent connection

$$\Gamma : \operatorname{Sym}^* E \to \mathcal{N}^+ \otimes \operatorname{Sym}^* E$$

and an isomorphism of $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}))$ with $H^0(\bar{B}(\operatorname{Sym}^* E))$ as Hopf algebras in $Conn^0_{\mathcal{N}}$.

Replacing $\mathcal{N}$ with its 1-minimal model, and noting that $Conn^0_{\mathcal{N}} \sim Conn^0_{\mathcal{N}\{1\}}$ we have the canonical structure of $H^0(\bar{B}(\operatorname{Sym}^* E))$ as a Hopf algebra in the category of co-modules over the co-Lie algebra $Q\mathcal{N} = \gamma_{\mathcal{N}}$ (remark 1.14.8). But this category is equivalent to the category of representations of $G_{\mathcal{N}}$, giving us another action of $G_{\mathcal{N}}$ on K.

Theorem 2.8.3 *The action of $G_{\mathcal{N}}$ on $K = \operatorname{Spec} H^0(\bar{B}(\operatorname{Sym}^* E))$ induced by the splitting s is the same as the action given by the flat nilpotent $\mathcal{N}$-connection Γ on $\operatorname{Sym}^* E$. In other words, there is an isomorphism*

$$K_s \cong \operatorname{Spec} H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}))$$

as $\mathbb{Q}$-group schemes with $G_{\mathcal{N}}$-action.

Proof It suffices to check that the two co-actions of the co-Lie algebra $\gamma_{\mathcal{N}}$ are the same, in fact, it suffices to check that the two co-actions of $\gamma_{\mathcal{N}}$ on the co-Lie algebra $\gamma_{\operatorname{Sym}^* E}$ of K are the same.

By Quillen's theorem (theorem 1.15.2(2)), we can identify the co-Lie algebras $\gamma_{\mathcal{A}}$, $\gamma_{\mathcal{N}}$ and $\gamma_{\operatorname{Sym}^* E}$ with $Q\mathcal{A}$, $Q\mathcal{N}$ and $Q\operatorname{Sym}^* E$, respectively. Since we are assuming $\mathcal{A}$ and $\mathcal{N}$ are both generalized nilpotent, $Q\mathcal{A}$, $Q\mathcal{N}$ and $Q\operatorname{Sym}^* E$ are the respective co-Lie algebras

$$d_{\mathcal{A}} : \mathcal{A}^1 \to \Lambda^2 \mathcal{A}^1,\ d_{\mathcal{N}} : \mathcal{N}^1 \to \Lambda^2 \mathcal{N}^1,\ d_E : E^1 \to \Lambda^2 E^1.$$

On the level of co-Lie algebras, the splitting s is just the decomposition of $\mathcal{A}^1 = (\operatorname{Sym}^* E \otimes \mathcal{N})^1$ as

$$\mathcal{A}^1 = (\operatorname{Sym}^* E)^1 \oplus \mathcal{N}^1.$$

The co-action of $\mathcal{N}^1$ on $\mathcal{A}^1$ determined by the splitting s is therefore given by $d_{\mathcal{A}}$ followed by the projection of $\Lambda^2 \mathcal{A}^1$ on $\mathcal{N}^1 \otimes \mathcal{A}^1$ via the isomorphism

$$\Lambda^2 \mathcal{A}^1 = \Lambda^2((\operatorname{Sym}^* E)^1 \oplus \mathcal{N}^1) \cong \Lambda^2 (\operatorname{Sym}^* E)^1 \oplus \mathcal{N}^1 \otimes (\operatorname{Sym}^* E)^1 \oplus \Lambda^2 \mathcal{N}^1.$$

This induces the co-action of $\mathcal{N}^1$ on $(\operatorname{Sym}^* E)^1$ by taking the composition

$$(\operatorname{Sym}^* E)^1 \to \mathcal{A}^1 \xrightarrow{d_{\mathcal{A}}} \Lambda^2 \mathcal{A}^1 \to \mathcal{N}^1 \otimes (\operatorname{Sym}^* E)^1.$$

Via our identifications, this gives us the co-action of $\gamma_{\mathcal{N}}$ on $\gamma_{\operatorname{Sym}^* E}$ determined by the section s.

On the other hand, the flat nilpotent connection Γ on $\operatorname{Sym}^* E$ giving the isomorphism of $H^0_{\mathcal{N}}(\bar{B}_{\mathcal{N}}(\mathcal{A}))$ with $H^0(\bar{B}(\operatorname{Sym}^* E))$ in $Conn^0_{\mathcal{N}}$ is just the restriction of $d_{\mathcal{A}}$ to $\operatorname{Sym}^* E$ followed by the projection of $\mathcal{A} = \mathcal{N} \otimes \operatorname{Sym}^* E$ to $\mathcal{N}^+ \otimes \operatorname{Sym}^* E$. However, by reasons of degree, the restriction of $d_{\mathcal{A}}$ to $(\operatorname{Sym}^* E)^1 = E^1$ decomposes as

$$d_{\mathcal{A}} : E^1 \to \Lambda^2 E^1 \oplus \mathcal{N}^1 \otimes E^1$$

from which it follows that $\Gamma : E^1 \to \mathcal{N}^1 \otimes E^1$ is the same as the co-action defined by s.

3 Motives over a base

This section summarizes the material we need from the work of Cisinski-Déglise [10].

3.1 Effective motives over a base

We summarize the main points of the construction of the category $DM^{\mathrm{eff}}(S)$ of effective motives over S from [10]; we will describe the category $DM(S)$ of motives over S in the next subsection. Although S is allowed to be a quite general scheme in [10], we restrict ourselves to the case of a base-scheme S that is separated, smooth and essentially of finite type over a field. We let $\mathbf{Sch}_S$ denote the category of finite type separated S-schemes and let $\mathbf{Sm}/S$ denote the full subcategory of $\mathbf{Sch}_S$ consisting of smooth S-schemes.

For $X, Y \in \mathbf{Sm}/S$, define the group of finite S-correspondences $c_S(X, Y)$ as the free abelian group on the integral closed subschemes $W \subset X \times_S Y$ with $W \to X$ finite and surjective over an irreducible component of X.

For X, Y, Z in $\mathbf{Sm}/S$, let p_{XY}, p_{YZ} and p_{XZ} be the evident projections from $X \times_S Y \times_S Z$. One checks that the formula

$$W \circ W' := p_{XZ*}(p^*_{XY}(W) \cdot p^*_{YZ}(W')) \in c_S(X, Z) \tag{3.1.1}$$

where $\cdot$ is the intersection product on $X \times_S Y \times_S Z$, is well-defined for all $W \in c_S(X, Y)$, $W' \in c_S(Y, Z)$; this follows from the fact that $\mathrm{supp}\,(W) \times_S Z \cap X \times_S \mathrm{supp}\,(W')$ is finite over X and each irreducible component of this intersection dominates a component of X. This is called the *composition of correspondences*.

We start with the category $SmCor(S)$. Objects are the same as $\mathbf{Sm}/S$, morphisms are

$$\mathrm{Hom}_{SmCor(S)}(X, Y) := c_S(X, Y)$$

with composition law given by the formula (3.1.1). Sending a morphism $f : X \to Y$ in $\mathbf{Sm}/S$ to the graph of f, $\Gamma_f \subset X \times_S Y$, defines an embedding $i_S : \mathbf{Sm}/S \to SmCor(S)$. Note that $SmCor(S)$ is an additive category, with direct sum induced by disjoint union.

Define the abelian category of *presheaves with transfer* on $\mathbf{Sm}/S$, $\mathrm{PST}(S)$, as the category of additive presheaves of abelian groups on $SmCor(S)$. We have the representable presheaves $\mathbb{Z}^{tr}_S(Z)$ for $Z \in \mathbf{Sm}/S$ defined by $\mathbb{Z}^{tr}_S(Z)(X) := c_S(X, Z)$ and pull-back maps given by the composition of correspondences. The full subcategory $Sh^{tr}_{\mathrm{Nis}}(S)$ of $\mathrm{PST}(S)$ has objects the presheaves P such that the restriction $P \circ i_S$ of P to a presheaf on $\mathbf{Sm}/S$ is a sheaf for the Nisnevich topology. For instance, the presheaves $\mathbb{Z}^{tr}_S(Z)$ are in $Sh^{tr}_{\mathrm{Nis}}(S)$.

Both $\mathrm{PST}(S)$ and $Sh^{tr}_{\mathrm{Nis}}(S)$ are Grothendieck abelian categories, with set of generators given by the objects $\mathbb{Z}^{tr}(X)$, $X \in \mathbf{Sm}/S$.

For an additive category A, we let $C(A)$ denote the category of unbounded complexes over A. One gives the category $C(Sh^{tr}_{\mathrm{Nis}}(S))$ the model structure of [9, example 1.6, theorem 1.7], that is, cofibrations are generated by maps of the form

$$\sigma_X[n] : \mathbb{Z}^{tr}(X)[n] \to D_X[n];\ X \in \mathbf{Sm}/S, n \in \mathbb{Z},$$

where D_X is the cone on the identity map $\mathbb{Z}^{tr}(X) \to \mathbb{Z}^{tr}(X)$, and $\sigma_X : \mathbb{Z}^{tr}(X) \to D_X$ is the canonical map. Here "generated" means that the class of cofibrations is the smallest collection of morphisms in $C(Sh^{tr}_{\mathrm{Nis}}(S))$ containing the maps $\sigma_X[n]$ and closed under push-outs, transfinite compositions and retracts. The weak equivalences are the quasi-isomophisms (for the Nisnevich topology) and the fibrations are as usual the morphisms having the right lifting property with respect to acyclic cofibrations. We denote this model structure by

$C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathrm{Nis}}$. In particular, the homotopy category of $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathrm{Nis}}$ is equivalent to the (unbounded) derived category $D(Sh^{tr}_{\mathrm{Nis}}(S))$.

The operation
$$\mathbb{Z}^{tr}_S(X)\otimes^{tr}_S\mathbb{Z}^{tr}_S(X'):=\mathbb{Z}^{tr}_S(X\times_S X')$$
extends to a tensor structure $\otimes^{tr}_S$ making $\mathrm{PST}(S)$ a tensor category: one forms the *canonical left resolution* $\mathcal{L}(\mathcal{F})$ of a presheaf $\mathcal{F}$ by taking the canonical surjection

$$\mathcal{L}_0(\mathcal{F}):=\bigoplus_{X\in\mathbf{Sm}/S,s\in\mathcal{F}(X)}\mathbb{Z}^{tr}_S(X)\xrightarrow{\phi_0}\mathcal{F}$$

setting $\mathcal{F}_1:=\ker\phi_0$ and iterating, giving the canonical resolution of $\mathcal{F}$ in terms of representable presheaves

$$\mathcal{L}(\mathcal{F})\to\mathcal{F}:=\ldots\to\mathcal{L}_1(\mathcal{F})\to\mathcal{L}_0(\mathcal{F})\to\mathcal{F}\to 0. \tag{3.1.2}$$

One then defines
$$\mathcal{F}\otimes^{tr}_S\mathcal{G}:=H_0(\mathcal{L}(\mathcal{F})\otimes^{tr}_S\mathcal{L}(\mathcal{G}))$$
noting that $\mathcal{L}(\mathcal{F})\otimes^{tr}_S\mathcal{L}(\mathcal{G})$ is defined since both complexes are degreewise direct sums of representable presheaves. One makes $Sh^{tr}_{\mathrm{Nis}}(S)$ a tensor category by taking the sheaf associated to the presheaf tensor product; we also denote this tensor product by $\otimes^{tr}_S$, using the context to distinguish the presheaf and sheaf tensor products.

Note that the objects $\mathbb{Z}^{tr}_S(X)$ of $Sh^{tr}_{\mathrm{Nis}}(S)$ are *weakly flat* in the sense of [9, §2.1] and that $\{\mathbb{Z}^{tr}_S(X),X\in\mathbf{Sm}/S\}$ is a set of weakly flat generators of $Sh^{tr}_{\mathrm{Nis}}(S)$, closed under $\otimes^{tr}_S$. Thus, by [9, proposition 2.3, proposition 2.8], the usual extension of $\otimes^{tr}_S$ to a tensor product on $C(Sh^{tr}_{\mathrm{Nis}}(S))$ makes $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathrm{Nis}}$ a closed symmetric monoidal model category, and $\otimes^{tr}_S$ defines a left-derived tensor product

$$\otimes^L_S:D(Sh^{tr}_{\mathrm{Nis}}(S))\times D(Sh^{tr}_{\mathrm{Nis}}(S))\to D(Sh^{tr}_{\mathrm{Nis}}(S)),$$

which makes $D(Sh^{tr}_{\mathrm{Nis}}(S))$ a triangulated tensor category.

Definition 3.1.1 ([9, example 3.15]) $DM^{\mathrm{eff}}(S)$ is the localization of the triangulated category $D(Sh^{tr}_{\mathrm{Nis}}(S))$ with respect to the localizing category generated by the complexes $\mathbb{Z}^{tr}_S(X\times\mathbb{A}^1)\to\mathbb{Z}^{tr}_S(X)$, $X\in\mathbf{Sm}/S$. Denote by $m^{\mathrm{eff}}_S(X)$ the image of $\mathbb{Z}^{tr}_S(X)$ in $DM^{\mathrm{eff}}(S)$.

Remark 3.1.2 The following facts are direct consequences of [9, proposition 3.5]:
1. $DM^{\mathrm{eff}}(S)$ is a triangulated tensor category with tensor product $\otimes_S$ induced from the tensor product $\otimes^L_S$ via the localization map

$$Q_S:D(Sh^{tr}_{\mathrm{Nis}}(S))\to DM^{\mathrm{eff}}(S),$$

and satisfying $m^{\mathrm{eff}}_S(X)\otimes_S m^{\mathrm{eff}}_S(Y)=m^{\mathrm{eff}}_S(X\times_S Y)$.

2. $C(Sh^{tr}_{\mathrm{Nis}}(S))$ has a model category structure $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$, defined as the Bousfield localization of $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathrm{Nis}}$ with respect to the set of complexes $\{\mathbb{Z}^{tr}_S(X\times\mathbb{A}^1)\to\mathbb{Z}^{tr}_S(X),X\in\mathbf{Sm}/S\}$, and the homotopy category of $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$ is equivalent to $DM^{\mathrm{eff}}(S)$.

3. Let $DM^{\mathrm{eff}}_\infty(S) \subset D(Sh^{tr}_{\mathrm{Nis}}(S))$ be the full subcategory consisting of complexes C which are $\mathbb{A}^1$-homotopy invariant, that is, the map

$$p^* : \mathbb{H}^n(X_{\mathrm{Nis}}, C) \to \mathbb{H}^n(X \times \mathbb{A}^1_{\mathrm{Nis}}, C)$$

is an isomorphism for all X and n. Then $DM^{\mathrm{eff}}_\infty(S)$ is a triangulated subcategory of $D(Sh^{tr}_{\mathrm{Nis}}(S))$, and the inclusion $DM^{\mathrm{eff}}_\infty(S) \to D(Sh^{tr}_{\mathrm{Nis}}(S))$ admits a left adjoint

$$L_{\mathbb{A}^1} : D(Sh^{tr}_{\mathrm{Nis}}(S)) \to DM^{\mathrm{eff}}_\infty(S),$$

which descends via the localization functor $D(Sh^{tr}_{\mathrm{Nis}}(S)) \to DM^{\mathrm{eff}}(S)$ to define an equivalence

$$L_{\mathbb{A}^1} : DM^{\mathrm{eff}}(S) \to DM^{\mathrm{eff}}_\infty(S)$$

of triangulated categories.

3.2 T^{tr}-spectra and the category of motives

We now recall the construction of the category $DM(S)$. This is given by "inverting" tensor product with the Lefschetz motive, done via the category of symmetric T^{tr}-spectra

Remark 3.2.1 Hovey [19] has formed a general machine for the construction of model structures on categories of spectra over a model category $\mathcal{M}$ with respect to an endofunctor T. Some of his results require the technical assumption that $\mathcal{M}$ be *weakly finitely generated*. This property of $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$ does not appear to be directly addressed in either [9] or [10], however, the arguments of [13, lemma 2.15, corollary 2.16] do show this. The main point is that the Brown-Gersten property and $\mathbb{A}^1$-homotopy invariance for a schemewise fibrant complex in $C(Sh^{tr}_{\mathrm{Nis}}(S))$ is implied by having the RLP with respect to the *finite* complexes corresponding to an elementary Nisnevich square, or a projection $\mathbb{A}^1 \times X \to X$. We will use the weak finite generation of $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$ without further mention in the sequel.

Definition 3.2.2 Let T^{tr} be the presheaf with transfers

$$T^{tr} := \mathrm{coker}(\mathbb{Z}^{tr}_S(S) \xrightarrow{i_{\infty *}} \mathbb{Z}^{tr}_S(\mathbb{P}^1))$$

and let $\mathbb{Z}_S(1)$ be the image in $DM^{\mathrm{eff}}(S)$ of $T^{tr}[-2]$. We often write $\mathbb{Z}^{tr}_S(n)$ for $(T^{tr}[-2])^{\otimes^{tr}_S n}$ and $\mathbb{Z}_S(n)$ for the image of $\mathbb{Z}^{tr}_S(n)$ in $DM^{\mathrm{eff}}(S)$.

Note that, as a summand of the cofibrant object $Z^{tr}_S(\mathbb{P}^1))$, T^{tr} is cofibrant.

Let $\mathbf{Spt}_{T^{tr}}(S)$ be the category of T^{tr} spectra in $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$ with the stable model structure: Objects are sequence $E := (E_0, E_1, \ldots)$, $E_n \in C(Sh^{tr}_{\mathrm{Nis}}(S))$, with bonding maps

$$\epsilon_n : E_n \otimes^{tr}_S T^{tr} \to E_{n+1}.$$

Morphisms are given by sequences of maps in $C(Sh^{tr}_{\mathrm{Nis}}(S))$ which strictly commute with the respective bonding maps. We will describe the model structure below.

We let $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)$ be the category of symmetric T^{tr} spectra in $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$ with the stable model structure. Objects are sequences $E := (E_0, E_1, \ldots)$, $E_n \in C(Sh^{tr}_{\mathrm{Nis}}(S))$, with E_n endowed with an action of the symmetric group $\mathfrak{S}_n$, together with bonding maps

$$\epsilon_n : E_n \otimes^{tr}_S T^{tr} \to E_{n+1}.$$

One requires in addition that, for all $n \geq 0$, $m \geq 1$, the iterated bonding map

$$E_n \otimes_S^{tr} (T^{tr})^{\otimes m} \xrightarrow{\epsilon_n \otimes \mathrm{id}_{(T^{tr})^{\otimes m}}} E_n \otimes_S^{tr} (T^{tr})^{\otimes m-1} \to \ldots \to E_{n+m-1} \otimes T^{tr} \xrightarrow{\epsilon_{n+m-1}} E_{n+m}$$

is $\mathfrak{S}_n \times \mathfrak{S}_m$ equivariant, with respect to the standard inclusion $\mathfrak{S}_n \times \mathfrak{S}_m \subset \mathfrak{S}_{n+m}$. Morphisms are given by sequences of maps $f = \{f_n\}$ in $C(Sh^{tr}_{\mathrm{Nis}}(S))$ which strictly commute with the respective bonding maps, and with f_n being $\mathfrak{S}_n$-equivariant for each n.

The model structure on the category of T^{tr}-spectra is defined by following the construction of Hovey [19]. For an object $A \in C(Sh^{tr}_{\mathrm{Nis}}(S))$, and integer $i \geq 0$, we have the object $A\{-i\}$ of $\mathbf{Spt}_{T^{tr}}(S)$, with $A\{-i\}_{i+n} = A \otimes (T^{tr})^{\otimes n}$, and $A\{-i\}_n = 0$ for $n < i$; sending A to $A\{-i\}$ defines a functor $(-)\{-i\}$. The *projective* model structure on $\mathbf{Spt}_{T^{tr}}(S)$ has generating cofibrations the maps of the form $f\{-i\}$ with f a cofibration in $C(Sh^{tr}_{\mathrm{Nis}}(S))$, and with weak equivalences and fibrations being those maps $f = \{f_n\}$ with each f_n a weak equivalence, resp. fibration. We let $\mathbf{Spt}_{T^{tr}}(S)_{proj}$ denote this model category.

Next, one defines the notion of a *T^{tr}-Ω spectrum*, this being a T^{tr}-spectrum $E = (E_0, E_1, \ldots)$ such that each E_n is fibrant in $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$, and such that the map $E_n \to \mathcal{H}om(T^{tr}, E_{n+1})$ adjoint to ϵ_n is a weak equivalence in $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$. A *stable weak equivalence* $f : A \to B$ is a map in $\mathbf{Spt}_{T^{tr}}(S)$ such that the induced map

$$f^* : \mathrm{Hom}_{\mathcal{H}(\mathbf{Spt}_{T^{tr}}(S)_{proj})}(B, E) \to \mathrm{Hom}_{\mathcal{H}(\mathbf{Spt}_{T^{tr}}(S)_{proj})}(A, E)$$

is an isomorphism for all T^{tr}-Ω spectra E. The model category $\mathbf{Spt}_{T^{tr}}(S)_s$ is the Bousfield localization of the model category $\mathbf{Spt}_{T^{tr}}(S)_{proj}$ with respect to stable weak equivalences.

In the symmetric setting, one does exactly the same, except that we use a symmetric version $A\{-i\}^{\mathfrak{S}}$ of $A\{-i\}$. Explicitly,

$$A\{-i\}^{\mathfrak{S}}_{n+i} := \mathfrak{S}_{n+i} \times_{\mathfrak{S}_n} A \otimes^{tr} (T^{tr})^{\otimes n},$$

with the evident bonding maps. This gives us the model category $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)_s$ with the stable model structure.

Definition 3.2.3 The "big" category of triangulated motives over S, $DM(S)$, is the homotopy category of $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)_s$. We write $DM(S)'$ for the homotopy category of $\mathbf{Spt}_{T^{tr}}(S)_s$.

Remarks 3.2.4

1. The homotopy categories of $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)_s$ and $\mathbf{Spt}_{T^{tr}}(S)_s$ are triangulated categories [10, proposition 3.4, definition 3.8, §4.12, §6.9]. In addition, one can define additive categories of T^{tr}-spectra and symmetric T^{tr}-spectra $\mathbf{Spt}(Sh^{tr}_{\mathrm{Nis}}(S))$ and $\mathbf{Spt}^{\mathfrak{S}}(Sh^{tr}_{\mathrm{Nis}}(S))$, so that

$$C(\mathbf{Spt}(Sh^{tr}_{\mathrm{Nis}}(S))) \cong \mathbf{Spt}_{T^{tr}}(S);\ C(\mathbf{Spt}^{\mathfrak{S}}(Sh^{tr}_{\mathrm{Nis}}(S))) \cong \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S),$$

giving $\mathbf{Spt}_{T^{tr}}(S)$ and $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)$ a dg structure.

2. Sending $A \in C(Sh^{tr}_{\mathrm{Nis}}(S))$ to the sequence $(A, A \otimes^{tr} T^{tr}, \ldots, A \otimes^{tr} (T^{tr})^{\otimes n}, \ldots)$ defines functors

$$\Sigma^\infty_T : C(Sh^{tr}_{\mathrm{Nis}}(S)) \to \mathbf{Spt}_{T^{tr}}(S)$$
$$\Sigma^\infty_T : C(Sh^{tr}_{\mathrm{Nis}}(S)) \to \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)$$

(the symmetric version uses the permutation action on $(T^{tr})^{\otimes n}$ and the trivial action on A), left-adjoint to the projection $(E_0, \ldots) \mapsto E_0$. These induce an adjoint pair of exact functors on the homotopy categories

$$\Sigma_t^\infty : DM^{\text{eff}}(S) \rightleftarrows DM(S)' : \Omega_t$$
$$\Sigma_t^\infty : DM^{\text{eff}}(S) \rightleftarrows DM(S) : \Omega_t$$

(see [10, §4.12]).
3. Forgetting the action of the symmetric groups defines a functor $u : \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S) \to \mathbf{Spt}_{T^{tr}}(S)$. By [10, theorem 6.10], this induces an equivalence of triangulated categories

$$u : DM(S)' \to DM(S).$$

We let

$$m_S : \mathbf{Sm}/S \to DM(S)$$

be the composition

$$\mathbf{Sm}/S \xrightarrow{m_S^{\text{eff}}} DM^{\text{eff}}(S) \xrightarrow{\Sigma_t^\infty} DM(S).$$

We will use the following fundamental result from [10].

Theorem 3.2.5 ([10, section 10.4]) *Suppose that S is in $\mathbf{Sm}/k$ for a field k, take X in $\mathbf{Sm}/S$, and let $m_k(X)$, $m_S(X)$ denote the motives of X in $DM(k)$, $DM(S)$, respectively. Then there is a natural isomorphism*

$$\text{Hom}_{DM(S)}(m_S(X), \mathbb{Z}_S(n)[m]) \cong \text{Hom}_{DM(k)}(m_k(X), \mathbb{Z}_k(n)[m])$$

3.3 Tensor product in $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)$

Let $\mathcal{C} = C(Sh^{tr}_{\text{Nis}}(S))$, and let $\mathcal{C}^{\mathfrak{S}}$ be the category of sequences $E = (E_0, E_1, \ldots)$, with E_n an object of $\mathcal{C}$ endowed with an $\mathcal{F}_n$-action; morphisms are sequences $f = \{f_n\}$ of morphisms in $\mathcal{C}$, with f_n $\mathfrak{S}_n$-equivariant.

For $E = (E_0, E_1, \ldots)$, $F = (F_0, F_1, \ldots)$ in $\mathcal{C}^{\mathfrak{S}}$, one defines

$$(\widetilde{E \otimes_S^{tr} F})_n := \oplus_{p+q=n, \alpha:\{1,\ldots,p\}\amalg\{1,\ldots,q\} \xrightarrow{\sim} \{1,\ldots,n\}} E_p \otimes_S^{tr} F_q,$$

where α runs over all bijections of sets. Using the evident operation of $\mathfrak{S}_n$ on the set of bijections $\{1, \ldots, p\} \amalg \{1, \ldots, q\} \xrightarrow{\sim} \{1, \ldots, n\}$, the $\mathfrak{S}_p \times \mathfrak{S}_q$ action on $E_p \otimes^{tr} F_q$ induces an $\mathfrak{S}_n$-action on $(\widetilde{E \otimes_S^{tr} F})_n$, giving us the object $\widetilde{E \otimes_S^{tr} F}$ of $\mathcal{C}^{\mathfrak{S}}$. This defines a symmetric monoidal structure on $\mathcal{C}^{\mathfrak{S}}$.

Let $\text{Sym}(T^{tr})$ be the sequence $n \mapsto (T^{tr})^{\otimes^{tr} n}$. Then $\text{Sym}(T^{tr})$ is a commutative monoid object in $\mathcal{C}^{\mathfrak{S}}$, and $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S))$ is just the category of (right) $\text{Sym}(T^{tr})$-modules in $\mathcal{C}^{\mathfrak{S}}$. Thus, (see [21, lemmas 2.2.2 and 2.2.8]) the symmetric monoidal structure on $\mathcal{C}^{\mathfrak{S}}$ induces a canonical symmetric monoidal structure on $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)$, which we denote by $\otimes_S^{tr}$.

By [19, theorem 8.11], the symmetric monoidal operation $\otimes_S^{tr}$ defines a tensor operation $\otimes_S$ on the homotopy category $DM(S)$, making $DM(S)$ a triangulated tensor category. In addition, the suspension spectra $\Sigma_t^\infty(\mathbb{Z}_S^{tr}(X))$ are flat (in the sense of [9, proposition 6.35]), and we have

Proposition 3.3.1 ([19, theorem 8.10]) *The functor*

$$-\otimes T^{tr} : DM(S) \to DM(S)$$

is an equivalence.

3.4 Motives with $\mathbb{Q}$-coefficients

We replace the category $Sh^{tr}_{\mathrm{Nis}}(X)$ with the category of sheaves of $\mathbb{Q}$-vector spaces $Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}}$, giving us the derived category $D(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$ and the $\mathbb{A}^1$-localization $DM^{\mathrm{eff}}(S)_{\mathbb{Q}}$. This latter category is the homotopy category of the model category $C(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})_{\mathbb{A}^1}$, defined exactly as $C(Sh^{tr}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$.

We have the evident $\mathbb{Q}$-linearization functors, e.g., from $Sh^{tr}_{\mathrm{Nis}}(S)$ to $Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}}$, which we denote as $M \mapsto M_{\mathbb{Q}}$, and we have isomorphisms

$$\mathrm{Hom}_{?}(M, N) \otimes \mathbb{Q} \cong \mathrm{Hom}_{?_{\mathbb{Q}}}(M_{\mathbb{Q}}, N_{\mathbb{Q}}).$$

We write $T^{tr}_{\mathbb{Q}}$ and $\mathbb{Q}^{tr}_S(n)$ for the image of T^{tr} and $\mathbb{Z}^{tr}_S(n)$ in $C(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$, and write $\mathbb{Q}_S(n)$ for the image of $\mathbb{Q}^{tr}_S(n)$ in $DM^{\mathrm{eff}}(S)_{\mathbb{Q}}$.

We have the model categories of $T^{tr}_{\mathbb{Q}}$-spectra and $T^{tr}_{\mathbb{Q}}$-symmetric spectra, $\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S)$ and $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S)$, with homotopy categories $DM(S)'_{\mathbb{Q}}$ and $DM(S)_{\mathbb{Q}}$, respectively.

One can also easily compare spectra and symmetric spectra: send $E = (E_0, E_1, \ldots)$ in $\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S)$ to the same sequence $E = (E_0, E_1, \ldots)$ with the same bonding maps; we denote this functor as

$$\iota : \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S) \to \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S).$$

The homtopy inverse sends a sequence $E = (E_0, E_1, \ldots)$ in $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S)$ to the sequence of $\mathfrak{S}_*$-invariants $E^{\mathfrak{S}_*} := (E_0, E_1, E_2^{\mathfrak{S}_2}, \ldots)$,

$$?^{\mathfrak{S}_*} : \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S) \to \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S).$$

Since $E^{\mathfrak{S}_*}$ is a summand of E in $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S)$, these operations give well-defined functors on the homotopy categories.

Proposition 3.4.1 *The functors*

$$\iota : DM(S)'_{\mathbb{Q}} \to DM(S)_{\mathbb{Q}}$$
$$?^{\mathfrak{S}_*} : DM(S)_{\mathbb{Q}} \to DM(S)'_{\mathbb{Q}}$$

are inverse equivalences.

Proof We use throughout the motivic model structures, without putting this explicitly into the notation. We recall the proof of the equivalence of $DM(S)'$ with $DM(S)$ as given by [19, theorems 10.1, 10.3]. This is done by comparing the model categories $\mathbf{Spt}_{T^{tr}}(\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S))$ and $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(\mathbf{Spt}_{T^{tr}}(S))$. Indeed, $\otimes T^{tr}$ is a equivalence on the respect homotopy categories, by [19, theorem 8.10] for $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)$ and by [19, theorem 10.3] for $\mathbf{Spt}_{T^{tr}}(S)$ (this is where one uses the fact that the cyclic permutation of $T^{tr} \otimes T^{tr} \otimes T^{tr}$ is homotopic to the identity). Thus, by [19, theorems 5.1 and 9.1], the infinite suspension functors

$$\Sigma^\infty_{T^{tr}} : \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S) \to \mathbf{Spt}_{T^{tr}}(\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S))$$
$$\Sigma^{\mathfrak{S}\infty}_{T^{tr}} : \mathbf{Spt}_{T^{tr}}(S) \to \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(\mathbf{Spt}_{T^{tr}}(S))$$

also induce equivalences on the homotopy categories. The equivalence $DM(S)' \sim DM(S)$ is then induced by the isomorphism τ

$$\mathbf{Spt}_{T^{tr}}(\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)) \cong \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(\mathbf{Spt}_{T^{tr}}(S))$$

defined by "exchanging indices": an object Y of the left-hand category is a doubly indexed collection of objects of $C(Sh^{tr}_{\mathrm{Nis}}(S))$, $Y = \{Y_{m,n}\}$, where $\mathfrak{S}_n$ acts on $Y_{m,n}$, the two bonding maps $Y_{m,n} \otimes T^{tr} \to Y_{m+1,n}$ and $Y_{m,n} \otimes T^{tr} \to Y_{m,n+1}$ are $\mathfrak{S}_n$-equiviariant, and the ℓ-fold iterated bonding map in the second variable is $\mathfrak{S}_n \times \mathfrak{S}_\ell$ equivariant. $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(\mathbf{Spt}_{T^{tr}}(S))$ has a similar description, with the symmetric variable being the first one, so sending $Y = \{Y_{m,n}\}$ to $Y' = \{Y'_{m,n}\}$, with $Y'_{m,n} := Y_{n,m}$ defines the isomorphism.

We apply our functor $?^{\mathfrak{S}_*}$ to $\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S))$ and $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S))$, giving the commutative diagram

$$\begin{array}{ccc} \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S)) & \xrightarrow{\quad\tau\quad} & \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S)) \\ {\scriptstyle \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(?^{\mathfrak{S}_*})} \searrow & & \swarrow {\scriptstyle ?^{\mathfrak{S}_*}} \\ & \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S)) & \end{array} \tag{3.4.1}$$

The composition

$$\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S) \xrightarrow{\Sigma^\infty_{T^{tr}_{\mathbb{Q}}}} \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S)) \xrightarrow{?^{\mathfrak{S}_*}} \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S))$$

sends E to the $T^{tr}_{\mathbb{Q}}$-spectrum

$$(E, E \otimes T^{tr}_{\mathbb{Q}}, \ldots, E \otimes [T^{tr\otimes n}_{\mathbb{Q}}]^{\mathfrak{S}_n}, \ldots).$$

But the inclusion of the summand $[T^{tr\otimes n}_{\mathbb{Q}}]^{\mathfrak{S}_n}$ in $T^{tr\otimes n}_{\mathbb{Q}}$ induces an isomorphism in $DM^{\mathrm{eff}}(S)$ (this follows from lemma 4.2.1 below), hence the evident map

$$(\Sigma^\infty_{T^{tr}_{\mathbb{Q}}} E)^{\mathfrak{S}_*} \to \Sigma^\infty_{T^{tr}_{\mathbb{Q}}} E$$

is a weak equivalence in $\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S))$. As

$$\Sigma_{T^{tr}_{\mathbb{Q}}} : \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S) \to \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S))$$

induces an equivalence of homotopy categories [19, theorems 5.1 and 10.3], we see that each map in the diagram (3.4.1) induces an equivalence between the respective homotopy categories. Combining this with the commutative diagram

$$\begin{array}{ccc} \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S) & \xrightarrow{?^{\mathfrak{S}_*}} & \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S) \\ {\scriptstyle \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(?^{\mathfrak{S}_*})\circ\Sigma^\infty_{T^{tr}_{\mathbb{Q}}}} \downarrow & & \downarrow {\scriptstyle \Sigma^\infty_{T^{tr}_{\mathbb{Q}}}} \\ \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S)) & = & \mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(\mathbf{Spt}_{T^{tr}_{\mathbb{Q}}}(S)) \end{array}$$

finishes the proof.

We will use proposition 3.4.1 to simplify the computation of tensor products in $DM(S)$.

We have the functor $-\otimes\mathbb{Q} : Sh^{tr}_{\mathrm{Nis}}(S) \to Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}}$, with $P\otimes\mathbb{Q}$ the sheaf associated to the presheaf $Y\mapsto P(Y)\otimes_{\mathbb{Z}}\mathbb{Q}$; clearly $-\otimes\mathbb{Q}$ extends to exact tensor functors

$$-\otimes\mathbb{Q} : DM^{\mathrm{eff}}(S)\to DM^{\mathrm{eff}}(S)_{\mathbb{Q}};\ -\otimes\mathbb{Q} : DM(S)\to DM(S)_{\mathbb{Q}}.$$

For $n\in\mathbb{Z}$, we let $\mathbb{Z}_S(n)$ denote the *Tate object* $\Sigma^n_t(m_S(S))[-2n]$, and set $\mathbb{Q}_S(n) := \mathbb{Z}_S(n)\otimes\mathbb{Q}$.

3.5 Geometric motives

Let k be a perfect field. We recall the category of effective geometric motives $DM^{\mathrm{eff}}_{gm}(k)$, from [15, chapter V], and the category of geometric motives $DM_{gm}(k) := DM^{\mathrm{eff}}_{gm}(k)[\otimes\mathbb{Z}(1)^{-1}]$, with $\mathbb{Z}(1)$ the Tate object of $DM^{\mathrm{eff}}_{gm}(k)$, represented by the complex $[\mathbb{P}^1]\to[\mathrm{Spec}\,k]$ with $[\mathbb{P}^1]$ in degree 2. Let

$$\iota : DM^{\mathrm{eff}}_{gm}(k)\to DM_{gm}(k)$$

be the canonical functor. We have the functor

$$M^{\mathrm{eff}}_{gm} : \mathbf{Sm}/k\to DM^{\mathrm{eff}}_{gm}(k)$$

inducing the functor $M_{gm} : \mathbf{Sm}/k\to DM_{gm}(k)$.

This has been extended in [9, example 5.5]

Definition 3.5.1 Let S be a smooth k-scheme. Let $\widetilde{DM}^{\mathrm{eff}}_{gm}(S)$ be the localization of triangulated category $K^b(SmCor(S))$ with respect to the thick subcategory generated by complexes of the form

(a) $[U\cap V]\xrightarrow{(i_{U*},-i_{V*})}[U]\oplus[V]\xrightarrow{j_{U*}+j_{V*}}[U\cup V]$, for U,V open subschemes of some $Y\in\mathbf{Sm}/S$.

(b) $[Y\times\mathbb{A}^1]\xrightarrow{p_*}[Y]$ for $Y\in\mathbf{Sm}/S$.

The maps in (a) are the evident open immersions, and the map p in (b) is the projection. $DM^{\mathrm{eff}}_{gm}(S)$ is by definition the pseudo-abelianization of $\widetilde{DM}^{\mathrm{eff}}_{gm}(S)$.

By [1], $DM^{\mathrm{eff}}_{gm}(S)$ has a canonical structure of a triangulated tensor category, so that the canonical functor $\pi : K^b(SmCor(S))\to DM^{\mathrm{eff}}_{gm}(S)$ is an exact tensor functor.

Cisinski-Déglise [10, definition 10.2] use the same approach to define category of geometric motives over S

$$DM_{gm}(S) := DM^{\mathrm{eff}}_{gm}(S)[-\otimes\mathbb{Z}_S(1)^{-1}],$$

where

$$\mathbb{Z}_S(1) := \mathrm{Cone}([S]\xrightarrow{i_\infty}[\mathbb{P}^1_S])[-2].$$

Let $\iota : DM^{\mathrm{eff}}_{gm}(S)\to DM_{gm}(S)$ be the canonical functor, let

$$M^{\mathrm{eff}}_{gm} : \mathbf{Sm}/S\to DM^{\mathrm{eff}}_{gm}(S)$$

be the functor induced by the graph embedding $\mathbf{Sm}/S\to SmCor(S)$ and let

$$M_{gm} : \mathbf{Sm}/S\to DM_{gm}(S)$$

be the composition $\iota\circ M^{\mathrm{eff}}_{gm}$.

Remark 3.5.2 Sending $Y \in \mathbf{Sm}/S$ to the representable presheaf with transfers $\mathbb{Z}_S^{tr}(Y)$ evidently extends to an exact tensor functor

$$i_S^{\text{eff}} : DM_{gm}^{\text{eff}}(S) \to DM^{\text{eff}}(S).$$

As $- \otimes \mathbb{Z}_S(1)$ is invertible on $DM(S)$ and $i_S^{\text{eff}}(\mathbb{Z}_S(1)) \cong \mathbb{Z}_S(1)$, i_S^{eff} extends canonically to an exact tensor functor

$$i_S : DM_{gm}(S) \to DM(S),$$

giving us the commutative diagram of exact tensor functors

$$\begin{array}{ccc} DM_{gm}^{\text{eff}}(S) & \xrightarrow{i_S^{\text{eff}}} & DM^{\text{eff}}(S) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \Sigma^\infty_{T^{tr}}} \\ DM_{gm}(S) & \xrightarrow[i_S]{} & DM(S) \end{array}$$

Cisinsk-Déglise show that the horizontal maps in this diagram are fully faithful embeddings, extending Voevodsky's embedding theorem [15, chapter V, theorem 3.2.6].

Theorem 3.5.3 ([10, §10.2]) *The functors*

$$i_S^{\text{eff}} : DM_{gm}^{\text{eff}}(S) \to DM^{\text{eff}}(S).$$

and

$$i_S : DM_{gm}(S) \to DM(S)$$

are full embeddings.

Remark 3.5.4 Voevodsky ([15, chapter V, theorem 3.4.1] and [38]) has also shown that the canonical functor

$$\iota : DM_{gm}^{\text{eff}}(k) \to DM_{gm}(k)$$

is a full embedding. The analog of this result for arbitrary $S \in \mathbf{Sm}/k$ appears to be unknown at present, however, a partial result follows from theorem 3.2.5.

3.6 Tate motives

We write $\mathbb{Z}_S(n)$ for $\mathbb{Z}_S(1)^{\otimes n}$ in $DM_{gm}^{\text{eff}}(S)$ or $DM_{gm}(S)$, and $\mathbb{Q}_S(n)$ for the image of $\mathbb{Z}_S(n)$ in the $\mathbb{Q}$-linearizations $DM_{gm}^{\text{eff}}(S)_{\mathbb{Q}}$ or $DM_{gm}(S)_{\mathbb{Q}}$. In $DM_{gm}(S)$ and $DM_{gm}(S)_{\mathbb{Q}}$, we have the objects $\mathbb{Z}_S(n)$, $\mathbb{Q}_S(n)$ for $n < 0$ as well. We have used the same notations for the corresponding objects in $DM^{\text{eff}}(S)$, $DM(S)$, $DM^{\text{eff}}(S)_{\mathbb{Q}}$ and $DM(S)_{\mathbb{Q}}$, but the context will make the meaning clear.

Definition 3.6.1 The *triangulated category of mixed Tate motives over* S, $\mathrm{DMT}_{gm}(S)$, is the smallest full triangulated subcategory of $DM_{gm}(S)_{\mathbb{Q}}$ containing the objects $\mathbb{Q}_S(n)$, $n \in \mathbb{Z}$, and closed under isomorphism in $DM_{gm}(S)_{\mathbb{Q}}$. Similarly, let $\mathrm{DMT}(S)$ be the smallest full triangulated subcategory of $DM(S)_{\mathbb{Q}}$ containing the objects $\mathbb{Q}_S(n)$, $n \in \mathbb{Z}$, and closed under isomorphism in $DM(S)_{\mathbb{Q}}$.

Since $\mathbb{Q}_S(n) \otimes \mathbb{Q}_S(m) \cong \mathbb{Q}_S(n+m)$, $\mathrm{DMT}_{gm}(S)$ and $\mathrm{DMT}(S)$ are tensor subcategories of $DM_{gm}(S)_{\mathbb{Q}}$ and $DM(S)_{\mathbb{Q}}$, respectively.

Proposition 3.6.2 *The restriction of the* $\mathbb{Q}$*-extension of*

$$i_S : DM_{gm}(S) \to DM(S)$$

to $\mathrm{DMT}_{gm}(S)$ *defines an equivalence*

$$i_S : \mathrm{DMT}_{gm}(S) \to \mathrm{DMT}(S)$$

of triangulated tensor categories.

Proof This is an immediate consequence of fact that $i_S(\mathbb{Z}_S(n)) \cong \mathbb{Z}_S(n)$, together with theorem 3.5.3.

Just as for the case of motives over a field, the category $\mathrm{DMT}(S)$ admits a canonical weight filtration, and, in case S satisfies the Beilinson-Soulé vanishing conjectures, a t-structure with heart generated by the Tate objects $\mathbb{Q}_S(n)$. In fact, the results of [31] apply directly, so we will content ourselves here with giving the relevant definitions.

Definition 3.6.3 Let $W_n\mathrm{DMT}(S)$ denote the full triangulated subcategory of $\mathrm{DMT}(S)$ generated by the Tate motives $\mathbb{Q}_S(-a)$ with $a \leq n$. Let $W_{[n,m]}\mathrm{DMT}(S)$ be the full triangulated subcategory of $\mathrm{DMT}(S)$ generated by the Tate motives $\mathbb{Q}_S(-a)$ with $n \leq a \leq m$, and let $W^{>n}\mathrm{DMT}(S)$ be the full triangulated subcategory of $\mathrm{DMT}(S)$ generated by the Tate motives $\mathbb{Q}_S(-a)$ with $a > n$.

Lemma 3.6.4 *For* $S \in \mathbf{Sm}/k$ *there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}_S(a), \mathbb{Q}_S(b)[m]) \cong H^m(S, \mathbb{Q}(b-a))$$

Proof Clearly, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}_S(a), \mathbb{Q}_S(b)[m]) &\cong \mathrm{Hom}_{DM(S)}(\mathbb{Z}_S(a), \mathbb{Q}_S(b)[m]) \\ &\cong \mathrm{Hom}_{DM(S)}(\mathbb{Z}_S(0), \mathbb{Q}_S(b-a)[m]) \\ &\cong \mathrm{Hom}_{DM(S)}(m_S(S), \mathbb{Z}_S(b-a)[m]) \otimes \mathbb{Q}\end{aligned}$$

By theorem 3.2.5, we have

$$\mathrm{Hom}_{DM(S)}(m_S(S), \mathbb{Z}_S(b-a)[m]) \cong \mathrm{Hom}_{DM(k)}(m_k(S), \mathbb{Z}_k(b-a)[m])$$

and by theorem 3.5.3 we have

$$\begin{aligned}\mathrm{Hom}_{DM(k)}(m_k(S), \mathbb{Z}_k(b-a)[m]) &\cong \mathrm{Hom}_{DM_{gm}(k)}(M_{gm}(S), \mathbb{Z}(b-a)[m]) \\ &=: H^m(S, \mathbb{Z}(b-a)).\end{aligned}$$

Lemma 3.6.5 $\mathrm{DMT}(S)$ *is a rigid tensor triangulated category.*

Proof The unit $\mathbb{1}$ for the tensor operation is $\mathbb{Q}_S(0)$. It suffices to check that the generators $\mathbb{Q}_S(n)$ of $\mathrm{DMT}(S)$ admit a dual (see e.g. [30, part I, IV.1.2]). Setting $\mathbb{Q}_S(n)^\vee = \mathbb{Q}_S(-n)$, with maps $\delta : \mathbb{1} \to \mathbb{Q}_S(n)^\vee \otimes \mathbb{Q}_S(n)$, $\epsilon : \mathbb{Q}_S(n) \otimes \mathbb{Q}_S(n)^\vee \to \mathbb{1}$ being the canonical isomorphisms shows that $\mathbb{Q}_S(n)$ has a dual.

Theorem 3.6.6 *1.* $(W_n\mathrm{DMT}(S), W^{>n}\mathrm{DMT}(S))$ *is a t-structure on* $\mathrm{DMT}(S)$ *with heart consisting of 0-objects.*

2. Denote the truncation functors for the t-structure $(W_n\mathrm{DMT}(S), W^{>n}\mathrm{DMT}(S))$ *by*

$$W_n : \mathrm{DMT}(S) \to W_n\mathrm{DMT}(S) \subset \mathrm{DMT}(S)$$
$$W^{>n} : \mathrm{DMT}(S) \to W^{>n}\mathrm{DMT}(S) \subset \mathrm{DMT}(S).$$

Then

(a) W_n *and* $W^{>n}$ *are exact*

(b) W_n *is right adjoint to the inclusion* $W_n\mathrm{DMT}(S) \to \mathrm{DMT}(S)$ *and* $W^{>n}$ *is left adjoint to the inclusion* $W^{>n}\mathrm{DMT}(S) \to \mathrm{DMT}(S)$.

(c) For each $n < m$ *there is an exact functor*

$$W_{[n+1,m]} : \mathrm{DMT}(S) \to W_{[n+1,m]}\mathrm{DMT}(S) \subset \mathrm{DMT}(S)$$

and a natural distinguished triangle

$$W_n \to W_m \to W_{[n+1,m]} \to W_n[1].$$

(d) $\mathrm{DMT}(S) = \cup_{n\in\mathbb{Z}} W_n\mathrm{DMT}(S) = \cup_{n\in\mathbb{Z}} W^{>n}\mathrm{DMT}(S)$.

Proof By lemma 3.6.4, we have an isomorphism

$$\begin{aligned}\mathrm{Hom}_{DM(S)_{\mathbb{Q}}}(\mathbb{Q}_S(a), \mathbb{Q}_S(b)[m]) &\cong H^m(S, \mathbb{Q}(b-a)) \\ &= \begin{cases} 0 & \text{for } b < a \\ 0 & \text{for } b = a, m \neq 0 \\ \mathbb{Q}\cdot\mathrm{id} & \text{for } b = a, m = 0. \end{cases}\end{aligned}$$

Thus, [31, lemma 1.2] applies to prove the theorem.

We denote the exact functor $W_{[n,n]} : \mathrm{DMT}(S) \to W_{[n,n]}\mathrm{DMT}(S)$ by gr_n^W and the category $W_{[n,n]}\mathrm{DMT}(S)$ by $\mathrm{gr}_n^W\mathrm{DMT}(S)$.

Remark 3.6.7 Since

$$\mathrm{Hom}_{\mathrm{DMT}(S)}(\mathbb{Q}_S(-n), \mathbb{Q}_S(-n)[m]) = \begin{cases} 0 & \text{for } m \neq 0 \\ \mathbb{Q}\cdot\mathrm{id} & \text{for } m = 0, \end{cases}$$

the category $\mathrm{gr}_n^W\mathrm{DMT}(S)$ is equivalent to $D^b(\mathbb{Q})$. Thus, we can define the $\mathbb{Q}$-vector space $H^n(\mathrm{gr}_n^W M)$ for M in $\mathrm{DMT}(S)$.

Definition 3.6.8 1. We say that S *satisfies the Beilinson-Soulé vanishing conjectures* if $H^m(S, \mathbb{Q}(n)) = 0$ for $m \leq 0$ and $n \neq 0$.

2. Let $\mathrm{DMT}(S)^{\leq 0}$ be the full subcategory of $\mathrm{DMT}(S)$ with objects those M such that $H^m(\mathrm{gr}^W_n M) = 0$ for all $m > 0$ and all $n \in \mathbb{Z}$. Let $\mathrm{DMT}(S)^{\geq 0}$ be the full subcategory of $\mathrm{DMT}(S)$ with objects M such that $H^m(\mathrm{gr}^W_n M) = 0$ for all $m < 0$ and all $n \in \mathbb{Z}$. Let $\mathrm{MT}(S) := \mathrm{DMT}(S)^{\leq 0} \cap \mathrm{DMT}(S)^{\geq 0}$.

Theorem 3.6.9 *Suppose S satisfies the Beilinson-Soulé vanishing conjectures. Then*

1. $(\mathrm{DMT}(S)^{\leq 0}, \mathrm{DMT}(S)^{\geq 0})$ *is a non-degenerate t-structure on* $\mathrm{DMT}(S)$ *with heart* $\mathrm{MT}(S)$ *containing the Tate motives $\mathbb{Q}_S(n)$, $n \in \mathbb{Z}$.*

2. $\mathrm{MT}(S)$ *is equal to the smallest abelian subcategory of* $\mathrm{MT}(S)$ *which contains the $\mathbb{Q}_S(n)$, $n \in \mathbb{Z}$, and which is closed under extensions in* $\mathrm{MT}(S)$*.*

3. The tensor operation in $\mathrm{DMT}(S)$ *restricted to* $\mathrm{MT}(S)$ *makes* $\mathrm{MT}(S)$ *a rigid $\mathbb{Q}$-linear abelian tensor category.*

4. The functor $\oplus_n \mathrm{gr}^W_n : \mathrm{MT}(S) \to \mathrm{Vec}_{\mathbb{Q}}$ is a fiber functor, making $\mathrm{MT}(S)$ *a neutral Tannakian category.*

Proof By lemma 3.6.4, the assumption that S satisfies the Beilinson-Soulé vanishing conjectures implies that

$$\mathrm{Hom}_{\mathrm{DMT}(S)_{\mathbb{Q}}}(\mathbb{Q}_S(a), \mathbb{Q}_S(b)[m]) = \begin{cases} 0 & \text{for } b > a, m \leq 0 \\ 0 & \text{for } b = a, m \neq 0 \end{cases}$$

With this, the result follows from [31, theorem 1.4, proposition 2.1].

4 Cycle algebras

Bloch's cycle complex $z^p(S, *)$ is defined using cycles on $S \times \Delta^n$, where Δ^n is the algebraic n-simplex

$$\Delta^n := \mathrm{Spec}\, k[t_0, \ldots, t_n]/(\sum_i t_i - 1).$$

One can also use cubes instead of simplices to define the various versions of the cycle complexes. The major advantage is that the product structure for the cubical complexes is easier to define and, with $\mathbb{Q}$-coefficients, one can construct cycle complexes which have a strictly commutative and associative product. This approach is used by Hanamura in his construction of a category of mixed motives, as well as in the construction of categories of Tate motives by Bloch [3], Bloch-Kriz [2], Kriz-May [26] and Joshua [25].

We combine the cubical version with the strictly functorial constructions of Friedlander-Suslin-Voevodsky to give a functorial version of the cycle complex. This allows us to extend the representation theorem of Spitzweck to give a description of mixed Tate motives over a smooth base in terms of cell modules over a cycle algebra.

4.1 Cubical complexes

We recall the definition of the cubical version of the Suslin-complex C_*^{Sus} from [15, Chap. V].

Let $(\square^1, \partial\square^1)$ denote the pair $(\mathbb{A}^1, \{0,1\})$, and $(\square^n, \partial\square^n)$ the n-fold product of $(\square^1, \partial\square^1)$. Explicitly, $\square^n = \mathbb{A}^n$, and $\partial\square^n$ is the divisor $\sum_{i=1}^n (x_i = 0) + \sum_{i=1}^n (x_i = 1)$, where $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{A}^n$. A *face* of $\square^n$ is a face of the normal crossing divisor $\partial\square^n$, i.e., a subscheme defined by equations of the form $x_{i_1} = \epsilon_1, \ldots, x_{i_s} = \epsilon_s$, with the ϵ_j in $\{0,1\}$. If a face F has codimension m in $\square^n$, we write $\dim F = n - m$.

For $\epsilon \in \{0,1\}$ and $j \in \{1, \ldots, n\}$ we let $\iota_{j,\epsilon} : \square^{n-1} \to \square^n$ be the closed embedding defined by inserting an ϵ in the jth coordinate. We let $\pi_j : \square^n \to \square^{n-1}$ be the projection which omits the jth factor.

Definition 4.1.1 Let S be a noetherian scheme and let $\mathcal{F}$ be presheaf on $\mathbf{Sm}/S$. Let $C_n^{\mathrm{cb}}(\mathcal{F})$ be the presheaf

$$C_n^{\mathrm{cb}}(\mathcal{F})(S) := \mathcal{F}(S \times \square^n) / \sum_{j=1}^n \pi_j^*(\mathcal{F}(S \times \square^{n-1})),$$

and let $C_*^{\mathrm{cb}}(\mathcal{F})$ be the complex with differential

$$d_n = \sum_{j=1}^n (-1)^{j-1} F(\iota_{j,1}) - \sum_{j=1}^n (-1)^{j-1} F(\iota_{j,0}).$$

We refer to the subgroup $\sum_{j=1}^n \pi_j^*(\mathcal{F}(S \times \square^{n-1}))$ of $\mathcal{F}(S \times \square^n)$ as the *degenerate* elements, written *degn*.

If $\mathcal{F}$ is a Nisnevich sheaf, then $C_*^{\mathrm{cb}}(\mathcal{F})$ is a complex of Nisnevich sheaves, and if $\mathcal{F}$ is a presheaf (resp. Nisnevich sheaf) with transfers, then $C_*^{\mathrm{cb}}(\mathcal{F})$ is a complex of presheaves (resp. Nisnevich sheaves) with transfers. We extend the construction to complexes of sheaves (with transfers) by taking the total complex of the evident double complex.

For a presheaf $\mathcal{F}$ on $\mathbf{Sm}/S$ and $Y \in \mathbf{Sm}/S$, let

$$C_n^{\mathrm{Alt}}(\mathcal{F})(Y) \subset C_n^{\mathrm{cb}}(\mathcal{F})(Y)_{\mathbb{Q}} = \mathcal{F}(Y \times \square^n)_{\mathbb{Q}} / degn$$

be the $\mathbb{Q}$-subspace consisting of the alternating elements of $\mathcal{F}(Y \times \square^n)_{\mathbb{Q}}$ with respect to the action of the symmetric group $\mathfrak{S}_n$ on $\square^n$, i.e., the elements x satisfying

$$(\mathrm{id} \times \sigma)^*(x) = \mathrm{sgn}(\sigma) \cdot x$$

for all $\sigma \in \mathfrak{S}_n$. Here $\mathfrak{S}_n$ acts on $\square^n = \mathbb{A}^n$ by permuting the coordinates. $Y \mapsto C_n^{\mathrm{Alt}}(\mathcal{F})(Y)$ evidently forms a sub-presheaf of $C_n^{\mathrm{cb}}(\mathcal{F})_{\mathbb{Q}}$, which we denote by $C_n^{\mathrm{Alt}}(\mathcal{F})$; in fact the $C_n^{\mathrm{Alt}}(\mathcal{F})$ form a subcomplex $C_*^{\mathrm{Alt}}(\mathcal{F}) \subset C_*^{\mathrm{cb}}(\mathcal{F})_{\mathbb{Q}}$. We extend this to complexes of presheaves by taking the total complex of the evident double complex.

Remark 4.1.2 Following Bloch [3], one can define the alternating complex as a subcomplex of $\mathcal{F}(Y \times \square^*)_{\mathbb{Q}}$, i.e., without taking the quotient by the degenerate cycles. For this, one extends the action of $\mathfrak{S}_n$ on $\square^n$ to an action of the semi-direct product $(\mathbb{Z}/2)^n \ltimes \mathfrak{S}_n$ where

$\mathbb{Z}/2$ acts on $\square^1$ by sending t to $1-t$. The sign representation of $\mathfrak{S}_n$ extends to a sign representation $(\mathbb{Z}/2)^n \ltimes \mathfrak{S}_n \to \{\pm 1\}$, and the subcomplex of $\mathcal{F}(Y \times \square^*)_{\mathbb{Q}}$ which is alternating with respect to these extended sign representations is isomorphic to our complex $C_*^{\text{Alt}}(\mathcal{F})$ via the projection $\mathcal{F}(Y \times \square^*)_{\mathbb{Q}} \to \mathcal{F}(Y \times \square^*)_{\mathbb{Q}}/degn$.

The arguments of e.g. [29, section 2.5] show

Lemma 4.1.3 *Let $\mathcal{F}$ be a complex of presheaves on $\mathbf{Sm}/S$.*

1. *There is a natural isomorphism $C_*^{\text{Sus}}(\mathcal{F}) \cong C_*^{\text{cb}}(\mathcal{F})$ in the derived category of presheaves on $\mathbf{Sm}/S$. If $\mathcal{F}$ is a complex of presheaves with transfer, we have an isomorphism $C_*^{\text{Sus}}(\mathcal{F}) \cong C_*^{\text{cb}}(\mathcal{F})$ in the derived category $D(\text{PST}(S))$.*
2. *The inclusion $C_*^{\text{Alt}}(\mathcal{F})(Y) \subset C_*^{\text{cb}}(\mathcal{F})_{\mathbb{Q}}(Y)$ is a quasi-isomorphism for all $Y \in \mathbf{Sm}/S$.*

Remark 4.1.4 One can define a cubical version of Bloch's cycle complex, following the pattern of definition 4.1.1. That is, define $\underline{z}^q(S,n)^{\text{cb}}$ to be the free abelian group on the codimension q subvarieties $W \subset S \times \square^n$ such that $W \cap S \times F$ has codimension q for every face $F \subset \square^n$, and let $z^q(S,n)^{\text{cb}}$ be the quotient of $\underline{z}^q(S,n)^{\text{cb}}$ by the "degenerate" cycles coming from $\underline{z}^q(S,n-1)^{\text{cb}}$ by pull-back. This gives us the complex $z^q(S,*)^{\text{cb}}$, which is quasi-isomorphic to the simplicial version $z^q(S,*)$ defined in [4].

Taking the subgroups of alternating cycles gives us the subcomplex

$$z^q(S,*)^{\text{Alt}} \subset z^q(S,*)^{\text{cb}}_{\mathbb{Q}},$$

quasi-isomorphic to $z^q(S,*)^{\text{cb}}_{\mathbb{Q}}$.

Call $\mathcal{F} \in C(Sh^{tr}_{\text{Nis}}(S))$ *quasi-fibrant* with respect to some model structure on $C(Sh^{tr}_{\text{Nis}}(S))$ if the map $\mathcal{F} \to \mathcal{F}^{\text{fib}}$ to a fibrant model is quasi-isomorphism of presheaves, that is, for each $Y \in \mathbf{Sm}/S$, the map on sections

$$\mathcal{F}(Y) \to \mathcal{F}^{\text{fib}}(Y)$$

is a quasi-isomorphism of complexes.

Lemma 4.1.5 *Let $\mathcal{F}$ be in $C(Sh^{tr}_{\text{Nis}}(S))$. Suppose that $C_*^{\text{cb}}(\mathcal{F})$ satisfies Nisnevich excision. Then $C_*^{\text{cb}}(\mathcal{F})$ is quasi-fibrant in model category $C(Sh^{tr}_{\text{Nis}}(S))_{\mathbb{A}^1}$.*

Proof Let $C_*^{\text{cb}}(\mathcal{F}) \to C_*^{\text{cb}}(\mathcal{F})^f$ be a fibrant model for $C_*^{\text{cb}}(\mathcal{F})$ in the model category $C(Sh^{tr}_{\text{Nis}}(S))_{\text{Nis}}$. Since $C_*^{\text{cb}}(\mathcal{F})$ satisfies Nisnevich excision, the map of complexes

$$C_*^{\text{cb}}(\mathcal{F})(Y) \to C_*^{\text{cb}}(\mathcal{F})^f(Y)$$

is a quasi-isomorphism for every $Y \in \mathbf{Sm}/S$. Thus, $C_*^{\text{cb}}(\mathcal{F})$ is quasi-fibrant in the model category $C(Sh^{tr}_{\text{Nis}}(S))_{\text{Nis}}$.

In addition, since the homotopy category of $C(Sh^{tr}_{\text{Nis}}(S))_{\text{Nis}}$ is equivalent to $D(Sh^{tr}_{\text{Nis}}(S))$, we have isomorphisms for every $Y \in \mathbf{Sm}/S$ and $n \in \mathbb{Z}$:

$$\begin{aligned} &\text{Hom}_{D(Sh^{tr}_{\text{Nis}}(S))}(\mathbb{Z}^{tr}_S(Y), C_*^{\text{cb}}(\mathcal{F})[n]) \\ &\qquad \cong \text{Hom}_{D(Sh^{tr}_{\text{Nis}}(S))}(\mathbb{Z}^{tr}_S(Y), C_*^{\text{cb}}(\mathcal{F})^f[n]) \\ &\qquad \cong \text{Hom}_{K(Sh^{tr}_{\text{Nis}}(S))}(\mathbb{Z}^{tr}_S(Y), C_*^{\text{cb}}(\mathcal{F})^f[n]) \\ &\qquad \cong H^n(C_*^{\text{cb}}(\mathcal{F})^f(Y)) \\ &\qquad \cong H^n(C_*^{\text{cb}}(\mathcal{F})(Y)). \end{aligned}$$

On the other hand, for every $\mathcal{F}$, the cubical complex construction $C_*^{\rm cb}(\mathcal{F})$ is homotopy invariant as a complex of presheaves, i.e.,

$$C_*^{\rm cb}(\mathcal{F})(Y) \to C_*^{\rm cb}(\mathcal{F})(Y \times \mathbb{A}^1)$$

is a quasi-isomorphism for each $Y \in \mathbf{Sm}/S$. Thus

$$\mathrm{Hom}_{D(Sh^{tr}_{\rm Nis}(S))}(\mathbb{Z}_S^{tr}(Y), C_*^{\rm cb}(\mathcal{F})[n]) \to \mathrm{Hom}_{D(Sh^{tr}_{\rm Nis}(S))}(\mathbb{Z}_S^{tr}(Y \times \mathbb{A}^1), C_*^{\rm cb}(\mathcal{F})[n])$$

is an isomorphism for all $Y \in \mathbf{Sm}/S$, i.e., $C_*^{\rm cb}(\mathcal{F})$ is $\mathbb{A}^1$-local. Thus $C_*^{\rm cb}(\mathcal{F})^f$ is also $\mathbb{A}^1$-local, hence $C_*^{\rm cb}(\mathcal{F})^f$ is quasi-fibrant in $C(Sh^{tr}_{\rm Nis}(S))_{\mathbb{A}^1}$, and thus $C_*^{\rm cb}(\mathcal{F})$ is quasi-fibrant in $C(Sh^{tr}_{\rm Nis}(S))_{\mathbb{A}^1}$ as well.

Example 4.1.6 Let W be a finite type k-scheme. We recall the presheaf with transfers $z_{\rm q.fin}(W)$ (also denoted $z_{equi}(W,0)$ in [15]) on $\mathbf{Sm}/k$. For $Y \in \mathbf{Sm}/k$, $z_{\rm q.fin}(W)(Y)$ is defined to be the free abelian group on integral closed subschemes $Z \subset Y \times_k W$ such that $Z \to Y$ is quasi-finite and dominant over a component of Y. The presheaf $z_{\rm q.fin}(W)(Y)$ is in fact a Nisnevich sheaf.

It follows from [15, chapter V, theorem 4.2.2(4)] and lemma 4.1.3 that one has a natural isomorphism for $Y \in \mathbf{Sm}/k$

$$H_n(C_*^{\rm cb}(z_{\rm q.fin}(W))(Y)) \cong \mathbb{H}^{-n}_{\rm Nis}(Y, C_*^{\rm cb}(z_{\rm q.fin}(W))),$$

and hence $C_*^{\rm cb}(z_{\rm q.fin}(W))$ satisfies Nisnevich excision as a complex of presheaves on $\mathbf{Sm}/S$. Thus $C_*^{\rm cb}(z_{\rm q.fin}(W))$ is quasi-fibrant in $C(Sh^{tr}_{\rm Nis}(S))_{\mathbb{A}^1}$.

Denote by $\mathbb{Z}_S^{tr}(\mathbb{P}^1/\infty)$ the sheaf defined by the exactness of the split exact sequence

$$0 \to \mathbb{Z}_S^{tr} \xrightarrow{i_{\infty *}} \mathbb{Z}_S^{tr}(\mathbb{P}^1) \to \mathbb{Z}_S^{tr}(\mathbb{P}^1/\infty) \to 0$$

Of course, $\mathbb{Z}_S^{tr}(\mathbb{P}^1/\infty) = \mathbb{Z}_S^{tr}(1)[2]$. Similarly, let $\mathbb{Z}_S^{tr}((\mathbb{P}^1/\infty)^r)$ be defined by the exactness of

$$\oplus_{j=1}^r \mathbb{Z}_S^{tr}((\mathbb{P}^1)^{r-1}) \xrightarrow{\sum_j i_{j,\infty *}} \mathbb{Z}^{tr}((\mathbb{P}^1)^r) \to \mathbb{Z}_S^{tr}((\mathbb{P}^1/\infty)^r) \to 0$$

where $i_{j,\infty} : (\mathbb{P}^1)^{r-1} \to (\mathbb{P}^1)^r$ inserts ∞ in the jth spot. Thus $\mathbb{Z}_S^{tr}((\mathbb{P}^1/\infty)^r)$ is isomorphic to $\mathbb{Z}_S^{tr}(r)[2r]$.

Remark 4.1.7 We used the notation T^{tr} for $\mathbb{Z}_S^{tr}(\mathbb{P}^1/\infty)$ in the context of "Tate spectra" (definition 3.2.2); we introduce this new notation to make clear the relation with the sheaf $z_{\rm q.fin}(\mathbb{A}^1)$.

4.2 The cycle cdga in $DM^{\rm eff}(S)$

For $Y \in \mathbf{Sm}/k$, we denote $\mathbb{Z}^{tr}_{\mathrm{Spec}\, k}(Y)$ by $\mathbb{Z}^{tr}(Y)$.

The symmetric group Σ_q acts on $\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q)$ by permuting the coordinates in $(\mathbb{P}^1)^q$. We let $\mathcal{N}(q) \subset C_*^{\rm Alt}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q)$ be the subsheaf of *symmetric* sections with respect to this action. This defines $\mathcal{N}(q)$ as an object of $C(Sh^{tr}_{\rm Nis}(k)_{\mathbb{Q}})$.

Lemma 4.2.1 *The inclusion $\mathcal{N}(q) \subset C_*^{\rm Alt}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q))$ is a quasi-isomorphism of complexes of presheaves on* $\mathbf{Sm}/k$.

Proof Fix $X \in \mathbf{Sm}/k$. We have the sequence of maps

$$C_*(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q))(X) \to C_*(z_{\rm q.fin}(\mathbb{A}^q))(X) \to z^q(X \times \mathbb{A}^q, *),$$

the first map induced by the inclusion $\mathbb{A}^q \subset (\mathbb{P}^1)^q$, the second by the inclusion of the quasi-finite cycles on $X \times \mathbb{A}^q \times \Delta^n$ to the cycles in good position on $X \times \mathbb{A}^q \times \Delta^n$. Both maps are quasi-isomorphisms: for the first, use the localization sequence of [15, chapter IV, corollary 5.12] together with [15, chapter IV, theorem 8.1]; for the second, use the duality theorem [15, chapter IV, theorem 7.4] and Suslin's comparison theorem [15, chapter VI, theorem 3.1].

Passing to the cubical versions, tensoring with $\mathbb{Q}$ and taking the alternating subcomplexes, it follows from lemma 4.1.3 and remark 4.1.4 that we have the sequence of quasi-isomorphisms

$$C_*^{\rm Alt}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q))(X) \to C_*^{\rm Alt}(z_{\rm q.fin}(\mathbb{A}^q))(X) \to z^q(X \times \mathbb{A}^q, *)^{\rm Alt}.$$

As the pull-back by the projection $p : X \times \mathbb{A}^q \to X$

$$z^q(X, *)^{\rm Alt} \to z^q(X \times \mathbb{A}^q, *)^{\rm Alt}$$

is also a quasi-isomorphism by the homotopy property for Bloch's higher Chow groups [4, theorem 2.1], $\mathfrak{S}_q$ acts trivially on $z^q(X \times \mathbb{A}^q, *)^{\rm Alt}$, in $D^-(\mathbf{Ab})$, and thus $\mathfrak{S}_q$ acts trivially on the cohomology of the complex $C_*^{\rm Alt}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q))(X)$. Since $C_*^{\rm Alt}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q))(X)$ is a complex of $\mathbb{Q}$-vector spaces, it follows that $\mathcal{N}(q)(X) \to C_*^{\rm Alt}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q))(X)$ is a quasi-isomorphism, as claimed.

For $X, Y \in \mathbf{Sm}/k$, the external product of correspondences gives the associative external product

$$C_n^{\rm cb}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q)(X) \otimes C_m^{\rm cb}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^p))(Y) \to C_{n+m}^{\rm cb}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^{p+q}))(X \times_k Y).$$

Taking $X = Y$ and pulling back by the diagonal $X \to X \times_k X$ gives the cup product of complexes of sheaves

$$\cup : C_*^{\rm cb}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^p)) \otimes C_*^{\rm cb}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^q)) \to C_*^{\rm cb}(\mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^{p+q})).$$

Taking the alternating projection with respect to the $\square^*$ and symmetric projection with respect to the $\mathbb{A}^*$ yields the associative, commutative product

$$\cdot : \mathcal{N}(p) \otimes \mathcal{N}(q) \to \mathcal{N}(p+q),$$

which makes $\mathcal{N} := \mathbb{Q} \oplus \oplus_{r\geq 1}\mathcal{N}(r)$ into an Adams graded cdga object in $C(Sh_{\rm Nis}(k)_{\mathbb{Q}})$. By abuse of notation, we write $\mathcal{N}(0)$ for the constant presheaf $\mathbb{Q}$.

Definition 4.2.2 For $S \in \mathbf{Sm}/k$, we let $\mathcal{N}_S(q)$ denote the restriction of $\mathcal{N}(q)$ to $SmCor(S)$; similarly define the Adams graded cdga object in $C(Sh_{\rm Nis}(S)_{\mathbb{Q}})$:

$$\mathcal{N}_S = \mathbb{Q} \oplus \oplus_{q\geq 1}\mathcal{N}_S(q).$$

Taking sections of $\mathcal{N}$ on S gives us the Adams graded cdga $\mathcal{N}(S)$. In fact, $\mathcal{N}_S$ is a presheaf of Adams graded cdgas over $\mathcal{N}(S)$, where for $f : X \to S$ in $\mathbf{Sm}/S$, the algebra structure comes from the pull-back map

$$f^* : \mathcal{N}(S) \to \mathcal{N}_S(X) = \mathcal{N}(X).$$

Remark 4.2.3 We will show in §4.3 how to make $\mathcal{N}_S$ into an Adams graded cdga in $C^-(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$, that is, we will extend the product map defined above to an associated graded-commutative product

$$\cdot : \mathcal{N}_S(p) \otimes^{tr}_S \mathcal{N}_S(q) \to \mathcal{N}(p+q).$$

4.3 Products and internal Hom in $Sh^{tr}_{\mathrm{Nis}}(S)$

It is convenient to give a more abstract construction of the product on $\mathcal{N}$, using canonical products on internal Hom complexes.

For $\mathcal{F} \in Sh^{tr}_{\mathrm{Nis}}(S)$ and $X \in \mathbf{Sm}/S$, let $\mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{F})$ denote the sheaf

$$\mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{F})(W) := \mathcal{F}(X \times_S W).$$

For fixed $\mathcal{F}$, sending X to $\mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{F})$ extends to a functor

$$\mathcal{H}om(\mathbb{Z}^{tr}_S(-), \mathcal{F}) : SmCor(S)^{\mathrm{op}} \to Sh^{tr}_{\mathrm{Nis}}(S).$$

Extend the definition of $\mathcal{H}om(-, \mathcal{F})$ to small direct sums by setting

$$\mathcal{H}om(\oplus_\alpha \mathbb{Z}^{tr}_S(S_\alpha), \mathcal{F}) := \prod_\alpha \mathcal{H}om(\mathbb{Z}^{tr}_S(S_\alpha), \mathcal{F}).$$

For $\mathcal{G} \in Sh^{tr}_{\mathrm{Nis}}(S)$, we have the canonical left resolution (3.1.2)

$$\ldots \to \mathcal{L}_1(\mathcal{G}) \to \mathcal{L}_0(\mathcal{G}) \to \mathcal{G} \to 0.$$

One defines $\mathcal{H}om(\mathcal{G}, \mathcal{F})$ as the kernel of

$$\mathcal{H}om(\mathcal{L}_0(\mathcal{G}), \mathcal{F}) \to \mathcal{H}om(\mathcal{L}_1(\mathcal{G}), \mathcal{F}).$$

We extend $\mathcal{H}om(\mathcal{G}, \mathcal{F})$ to $\mathcal{F} \in C(Sh^{tr}_{\mathrm{Nis}}(k))$, $\mathcal{G} \in C^b(Sh^{tr}_{\mathrm{Nis}}(k))$ by taking the extended total complex of the evident double complex, giving the bi-functor

$$\mathcal{H}om(-,-) : C^b(Sh^{tr}_{\mathrm{Nis}}(k)) \times C(Sh^{tr}_{\mathrm{Nis}}(k)) \to C(Sh^{tr}_{\mathrm{Nis}}(k)).$$

Concretely,

$$\mathcal{H}om(\mathcal{G}, \mathcal{F})^n := \oplus_{m\in\mathbb{Z}} \mathcal{H}om(\mathcal{G}^m, \mathcal{F}^{m+n});$$

the sum is finite since G is in $C^b(Sh^{tr}_{\mathrm{Nis}}(k))$.

The isomorphism

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}^{tr}_S(W), \mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{F})) &\cong \mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{F})(W) = \mathcal{F}(X \times_k W) \\ &\cong \mathrm{Hom}(\mathbb{Z}^{tr}_S(X \times_k W), \mathcal{F}) = \mathrm{Hom}(\mathbb{Z}^{tr}_S(X) \otimes \mathbb{Z}^{tr}_S(W), \mathcal{F})\end{aligned}$$

extends to give an adjunction of Hom complexes, for $\mathcal{G} \in C^b(Sh^{tr}_{\mathrm{Nis}}(k))$, $\mathcal{F}, \mathcal{H} \in C(Sh^{tr}_{\mathrm{Nis}}(k))$,

$$\mathrm{Hom}_{C(Sh^{tr}_{\mathrm{Nis}}(k))}(\mathcal{H}, \mathcal{H}om(\mathcal{G}, \mathcal{F})) \cong \mathrm{Hom}_{C(Sh^{tr}_{\mathrm{Nis}}(k))}(\mathcal{G} \otimes^{tr}_S \mathcal{H}, \mathcal{F}).$$

This in turn formally gives an adjunction (for $\mathcal{G}, \mathcal{H} \in C^b(Sh^{tr}_{\mathrm{Nis}}(k))$)

$$\mathcal{H}om(\mathcal{H}, \mathcal{H}om(\mathcal{G}, \mathcal{F})) \cong \mathcal{H}om(\mathcal{G} \otimes^{tr}_S \mathcal{H}, \mathcal{F}).$$

Similarly, we have a canonical map

$$- \otimes \mathrm{id}_{\mathcal{H}} : \mathcal{H}om(\mathcal{G}, \mathcal{F}) \to \mathcal{H}om(\mathcal{G} \otimes^{tr}_S \mathcal{H}, \mathcal{F} \otimes^{tr} \mathcal{H})$$

which, via the adjunction

$$\begin{aligned}
&\mathrm{Hom}(\mathcal{H}om(\mathcal{G}, \mathcal{F}) \otimes^{tr}_S \mathcal{H}, \mathcal{H}om(\mathcal{G}, \mathcal{F} \otimes^{tr}_S \mathcal{H})) \\
&\qquad \cong \mathrm{Hom}(\mathcal{H}om(\mathcal{G}, \mathcal{F}), \mathcal{H}om(\mathcal{H}, \mathcal{H}om(\mathcal{G}, \mathcal{F} \otimes^{tr}_S \mathcal{H}))) \\
&\qquad\qquad \cong \mathrm{Hom}(\mathcal{H}om(\mathcal{G}, \mathcal{F}), \mathcal{H}om(\mathcal{G} \otimes^{tr}_S \mathcal{H}, \mathcal{F} \otimes^{tr}_S \mathcal{H})),
\end{aligned}$$

gives a canonical product map

$$\mathcal{H}om(\mathcal{G}, \mathcal{F}) \otimes^{tr}_S \mathcal{H} \to \mathcal{H}om(\mathcal{G}, \mathcal{F} \otimes^{tr}_S \mathcal{H}).$$

The identity map on $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ gives by adjunction the evaluation map

$$\mathrm{ev}_{\mathcal{A}} : \mathcal{A} \otimes^{tr}_S \mathcal{H}om(\mathcal{A}, \mathcal{B}) \to \mathcal{B}$$

The map

$$\mathrm{ev}_{\mathcal{A}} \otimes^{tr}_S \mathrm{ev}_{\mathcal{C}} : \mathcal{A} \otimes^{tr}_S \mathcal{H}om(\mathcal{A}, \mathcal{B}) \otimes^{tr}_S \mathcal{C} \otimes^{tr}_S \mathcal{H}om(\mathcal{C}, \mathcal{D}) \to \mathcal{B} \otimes^{tr}_S \mathcal{D}$$

gives, via the adjunction

$$\begin{aligned}
&\mathrm{Hom}(\mathcal{A} \otimes^{tr}_S \mathcal{H}om(\mathcal{A}, \mathcal{B}) \otimes^{tr}_S \mathcal{C} \otimes^{tr}_S \mathcal{H}om(\mathcal{C}, \mathcal{D}), \mathcal{B} \otimes^{tr}_S \mathcal{D}) \\
&\qquad \cong \mathrm{Hom}(\mathcal{H}om(\mathcal{A}, \mathcal{B}) \otimes^{tr}_S \mathcal{H}om(\mathcal{C}, \mathcal{D}), \mathcal{H}om(\mathcal{A} \otimes^{tr}_S \mathcal{C}, \mathcal{B} \otimes^{tr}_S \mathcal{D})),
\end{aligned}$$

the external product map

$$\mathcal{H}om(\mathcal{A}, \mathcal{B}) \otimes^{tr}_S \mathcal{H}om(\mathcal{C}, \mathcal{D}) \to \mathcal{H}om(\mathcal{A} \otimes^{tr}_S \mathcal{C}, \mathcal{B} \otimes^{tr}_S \mathcal{D}).$$

Taking $\mathcal{A} = \mathcal{C} = Z^{tr}(X)$ and pulling back by the diagonal

$$\delta_X : \mathbb{Z}^{tr}(X) \to \mathbb{Z}^{tr}(X \times_S X) = \mathbb{Z}^{tr}(X) \otimes^{tr}_S \mathbb{Z}^{tr}(X)$$

gives us the cup product map

$$\cup_{\mathcal{F},\mathcal{G}} : \mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{F}) \otimes^{tr}_S \mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{G}) \to \mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{F} \otimes^{tr}_S \mathcal{G}),$$

defined for all $\mathcal{F}, \mathcal{G} \in C(Sh^{tr}_{\mathrm{Nis}}(S))$.

Given $\mathcal{F}, \mathcal{G} \in C(Sh^{tr}_{\mathrm{Nis}}(S))$, we can restrict $\mathcal{F}$ and $\mathcal{G}$ to complexes of Nisnevich sheaves on $\mathbf{Sm}/S$, where we have the usual tensor product and internal Hom of sheaves, with natural maps (of complexes of sheaves on $\mathbf{Sm}/S$)

$$\mathcal{F} \otimes^{sh} \mathcal{G} \to \mathcal{F} \otimes^{tr}_S \mathcal{G}, \ \mathcal{H}om(\mathcal{G}, \mathcal{F}) \to \mathcal{H}om^{sh}(\mathcal{G}, \mathcal{F}).$$

We note that the respective adjunction isomorphisms are compatible with the restriction maps from $\mathrm{Hom}_{C(Sh^{tr}_{\mathrm{Nis}}(S))}$ to $\mathrm{Hom}_{C(Sh_{\mathrm{Nis}}(S))}$ and the above comparison maps. In particular, the various products described above are compatible with their counterparts for Nisnevich sheaves on $\mathbf{Sm}/S$.

Remark 4.3.1 In general, the functor $\mathcal{H}om(\mathbb{Z}^{tr}_S(X), -)$ does not transform quasi-isomorphisms to quasi-isomorphisms, so to get a well defined functor

$$D(Sh^{tr}_{\mathrm{Nis}}(S)) \to D(Sh^{tr}_{\mathrm{Nis}}(S)),$$

one needs to pass to the right-derived functor $R\mathcal{H}om(\mathbb{Z}^{tr}_S(X), -)$. However, if a complex $\mathcal{F} \in C(Sh^{tr}_{\mathrm{Nis}}(S))$ satisfies Nisnevich excision, then the canonical map

$$\mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{F}) \to R\mathcal{H}om(\mathbb{Z}^{tr}_S(X), \mathcal{F})$$

is an isomorphism in $D(Sh^{tr}_{\mathrm{Nis}}(S))$. In the examples of interest, we will usually apply $\mathcal{H}om(\mathbb{Z}^{tr}_S(X), -)$ to complexes satisfying Nisnevich excision, so we will suppress the use of the derived version $R\mathcal{H}om(\mathbb{Z}^{tr}_S(X), -)$ in order that we may have a concrete model on the level of complexes.

Example 4.3.2 Take $S \in \mathbf{Sm}/k$. For $\mathcal{F} \in C(Sh^{tr}_{\mathrm{Nis}}(S))$ we have

$$C^{\mathrm{cb}}_n(\mathcal{F}) = \mathcal{H}om(\mathbb{Z}^{tr}_S(\square^n), \mathcal{F}).$$

Thus, the product map

$$C^{\mathrm{cb}}_*(\mathcal{F}) \otimes C^{\mathrm{cb}}_*(\mathcal{G}) \to C^{\mathrm{cb}}_*(\mathcal{F} \otimes \mathcal{G}) \to C^{\mathrm{cb}}_*(\mathcal{F} \otimes^{tr}_S \mathcal{G})$$

extends to a product

$$C^{\mathrm{cb}}_*(\mathcal{F}) \otimes^{tr}_S C^{\mathrm{cb}}_*(\mathcal{G}) \to C^{\mathrm{cb}}_*(\mathcal{F} \otimes^{tr}_S \mathcal{G})$$

via the external products

$$\begin{aligned}\mathcal{H}om(\mathbb{Z}^{tr}_S(\square^n), \mathcal{F}) \otimes^{tr}_S \mathcal{H}om(\mathbb{Z}^{tr}_S(\square^m), \mathcal{G}) \\ \to \mathcal{H}om(\mathbb{Z}^{tr}_S(\square^n) \otimes^{tr}_S \mathbb{Z}^{tr}(\square^m), \mathcal{F} \otimes^{tr}_S \mathcal{G}) \\ = \mathcal{H}om(\mathbb{Z}^{tr}_S(\square^{n+m}), \mathcal{F} \otimes^{tr}_S \mathcal{G}).\end{aligned}$$

As $\mathbb{Z}^{tr}_S((\mathbb{P}^1/\infty)^n) \otimes^{tr}_S \mathbb{Z}^{tr}_S((\mathbb{P}^1/\infty)^m) = \mathbb{Z}^{tr}((\mathbb{P}^1/\infty)^{n+m})$, we thus have the associative product

$$C^{\mathrm{cb}}_*(\mathbb{Z}^{tr}_S((\mathbb{P}^1/\infty)^n)) \otimes^{tr}_S C^{\mathrm{cb}}_*(\mathbb{Z}^{tr}_S((\mathbb{P}^1/\infty)^m)) \to C^{\mathrm{cb}}_*(\mathbb{Z}^{tr}_S((\mathbb{P}^1/\infty)^{n+m})).$$

Applying the appropriate alternating and symmetric projections, we have the commutative and associative product

$$\mathcal{N}_S(n) \otimes^{tr}_S \mathcal{N}_S(m) \to \mathcal{N}_S(n+m)$$

making $\mathcal{N}_S$ an Adams graded cdga object of $C(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$.

Passing to the derived category, and composing with the canonical natural transformation

$$- \otimes^L - \to - \otimes^{tr} -,$$

makes $\mathcal{N}_S$ an Adams graded commutative ring object of $D(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$.

Thus, we may apply the localization functor

$$D(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}}) \to DM^{\mathrm{eff}}(S)_{\mathbb{Q}}$$

to the product defined above, giving us the product map

$$\mu_{n,m} : \mathcal{N}_S(n) \otimes \mathcal{N}_S(m) \to \mathcal{N}_S(n+m)$$

in $DM^{\mathrm{eff}}(S)_{\mathbb{Q}}$, making $\mathcal{N}_S$ an Adams graded commutative ring object of $DM^{\mathrm{eff}}(S)_{\mathbb{Q}}$.

Lemma 4.3.3 *In $DM^{\mathrm{eff}}(S)_{\mathbb{Q}}$, we have a canonical isomorphism*

$$\mathbb{Q}_S(r) \to \mathcal{N}_S(r)$$

giving a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}_S(n) \otimes \mathbb{Q}_S(m) & = & \mathbb{Q}_S(n+m) \\ \downarrow & & \downarrow \\ \mathcal{N}_S(n) \otimes \mathcal{N}_S(m) & \xrightarrow[\mu_{n,m}]{} & \mathcal{N}_S(n+m), \end{array}$$

of isomorphisms in $DM^{\mathrm{eff}}(S)_{\mathbb{Q}}$.

Proof By definition $\mathbb{Z}_S^{tr}(1)[2] = \mathbb{Z}_S^{tr}(\mathbb{P}^1/\infty)$. As $\mathbb{Z}_S^{tr}(W) \otimes_S^{tr} \mathbb{Z}_S^{tr}(X) = \mathbb{Z}_S^{tr}(W \times_S X)$, we thus have $\mathbb{Z}_S^{tr}(n)[2n] = \mathbb{Z}_S^{tr}((\mathbb{P}^1/\infty)^n)$.

Take $\mathcal{F} \in Sh_{\mathrm{Nis}}^{tr}(k)$. By [15, the proof of proposition 3.2.3, chap. V] and lemma 4.1.3, the canonical map $\mathcal{F} = C_0^{\mathrm{cb}}(\mathcal{F}) \to C_*^{\mathrm{cb}}(\mathcal{F})$ becomes an isomorphism after applying the localization functor $RC_* : D^-(Sh_{\mathrm{Nis}}^{tr}(k)) \to DM_-^{\mathrm{eff}}(k)$. Thus, the cone of $\mathcal{F} \to C_*^{\mathrm{cb}}(\mathcal{F})$ is in the localizing subcategory of $D^-(Sh_{\mathrm{Nis}}^{tr}(k))$ generated by the complexes $\mathbb{Z}^{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}^{tr}(X)$, $X \in \mathbf{Sm}/k$.

Let $p : S \to \operatorname{Spec} k$ be the structure morphism. Sending $(f : X \to S) \in \mathbf{Sm}/S$ to $pf : X \to \operatorname{Spec} k$ defines the functor

$$p : \mathbf{Sm}/S \to \mathbf{Sm}/k.$$

Noting that $X \times_S Z$ is a closed subscheme of $p(X) \times_k p(Z)$, we see that p extends to a faithful functor

$$p : SmCor(S) \to SmCor(k),$$

inducing the exact restriction functor

$$p^* : Sh_{\mathrm{Nis}}^{tr}(k) \to Sh_{\mathrm{Nis}}^{tr}(S).$$

We note that, for $\mathcal{F} \in Sh_{\mathrm{Nis}}^{tr}(k)$, we have $p^* C_*^{\mathrm{cb}}(\mathcal{F}) = C_*^{\mathrm{cb}}(p^*\mathcal{F})$. Furthermore, we have $p^*(\mathbb{Z}^{tr}(X)) = \mathbb{Z}_S^{tr}(X \times_k S)$. Thus, the fact that $\mathrm{Cone}(\mathcal{F} \to C_*^{\mathrm{cb}}(\mathcal{F}))$ is in the localizing subcategory of $D^-(Sh_{\mathrm{Nis}}^{tr}(k))$ generated by the complexes $\mathbb{Z}^{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}^{tr}(X)$, $X \in \mathbf{Sm}/k$, implies that $\mathrm{Cone}(p^*\mathcal{F} \to C_*^{\mathrm{cb}}(p^*\mathcal{F}))$ is in the localizing category generated by the complexes $\mathbb{Z}_S^{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_S^{tr}(X)$, $X \in \mathbf{Sm}/S$. Thus $p^*\mathcal{F} \to C_*^{\mathrm{cb}}(p^*\mathcal{F})$ becomes an isomorphism after applying the localization functor $D(Sh_{\mathrm{Nis}}^{tr}(S)) \to DM^{\mathrm{eff}}(S)$.

As a particular case, the map

$$Z_S^{tr}(n)[2n] = \mathbb{Z}_S^{tr}((\mathbb{P}^1/\infty)^n) \to C_*^{\mathrm{cb}}((\mathbb{P}^1/\infty)^n)$$

induces an isomorphism

$$\mathbb{Z}_S(n)[2n] \to C_*^{\mathrm{cb}}((\mathbb{P}^1/\infty)^n)$$

in $DM^{\mathrm{eff}}(S)$. By lemma 4.2.1, composing this map with the canonical projection

$$C_*^{\mathrm{cb}}((\mathbb{P}^1/\infty)^n)) \otimes \mathbb{Q} \to \mathcal{N}_S(n)[2n]$$

induces an isomorphism

$$\mathbb{Q}_S(n) \to \mathcal{N}_S(n)$$

in $DM^{\rm eff}(S)_{\mathbb{Q}}$.

It follows directly from the definition of the products $\mu_{n,m}$ that the diagram

$$\begin{array}{ccc} \mathbb{Z}_S^{tr}(n) \otimes_S^{tr} \mathbb{Z}_S^{tr}(m) & = & \mathbb{Z}_S^{tr}(n+m) \\ \downarrow & & \downarrow \\ \mathcal{N}_S(n) \otimes_S^{tr} sN_S(m) & \xrightarrow[\mu_{n,m}]{} & \mathcal{N}_S(n+m) \end{array}$$

commutes in $C(Sh^{tr}_{\rm Nis}(S))$. Applying the localization functor, we see that

$$\begin{array}{ccc} \mathbb{Q}_S(n) \otimes \mathbb{Q}_S(m) & = & \mathbb{Q}_S(n+m) \\ \downarrow & & \downarrow \\ \mathcal{N}_S(n) \otimes \mathcal{N}_S(m) & \xrightarrow[\mu_{n,m}]{} & \mathcal{N}_S(n+m), \end{array}$$

commutes in $DM^{\rm eff}(S)_{\mathbb{Q}}$.

4.4 Equi-dimensional cycles

We consider the case $S = \operatorname{Spec} k$.

Definition 4.4.1 Let X be in $\mathbf{Sm}/k$, $r \geq 0$ an integer. The sheaf $z_{equi}(X,r)$ has sections over $T \in \mathbf{Sm}/k$ the free abelian group on the integral closed subschemes $W \subset T \times_k X$ with $W \to T$ dominant and of pure relative dimension r over some irreducible component of T. Acting by correspondences in the evident manner defines $z_{equi}(X,r)$ as an object in $Sh^{tr}_{\rm Nis}(k)$.

For $r = 0$, we have the evident map

$$\mathbb{Z}^{tr}(X) \to z_{equi}(X,0)$$

which is an isomorphism if X is proper over k. Similarly, if $f : Z \to X$ is a dominant equi-dimensional morphism of relative dimension d, the pull-back of cycles from $T \times_k X$ to $T \times_k U$ defines a map

$$f^* : z_{equi}(X,r) \to z_{equi}(Z, r+d).$$

If $f : Z \to X$ is proper, we have the push-forward map

$$f_* : z_{equi}(Z,r) \to z_{equi}(X,r)$$

and if $j : U \to X$ is an open immersion with closed complement $i : W \to X$, the sequence

$$0 \to z_{equi}(W,r) \xrightarrow{i_*} z_{equi}(X,r) \xrightarrow{j^*} z_{equi}(U,r)$$

is exact.

Taking products of cycles gives the pairing

$$\boxtimes : z_{equi}(X,r)(T) \otimes z_{equi}(X',r')(T') \to z_{equi}(X \times_k X', r+r')(T \times_k T');$$

for $T = T'$, one pulls back by the diagonal $T \to T \times_k T$ to define the pairing

$$\cup_{X,X'}(T) : z_{equi}(X,r)(T) \otimes z_{equi}(X',r')(T) \to z_{equi}(X \times_k X', r+r')(T).$$

Lemma 4.4.2 *The pairings $\cup_{X,X'}(T)$ extend to a pairing*

$$\cup_{X,X'} : z_{equi}(X,r) \otimes^{tr} z_{equi}(X',r') \to z_{equi}(X \times_k X', r+r').$$

Proof Take $W \in z_{equi}(X,r)(T)$, $W' \in z_{equi}(X',r')(T')$. We let

$$\phi_{W \boxtimes W'} : \mathbb{Z}^{tr}(T \times_k T') \to z_{equi}(X \times_k X', r+r')$$

be the map corresponding to $W \boxtimes W' \in z_{equi}(X \times_k X', r+r')(T \times_k T')$. Thus we have the map

$$\oplus\phi_{W\boxtimes W'} : \bigoplus_{\substack{T \in \mathbf{Sm}/k, W \in z_{equi}(X,r)(T) \\ T' \in \mathbf{Sm}/k, W' \in z_{equi}(X',r')(T')}} \mathbb{Z}^{tr}(T \times_k T') \to z_{equi}(X \times_k X', r+r'),$$

i.e., a map

$$\tilde{\cup} : \mathcal{L}_0(z_{equi}(X,r)) \otimes^{tr} \mathcal{L}_0(z_{equi}(X',r')) \to z_{equi}(X \times_k X', r+r').$$

It is a simple matter to check that $\tilde{\cup}$ descends to a map on the quotient $H_0(\mathcal{L}(z_{equi}(X,r)) \otimes^{tr} \mathcal{L}(z_{equi}(X',r')))$, giving the desired pairing.

5 $\mathcal{N}(S)$-modules and motives

We relate the category of Tate motives over $S \in \mathbf{Sm}/k$ to the derived category of dg modules over the cycle algebra $\mathcal{N}(S)$.

5.1 The contravariant motive

We define a functor

$$h_S : \mathbf{Sm}/S^{\mathrm{op}} \to DM(S)$$

as follows: For $X \to S$ in $\mathbf{Sm}/S$ we have the internal Hom presheaf on $SmCor(S)$

$$\mathcal{H}om(\mathbb{Z}^{tr}_S(X), C^{\mathrm{Sus}}_*(\mathbb{Z}^{tr}_S(n)[2n]))(W) := C_*(\mathbb{Z}^{tr}_S(n)[2n])(X \times_S W).$$

The multiplication

$$\mathbb{Z}^{tr}_S(n)[2n] \otimes^{tr}_S \mathbb{Z}^{tr}_S(1)[2] \to \mathbb{Z}^{tr}_S(n+1)[2n+2]$$

together with the canonical map $T^{tr} := \mathbb{Z}^{tr}(1)[2] \to C_*(\mathbb{Z}^{tr}_S(1)[2])$ gives rise to the bonding maps

$$\mathcal{H}om(\mathbb{Z}^{tr}_S(X), C_*(\mathbb{Z}^{tr}_S(n)[2n])) \otimes^{tr}_S T^{tr} \to \mathcal{H}om(\mathbb{Z}^{tr}_S(X), C_*(\mathbb{Z}^{tr}_S(n+1)[2n+2])).$$

Noting that $\mathbb{Z}^{tr}_S(n)[2n] = (\mathbb{Z}^{tr}_S(1)[2])^{\otimes^{tr} n}$, the commutativity constraints for the tensor structure define a $\mathfrak{S}_n$ action on $\mathcal{H}om(\mathbb{Z}^{tr}_S(X), C_*(\mathbb{Z}^{tr}_S(n)[2n]))$, giving us the symmetric T^{tr} spectrum $h_S(X) \in \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}}(S)$:

$$h_S(X) := (\mathcal{H}om(\mathbb{Z}^{tr}_S(X), C_*(\mathbb{Z}^{tr}_S)), \ldots, \mathcal{H}om(\mathbb{Z}^{tr}_S(X), C_*(\mathbb{Z}^{tr}_S(n)[2n])), \ldots).$$

Using the action of correspondences on $\mathbb{Z}_S^{tr}(X)$, one sees immediately that h_S extends to a functor

$$h_S : SmCor(S)^{\mathrm{op}} \to \mathbf{Spt}_{T^{tr}}^{\mathfrak{S}}(S),$$

which in turn extends to

$$C^b(h_S) : C^b(SmCor(S)^{\mathrm{op}}) \to \mathbf{Spt}_{T^{tr}}^{\mathfrak{S}}(S).$$

Passing to the respective homotopy categories gives the exact functor

$$K^b(h_S) : K^b(SmCor(S))^{\mathrm{op}} \to DM(S).$$

Lemma 5.1.1 *The functor $K^b(h_S) : K^b(SmCor(S))^{\mathrm{op}} \to DM(S)$ descends to an exact functor*

$$h_{gm}^{\mathrm{eff}} : DM_{gm}^{\mathrm{eff}}(S)^{\mathrm{op}} \to DM(S).$$

Proof By [15, chapter IV, theorem 8.1], the natural map

$$H^m(C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n)[2n])(X)) \to \mathbb{H}^m(X_{\mathrm{Nis}}, C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n)[2n]))$$

is an isomorphism for all $X \in \mathbf{Sm}/k$ and all m. Thus, by the Mayer-Vietoris property for hypercohomology, the total complex associated to the following term-wise exact sequence of complexes

$$\begin{aligned} &C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n)[2n])(T \times_S (U \cap V)) \\ &\qquad \to C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n)[2n])(T \times_S U) \oplus C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n)[2n])(T \times_S V) \\ &\qquad\qquad \to C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n)[2n])(T \times_S (U \cup V)) \end{aligned}$$

is acyclic for all U, V as in (a), for all $T \in \mathbf{Sm}/S$ and for all $n \geq 0$. Thus $K^b(h_S)$ maps the complexes in (a) to an object $\cong 0$ in $DM(S)$.

Similarly, since $C_*^{\mathrm{Sus}}(\mathbb{Z}_S^{tr}(n)[2n])$ is in $DM_-^{\mathrm{eff}}(k) \subset D^-(Sh_{\mathrm{Nis}}^{tr}(k))$, the map

$$p^* : C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n)[2n])(T \times_S X) \to C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n)[2n])(T \times_S X \times \mathbb{A}^1)$$

is a quasi-isomorphism for all $T, X \in \mathbf{Sm}/k$ and all $n \geq 0$. Thus $K^b(h_S)$ maps the complexes in (b) to an object $\cong 0$ in $DM(k)$, giving us the exact functor

$$\tilde{h}_{gm}^{\mathrm{eff}} : \widetilde{DM}_{gm}^{\mathrm{eff}}(S)^{\mathrm{op}} \to DM(S).$$

As arbitrary direct sums exist in $DM(S)$, that category is pseudo-abelian, hence $\tilde{h}_{gm}^{\mathrm{eff}}$ extends canonically to the pseudo-abelian hull

$$h_{gm}^{\mathrm{eff}} : DM_{gm}^{\mathrm{eff}}(S)^{\mathrm{op}} \to DM(S).$$

Lemma 5.1.2 *h_{gm}^{eff} is a lax tensor functor. If $S = \mathrm{Spec}\, k$, with k a perfect field admitting resolution of singularities, then h_{gm}^{eff} is a tensor functor.*

Proof We have the pairing

$$C_*^{\text{Sus}}(\mathbb{Z}_S^{tr}(n)[2n])\otimes_S^{tr} C_*^{\text{Sus}}(\mathbb{Z}_S^{tr}(m)[2m])\to C_*^{\text{Sus}}(\mathbb{Z}_S^{tr}(n+m)[2(n+m)])$$

induced by the identity pairing

$$\mathbb{Z}_S^{tr}(n)[2n]\otimes_S^{tr}\mathbb{Z}_S^{tr}(m)[2m]\to\mathbb{Z}_S^{tr}(n+m)[2(n+m)]).$$

Thus, for $X, X'\in\mathbf{Sm}/k$, we have the pairing

$$\begin{aligned}\mathcal{H}om(\mathbb{Z}_S^{tr}(X), C_*^{\text{Sus}}(\mathbb{Z}_S^{tr}(n)[2n]))\otimes_S^{tr}\mathcal{H}om(\mathbb{Z}_S^{tr}(X'), C_*^{\text{Sus}}(\mathbb{Z}_S^{tr}(m)[2m])\\ \to\mathcal{H}om(\mathbb{Z}_S^{tr}(X\times_S X'), C_*^{\text{Sus}}(\mathbb{Z}_S^{tr}(n+m)[2(n+m)])),\end{aligned}\quad(5.1.1)$$

giving rise to the commutative diagram

$$\begin{array}{ccc}H_X(n)\otimes_S^{tr}T^{tr}\otimes_S^{tr}H_{X'}(m) & \longrightarrow & H_X(n+1)\otimes_S^{tr}H_{X'}(m)\\ \downarrow & & \downarrow\\ H_{X\times_S X'}(n+m)\otimes_S^{tr}T^{tr} & \longrightarrow & H_{X\times X'}(n+m+1)\end{array}$$

where

$$\begin{aligned}H_X(n)&:=\mathcal{H}om(\mathbb{Z}_S^{tr}(X), C_*^{\text{Sus}}(\mathbb{Z}_S^{tr}(n)[2n]))\\ H_{X'}(m)&:=\mathcal{H}om(\mathbb{Z}_S^{tr}(X'), C_*^{\text{Sus}}(\mathbb{Z}_S^{tr}(m)[2m])\\ H_{X\times X'}(l)&:=\mathcal{H}om(\mathbb{Z}_S^{tr}(X\times_S X'), C_*^{\text{Sus}}(\mathbb{Z}_S^{tr}(l)[2l]))\end{aligned}$$

Replacing X and X' with arbitrary objects in $C^b(SmCor(k))$, this yields the natural transformation

$$\psi_{M,N}: h_S(M)\otimes h_k(N)\to h_S(M\otimes N),$$

making h_{gm}^{eff} a lax tensor functor.

To show that h_{gm}^{eff} is a tensor functor in case $S=\operatorname{Spec}k$, it suffices to show that $\psi_{X,X'}$ is an isomorphism for $X, X'\in\mathbf{Sm}/k$. For this, it suffices to show that the pairing (5.1.1) induces an isomorphism in $DM_-^{\text{eff}}(k)$

$$\begin{aligned}\mathcal{H}om(\mathbb{Z}^{tr}(X), C_*^{\text{Sus}}(\mathbb{Z}^{tr}(n)[2n]))\otimes\mathcal{H}om(\mathbb{Z}^{tr}(X'), C_*^{\text{Sus}}(\mathbb{Z}^{tr}(m)[2m])\\ \to\mathcal{H}om(\mathbb{Z}^{tr}(X\times_k X'), C_*^{\text{Sus}}(\mathbb{Z}^{tr}(n+m)[2(n+m)])).\end{aligned}$$

for n, m sufficiently large.

To see this, take $S, T, X\in\mathbf{Sm}/k$, and let $d_X=\dim_k X$. We have the map

$$\rho_X: C_*^{\text{Sus}}(z_{equi}(S,r))(X\times T)\to C_*^{\text{Sus}}(z_{equi}(S\times X, r+d_X))(T)$$

which sends a cycle W on $X\times T\times\Delta^p\times S$ of relative dimension r over $X\times T$, to the same cycle, now of relative dimension $r+d_X$ over T. By [15, chapter IV, theorem 8.1], the canonical map

$$H^n(C_*^{\text{Sus}}(z_{equi}(S,r))(T))\to\mathbb{H}^n(T_{\text{Nis}}, C_*^{\text{Sus}}(z_{equi}(S,r))_{\text{Nis}})$$

is an isomorphism for every n and $r \geq 0$. Thus, it follows from [15, chapter IV, theorem 8.2] that the map ρ_X is a quasi-isomorphism for all $r \geq 0$.

Noting that $C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^n, 0)) = C_*^{\mathrm{Sus}}(\mathbb{Z}(n)[2n])$, we have the quasi-isomorphism

$$\rho_X : \mathcal{H}om(\mathbb{Z}^{tr}(X), C_*^{\mathrm{Sus}}(\mathbb{Z}(n)[2n])) \to C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^n \times X, d_X))$$

Finally, by [15, chapter IV, theorem 8.3(2)], the pull-back by the projection $X \times (\mathbb{P}^1)^n \to X \times (\mathbb{P}^1)^{n-1}$ induces a natural quasi-isomorphism

$$C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^{n-1} \times X, m-1))(T) \cong C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^n \times X, m))(T)$$

for all $n, m \geq 1$. Thus, for $n \geq d_X$ we have the diagram of quasi-isomorphisms

$$\begin{array}{ccc} \mathcal{H}om(\mathbb{Z}^{tr}(X), C_*^{\mathrm{Sus}}(\mathbb{Z}(n)[2n])) & \longrightarrow & C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^n \times X, d_X)) \\ & & \uparrow \\ & & C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^{n-d_X} \times X, 0)) \end{array}$$

and similarly for S. Thus, these quasi-isomorphisms give isomorphisms in $DM_-^{\mathrm{eff}}(k)$

$$\begin{aligned} &\mathcal{H}om(\mathbb{Z}^{tr}(X), C_*^{\mathrm{Sus}}(\mathbb{Z}(n)[2n])) \cong C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^{n-d_X} \times X, 0)) \\ &\mathcal{H}om(\mathbb{Z}^{tr}(S), C_*^{\mathrm{Sus}}(\mathbb{Z}(m)[2m])) \cong C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^{m-d_S} \times S, 0)) \\ &\mathcal{H}om(\mathbb{Z}^{tr}(S \times_k X), C_*^{\mathrm{Sus}}(\mathbb{Z}(n+m)[2(n+m)])) \\ &\qquad \cong C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^{n+m-d_S-d_X} \times S \times_k X, 0)) \end{aligned}$$

for $m \geq d_S$, $n \geq d_X$. One checks that these isomorphisms are compatible with the pairings

$$\begin{aligned} &\mathcal{H}om(\mathbb{Z}^{tr}(S), C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n)[2n])) \otimes \mathcal{H}om(\mathbb{Z}^{tr}(X), C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(m)[2m]) \\ &\qquad \to \mathcal{H}om(\mathbb{Z}^{tr}(S \times_k X), C_*^{\mathrm{Sus}}(\mathbb{Z}^{tr}(n+m)[2(n+m)])) \end{aligned}$$

$$\begin{aligned} &C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^{n-d_X} \times X, 0)) \otimes C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^{m-d_S} \times S, 0)) \\ &\qquad \to C_*^{\mathrm{Sus}}(z_{equi}((\mathbb{P}^1/\infty)^{n+m-d_S-d_X} \times S \times_k X, 0)) \end{aligned}$$

But by [15, chapter V, proposition 4.1.7], this last pairing is an isomorphism in $DM_-^{\mathrm{eff}}(k)$, completing the proof.

Lemma 5.1.3 *Take $S \in \mathbf{Sm}/k$ and consider the functor $h_{gm}^{\mathrm{eff}} : DM_{gm}^{\mathrm{eff}}(S)^{\mathrm{op}} \to DM(S)$.*

1. There is a natural isomorphism

$$h_{gm}^{\mathrm{eff}}(M(1)) \cong h_{gm}^{\mathrm{eff}}(M)(-1).$$

2. The functor $h_{gm}^{\mathrm{eff}} : DM_{gm}^{\mathrm{eff}}(S)^{\mathrm{op}} \to DM(S)$ extends to an exact lax tensor functor

$$h_{gm} : DM_{gm}(S)^{\mathrm{op}} \to DM(S).$$

3. If $S = \operatorname{Spec} k$, and k is a perfect field admitting resolution of singularities, then h_{gm} is a tensor functor.

Proof By lemma 5.1.2, h^{eff}_{gm} is a lax tensor functor, and is a tensor functor if $S = \operatorname{Spec} k$. Since $DM_{gm}(S) = DM^{\text{eff}}_{gm}(S)[\otimes\mathbb{Z}(1)^{-1}]$ and $\otimes\mathbb{Z}(1)$ is invertible on $DM(S)$, it suffices to prove (1).

Since h^{eff}_{gm} is a lax tensor functor, we have the natural map

$$\psi_M : h^{\text{eff}}_{gm}(M) \otimes h^{\text{eff}}_{gm}(\mathbb{Z}_S(1)) \to h^{\text{eff}}_{gm}(M(1)).$$

As $DM^{\text{eff}}_{gm}(S)$ is generated by the motives $M^{\text{eff}}_{gm}(X)$, for $X \in \mathbf{Sm}/S$, it suffices to show

(a) $h^{\text{eff}}_{gm}(\mathbb{Z}_S(1)) \cong \mathbb{Z}_S(-1)$.

(b) $\psi_{M^{\text{eff}}_{gm}(X)}$ is an isomorphism for all $X \in \mathbf{Sm}/S$.

For (a), by definition, $h^{\text{eff}}_{gm}(\mathbb{Z}_S(1)[2])$ is represented by the T^{tr}-spectrum with nth term $\mathcal{H}om(\mathbb{Z}^{tr}_S(\mathbb{P}^1/\infty), C^{\text{Sus}}_*(\mathbb{Z}_S(n)[2n]))$.This presheaf on $\mathbf{Sm}/S$ is isomorphic to the restriction of the presheaf $\mathcal{H}om(\mathbb{Z}^{tr}(S \times \mathbb{P}^1/S \times \infty), C^{\text{Sus}}_*(\mathbb{Z}(n)[2n]))$ on $\mathbf{Sm}/k$. Similarly, $\mathbb{Z}_S(-1)[-2]$ is represented by the T^{tr}-spectrum with nth term the restriction to $\mathbf{Sm}/S$ of the presheaf $\mathcal{H}om(\mathbb{Z}^{tr}(S), C^{\text{Sus}}_*(\mathbb{Z}(n-1)[2(n-1)]))$, with bonding maps induced by the multiplication $C^{\text{Sus}}_*(\mathbb{Z}(n-1)[2(n-1)]) \otimes^{tr} \mathbb{Z}(1)[2] \to C^{\text{Sus}}_*(\mathbb{Z}(n)[2n])$.

As in the proof of lemma 5.1.2, we have the diagram of quasi-isomorphisms of presheaves on $\mathbf{Sm}/k$ (for $n \geq 1$)

$$\begin{array}{c}
\mathcal{H}om(\mathbb{Z}^{tr}(S \times \mathbb{P}^1/S \times \infty), C^{\text{Sus}}_*(\mathbb{Z}(n)[2n])) \\
\searrow \\
\mathcal{H}om(\mathbb{Z}^{tr}(S, C^{\text{Sus}}_*(z_{equi}((\mathbb{P}^1/\infty)^n \times (\mathbb{P}^1/\infty), 1))) \\
\uparrow \\
\mathcal{H}om(\mathbb{Z}^{tr}(S, C^{\text{Sus}}_*(z_{equi}((\mathbb{P}^1/\infty)^{n-1}, 0))) \\
\| \\
\mathcal{H}om(\mathbb{Z}^{tr}(S, C^{\text{Sus}}_*(\mathbb{Z}(n-1)[2(n-1)])),
\end{array}$$

compatible with bonding maps, proving (a).

For (b), $h^{\text{eff}}_{gm}(M^{\text{eff}}_{gm}(X))$ is represented by the T^{tr}-spectrum with nth term the presheaf $\mathcal{H}om(\mathbb{Z}^{tr}_S(X), C^{\text{Sus}}_*(\mathbb{Z}_S(n)[2n]))$ and $h^{\text{eff}}_{gm}(M^{\text{eff}}_{gm}(X)(1)[2])$ is represented by the T^{tr}-spectrum with nth term $\mathcal{H}om(\mathbb{Z}^{tr}_S(X \times \mathbb{P}^1/X \times \infty), C^{\text{Sus}}_*(\mathbb{Z}_S(n)[2n]))$. We note that the presheaf $\mathcal{H}om(\mathbb{Z}^{tr}_S(X), C^{\text{Sus}}_*(\mathbb{Z}_S(n)[2n]))$ is as above the restriction to $SmCor(S)$ of the presheaf $\mathcal{H}om(\mathbb{Z}^{tr}(X), C^{\text{Sus}}_*(\mathbb{Z}(n)[2n]))$ on $SmCor(k)$, where we consider X as in $\mathbf{Sm}/k$ via the composition $X \to S \to \operatorname{Spec} k$.

Similarly, $\mathcal{H}om(\mathbb{Z}^{tr}_S(X \times \mathbb{P}^1/X \times \infty), C^{\text{Sus}}_*(\mathbb{Z}_S(n)[2n]))$ is the restriction to $SmCor(S)$ of the presheaf $\mathcal{H}om(\mathbb{Z}^{tr}(X \times \mathbb{P}^1/X \times \infty), C^{\text{Sus}}_*(\mathbb{Z}(n)[2n]))$ on $SmCor(k)$. The same proof as for (a), replacing S with X, proves (b).

We call h_{gm} the *dual motive functor*. Recall from theorem 3.5.3 the full tensor embedding

$$i_S : DM_{gm}(S) \to DM(S).$$

Our terminology for h_{gm} is justified by

Proposition 5.1.4 *Let k be a perfect field admitting resolution of singularities. There is a natural isomorphism of $h_{gm} : DM_{gm}(k)^{\mathrm{op}} \to DM(k)$ with the composition*

$$DM_{gm}(k)^{\mathrm{op}} \xrightarrow{\vee} DM_{gm}(k) \xrightarrow{i_S} DM(k).$$

Proof For $X \in \mathbf{Sm}/k$, we denote $C_*(z_{equi}(X, 0))$ by $C^c_*(X)$ and let $M^c_{gm}(X)$ denote the image of $C^c_*(X)$ in $DM^{\mathrm{eff}}_-(k)$.

For $X \in \mathbf{Sm}/k$ of dimension d, one has the dual motive $M_{gm}(X)^\vee$ in $DM_{gm}(k)$, since k admits resolution of singularities. Also, $M_{gm}(X)^\vee(d)[2d]$ is in $DM^{\mathrm{eff}}_{gm}(k)$ and the image of $M_{gm}(X)^\vee(d)[2d]$ in $DM^{\mathrm{eff}}_-(k)$ is canonically isomorphic to $M^c_{gm}(X)$ (see [15, chapter V, section 4.3]). Letting $\Sigma^\infty_t M^c_{gm}(X)(-d)[-2d]$ denote the T^{tr} spectrum

$$(0, \ldots, 0, C^c_*(X), C^c_*(X)(1)[2], \ldots)$$

with $d-1$ 0's, we see that in $DM(k)$, $\Sigma^\infty_t M_{gm}(X)$ has a dual, namely, the object represented by $\Sigma^\infty_t M^c_{gm}(X)(-d)[-2d]$.

The restriction by the open immersion $\mathbb{A}^n \to (\mathbb{P}^1)^n$ induces a quasi-isomorphism of presheaves

$$\mathcal{H}om(\mathbb{Z}^{tr}_k(X), C_*(\mathbb{Z}^{tr}_k(n)[2n])) \to \mathcal{H}om(\mathbb{Z}^{tr}_S(X), C^c_*(\mathbb{A}^n)).$$

By the duality theorem [15, chapter IV, theorem 7.1], the inclusion of complexes of presheaves

$$\mathcal{H}om(\mathbb{Z}^{tr}(X), C^c_*(\mathbb{A}^n)) \to C_*(z_{equi}(X \times \mathbb{A}^n, d))$$

is a quasi-isomorphism of complexes of presheaves, as is each morphism in the sequence

$$C^c_*(X \times \mathbb{A}^{n-d}) \to \mathcal{H}om(\mathbb{Z}^{tr}(\mathbb{A}^d), C^c_*(X \times \mathbb{A}^{n-d})) \to C_*(z_{equi}(X \times \mathbb{A}^n, d))$$

for all $n \geq d$.

By [15, chapter V, corollary 4.1.8] we have $M^c_{gm}(X \times \mathbb{A}^n) \cong M^c_{gm}(X)(n)[2n]$ for all $n \geq 0$ Thus we have the canonical isomorphisms in $DM^{\mathrm{eff}}_-(k)$:

$$C^c_*(X)(n-d)[2n-2d] \cong C^c(X \times \mathbb{A}^{n-d}) \cong \mathcal{H}om(\mathbb{Z}^{tr}_k(X), C_*(\mathbb{Z}^{tr}_k(n)[2n])),$$

for all $n \geq d$. One checks that this isomorphism is compatible with the bonding morphisms for $\Sigma^\infty_t M^c_{gm}(X)(-d)[-2d]$ and $h_k(X)$, giving the desired isomorophism $M_{gm}(X)^\vee \cong h_k(X)$ in $DM(k)$.

Finally, we may consider the $\mathbb{Q}$-extension of h_{gm}

$$h_{gm} : DM^{\mathrm{op}}_{gm}(S)_{\mathbb{Q}} \to DM(S)_{\mathbb{Q}}.$$

Proposition 5.1.5 *The restriction of h_{gm} to $\mathrm{DMT}_{gm}(S)^{\mathrm{op}}$ defines a tensor functor*

$$h_{gm} : \mathrm{DMT}_{gm}(S)^{\mathrm{op}} \to \mathrm{DMT}(S)$$

with $h_{gm}(\mathbb{Q}_S(n)) \cong \mathbb{Q}_S(-n)$.

Proof This follows directly from lemma 5.1.3 and the fact that $h_{gm}(\mathbb{Q}_S) \cong \mathbb{Q}_S$.

5.2 The dual motive and cycle complexes

We let

$$h_S : K(SmCor(S)^{\mathrm{op}}) \to DM(S)$$

be the exact functor induced by the composition

$$C(SmCor(S)^{\mathrm{op}}) \xrightarrow{C(h_S)} \mathbf{Spt}_{T^{tr}}(S) \to DM(S).$$

We can use the cycle complex construction $\mathcal{N}_S$ (definition 4.2.2) to define a $\mathbb{Q}$ version of h_S. Indeed, for $X \in \mathbf{Sm}/S$, set

$$\mathfrak{h}_S(X)(n) := \mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(n)).$$

The composition

$$\mathbb{Z}_S^{tr}(1)[2] \to C_*^{\mathrm{cb}}(\mathbb{Z}_S^{tr}(1)[2]) \to \mathcal{N}_S(1)$$

together with the multiplication in $\mathcal{N}_S$ induces bonding maps

$$\epsilon_n : \mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(n)) \otimes_S^{tr} T_S^{tr} \to \mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(n+1)),$$

giving us the symmetric T^{tr}-spectrum

$$\mathfrak{h}_S(X) := (\mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(0)), \mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(1)), \ldots)$$

(with trivial $\mathfrak{S}_*$-action). Sending X to $\mathfrak{h}_S(X)$ gives an exact functor

$$\mathfrak{h}_S : K(SmCor(S))^{\mathrm{op}} \to DM(S)_{\mathbb{Q}}.$$

We have the canonical isomorphism in $D(\mathbb{Q})$

$$\mathcal{N}(n)(X) \cong C_*(\mathbb{Z}_S^{tr}(n)[2n])(X)_{\mathbb{Q}}.$$

This gives an isomorphism (in $D(\mathrm{PST}(S))_{\mathbb{Q}}$)

$$\mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(n)) \cong \mathcal{H}om(\mathbb{Z}_S^{tr}(X), C_*(\mathbb{Z}_S^{tr}(n)[2n]))_{\mathbb{Q}} =: h_S(X)_{\mathbb{Q}},$$

which induces a canonical isomorphism

$$\mathfrak{h}_S(X) \cong h_S(X)_{\mathbb{Q}}$$

natural in X, in fact an isomorphism of functors

$$\mathfrak{h}_S \cong h_{S\mathbb{Q}} : K(SmCor(S))^{\mathrm{op}} \to DM(S)_{\mathbb{Q}}. \tag{5.2.1}$$

Lemma 5.2.1 *For each $r \geq 0$, the presheaf $\mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(r))$ is quasi-fibrant in the model category $C(Sh_{\mathrm{Nis}}^{tr}(S)_{\mathbb{Q}})_{\mathbb{A}^1}$, that is, $\mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(r))$ satisfies Nisnevich excision and $\mathbb{A}^1$-homotopy invariance,*

Proof This follows from lemma 4.2.1 and the comments in example 4.1.6.

5.3 Cell modules and Tate motives

Recall the Adams graded cdga $\mathcal{N}(S)$ gotten by evaluating the presheaf $\mathcal{N} := \mathcal{N}_k$ of Adams graded cdgas at $S \in \mathbf{Sm}/k$. The identity $\mathcal{N}(S) = \mathcal{N}_S(S)$ makes the presheaf $\mathcal{N}_S$ on $\mathbf{Sm}/S$ a presheaf of $\mathcal{N}(S)$ algebras.

For fixed r, sending $M \in \mathcal{CM}_{\mathcal{N}(S)}$ to the weight r summand $(M \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(r)$ of $M \otimes_{\mathcal{N}(S)} \mathcal{N}_S$ defines a dg functor

$$\mathcal{M}_S(r)^{dg} : \mathcal{CM}_{\mathcal{N}(S)} \to C(Sh^{tr}_{\mathrm{Nis}}(S)),$$

and thus an exact functor

$$\mathcal{M}_S(r) : \mathcal{KCM}_{\mathcal{N}(S)} \to D(Sh^{tr}_{\mathrm{Nis}}(S))$$

For $M \in \mathcal{CM}_{\mathcal{N}(S)}$, the multiplication in $\mathcal{N}_S$ gives us the map in $C^-(Sh^{tr}_{\mathrm{Nis}}(S))$

$$M \otimes_{\mathcal{N}(S)} \mathcal{N} \otimes^{tr}_S \mathcal{N}_S(1) \to M \otimes_{\mathcal{N}(S)} \mathcal{N}_S;$$

restricting to the summand $(M \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(r)$ and composing with the canonical map $T^{tr}_{\mathbb{Q}} \to \mathcal{N}_S(1)$ gives us the map in $C(Sh^{tr}_{\mathrm{Nis}}(S))$

$$(M \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(r) \otimes^{tr}_S T^{tr}_{\mathbb{Q}} \xrightarrow{\epsilon_r(M)} (M \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(r+1).$$

Sending $M \in \mathcal{CM}_{\mathcal{N}(S)}$ to the sequence

$$\mathcal{M}^{dg}_S(M) := ((M \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(0), (M \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(1), \ldots)$$

with bonding maps $\epsilon_r(M)$ (and trivial $\mathfrak{S}_n$-action) defines the dg functor

$$\mathcal{M}^{dg}_S : \mathcal{CM}_{\mathcal{N}(S)} \to \mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S),$$

giving the exact functor on the respective homotopy categories

$$\mathcal{M}_S : \mathcal{KCM}_{\mathcal{N}(S)} \to DM(S)_{\mathbb{Q}}.$$

Lemma 5.3.1 *1.* $\mathcal{M}(\mathcal{N}(S)\lt n\gt) \cong \mathbb{Q}_S(n)$.

2. There are natural isomorphisms

$$\mathcal{M}(M \otimes \mathbb{Q}(n)) \cong \mathcal{M}(M) \otimes \mathbb{Q}_S(n);$$

3. The restriction of $\mathcal{M}$ *to* $\mathcal{KCM}^f_{\mathcal{N}(S)}$ *is a tensor functor.*

Proof For (1), we note that

$$\mathcal{M}(r)^{dg}(\mathcal{N}(S)\lt n\gt) = \mathcal{N}_S(r+n).$$

As the canonical map $\mathbb{Q}^{tr}_S(r+n) \to \mathcal{N}_S(r+n)$ induces an isomorphism

$$\mathbb{Q}_S(r+n) \cong \mathcal{N}_S(r+n)$$

in $DM^{\mathrm{eff}}(S)$, we have the canonical isomorphism

$$\mathcal{M}(r)(\mathcal{N}(S)<n>) \cong \mathbb{Q}_S(n+r)$$

compatible with the respective bonding maps, proving (1).

(2) follows by noting

$$\mathcal{M}(r)^{dg}(M \otimes \mathcal{N}(S)<n>) = \mathcal{M}(r+n)^{dg}(M)$$

for $r + n \geq 0$.

For (3), we have canonical maps in $C^-(Sh^{tr}_{\mathrm{Nis}}(S))$

$$\begin{aligned}
&(M \otimes_{\mathcal{N}(S)} \mathcal{N}_S) \otimes^{tr}_S (M' \otimes_{\mathcal{N}(S)} \mathcal{N}) \\
&\qquad \to (M \otimes_{\mathcal{N}(S)} M') \otimes_{\mathcal{N}(S)} (\mathcal{N}_S \otimes^{tr}_S \mathcal{N}_S) \\
&\qquad\qquad \xrightarrow{\mathrm{id}\otimes\mu} (M \otimes_{\mathcal{N}(S)} M') \otimes_{\mathcal{N}(S)} \mathcal{N}_S
\end{aligned}$$

where μ is the multiplication. On the respective Adams graded summands, this induces

$$\begin{aligned}
&(M \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(r) \otimes^{tr}_S (M \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(s) \\
&\qquad \xrightarrow{\rho_{M,M'}(r,s)} ((M \otimes_{\mathcal{N}(S)} M') \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(r+s).
\end{aligned}$$

The maps $\rho_{M,M'}(r,s)$ are compatible with the bonding maps, giving us the natural transformation

$$\rho_{M,M'} : \mathcal{M}(M) \otimes \mathcal{M}(M') \to \mathcal{M}(M \otimes M')$$

in $\mathbf{Spt}^{\mathfrak{S}}(S)_{\mathbb{Q}}$, making the functor $\mathcal{M}$ a lax tensor functor.

If $M = \mathcal{N}(S)<a>$ and $M' = \mathcal{N}(S)<b>$ it is a simple matter to check that $\rho_{M,M'}$ is just the canonical isomorphism

$$\mathbb{Q}_S(a) \otimes \mathbb{Q}_S(b) \to \mathbb{Q}_S(a+b);$$

it follows by induction on the length of the weight filtration that $\rho_{M,M'}$ is an isomorphism for all $M, M' \in \mathcal{KCM}^f_{\mathcal{N}(S)}$.

The following result extends Spitzweck's representation theorem (see [29, section 5]) from fields to $S \in \mathbf{Sm}/k$.

Theorem 5.3.2 *Let S be in $\mathbf{Sm}/k$. There is an exact functor*

$$\mathcal{M}_S : \mathcal{D}_{\mathcal{N}(S)} \to DM(S)_{\mathbb{Q}}$$

with $\mathcal{M}_S(\mathbb{Q}(n)) \cong \mathbb{Q}_S(n)$; $\mathcal{M}_S$ is a lax tensor functor. In addition

1. *The restriction of $\mathcal{M}_S$ to*
$$\mathcal{M}^f_S : \mathcal{D}^f_{\mathcal{N}(S)} \to DM(S)_{\mathbb{Q}}$$
defines an equivalence of $\mathcal{D}^f_{\mathcal{N}(S)}$ with $\mathrm{DMT}(S)$, as tensor triangulated categories, natural in S.

2. *$\mathcal{M}^f$ transforms the weight filtration in $\mathcal{D}^f_{\mathcal{N}(S)}$ to that in $\mathrm{DMT}(S)$.*

3. Suppose that S satisfies the Beilinson-Soulé vanishing conjectures. Then $\mathcal{M}^f$ is a functor of triangulated categories with t-structure. In particular, $\mathcal{M}^f$ intertwines the respective truncation functors and induces an equivalence of Tannakian categories

$$H^0(\mathcal{M}^f) : \mathcal{H}^f_{\mathcal{N}(S)} \to \mathrm{MT}(S),$$

which identifies $\mathcal{D}^f_{\mathcal{N}(S)}$ with $\mathrm{DMT}(S)$*.*

Proof Using the equivalence $\mathcal{D}_{\mathcal{N}(S)} \sim \mathcal{KCM}_{\mathcal{N}(S)}$, we just use the functor $\mathcal{M} : \mathcal{KCM}_{\mathcal{N}(S)} \to DM(S)_{\mathbb{Q}}$ to define $\mathcal{M}_S$. Similarly, the equivalence $\mathcal{D}^f_{\mathcal{N}(S)} \sim \mathcal{KCM}^f_{\mathcal{N}(S)}$ and lemma 5.3.1 proves that the restriction of $\mathcal{M}_S$ to $\mathcal{D}^f_{\mathcal{N}(S)}$ is a tensor functor with $\mathcal{M}_S(\mathbb{Q}(n)) \cong \mathbb{Q}_S(n)$.

We have

$$\mathrm{Hom}_{\mathcal{D}^f_{\mathcal{N}(S)}}(\mathbb{Q}(n), \mathbb{Q}(m+n)[p]) \cong \mathrm{Hom}_{\mathcal{KCM}^f_{\mathcal{N}(S)}}(\mathbb{Q}(n), \mathbb{Q}(m+n)[p]) \cong H^p(\mathcal{N}(S)(m)).$$

By lemmas 4.1.3 and 4.2.1, we have

$$H^p(\mathcal{N}(S)(m)) \cong H^p(C^{\mathrm{Sus}}_*(\mathbb{Z}(m))(S)_{\mathbb{Q}}).$$

By Voevodsky's results [15, chapter V, theorem 4.22, proposition 4.2.3], we have

$$H^p(S, \mathbb{Q}(m)) \cong H^p(C^{\mathrm{Sus}}_*(\mathbb{Z}(m))(S)_{\mathbb{Q}}).$$

Finally, by theorem 3.2.5 and theorem 3.5.3 we have

$$H^p(S, \mathbb{Q}(m)) := \mathrm{Hom}_{DM_{gm}(k)_{\mathbb{Q}}}(M_{gm}(S)_{\mathbb{Q}}, \mathbb{Q}(m)[p]) \cong \mathrm{Hom}_{DM(S)_{\mathbb{Q}}}(\mathbb{Q}_S(n), \mathbb{Q}_S(n+m)[p]),$$

giving us the isomorphism

$$\mathrm{Hom}_{\mathcal{D}^f_{\mathcal{N}(S)}}(\mathbb{Q}(n), \mathbb{Q}(m+n)[p]) \cong \mathrm{Hom}_{DM(S)_{\mathbb{Q}}}(\mathbb{Q}_S(n), \mathbb{Q}_S(n+m)[p]).$$

It is not hard to check that this isomorphism is induced by the functor $\mathcal{M}_S$. By induction on the length of the weight filtration, this shows that $\mathcal{M}^f_S$ gives an equivalence of $\mathcal{D}^f_{\mathcal{N}(S)}$ with the essential image of $\mathcal{M}^f_S$, that is, with $\mathrm{DMT}(S)$. This proves (1).

It is clear that $\mathcal{M}^f_S$ sends the subcategory $W_n\mathcal{D}^f_{\mathcal{N}(S)}$ to $W_n\mathrm{DMT}(S)$; this together with (1) proves (2). For (3), the t-structures on $\mathcal{D}^f_{\mathcal{N}(S)}$, resp. $\mathrm{DMT}(S)$ are defined by using the equivalence of $W_{[n,n]}\mathcal{D}^f_{\mathcal{N}(S)}$, resp. $W_{[n,n]}\mathrm{DMT}(S)$ with $D^b(\mathrm{Vec}(\mathbb{Q}))$, induced by sending a vector space V to $V \otimes \mathbb{Q}(-n)$, resp. $V \otimes \mathbb{Q}_S(-n)$. As this is clearly compatible with $\mathcal{M}^f_S$, we have proved (3).

5.4 Motives and $\mathcal{N}_S$-modules

Take $S \in \mathbf{Sm}/k$. We begin by defining the category $\mathcal{M}_{\mathcal{N}_S}$ of Adams graded dg modules over the sheaf (on $\mathbf{Sm}/S$) of cdgas $\mathcal{N}_S$.

Objects in $\mathcal{M}_{\mathcal{N}_S}$ are Adams graded dg objects in $C(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$, that is, (M, d_M), where $M := \oplus_r (M(r)^*, d_M(r))$, with each $(M(r)^*, d_M(r)) \in C(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$. In addition, with respect to the Adams grading r and the cohomological grading $*$, M is a bi-graded module over $\mathcal{N}_S$ in $C(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$, that is, we have module action

$$m : \mathcal{N}_S \otimes^{tr}_S M \to M$$

which is a bi-graded map in $C(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})$. We have the Tate twist operator $M \mapsto M{<}s{>}$ on $\mathcal{M}_{\mathcal{N}_S}$, with $M{<}s{>}(r) := M(s + r)$.

Let $\mathcal{D}_{\mathcal{N}_S}$ denote the derived category of $\mathcal{M}_{\mathcal{N}_S}$, i.e, localize the homotopy category $K\mathcal{M}_{\mathcal{N}_S}$ with respect to the full subcategory of complexes M such that each $M(r)$ has vanishing cohomology sheaves (for the Nisnevich topology). We let $\mathcal{D}^f_{\mathcal{N}_S}$ be the full triangulated subcategory of $\mathcal{D}_{\mathcal{N}_S}$ generated by the objects $\mathcal{N}_S{<}n{>}$, $n \in \mathbb{Z}$.

As $\mathcal{N}_S$ is a presheaf of $\mathcal{N}(S)$-algebras, we have an action of $\mathcal{M}_{\mathcal{N}(S)}$ on $\mathcal{M}_{\mathcal{N}_S}$: given an $\mathcal{N}(S)$-module N and an $M \in \mathcal{M}_{\mathcal{N}_S}$, we may form the sheaf tensor product

$$N \otimes_{\mathcal{N}(S)} M.$$

Restricting to $\mathcal{CM}_{\mathcal{N}(S)}$ gives the bi-exact functor

$$\otimes_{\mathcal{N}(S)} : \mathcal{KCM}_{\mathcal{N}(S)} \times \mathcal{D}_{\mathcal{N}_S} \to \mathcal{D}_{\mathcal{N}_S};$$

via the equivalence $\mathcal{KCM}_{\mathcal{N}(S)} \to \mathcal{D}_{\mathcal{N}(S)}$, we have the bi-exact functor

$$\otimes^L_{\mathcal{N}(S)} : \mathcal{D}_{\mathcal{N}(S)} \times \mathcal{D}_{\mathcal{N}_S} \to \mathcal{D}_{\mathcal{N}_S}.$$

Clearly $\otimes^L_{\mathcal{N}(S)}$ restricts to

$$\otimes^L_{\mathcal{N}(S)} : \mathcal{D}^f_{\mathcal{N}(S)} \times \mathcal{D}^f_{\mathcal{N}_S} \to \mathcal{D}^f_{\mathcal{N}_S}.$$

We have the exact functor

$$\widetilde{\mathcal{M}}_S : K\mathcal{M}_{\mathcal{N}_S} \to DM(S).$$

defined by sending an $\mathcal{N}_S$-module M to the sequence of Adams graded summands

$$\widetilde{\mathcal{M}}_S(M) := (M(0), M(1), \ldots)$$

with bonding maps $M(n) \otimes^{tr}_S T^{tr}_{\mathbb{Q}} \to M(n+1)$ given by the multiplication $M(n) \otimes^{tr}_S \mathcal{N}_S(1) \to M(n + 1)$ and the canonical map $T^{tr}_{\mathbb{Q}} \to \mathcal{N}(1)$. If $M' \to M$ is a quasi-isomorphism of $\mathcal{N}_S$-modules, then $M'(n) \to M(n)$ is a weak equivalence in $C(Sh^{tr}_{\mathrm{Nis}}(S)_{\mathbb{Q}})_{\mathrm{Nis}}$, hence $\widetilde{\mathcal{M}}_S$ descends to an exact functor

$$\widetilde{\mathcal{M}}_S : \mathcal{D}_{\mathcal{N}_S} \to DM(S)_{\mathbb{Q}}.$$

We note that $\mathcal{D}_{\mathcal{N}_S}$ is pseudo-abelian.

Proposition 5.4.1 *There is an exact functor*

$$\mathfrak{h}^{\mathcal{N}}_S : DM_{gm}(S)^{\mathrm{op}}_{\mathbb{Q}} \to \mathcal{D}_{\mathcal{N}_S}$$

such that $\mathfrak{h}_S : DM_{gm}(S)^{\mathrm{op}}_{\mathbb{Q}} \to DM(S)_{\mathbb{Q}}$ *is isomorphic to* $\widetilde{\mathcal{M}}_S \circ \mathfrak{h}^{\mathcal{N}}_S$. *In addition, we have* $\mathfrak{h}^{\mathcal{N}}_S(\mathbb{Q}_S) \cong \mathcal{N}_S$ *and*

$$\mathfrak{h}^{\mathcal{N}}_S(M(1)) \cong \mathfrak{h}^{\mathcal{N}}_S(M) \otimes_{\mathcal{N}_S} \mathcal{N}_S{<}{-1}{>}$$

for all $M \in DM_{gm}(S)_{\mathbb{Q}}$.

Proof Take $X \in \mathbf{Sm}/S$. Recalling that $\mathfrak{h}_S(X)_r = \mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(r))$, define

$$\mathfrak{h}_S^{\mathcal{N}}(X)(r) := \mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(r))$$

giving us the Adams graded object $\mathfrak{h}_S^{\mathcal{N}}(X) := \oplus_{r\geq 0}\mathfrak{h}_S^{\mathcal{N}}(X)(r)$ of $C^-(Sh_{\mathrm{Nis}}^{tr}(S))$. We note that the multiplication

$$\mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(r)) \otimes_S \mathcal{N}_S(s) \to \mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(r+s))$$

extends canonically to a map of complexes

$$\mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(r)) \otimes_S^{tr} \mathcal{N}_S(s) \to \mathcal{H}om(\mathbb{Q}_S^{tr}(X), \mathcal{N}_S(r+s)),$$

giving $\mathfrak{h}_S^{\mathcal{N}}(X)$ the structure of an $\mathcal{N}_S$-module. We thus have the object $\mathfrak{h}_S^{\mathcal{N}}(X)$ of $\mathcal{M}_{\mathcal{N}_S}$ for every $X \in \mathbf{Sm}/S$.

As $\mathfrak{h}_S(X)_n$ is just $\mathfrak{h}_S^{\mathcal{N}}(X)(n)$, it follows from the construction of $\mathfrak{h}_S$ that sending $X \in \mathbf{Sm}/S$ to $\mathfrak{h}_S^{\mathcal{N}}(X)$ extends to an exact functor

$$\mathfrak{h}_S^{\mathcal{N}} : K^b(SmCor(S))^{\mathrm{op}} \to \mathcal{D}_{\mathcal{N}_S}.$$

By the quasi-isomorphisms established in the proof of lemma 5.1.1, $\mathfrak{h}_S^{\mathcal{N}}$ descends further to an exact functor

$$\mathfrak{h}_S^{\mathcal{N}} : DM_{gm}^{\mathrm{eff}}(S)^{\mathrm{op}} \to \mathcal{D}_{\mathcal{N}_S}.$$

It follows from the isomorphism

$$h_{gm}^{\mathrm{eff}}(M \otimes \mathbb{Z}(1)) \cong h_{gm}^{\mathrm{eff}}(M) \otimes \mathbb{Z}(-1)$$

established in the proof of lemma 5.1.3 and the isomorphism (5.2.1) that $\mathfrak{h}_S^{\mathcal{N}}$ extends canonically to an exact functor

$$\mathfrak{h}_S^{\mathcal{N}} : DM_{gm}(S)^{\mathrm{op}} \to \mathcal{D}_{\mathcal{N}_S}$$

satisfying

$$\mathfrak{h}_S^{\mathcal{N}}(M(1)) \cong \mathfrak{h}_S^{\mathcal{N}}(M) \otimes_{\mathcal{N}_S} \mathcal{N}_S{<}{-1}{>}.$$

As $\mathcal{D}_{\mathcal{N}_S}$ is a $\mathbb{Q}$-linear category, this functor extends canonically to

$$\mathfrak{h}_S^{\mathcal{N}} : DM_{gm}(S)_{\mathbb{Q}}^{\mathrm{op}} \to \mathcal{D}_{\mathcal{N}_S}.$$

Remark 5.4.2 As a particular case, proposition 5.4.1 tells us that

$$\mathfrak{h}_S^{\mathcal{N}}(\mathbb{Q}_S(n)) \cong \mathcal{N}_S{<}{-n}{>}$$

for all $n \in \mathbb{Z}$ and thus $\mathfrak{h}_S^{\mathcal{N}}$ restricts to

$$\mathfrak{h}_S^{\mathcal{N}} : \mathrm{DMT}_{gm}(S)^{\mathrm{op}} \to \mathcal{D}_{\mathcal{N}_S}^{f}.$$

Similarly, it is easy to see that $\widetilde{\mathcal{M}}_S(\mathcal{N}_S{<}{-n}{>}) \cong \mathbb{Q}_S(-n)$ and thus

$$\widetilde{\mathcal{M}}_S : \mathcal{D}_{\mathcal{N}_S} \to DM(S)_{\mathbb{Q}}$$

restricts to

$$\widetilde{\mathcal{M}}_S : \mathcal{D}_{\mathcal{N}_S}^{f} \to \mathrm{DMT}(S).$$

We have the global sections functor

$$\Gamma(S,-) : C(Sh^{tr}_{\mathrm{Nis}}(S)) \to C(\mathbf{Ab})$$

with $\Gamma(S,\mathcal{F}) := \mathcal{F}(S)$. Applying $\Gamma(S,-)$ to each $M(r)$ gives us the global sections functor

$$\Gamma(S,-) : \mathcal{M}_{\mathcal{N}_S} \to \mathcal{M}_{\mathcal{N}(S)}.$$

It is not hard to show that category $\mathcal{M}_{\mathcal{N}_S}$ has enough $\Gamma(S,-)$-acyclic objects (take for example the Godement resolution), hence $\Gamma(S,-)$ admits the right-derived functor

$$R\Gamma : \mathcal{D}_{\mathcal{N}_S} \to \mathcal{D}_{\mathcal{N}(S)}$$

with $\Gamma(S,M) \to R\Gamma(S,M)$ an isomorphism in $\mathcal{D}_{\mathcal{N}(S)}$ if M satisfies Nisnevich excision.

Finally, we have the evident natural map, for $M \in \mathcal{D}_{\mathcal{N}_S}$,

$$\phi_M : R\Gamma(S,M) \otimes^L_{\mathcal{N}(S)} \mathcal{N}_S \to M$$

5.5 From cycle algebras to motives

Let $p : X \to S$ be in $\mathbf{Sm}/S$, giving us the map of cycle algebras

$$p^* : \mathcal{N}(S) \to \mathcal{N}(X);$$

in particular, we may consider $\mathcal{N}(X)$ as a dg module over $\mathcal{N}(S)$.

Lemma 5.5.1 *Suppose that $M_{gm}(X)_{\mathbb{Q}} \in DM_{gm}(S)_{\mathbb{Q}}$ is in the Tate subcategory $\mathrm{DMT}_{gm}(S)$. Then $\mathcal{N}(X)$ is in $\mathcal{D}^f_{\mathcal{N}(S)}$.*

Proof Note that

$$\mathcal{N}(X)(r) = \mathcal{H}om(\mathbb{Q}^{tr}_S(X), \mathcal{N}_S(r))(S),$$

giving us the canonical isomorphism in $\mathcal{M}_{\mathcal{N}(S)}$

$$\mathcal{N}(X) \cong \Gamma(S, \mathfrak{h}^{\mathcal{N}}_S(M_{gm}(X))).$$

By lemma 5.2.1, the presheaf $\mathcal{H}om(\mathbb{Q}^{tr}_S(X), \mathcal{N}_S(r))$ satisfies Nisnevich excision, hence the natural map

$$\Gamma(S, \mathfrak{h}^{\mathcal{N}}_S(M_{gm}(X))) \to R\Gamma(S, \mathfrak{h}^{\mathcal{N}}_S(M_{gm}(X)))$$

is an isomorphism in $\mathcal{D}_{\mathcal{N}(S)}$

Therefore, the image of $\mathcal{N}(X)$ in $\mathcal{D}_{\mathcal{N}(S)}$ is given by applying the composition of functors

$$\mathbf{Sm}/S \xrightarrow{M_{gm}} DM_{gm}(S)_{\mathbb{Q}} \xrightarrow{\mathfrak{h}^{\mathcal{N}}_S} \mathcal{D}_{\mathcal{N}_S} \xrightarrow{R\Gamma(S,-)} \mathcal{D}_{\mathcal{N}(S)}$$

to X. Thus, if $M_{gm}(X) \cong M$ in $DM_{gm}(S)_{\mathbb{Q}}$, we have the isomorphism

$$\mathcal{N}(X) \cong R\Gamma(S, \mathfrak{h}^{\mathcal{N}}_S(M))$$

in $\mathcal{D}_{\mathcal{N}(S)}$. Therefore, it suffices to show that $R\Gamma(S,-) \circ \mathfrak{h}^{\mathcal{N}}_S$ maps $\mathrm{DMT}_{gm}(S)$ into the full subcategory $\mathcal{D}^f_{\mathcal{N}(S)}$ of $\mathcal{D}_{\mathcal{N}(S)}$.

But by remark 5.4.2,

$$\mathfrak{h}^{\mathcal{N}}_S(\mathbb{Q}_S(n)) \cong \mathcal{N}_S{<}{-n}{>}$$

and $R\Gamma(S, \mathcal{N}_S{<}{-n}{>}) \cong \mathcal{N}(S){<}{-n}{>}$. Thus, $R\Gamma(S,-) \circ \mathfrak{h}^{\mathcal{N}}_S(\mathbb{Q}_S(n))$ is in $\mathcal{D}^f_{\mathcal{N}(S)}$; the general case follows easily by induction on the length of the weight filtration.

Since $\mathcal{KCM}^f_{\mathcal{N}(S)} \to \mathcal{D}^f_{\mathcal{N}(S)}$ is an equivalence, we thus have

Proposition 5.5.2 *Take $X \in \mathbf{Sm}/S$. Suppose that $M_{gm}(X)_{\mathbb{Q}} \in DM_{gm}(S)_{\mathbb{Q}}$ is in the Tate subcategory* $\mathrm{DMT}_{gm}(S)$*. Then there is a finite $\mathcal{N}(S)$-cell module $\mathfrak{cm}_S(X)$ and a quasi-isomorphism of dg $\mathcal{N}(S)$-modules $\mathfrak{cm}_S(X) \to \mathcal{N}(X)$.*

We now suppose that $M_{gm}(X)_{\mathbb{Q}}$ is in $\mathrm{DMT}_{gm}(S)$, so that the finite $\mathcal{N}(S)$-cell module $\mathfrak{cm}_S(X)$ is defined (uniquely up to homotopy equivalence, we fix a choice once and for all). We proceed to define a natural transformation

$$\psi_X : \mathcal{M}_S(\mathfrak{cm}_S(X)) \to \mathfrak{h}_S(X).$$

Recall that $\mathfrak{h}(X)$ is the symmetric $T^{tr}_{\mathbb{Q}}$ spectrum defined by the sequence

$$\mathfrak{h}(X)_n := \mathcal{H}om(\mathbb{Q}^{tr}_S(X), \mathcal{N}_S(n))$$

with bonding maps induced by the multiplication in $\mathcal{N}_S$ and the structure map $T^{tr}_{\mathbb{Q}} \to \mathcal{N}_S(1)$, while $\mathcal{M}_S(\mathfrak{cm}_S(X))$ is given by the sequence

$$\mathcal{M}_S(\mathfrak{cm}_S(X))_n := \mathcal{M}_S(n)(\mathfrak{cm}_S(X)) := (\mathfrak{cm}_S(X) \otimes_{\mathcal{N}(S)} \mathcal{N}_S)(n)$$

and with bonding maps also given by the multiplication with $T^{tr}_{\mathbb{Q}} \to \mathcal{N}_S(1)$.

Now take $W \in \mathbf{Sm}/S$. Then

$$\mathcal{H}om(\mathbb{Q}^{tr}_S(X), \mathcal{N}_S(r))(W) := \mathcal{N}_S(X \times_S W)(r).$$

Using the external products in $\mathcal{N}_S$, we thus have the canonical map of Adams graded complexes

$$\tilde{\psi}(W) : \mathcal{N}(X) \otimes_{\mathcal{N}(S)} \mathcal{N}(W) \to \mathcal{N}(X \times_S W).$$

The maps $\tilde{\psi}(W)$ clearly define a map of Adams graded complexes of presheaves with transfer

$$\tilde{\psi}_X : \mathcal{N}(X) \otimes_{\mathcal{N}(S)} \mathcal{N}_S \to \oplus_{r\geq 0}\mathcal{H}om(\mathbb{Q}^{tr}_S(X), \mathcal{N}_S(r));$$

restricting to the component of Adams weight r gives the map of complexes of presheaves with transfer

$$\tilde{\psi}_X(r) : [\mathcal{N}(X) \otimes_{\mathcal{N}(S)} \mathcal{N}_S](r) \to \mathcal{H}om(\mathbb{Q}^{tr}_S(X), \mathcal{N}_S(r)).$$

It is easy to see that $\tilde{\psi}_X$ respects the action (on the right) by $\mathcal{N}_S$.

Composing $\tilde{\psi}_X(r)$ with the structure map

$$\mathfrak{cm}_S(X) \otimes_{\mathcal{N}(S)} \mathcal{N}_S \xrightarrow{\rho_X \otimes \mathrm{id}} \mathcal{N}(X) \otimes_{\mathcal{N}(S)} \mathcal{N}_S$$

gives us the map

$$\psi_X(r) : [\mathfrak{cm}_S(X) \otimes_{\mathcal{N}(S)} \mathcal{N}_S](r) \to \mathcal{H}om(\mathbb{Q}^{tr}_S(X), \mathcal{N}_S(r)).$$

also respecting the right $\mathcal{N}_S$ action. Thus, the maps $\psi_X(r)$ define a map of the symmetric $T^{tr}_{\mathbb{Q}}$-spectrum $\mathcal{M}_S(\mathfrak{cm}_S(X))$ to the symmetric $T^{tr}_{\mathbb{Q}}$-spectrum $\mathfrak{h}_S(X)$

$$\psi_X : \mathcal{M}_S(\mathfrak{cm}_S(X)) \to \mathfrak{h}_S(X),$$

as desired.

Our main result is

Theorem 5.5.3 *Suppose that $M_{gm}(X)_{\mathbb{Q}}$ is in $\mathrm{DMT}_{gm}(S)$. Then*

$$\psi_X : \mathcal{M}_S(\mathfrak{cm}_S(X)) \to \mathfrak{h}_S(X)$$

is an isomorphism.

Proof We have the diagram

$$\begin{array}{ccc} DM_{gm}(S)^{\mathrm{op}} & \xrightarrow{\mathfrak{h}_S^{\mathcal{N}}} & \mathcal{D}_{\mathcal{N}_S} \\ & \searrow^{\mathfrak{h}_S} & \downarrow{\scriptstyle \widetilde{\mathcal{M}}_S} \\ & & DM(S)_{\mathbb{Q}} \end{array}$$

commutative up to natural isomorphism. We have as well the finite version of $\mathfrak{h}_S^{\mathcal{N}}$,

$$\mathfrak{h}_S^{\mathcal{N}} : \mathrm{DMT}_{gm}(S) \to \mathcal{D}^f_{\mathcal{N}_S}$$

and diagram

$$\begin{array}{ccc} \mathrm{DMT}_{gm}(S)^{\mathrm{op}} & \xrightarrow{\mathfrak{h}_S^{\mathcal{N}}} & \mathcal{D}^f_{\mathcal{N}_S} \\ & \searrow^{\mathfrak{h}_S} & \downarrow{\scriptstyle \widetilde{\mathcal{M}}_S} \\ & & \mathrm{DMT}(S) \end{array},$$

compatible with the first diagram via the inclusions $\mathrm{DMT}_{gm}(S) \to DM_{gm}(S)_{\mathbb{Q}}$, $\mathrm{DMT}(S) \to DM(S)_{\mathbb{Q}}$ and $\mathcal{D}^f_{\mathcal{N}_S} \to \mathcal{D}_{\mathcal{N}_S}$.

In particular, we have the isomorphism

$$\mathfrak{h}_S(X) \cong \widetilde{\mathcal{M}}_S(\mathfrak{h}_S^{\mathcal{N}}(M_{gm}(X))).$$

Similarly, we have the functor

$$R\Gamma(S,-) : \mathcal{D}_{\mathcal{N}_S} \to \mathcal{D}_{\mathcal{N}(S)}.$$

Since $R\Gamma(S,\mathcal{N}_S<r>) \cong \Gamma(S,\mathcal{N}_S<r>) = \mathcal{N}(S)<r>$, it follows that $R\Gamma(S,-)$ restricts to an exact functor

$$R\Gamma^f(S,-) : \mathcal{D}^f_{\mathcal{N}_S} \to \mathcal{D}^f_{\mathcal{N}(S)}.$$

From the proof of lemma 5.5.1 we have

$$\mathfrak{cm}_S(X) \cong R\Gamma^f(S, \mathfrak{h}_S^{\mathcal{N}}(M_{gm}(X)))$$

in $\mathcal{D}^f_{\mathcal{N}(S)}$.

For $\mathcal{F} \in \mathcal{M}_{\mathcal{N}_S}$, we have the canonical map

$$\phi_{\mathcal{F}} : \Gamma(S,\mathcal{F}) \otimes_{\mathcal{N}(S)} \mathcal{N}_S \to \mathcal{F}$$

inducing the natural map

$$\phi^L_{\mathcal{F}} : R\Gamma(S,\mathcal{F}) \otimes^L_{\mathcal{N}(S)} \mathcal{N}_S \to \mathcal{F}$$

in $\mathcal{D}_{\mathcal{N}_S}$.

For $\mathcal{F} \in \mathcal{D}^f_{\mathcal{N}_S} \subset \mathcal{D}_{\mathcal{N}_S}$, $\phi^L_{\mathcal{F}}$ restricts to the natural transformation

$$\phi^L_{\mathcal{F}} : R\Gamma^f(S, \mathcal{F}) \otimes^L_{\mathcal{N}(S)} \mathcal{N}_S \to \mathcal{F}$$

in $\mathcal{D}^f_{\mathcal{N}_S}$. This gives us the natural transformation

$$\widetilde{\mathcal{M}}_S(\phi^L_{\mathcal{F}}) : \widetilde{\mathcal{M}}_S(R\Gamma^f(S, \mathcal{F}) \otimes^L_{\mathcal{N}(S)} \mathcal{N}_S) \to \widetilde{\mathcal{M}}_S(\mathcal{F})$$

in $\mathrm{DMT}(S)$. In particular, for $M \in \mathrm{DMT}_{gm}(S) \subset DM_{gm}(S)_{\mathbb{Q}}$, we have the natural transformation

$$\psi_M := \widetilde{\mathcal{M}}_S(\phi^L_{\mathfrak{h}^{\mathcal{N}}_S(M)}) : \widetilde{\mathcal{M}}_S(R\Gamma^f(S, \mathfrak{h}^{\mathcal{N}}_S(M)) \otimes^L_{\mathcal{N}(S)} \mathcal{N}_S) \to \widetilde{\mathcal{M}}_S(\mathfrak{h}^{\mathcal{N}}_S(M))$$

in $\mathrm{DMT}(S)$.

In case $M = M_{gm}(X)_{\mathbb{Q}}$ for some $X \in \mathbf{Sm}/S$ with $M_{gm}(X)_{\mathbb{Q}} \in \mathrm{DMT}_{gm}(S)$, we have

$$R\Gamma^f(S, \mathfrak{h}^{\mathcal{N}}_S(M)) \cong R\Gamma^f(S, \mathfrak{h}^{\mathcal{N}}_S(M_{gm}(X))) \cong \mathfrak{cm}_S(X)$$

in $\mathcal{D}^f_{\mathcal{N}(S)}$,

$$\widetilde{\mathcal{M}}_S(R\Gamma^f(S, \mathfrak{h}^{\mathcal{N}}_S(M)) \otimes^L_{\mathcal{N}(S)} \mathcal{N}_S) \cong \mathcal{M}_S(\mathfrak{cm}_S(X))$$

and

$$\widetilde{\mathcal{M}}_S(\mathfrak{h}^{\mathcal{N}}_S(M)) \cong \mathfrak{h}_S(X)$$

in $DM(S)_{\mathbb{Q}}$, and, via these isomorphisms, ψ_M corresponds to ψ_X.

Thus, it suffices to show that ψ_M is an isomorphism for all $M \in \mathrm{DMT}_{gm}(S)$; as usual, we reduce to the case of $M = \mathbb{Q}_S(n)$ by induction on the length of the weight filtration. For $M = \mathbb{Q}_S(n)$, we have

$$\begin{aligned} \mathfrak{h}^{\mathcal{N}}_S(\mathbb{Q}_S(n)) &= \mathcal{N}_S{<}{-n}{>} \\ R\Gamma^f(S, \mathfrak{h}^{\mathcal{N}}_S(\mathbb{Q}_S(n))) &= R\Gamma^f(S, \mathcal{N}_S{<}{-n}{>}) \\ &\cong \Gamma(S, \mathcal{N}_S{<}{-n}{>}) = \mathcal{N}(S){<}{-n}{>} \end{aligned}$$

so

$$\phi^L_{\mathfrak{h}^{\mathcal{N}}(\mathbb{Q}_S(n))} : R\Gamma^f(S, \mathfrak{h}^{\mathcal{N}}_S(\mathbb{Q}_S(n))) \otimes^L_{\mathcal{N}(S)} \mathcal{N}_S \to \mathfrak{h}^{\mathcal{N}}_S(\mathbb{Q}_S(n))$$

is already an isomorphism in $\mathcal{D}^f_{\mathcal{N}_S}$.

5.6 The cell algebra of an S-scheme

We now assume that $\mathcal{N}(S)$ is cohomologically connected.

Let $p : X \to S$ be in $\mathbf{Sm}/S$ with a section $s : S \to X$. We thus have the map of cycle algebras $p^* : \mathcal{N}(S) \to \mathcal{N}(X)$ making $\mathcal{N}(X)$ a cdga over $\mathcal{N}(S)$ with augmentation $s^* : \mathcal{N}(X) \to \mathcal{N}(S)$. Let $\mathcal{N}(X)_S\{\infty\} \to \mathcal{N}(X)$ be the relative minimal model of $\mathcal{N}(X)$ over $\mathcal{N}(S)$. In particular, $\mathcal{N}(X)_S\{\infty\}$ is a cell module over $\mathcal{N}(S)$. In addition, the multiplication

$$\mathcal{N}(X)_S\{\infty\} \otimes \mathcal{N}(X)_S\{\infty\} \to \mathcal{N}(X)_S\{\infty\}$$

given by the cdga structure on $\mathcal{N}(X)_S\{\infty\}$ descends to

$$\mu_X : \mathcal{N}(X)_S\{\infty\} \otimes_{\mathcal{N}(S)} \mathcal{N}(X)_S\{\infty\} \to \mathcal{N}(X)_S\{\infty\}.$$

Definition 5.6.1 The *motivic cell algebra of* X,

$$\mathfrak{ca}_S(X) \in \mathcal{CM}_{\mathcal{N}(S)}$$

is $\mathcal{N}(X)_S\{\infty\}$, considered as a cell module over $\mathcal{N}(S)$.

The same construction we used to define the map $\mathcal{M}_S(\mathfrak{cm}_S(X)) \to \mathfrak{h}_S(X)$ gives us the map in $\mathbf{Spt}^{\mathfrak{S}}_{T^{tr}_{\mathbb{Q}}}(S)$

$$\psi_X : \mathcal{M}_S(\mathfrak{ca}_S(X)) \to \mathfrak{h}_S(X). \tag{5.6.1}$$

Theorem 5.6.2 *Suppose that* $M_{gm}(X)_{\mathbb{Q}}$ *is in* $\mathrm{DMT}_{gm}(S)$ *and that* X *satisfies the Beilinson-Soulé vanishing conjectures. Then*

$$\psi_X : \mathcal{M}_S(\mathfrak{ca}_S(X)) \to \mathfrak{h}_S(X)$$

is an isomorphism.

Proof Suppose we knew that $\mathfrak{ca}_S(X) \to \mathcal{N}(X)$ is a quasi-isomorphism. As $\mathfrak{ca}_S(X)$ is a generalized nilpotent $\mathcal{N}(S)$-algebra, $\mathfrak{ca}_S(X)$ is an $\mathcal{N}(S)$-cell module. Thus, we can take $\mathfrak{cm}_S(X) \to \mathcal{N}(X)$ to be $\mathfrak{ca}_S(X) \to \mathcal{N}(X)$, and the proposition follows from theorem 5.5.3. We now show that $\mathfrak{ca}_S(X) \to \mathcal{N}(X)$ is a quasi-isomorphism.

Recall that the Beilinson-Soulé vanishing conjectures for X are just saying that $\mathcal{N}(X)$ is cohomologically connected. Using the section $s : S \to X$, we see that S also satisfies the Beilinson-Soulé vanishing conjectures, hence $\mathcal{N}(S)$ is cohomologically connected. The structure map $\mathcal{N}(X)_S\{\infty\} \to \mathcal{N}(X)$ is thus a quasi-isomorphism by remark 2.4.11.

6 Motivic π_1

We can now put all our constructions together to give a description of the Deligne-Goncharov motivic π_1 in terms of a relative bar construction. In this section, we assume k admits resolution of singularities.

6.1 Cosimplicial constructions

Fix a base-field k and an $S \in \mathbf{Sm}/k$. We have the action of finite sets on $\mathbf{Sch}_S$ by

$$X^{A/S} := \prod_{a \in A} X$$

for $X \in \mathbf{Sch}_S$ and A a finite set, where $\prod$ means product over S. As this defines a functor

$$X^{?/S} : \mathbf{Sets}^{\mathrm{op}}_{fin} \to \mathbf{Sch}_S$$

we have an induced functor (also denoted $X^{?/S}$) from simplicial objects in finite sets to cosimplicial schemes. In case A is the set $\{1, \ldots, n\}$ we write $X^{n/S}$ for $X^{A/S}$.

Examples 6.1.1 1. We have the simplicial object in finite sets $[0,1]$:

$$[0,1]([n]) := \mathrm{Hom}_{\mathbf{Ord}}([n],[1])$$

giving us the *cosimplicial path space of* X, $X^{[0,1]/S}$. The two inclusions $i_0, i_1 : [0] \to [1]$ induce the projection

$$\pi : X^{[0,1]/S} \to X^{\{0,1\}/S}.$$

Explicitly, $X^{\{0,1\}/S}$ is the constant cosimplicial scheme $X \times_S X$. $X^{[0,1]/S}$ has n-cosimplices $X^{n+2/S}$ with the ith coface map given by the diagonal

$$(t_0, \ldots, t_n) \mapsto (t_0, \ldots, t_{i-1}, t_i, t_i, t_{i+1}, \ldots, t_n)$$

and the codegeneracies given by projections. The structure morphism π is given by the projection $X^{n+2/S} \to X^{2/S}$ on the first and last factor.

2. Suppose we have sections $a, b : S \to X$, giving the map $i_{b,a} : S \to X \times_S X$. The *pointed path space* $\mathcal{P}_{b,a}(X/S)$ is

$$\mathcal{P}_{b,a}(X/S) := S \times_{i_{b,a},\pi} X^{[0,1]/S}.$$

We write $\mathcal{P}_a(X/S)$ for $\mathcal{P}_{a,a}(X/S)$.

In case $S = \mathrm{Spec}\, k$, we sometimes delete the mention of S in the notation, writing, e.g., X^A for $X^{A/\mathrm{Spec}\, k}$.

Remark 6.1.2 Suppose that S and X both satisfy the Beilinson-Soulé vanishing conjectures and that $M_{gm}(X)_{\mathbb{Q}}$ is in $\mathrm{DMT}_{gm}(S)$. Then $X^{n/S}$ also satisfies the Beilinson-Soulé vanishing conjectures for all $n \geq 1$.

Indeed, by theorem 5.6.2, the canonical map

$$\mathcal{N}(X)_S\{\infty\} \to \mathcal{N}(X)$$

is a quasi-isomorphism.

It follows by induction on the length of the weight filtration for $M_{gm}(X)_{\mathbb{Q}}$ that

$$H^*(X^{n/S}, \mathbb{Q}(*)) \cong H^*(X, \mathbb{Q}(*))^{\otimes^L_{H^*(S,\mathbb{Q}(*))} n}$$

and thus, the natural map

$$\mathcal{N}(X)^{\otimes^L_{\mathcal{N}(S)} n} \to \mathcal{N}(X^{n/S})$$

is a quasi-isomorphism, hence

$$\mathcal{N}(X)_S\{\infty\}^{\otimes_{\mathcal{N}(S)} n} \to \mathcal{N}(X^{n/S})$$

is a quasi-isomorphism. But then $\mathcal{N}(X^{n/S})$ is cohomologically connected, that is, $X^{n/S}$ satisfies the Beilinson-Soulé vanishing conjectures.

6.2 The motive of a cosimplicial scheme

Let $X^\bullet : \mathbf{Ord} \to \mathbf{Sm}/S$ be a smooth cosimplicial S-scheme, $[i] \mapsto X[i] \in \mathbf{Sm}/S$. Modifying the construction of Deligne-Goncharov, we define $h_S(X^\bullet)$ as an object in $DM(S)$.

Let $\mathbb{Z}\mathbf{Sm}/S$ be the additive category generated by $\mathbf{Sm}/S$: objects are denoted $\mathbb{Z}(X)$ for $X \in \mathbf{Sm}/S$, for X irreducible, $\mathrm{Hom}_{\mathbb{Z}\mathbf{Sm}/S}(\mathbb{Z}(X'), \mathbb{Z}(X))$ is the free abelian group on $\mathrm{Hom}_{\mathbf{Sm}/S}(X', X)$ and disjoint union is direct sum. The embedding $\mathbf{Sm}/S \to SmCor(S)$ extends by $\mathbb{Z}$-linearity to an embedding $\mathbb{Z}\mathbf{Sm}/S \to SmCor(S)$.

For a smooth cosimplicial S-scheme $X^\bullet$, let $\mathbb{Z}(X^\bullet) \in C(\mathbb{Z}\mathbf{Sm}/S^{\mathrm{op}})$ denote the complex with

$$\mathbb{Z}(X^\bullet)^n := \mathbb{Z}(X^{-n})$$

and with differential the usual alternating sum of the coface maps (in the opposite category). We consider $\mathbb{Z}(X^\bullet)$ as an object of $C(SmCor(S)^{\mathrm{op}})$ via the embedding $\mathbb{Z}\mathbf{Sm}/S \to SmCor(S)$.

The category $DM(S)$ is large enough to define the object $h_S(X^\bullet)$ directly.

Definition 6.2.1 For a cosimplicial scheme $X^\bullet$, define $h_S(X^\bullet)$ by

$$h_S(X^\bullet) := h_S(\mathbb{Z}(X^\bullet)),$$

where

$$h_S : K(SmCor(S)^{\mathrm{op}}) \to DM(S)$$

is the exact functor defined in §5.2. Sending $X^\bullet$ to $h_S(X^\bullet)$ extends to a functor

$$h_S : [\mathbf{Sm}/S^{\mathbf{Ord}}]^{\mathrm{op}} \to DM(S).$$

We now relate this construction to the ind-object construction of Deligne-Goncharov [12]. For each n, one has the complex $C^*(\Delta_n, X^\bullet) \in C^b(\mathbb{Z}\mathbf{Sm}/S)$ with

$$C^i(\Delta^n, X^\bullet) := \oplus_{g:[i]\hookrightarrow[n]}\mathbb{Z}(X([i])),$$

where the sum is over all injective maps $g : [i] \to [n]$ in $\mathbf{Ord}$. The boundary

$$d^i : C^i(\Delta^n, X^\bullet) \to C^{i+1}(\Delta^n, X^\bullet)$$

is defined as follows: For $0 \le j \le i+1$, we have the coface map $\delta^i_j : [i] \to [i+1]$ (see section 1.2). Fix an injection $g : [i+1] \to [n]$. Define

$$\delta^{i,g}_{j*} : C^i(\Delta^n, X^\bullet) \to C^{i+1}(\Delta^n, X^\bullet)$$

by projecting $C^i(\Delta^n, X^\bullet)$ to the component $\mathbb{Z}(X[i])$ indexed by $g \circ \delta^i_j$ followed by the map

$$X(\delta^i_j) : X[i] \to X[i+1]$$

and then the inclusion $\mathbb{Z}(X[i+1]) \to C^{i+1}(\Delta^n, X^\bullet)$ indexed by g. Set

$$d^i := \sum_{j,g} \mathrm{sgn}(j, g) \cdot \delta^{i,g}_{j*}$$

where $\mathrm{sgn}(j,g)$ is the sign of the shuffle permutation of $[n]$ given by $(g\circ\delta^i_j([i])^c, g\circ\delta^i_j([i]))$.

Projecting on the factors g with 0 in the image of g defines a map of complexes

$$\pi_{n+1,n} : C^*(\Delta_{n+1}, X^\bullet) \to C^*(\Delta_n, X^\bullet)$$

giving us a projective system in $C^b(\mathbb{Z}\mathbf{Sm}/S)$. Reindexing so that C^n is now in degree $-n$ gives an inductive system in $C^b(\mathbb{Z}\mathbf{Sm}/S^{\mathrm{op}})$

$$\ldots \to C_*(\Delta_n, X^\bullet) \to C_*(\Delta_{n+1}, X^\bullet) \to \ldots$$

Definition 6.2.2 $h^{\mathrm{ind}}_S(X^\bullet)$ is the ind-object of $DM(S)$ defined by the ind-system

$$n \mapsto h_S(C_*(\Delta_n, X^\bullet))$$

Remark 6.2.3 Suppose that $S = \mathrm{Spec}\,k$, where k is a perfect field admitting resolution of singularities. We have the sequence of functors

$$\mathbf{Sm}/k \xrightarrow{M_{gm}} DM_{gm}(k) \xrightarrow{\vee} DM_{gm}(k) \xrightarrow{i} DM(k),$$

with $^\vee$ the duality involution and $i : DM_{gm}(k) \to DM(k)$ the full embedding $\Sigma^\infty_t \circ \mathbb{Z}^{tr}$ of theorem 3.5.3. We let $H_{gm} : \mathbf{Sm}/k \to DM_{gm}(k)$ be the functor $X \mapsto M_{gm}(X)^\vee$ and write H_{gm} as well for the extension to an exact functor

$$H_{gm} : K^b(\mathbb{Z}\mathbf{Sm}/k) \to DM_{gm}(k).$$

By proposition 5.1.4 we have a natural isomorphism

$$h_{gm} \circ M_{gm} \cong i \circ H_{gm}.$$

For $X^\bullet$ a smooth cosimplicial k-scheme, let $H^{\mathrm{ind}}_{gm}(X^\bullet)$ be the ind-object

$$n \mapsto H_{gm}(C_*(\Delta_n, X^\bullet))$$

of $DM_{gm}(k)$. Then $H^{\mathrm{ind}}_{gm}(X^\bullet)$ is the ind-object associated to $X^\bullet$, as defined in [12, §3.12], and we have a natural isomorphism of ind-objects of $DM(k)$

$$i(H^{\mathrm{ind}}_{gm}(X^\bullet)) \cong h^{\mathrm{ind}}_k(X^\bullet).$$

Taking the sum of the identity maps defines a map

$$q_n : C_*(\Delta_n, X^\bullet) \to \mathbb{Z}(X^\bullet)$$

in $C(\mathbb{Z}\mathbf{Sm}/S^{\mathrm{op}})$, giving a map of the ind-system $n \mapsto C_*(\Delta_n, X^\bullet)$ to $\mathbb{Z}(X^\bullet)$.

Lemma 6.2.4 *Let $F : \mathbb{Z}\mathbf{Sm}/S^{\mathrm{op}} \to \mathcal{A}$ be an additive functor to a pseudo-abelian category, closed under filtered inductive limits. Then*

$$\varinjlim_n F(C_*(\Delta_n, X^\bullet)) \to F(\mathbb{Z}(X^\bullet))$$

is a homotopy equivalence in $C(\mathcal{A})$.

For a proof, see [29] or [12, proposition 3.10].

Proposition 6.2.5 *We have a natural isomorphism in $DM(S)$*

$$\varinjlim_n h_S^{\rm ind}(C_*(\Delta_n, X^\bullet)) \cong h_S(X^\bullet)$$

Proof This follows directly from lemma 6.2.4.

Finally, we may replace h_S with the functor $\mathfrak{h}_S$. Sending $X^\bullet$ to $\mathfrak{h}_S(X^\bullet) := \mathfrak{h}_S(\mathbb{Z}(X^\bullet))$ extends to the functor

$$\mathfrak{h}_S : [\mathbf{Sm}/S^{\mathbf{Ord}}]^{\rm op} \to DM(S)_\mathbb{Q},$$

the natural isomorphism (5.2.1) $h_{S\mathbb{Q}} \cong \mathfrak{h}_S$ gives natural isomorphisms

$$\phi_{X^\bullet} : h_S(X^\bullet)_\mathbb{Q} \to \mathfrak{h}_S(X^\bullet).$$

Similarly, we have natural isomorphisms:

$$h_S(C_*(\Delta_n, X^\bullet))_\mathbb{Q} \to \mathfrak{h}_S((C_*(\Delta_n, X^\bullet))$$

and

$$\varinjlim_n h_S(C_*(\Delta_n, X^\bullet)) \cong \mathfrak{h}_S(X^\bullet).$$

6.3 Motivic π_1

Let X be a smooth S-scheme with a section $x : S \to X$. This gives us the ind-system in $DM(S)_\mathbb{Q}$

$$n \mapsto h_S(C_*(\Delta_n, \mathcal{P}_x(X)))_\mathbb{Q}$$

as well as the object $h_S(\mathcal{P}_x)_\mathbb{Q} \in DM(S)_\mathbb{Q}$ with isomorphism

$$\varinjlim_n h_S(C_*(\Delta_n, \mathcal{P}_x(X)))_\mathbb{Q} \cong h_S(\mathcal{P}_x)_\mathbb{Q}.$$

Suppose that $M_{gm}(X)_\mathbb{Q}$ is in $\mathrm{DMT}_{gm}(S)$. As $\mathrm{DMT}_{gm}(S)$ is a tensor subcategory of $DM_{gm}(S)_\mathbb{Q}$ and as $M_{gm}(X^{n/S}) = M_{gm}(X)^{\otimes n}$, it follows that $M_{gm}(X^{n/S})_\mathbb{Q}$ is in $\mathrm{DMT}_{gm}(S)$ for all $n \geq 0$. Since the individual terms in $C_*(\Delta_n, \mathcal{P}_x(X))$ are all direct sums of self-products of X, the motive $M_{gm}(C_*(\Delta_n, \mathcal{P}_x(X)))_\mathbb{Q}$ is in $\mathrm{DMT}_{gm}(S)$ for all n, and thus $\mathfrak{h}_S(C_*(\Delta_n, \mathcal{P}_x(X))) = \mathfrak{h}_S(M_{gm}(C_*(\Delta_n, \mathcal{P}_x(X)))_\mathbb{Q})$ is in $\mathrm{DMT}(S)$ for all n.

If S satisfies the Beilinson-Soulé vanishing conjectures, we have the truncation functor

$$H^0_{mot} : \mathrm{DMT}(S) \to \mathrm{MT}(S).$$

Thus we have the ind-system $\chi(X, x)_*$ in $\mathrm{MT}(S)$

$$n \mapsto H^0_{mot}(\mathfrak{h}_S(C_*(\Delta_n, \mathcal{P}_x(X)))) := \chi(X, x)_n.$$

Suppose that $S = \operatorname{Spec} k$. Deligne-Goncharov [12], following Wojtkowiak [39], note that the standard structures of product, coproduct and antipode in the classical bar construction make the ind-system $\chi(X, x)_*$ into an ind-Hopf algebra object in $\mathrm{MT}(k)$; we note that the same operations make $\chi(X,x)_*$ into an ind-Hopf algebra object in $\mathrm{MT}(S)$ as long as the ind-system is defined, that is, if S satisfies the Beilinson-Soulé vanishing conjectures and $M_{gm}(X)_{\mathbb{Q}}$ is in $\mathrm{DMT}_{gm}(S)$.

Returning to the case $S = \operatorname{Spec} k$, if X is the complement of a finite set of k-points of $\mathbb{P}^1_k$, Deligne and Goncharov define $\pi_1^{mot}(X,y)$ to be the dual group scheme object in pro-$\mathrm{MT}(k)$. They also generalize the definition of $\pi_1^{mot}(X,y)$ to the case where X is a smooth uni-rational variety defined over k and where y is a *tangential base-point*: they show in [12, theéorème 4.13] that a suitable object of Deligne's realization category comes from the mixed Artin-Tate category $\mathrm{MAT}(k)$ (which is larger than $\mathrm{MT}(k)$ as it takes into account trivial motives defined over a finite extension of k). However, in this case, they do not give a direct construction as a motive in $DM_{gm}(k)$. We extend their definition in the following direction:

Definition 6.3.1 Suppose that S and X both satisfy the Beilinson-Soulé vanishing conjectures, and that $M_{gm}(X)_{\mathbb{Q}}$ is in $\mathrm{DMT}(S)$. Let $x : S \to X$ be a section. Define $\pi_1^{mot}(X,x)$ to be the group scheme object in pro-$\mathrm{MT}(S)$ dual to the ind-Hopf algebra object $\chi(X,x)_*$ of $\mathrm{MT}(S)$.

Remark 6.3.2 Deligne-Goncharov work in the geometric category $\mathrm{DMT}_{gm}(k)$ rather than in $\mathrm{DMT}(k)$. However, since $i_k : \mathrm{DMT}_{gm}(k) \to \mathrm{DMT}(k)$ is an equivalence, we can just as well work in $\mathrm{DMT}(k)$.

6.4 Simplicial constructions

Let $\mathcal{A} \xrightarrow{\epsilon} \mathcal{N}$ be an augmented cdga over a cdga $\mathcal{N}$. Recall from section 2.5 the simplicial version of the relative bar construction

$$B^{pd}_\bullet(\mathcal{A}/\mathcal{N},\epsilon) := \mathcal{A}^{\otimes_{\mathcal{N}}[0,1]} \otimes_{\mathcal{A}\otimes\mathcal{A}} \mathcal{N}.$$

The total complex associated to the simplicial object $n \mapsto B^{pd}_n(\mathcal{A}/\mathcal{N},\epsilon)$ is the relative bar complex $\bar{B}^{pd}_{\mathcal{N}}(\mathcal{A},\epsilon)$.

Using the opposite of the construction described in section 6.2, we have the ind-system of "finite" complexes $C_*(\Delta_n, B^{pd}_\bullet(\mathcal{A}/\mathcal{N},\epsilon))$, and a homotopy equivalence

$$\varinjlim_n C_*(\Delta_n, B^{pd}_\bullet(\mathcal{A}/\mathcal{N},\epsilon)) \to \bar{B}^{pd}_{\mathcal{N}}(\mathcal{A},\epsilon).$$

Replacing $\mathcal{A}$ with its relative minimal model over $\mathcal{N}$ (assuming for this that $\mathcal{N}$ is cohomologically connected), we have the refined version of the simplicial bar construction, $B_\bullet(\mathcal{A}/\mathcal{N},\epsilon)$, the associated complex $\bar{B}_{\mathcal{N}}(\mathcal{A},\epsilon)$, the approximations $C_*(\Delta_n, B_\bullet(\mathcal{A}/\mathcal{N},\epsilon))$ and the homotopy equivalence

$$\varinjlim_n C_*(\Delta_n, B_\bullet(\mathcal{A}/\mathcal{N},\epsilon)) \to \bar{B}_{\mathcal{N}}(\mathcal{A},\epsilon).$$

6.5 The comparison theorem

Take $X \in \mathbf{Sm}/S$. with section $x : S \to X$. We apply the construction of the preceeding section to the augmented cdga $\mathcal{N}(X)$ over $\mathcal{N}(S)$:

$$\mathcal{N}(X) \underset{p^*}{\overset{x^*}{\rightleftarrows}} \mathcal{N}(S).$$

Assuming that $\mathcal{N}(S)$ is cohomologically connected, we have the relative minimal model $\mathcal{N}_\infty(X/S) := \mathcal{N}_k(X)\{\infty\}_{\mathcal{N}(S)}$, which is an augmented $\mathcal{N}(S)$-algebra via $x^* : \mathcal{N}_\infty(X/S) \to \mathcal{N}(S)$. The multiplication in $\mathcal{N}_\infty(X/S)$ gives the natural maps

$$\mu_n : \mathcal{N}_\infty(X/S)^{\otimes_{\mathcal{N}(S)} n} \to \mathcal{N}(X^{n/S})$$

which thus gives natural maps in $DM(S)_\mathbb{Q}$

$$\phi_n(X,x) : \mathcal{M}_S(C_*(\Delta_n, B_\bullet(\mathcal{N}(X)/\mathcal{N}(S), x^*))) \to \mathfrak{h}_S(C_*(\Delta_n, \mathcal{P}_x(X)))$$

and

$$\phi(X,x) : \mathcal{M}_S(\bar{B}_{\mathcal{N}(S)}(\mathcal{N}(X), x^*)) \to \mathfrak{h}_S(\mathcal{P}_x(X))).$$

The maps $\phi_n(X,x)$ give a map of ind-Hopf algebra objects in $DM(X)$.

Theorem 6.5.1 *Suppose that $M_{gm}(X)_\mathbb{Q}$ is in* $\mathrm{DMT}_{gm}(S)$ *and X satisfies the Beilinson-Soulé vanishing conjectures. Then both $\phi_n(X,x)$ and $\phi(X,x)$ are isomorphisms in $DM(S)_\mathbb{Q}$.*

Proof Note that the Beilinson-Soulé vanishing conjectures for X imply the vanishing conjectures for S, hence $\mathcal{N}_(S)$ is cohomologically connected and thus the relative bar complex $\bar{B}_{\mathcal{N}(S)}(\mathcal{N}(X), x^*)$ is defined.

As $\phi(X,x)$ is identified with the filtered homotopy colimit of the maps $\phi_n(X,x)$, it suffices to show that $\phi_n(X,x)$ is an isomorphism for each n. But on the individual terms in the complexes defining $C_*(\Delta_n, B_\bullet(\mathcal{N}(X)/\mathcal{N}(S), x^*))$ and $C_*(\Delta_n, \mathcal{P}_x(X))$, $\phi_n(X,x)$ is the map

$$\phi_n(X,x)_n : \mathcal{M}_S(\mathcal{N}_\infty(X/S)^{\otimes_{\mathcal{N}(S)} n}) \to \mathcal{H}om(\mathbb{Q}^{tr}(X^{n/S}), \mathcal{N}_S) = \mathfrak{h}_S(X^{n/S})$$

induced by the maps $\psi_{X^{n/S}} \circ \mu_n$ (see(5.6.1) to recall the definition of $\psi_{X^{n/S}}$).

Since $\mathrm{DMT}_{gm}(S)$ is a full tensor subcategory of $DM_{gm}(S)_\mathbb{Q}$, closed under isomorphism, our assumption $M_{gm}(X)_\mathbb{Q} \in \mathrm{DMT}_{gm}(S)$ implies $M_{gm}(X^{n/S})_\mathbb{Q}$ is in $\mathrm{DMT}_{gm}(S)$ for all $n \geq 0$. By remark 6.1.2, $X^{n/S}$ satisfies the Beilinson-Soulé vanishing conjectures for all $n \geq 0$. Therefore, it follows from theorem 5.6.2 that $\psi_{X^{n/S}}$ is an isomorphism for all $n \geq 0$.

In addition, the structure map μ_1 is a quasi-isomorphism since $\mathcal{N}(X)$ is cohomologically connected. As mentioned in remark 6.1.2, the motivic cohomology of $X^{n/S}$ satisfies a Künneth formula (over the motivic cohomology of S) for each n. Thus μ_n is a quasi-isomorphism for each n, and hence $\phi_n(S,x)_n$ is an isomorphism for each n.

Corollary 6.5.2 *Suppose that $M_{gm}(X)_\mathbb{Q}$ is in* $\mathrm{DMT}(S)$ *and X satisfies the Beilinson-Soulé vanishing conjectures. Then we have canonical isomorphisms of ind-Hopf algebras in* $\mathrm{MT}(k)$*,*

$$n \mapsto [\mathcal{M}_S(H^0_{\mathcal{N}(S)}(C_*(\Delta_n, B_\bullet(\mathcal{N}(X)/\mathcal{N}(X), x^*)) \xrightarrow{H^0(\phi_n(X,x))} H^0_{mot}(\mathfrak{h}_S(C_*(\Delta_n, \mathcal{P}_x(X)))].$$

Proof This follows from theorem 6.5.1 and theorem 5.3.2.

6.6 The fundamental exact sequence

Let $p : X \to S$ be in $\mathbf{Sm}/S$. We have the exact functor of triangulated tensor categories $p^* : DM(S) \to DM(X)$; since $p^*(\mathbb{Z}_S(n)) \cong \mathbb{Z}_X(n)$, p^* induces the exact tensor functor

$$p^* : \mathrm{DMT}(S) \to \mathrm{DMT}(X).$$

Similarly, if $x : S \to X$ is a section, we have

$$x^* : \mathrm{DMT}(X) \to \mathrm{DMT}(S).$$

Both p^* and x^* are compatible with the weight filtrations; we have the analogous functors on the "geometric" Tate categories DMT_{gm}.

Similarly, the maps p and x induce maps of cdgas

$$p^* : \mathcal{N}(S) \to \mathcal{N}(X);\ x^* : \mathcal{N}(X) \to \mathcal{N}(S)$$

and thus exact tensor functors

$$p^* : D^f_{\mathcal{N}(S)} \to D^f_{\mathcal{N}(X)},\ x^* : D^f_{\mathcal{N}(X)} \to D^f_{\mathcal{N}(S)}.$$

Recall that the equivalence $\mathcal{M}^f_S$ of theorem 5.3.2 is natural in S, so we have natural isomorphisms

$$\mathcal{M}^f_X \circ p^* \cong p^* \circ \mathcal{M}^f_S;\ \mathcal{M}^f_S \circ x^* \cong x^* \circ \mathcal{M}^f_X.$$

Now suppose that X satisfies the Beilinson-Soulé vanishing conjectures; this property is inherited by S using the splitting x^*. Thus we have the functors p^* and x^* between the Tannakian categories $\mathrm{MT}(X)$ and $\mathrm{MT}(S)$, with p^* and x^* respecting the fiber functors gr^W_*. Similarly, we have functors p^* and x^* for the Tannakian categories $\mathcal{H}^f_{\mathcal{N}(X)}$ and $\mathcal{H}^f_{\mathcal{N}(S)}$, respecting the fiber functors gr^W_*. Finally, $H^0(\mathcal{M}^f_X)$ and $H^0(\mathcal{M}^f_S)$ give an equivalence between these two structures.

Let $G(\mathrm{MT}(X), \mathrm{gr}^W_*)$, $G(\mathrm{MT}(S), \mathrm{gr}^W_*)$ denote the Tannaka groups (more precisely, pro-group schemes over $\mathbb{Q}$) of $(\mathrm{MT}(X), \mathrm{gr}^W_*)$ and $(\mathrm{MT}(S), \mathrm{gr}^W_*)$. We sometimes omit the "base-point" gr^W_* from the notation.

The functors p^* and x^* gives maps of pro-group schemes over $\mathbb{Q}$

$$p_* : G(\mathrm{MT}(X), \mathrm{gr}^W_*) \to G(\mathrm{MT}(S), \mathrm{gr}^W_*)),\ x_* : G(\mathrm{MT}(S), \mathrm{gr}^W_*) \to G(\mathrm{MT}(X), \mathrm{gr}^W_*).$$

Letting $K = \ker p_*$, we thus have the split exact sequence

$$1 \longrightarrow K \longrightarrow G(\mathrm{MT}(X), \mathrm{gr}^W_*) \underset{x_*}{\overset{p_*}{\rightleftarrows}} G(\mathrm{MT}(S), \mathrm{gr}^W_*) \longrightarrow 1$$

of pro-group schemes over $\mathbb{Q}$. Via the splitting x_*, $G(\mathrm{MT}(S))$ acts by conjugation on K. Thus the pro-affine Hopf algebra $\mathbb{Q}[K]$ is a $G(\mathrm{MT}(S))$-representation. Tannaka duality yields the corresponding ind object in $\mathrm{MT}(S)$, and its dual is a pro-group scheme object in $\mathrm{MT}(S)$, which we denote by K_x. As we have seen above, the Deligne-Goncharov motivic fundamental group $\pi_1^{mot}(X, x)$, is also a pro-group scheme object in $\mathrm{MT}(S)$.

Theorem 6.6.1 *Let X be in $\mathbf{Sm}/S$ with section $x : S \to X$. Suppose that X satisfies the Beilinson-Soulé vanishing conjectures and that the motive $M_{gm}(X)_{\mathbb{Q}} \in DM_{gm}(S)_{\mathbb{Q}}$ is in $\mathrm{DMT}_{gm}(S)$. Then there is a natural isomorphism*

$$\pi_1^{mot}(X, x) \cong K_x$$

as pro-group objects in $\mathrm{MT}(S)$.

Proof As we have seen above, we may identify $G(\mathrm{MT}(X))$ and $G(\mathrm{MT}(S))$ with the Tannaka groups of the categories $\mathcal{H}^f_{\mathcal{N}(X)}$ and $\mathcal{H}^f_{\mathcal{N}(S)}$, respectively. By theorem 1.15.2, this gives an isomorphism of K with the kernel of the map of pro-groups schemes over $\mathbb{Q}$:

$$p_* : \mathrm{Spec}\,(H^0(\bar{B}(\mathcal{N}(X)))) \to \mathrm{Spec}\,(H^0(\bar{B}(\mathcal{N}(S))))$$

induced by

$$H^0(\bar{B}(p^*)) : H^0(\bar{B}(\mathcal{N}(S))) \to H^0(\bar{B}(\mathcal{N}(X)))$$

Similarly, the splitting x_* becomes identified with

$$x_* : \mathrm{Spec}\,(H^0(\bar{B}(\mathcal{N}(S)))) \to \mathrm{Spec}\,(H^0(\bar{B}(\mathcal{N}(X)))).$$

By lemma 2.8.2 and theorem 2.8.3, we have the identification

$$K_x \cong \mathrm{Spec}\,(H^0_{\mathcal{N}(S)}(\bar{B}_{\mathcal{N}(S)}(\mathcal{N}(X), x^*)))$$

as group schemes in $\mathcal{H}_{\mathcal{N}(S)}$, hence as pro-group schemes in $\mathcal{H}^f_{\mathcal{N}(S)}$.

But by theorem 6.5.1, the equivalence

$$H^0(\mathcal{M}^f) : \mathcal{H}^f_{\mathcal{N}(S)} \to \mathrm{MT}(S)$$

identifies $\mathrm{Spec}\,(H^0_{\mathcal{N}(S)}(\bar{B}_{\mathcal{N}(S)}(\mathcal{N}(X), x^*)))$ with $\pi_1^{mot}(X, x)$, completing the proof.

Corollary 6.6.2 *Let k be a number field and $S \subset \mathbb{P}^1(k)$ a finite set of k-points of $\mathbb{P}^1$. Set $X := \mathbb{P}^1_k \setminus S$ and let $a \in X(k)$ be a k-point. Then both k and X satisfy the Beilinson-Soulé vanishing conjectures. Furthermore, there is an isomorphism*

$$\pi_1^{mot}(X, a) \cong K_a$$

as pro-group objects in $\mathrm{MT}(k)$.

Proof k satisfies the Beilinson-Soulé vanishing conjectures by Borel's theorem on the rational K-groups of k [6]. For X, we have the Gysin distinguished triangle

$$M_{gm}(X) \to M_{gm}(\mathbb{P}^1) \to \oplus_{y \in S}\mathbb{Z}(1)[2] \to M_{gm}(X)[1].$$

Taking motivic cohomology gives the long exact sequence

$$\ldots \to \oplus_{x\in S} H^{p-2}(k, \mathbb{Z}(q-1)) \to H^p(k, \mathbb{Z}(q)) \oplus H^{p-2}(k, \mathbb{Z}(q-1))$$
$$\to H^p(X, \mathbb{Z}(q)) \xrightarrow{\partial} \oplus_{x\in S} H^{p-1}(k, \mathbb{Z}(q-1)) \to \ldots$$

Thus the vanishing conjectures for k imply the vanishing conjectures for X. In addition, since $M_{gm}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$, the Gysin exact triangle shows that $M_{gm}(X)_{\mathbb{Q}}$ is in $\mathrm{DMT}(k)$.

We may therefore apply theorem 6.6.1 to give the isomorphism

$$\pi_1^{mot}(X, a) \cong K_a.$$

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