Tate motives and the vanishing conjectures for algebraic K-theory

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Introduction

The existence of an abelian category $\mathbf{TM}_k$ of $\mathbb{Q}$-mixed Tate motives over a field k, whose Ext-groups compute the weight graded pieces of the rational algebraic K-theory of k, implies the following weak form of the vanishing conjectures of Soulé and Beilinson:

Conjecture. *$K_{2q-p}(k)^{(q)} = 0$ for $p < 0$*

If $\mathbf{TM}_k$ has as well a weight filtration structure, the stronger version of Soulé-Beilinson vanishing:

Conjecture. *$K_{2q-p}(k)^(q) = 0$ for $p \leq 0$ and $q > 0$*

is true. On the other hand, we have constructed in [L] a triangulated category $\mathbf{DM}_k$ with objects $\mathbb{Z}(n)$, and with a natural isomorphism

$$\mathrm{Hom}_{\mathbf{DM}_k}(\mathbb{Z}(0), \mathbb{Z}(q)[p]) \otimes \mathbb{Q} \to K_{2q-p}(k)^{(q)}.$$

Letting $\mathbf{DTM}_k$ be the triangulated subcategory of $\mathbf{DM}_k \otimes \mathbb{Q}$ generated by the objects $\mathbb{Z}(n)$, it is natural to attempt to apply the methods of *Faisceaux pervers* [BBD] to construct the category $\mathbf{TM}_k$ as a subcategory of $\mathbf{DTM}_k$. This is the main purpose of this paper.

We will show that, without any assumptions on k, the category $\mathbf{DTM}_k$ has a natural weight structure. Furthermore, if we assume the strong version of the vanishing conjecture above for k, there is a t-structure on $\mathbf{DTM}_k$ with heart $\mathbf{TM}_k$ generated as an abelian category by the Tate objects. In addition, there is a natural duality on $\mathbf{TM}_k$, making $\mathbf{TM}_k$ a Tannakian category. The weight structure on $\mathbf{DTM}_k$ gives rise to a canonical weight filtration on $\mathbf{TM}_k$; the functor $\oplus gr^W_{2a}$ is then a fiber functor from $\mathbf{TM}_k$ to the Tannakian category of finite dimensional graded $\mathbb{Q}$–vector spaces.

What is missing from this picture is the desired relationship between the Ext-groups in $\mathbf{TM}_k$ and the shifted Hom-groups in $\mathbf{DTM}_k$. By our Theorem 4.2, there is, for each $p \geq 1$, a natural map

$$\phi_p \colon \mathrm{Ext}^p_{\mathbf{TM}_k}(M, N) \to \mathrm{Hom}_{\mathbf{DTM}_k}(M, N[p])$$

(assuming the vanishing conjecture for k). ϕ_1 is an isomorphism, and ϕ_2 is injective. Taking $M = \mathbb{Q}(0)$, $N = \mathbb{Q}(q)$, and combining with the isomorphism above, we have the maps

$$\tau_{q,p} \colon \mathrm{Ext}^p_{\mathbf{TM}_k}(\mathbb{Q}(0), \mathbb{Q}(q)) \to K_{2q-p}(k)^{(q)}.$$

If k is a number field, the vanishing conjecture is true; in addition the maps $\tau_{q,p}$ are isomorphisms (see Cor. 4.3). Presumably, the conjecture of Suslin:

For F an algebraically closed field with subfield of constants F_0, $K_*(F)$ is generated by $K^M_*(F)$ and $K_*(F_0)$,

would suffice to show that the $\tau_{q,p}$ are surjective.. Similarly, it would probably suffice that Goncharov's complexes [G] compute the weight graded pieces of $K_*(k) \otimes \mathbb{Q}$. It would be interesting to find a criterion for the maps $\tau_{q,p}$ to be isomorphisms, without directly refering to Suslin's conjecture or Goncharov's complexes.

§1. Categories of Tate type

We begin with a discussion of weight filtrations on certain triangulated $\mathbb{Q}$-categories. This is very much in the spirit of [BGS]; as many of the results we require do not appear explictly in that paper, we include this discussion for the reader's convenience. For the basic notions concerning triangulated categories, we refer the reader to Verdier [V]. For the foundational aspects of tensor categories, we will use Deligne [D] and Saavedra-Rivano [S]. By a triangulated tensor category D over a commutative ring A, we mean a category which is both a triangulated category, and a tensor category over A, with the property:

Let (X, Y, Z, u, v, w) be an exact triangle. Then, for all W in D, $(X\otimes W, Y\otimes W, Z\otimes W, u\otimes id, v\otimes id, w\otimes id)$ is an exact triangle.

Definition 1.1. A triangulated tensor category *of Tate type* (over $\mathbb{Q}$) is a triangulated tensor category T over $\mathbb{Q}$, generated by objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$, together with isomorphisms

$$\mathbb{Q}(n)\otimes\mathbb{Q}(m)\to\mathbb{Q}(n+m),$$

such that

i) $\mathrm{Hom}_T(\mathbb{Q}(n)[a], \mathbb{Q}(m)[b]) = 0$; if $n > m$

ii) $\mathrm{Hom}_T(\mathbb{Q}(n)[a], \mathbb{Q}(n)[b]) = 0$; if $a \neq b$

iii) $\mathrm{Hom}_T(\mathbb{Q}(n), \mathbb{Q}(n)) = \mathbb{Q}\cdot id.$

vi) the isomorphisms

$$\mathbb{Q}(n)\otimes\mathbb{Q}(m)\to\mathbb{Q}(n+m)$$

satisfy the usual compatiblities of associativity and commutativity.

□

Let T be a triangulated $\mathbb{Q}$-tensor category of Tate type. We let $T_{[a,b]}$ be the strictly full triangulated subcategory of T generated by the objects $\mathbb{Q}(n)$, for $a \leq -2n \leq b$; we allow $a = -\infty$, $b = \infty$, and we denote $T_{[a,a]}$ by T_a. The axioms (ii) and (iii) above readily imply that, for a even, the category T_a is equivalent to the derived category of the category of finite dimensional $\mathbb{Q}$-vector spaces (and is zero for a odd).

Definition 1.2. ([BBD], Def. 1.1.1) A *t-structure* $(T^{\leq 0}, T^{\geq 0})$ on a triangulated category T consists of strictly full subcategories $T^{\leq 0}$, $T^{\geq 0}$ of T such that

i) $T^{\leq 0}[1]\subset T^{\leq 0}$ and $T^{\geq 0}[-1]\subset T^{\geq 0}$

ii) For X in $T^{\leq 0}$, Y in $T^{\geq 0}[-1]$, we have $\mathrm{Hom}_T(X, Y) = 0$

iii) each object X of T fits into an exact triangle

$$X^{\leq 0}\to X\to X^{>0}\to X^{\leq 0}[1]$$

with $X^{\leq 0}$ in $T^{\leq 0}$ and $X^{>0}$ in $T^{\geq 0}[-1]$

The t-structure is called *non-degenerate* if

iv) The intersections $\cap_n T^{\geq 0}[n]$ and $\cap_n T^{\leq 0}[n]$ consist only of zero objects. □

We denote $T^{\leq 0}[-n]$ by $T^{\leq n}$, $T^{\geq 0}[-n]$ by $T^{\geq n}$, $T^{\leq n-1}$ by $T^{<n}$ and $T^{\geq n+1}$ by $T^{>n}$. The *heart* of a t-structure $(T^{\leq 0}, T^{\geq 0})$ on T is the full subcategory $T^{\leq 0}\cap T^{\geq 0}$. If T is a triangulated category, A an abelian subcategory of T, we say that A is *admissible* if a sequence

$$0\to M'\xrightarrow{i} M\xrightarrow{j} M''\to 0$$

is exact in A if and only if there is an exact triangle

$$M' \xrightarrow{i} M \xrightarrow{j} M'' \to M'[1]$$

in T.

Let $(T^{\leq 0}, T^{\geq 0})$ be a t-structure on a triangulated category T. Theorem 1.3.6 of [BBD] states that the heart A of $(T^{\leq 0}, T^{\geq 0})$ is a full admissible abelian subcategory of T. In addition, by ([BBD], Prop. 1.3.3), the triangle $X^{\leq 0} \to X \to X^{>0} \to X^{\leq 0}[1]$ of Definition 1.2(iii) is uniquely determined by X, up to unique isomorphism. Sending X to $X^{>0}$ determines an exact functor

$$\tau_{>0}: T \to T^{>0},$$

left adjoint to the inclusion $T^{>0} \to T$; sending X to $X^{\leq 0}$ determines an exact functor

$$\tau_{\leq 0}: T \to T^{\leq 0},$$

right adjoint to the inclusion $T^{\leq 0} \to T$.

Lemma 1.1. *Let D be a triangulated category, with exact triangles*

$$Y_i \xrightarrow{h_i} Z_i \xrightarrow{f_i} X_i \xrightarrow{g_i} Y_i[1],$$

i=1,2, and

$$Z_1 \xrightarrow{f} Z_2 \to Z \to Z_1[1].$$

Suppose $Hom_D(Y_1, X_2) = 0$. Then there are exact triangles

$$Y_1 \to Y_2 \to Y_3 \to Y_1[1]$$

$$X_1 \to X_2 \to X_3 \to X_1[1]$$

and

$$Y_3 \to Z \to X_3 \to Y_3[1].$$

Proof. Since $\mathrm{Hom}_D(Y_1, X_2) = 0$, we have a commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{h_1} & Z_1 \\ \alpha \downarrow & & f \downarrow \\ Y_2 & \xrightarrow{h_2} & Z_2. \end{array}$$

This we can complete (cf. Verdier [V] or [BBD], Prop. 1.1.11) to a diagram with exact rows and columns, and with all the squares commutative, except the square marked with a $*$, which is anti-commutative:

$$\begin{array}{ccccccc} Y_1 & \xrightarrow{h_1} & Z_1 & \xrightarrow{f_1} & X_1 & \xrightarrow{g_1} & Y_1[1] \\ \alpha \downarrow & & f \downarrow & & \downarrow & & \alpha[1] \downarrow \\ Y_2 & \xrightarrow{h_2} & Z_2 & \xrightarrow{f_2} & X_2 & \xrightarrow{g_2} & Y_2[1] \\ \beta \downarrow & & \downarrow & & \downarrow & & \beta[1] \downarrow \\ Y_3 & \to & Z_3 & \to & X_3 & \to & Y_3[1] \\ \gamma \downarrow & & \downarrow & & \downarrow & * & \gamma[1] \downarrow \\ Y_1[1] & \xrightarrow{h_1[1]} & Z_1[1] & \xrightarrow{f_1[1]} & X_1[1] & \xrightarrow{g_1[1]} & Y_1[2]. \end{array}$$

The identity maps on Z_1 and Z_2 give the isomorphism of triangles

$$\begin{array}{ccccccc} Z_1 & \xrightarrow{f} & Z_2 & \to & Z & \to & Z_1[1] \\ || & & || & & g\downarrow & & || \\ Z_1 & \xrightarrow{f} & Z_2 & \to & Z_3 & \to & Z_1[1], \end{array}$$

completing the proof. □

Lemma 1.2. *Let T be a triangulated $\mathbb{Q}$-tensor category of Tate type, and let $a \le b \le c$ be integers (we also allow $a = -\infty$, $c = \infty$). Then $(T_{[a,b-1]}, T_{[b,c]})$ is a t-structure on $T_{[a,c]}$.*

Proof. Since

$$T_{(-\infty,\infty)} = \cup_{-\infty<a\le c<\infty} T_{[a,c]}$$

$$T_{(-\infty,b-1]} = \cup_{-\infty<a<b} T_{[a,b-1]}$$

and

$$T_{[b,\infty)} = \cup_{b\le c<\infty} T_{[b,c]},$$

it suffices to prove the lemma for $a > -\infty$ and $c < \infty$.

We proceed by induction on $b-a-1$ and $c-b$. We first prove (again, by induction on $b-a-1$ and $c-b$) that $\mathrm{Hom}_T(Y, X) = 0$ for X in $T_{[b+m,c+m]}$, Y in $T_{[a-n,b-n-1]}$ and $n, m \ge 0$. Indeed, if $b = c = a+1$, this is just Def. 1.1(i). By induction, $(T_{[a,b-2]}, T_{b-1})$ is a t-structure on $T_{[a,b-1]}$ and $(T_{[b,c-1]}, T_c)$ is a t-structure on $T_{[b,c]}$. Now let X be in $T_{[a,b-1]}$, Y in $T_{[b,c]}$. Thus we have exact triangles

$$X^{\le b-2} \to X \to X^{>b-2} \to X^{\le b-2}[1]$$

and

$$Y^{\le c-1} \to Y \to Y^{>c-1} \to Y^{\le c-1}[1],$$

with $X^{\le b-2}$ in $T_{[a,b-2]}$, $X^{>b-2}$ in T_{b-1}, etc. The vanishing of $\mathrm{Hom}_T(Y, X)$ follows from the long exact sequences of Hom's associated to the two triangles above, together with our induction assumption. In particular, we have verified Definition 1.2(ii)

We now verify Definition 1.2(iii). Let W be the strictly full additive subcategory of $T_{[a,c]}$ generated by objects Z of $T_{[a,c]}$ which fit into an exact triangle

$$Y \to Z \to X \to Y[1],$$

with Y in $T_{[a,b-1]}$ and X in $T_{[b,c]}$. Clearly W is graded and contains $T_{[a,b-1]}$ and $T_{[b,c]}$. Thus, we need only show that if two members of a triangle in $T_{[a,c]}$ are in W, then so is the third.

Suppose then we have Y_i in $T_{[a,b-1]}$, X_i in $T_{[b,c]}$, exact triangles

$$Y_i \xrightarrow{h_i} Z_i \xrightarrow{f_i} X_i \xrightarrow{g_i} Y_i[1],$$

$i = 1, 2$, and an exact triangle

$$Z_1 \xrightarrow{f} Z_2 \to Z \to Z_1[1]$$

By the first part of the proof, we have $\mathrm{Hom}_T(Y_1[n], X_2[m]) = 0$ for all n, m. By Lemma 1.1, we have triangles

$$Y_1 \to Y_2 \to Y_3 \to Y_1[1]$$

$$X_1 \to X_2 \to X_3 \to X_1[1]$$

and

$$Y_3 \to Z \to X_3 \to Y_3[1].$$

Since $T_{[a,b-1]}$ and $T_{[b,c]}$ are strictly full triangulated subcategories of T, Y_3 is in $T_{[a,b-1]}$ and X_3 is in $T_{[b,c]}$. Thus, Z is in W, as desired.

Since $T_{[a,b-1]}$ and $T_{[b,c]}$ are graded subcategories of T, Definition 1.2(i) is immediately verified. This completes the proof. □

Definition 1.3. Denote the truncation functors $\tau^{\leq 0}$, $\tau^{>0}$ for the t-structure $(T_{(-\infty,b]}, T_{[b+1,\infty)})$ on $T_{(-\infty,\infty)}$ by

$$W_{\leq b}: T_{(-\infty,\infty)} \to T_{(-\infty,b]}$$

and

$$W^{>b}: T_{(-\infty,\infty)} \to T_{[b+1,\infty)}.$$

□

For each X in T, we have the exact triangle in T:

$$W_{\leq b}(X) \to X \to W^{>b}(X) \to W_{\leq b}(X)[1]; \tag{1.1}$$

this gives us the functor from T to exact triangles in T:

$$W_{\leq b}(?) \to (?) \to W^{>b}(?) \to W_{\leq b}(?)[1]; \tag{1.2}$$

By the uniqueness of the triangle (1.1), the functors $W_{\leq b}$ and $W^{>b}$ map $T_{[a,c]}$ into $T_{[a,b]}$ and $T_{[b+1,c]}$, respectively. For $a < b$, we have the canonical isomorphisms

$$W_{\leq a}(W_{\leq b}(?)) \to W_{\leq a}(?); \quad W^{>b}(W^{>a}(?)) \to W^{>b}(?), \tag{1.3}$$

and the map of triangles

$$\begin{array}{ccccccc} W_{\leq a}(?) & \to & (?) & \to & W^{>a}(?) & \to & W_{\leq a}(?)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W_{\leq b}(?) & \to & (?) & \to & W^{>b}(?) & \to & W_{\leq b}(?)[1] \end{array} \tag{1.4}$$

We write $W^{\geq b}$ for $W^{>b-1}$. Let $W_{[a,b]}(Z)$ denote $W^{\geq a}(W_{\leq b}(Z))$, and write $gr^W_a(Z)$ for $W_{[a,a]}(Z)$. This determines functors

$$W_{[a,b]}: T \to T_{[a,b]};$$

the restriction of $W_{[a,b]}$ to $T_{[a,b]}$ is isomorphic to the identity. If a is odd, then we set $\mathbb{Q}(-a/2)$ equal to the zero object.

Lemma 1.3. *The functor*

$$\oplus_{i=a}^{b} gr_i^W : T_{[a,b]} \to \oplus_{i=a}^{b} T_i$$

is an exact tensor functor.

Proof. We have already remarked that the functors gr_i^W are exact. The isomorphisms

$$\mathbb{Q}(n) \otimes \mathbb{Q}(m) \to \mathbb{Q}(n+m)$$

give rise to functorial isomorphisms

$$gr_{-2n}^W(X) \otimes gr_{-2m}^W(X) \to gr_{-2(n+m)}^W(X),$$

showing that $\oplus_{i=a}^{b} gr_i^W$ is a tensor functor. □

Definition 1.4. Let T be a triangulated category of Tate type. Let a be even, and let $T_a^{\geq 0}$ be the full subcategory of T_a generated by objects $\mathbb{Q}(-a/2)[n]$ for $n \leq 0$. Similarly, let $T_a^{\leq 0}$ be the full subcategory of T_a generated by objects $\mathbb{Q}(-a/2)[n]$ for $n \geq 0$. For $a \leq b$, let $T_{[a,b]}^{\geq 0}$ be the full subcategory of $T_{[a,b]}$ generated by objects X of $T_{[a,b]}$ with $gr_c^W(X)$ in $T_c^{\geq 0}$ for $a \leq c \leq b$. Let $T_{[a,b]}^{\leq 0}$ be the full subcategory of $T_{[a,b]}$ generated by objects X of $T_{[a,b]}$ with $gr_c^W(X)$ in $T_c^{\leq 0}$ for $a \leq c \leq b$. □

We note that the two definitions of $T_a^{\leq 0}$ and $T_a^{\geq 0}$ are the same, and that $(T_a^{\leq 0}, T_a^{\geq 0})$ is the standard t-structure on T_a, under the equivalence

$$V \mapsto V \otimes \mathbb{Q}(-a/2)$$

of T_a with the bounded derived category of the category $\mathbf{V}_{\mathbb{Q}}$ of finite dimensional $\mathbb{Q}$-vector spaces.. The functors $W_{[c,d]}$ map $T_{[a,b]}^{\leq 0}$ to $T_{[c,d]}^{\leq 0}$ and $T_{[a,b]}^{\geq 0}$ to $T_{[c,d]}^{\geq 0}$. We let $\mathbf{GrV}_{\mathbb{Q}}$ denote the tensor category of graded, finite dimensional $\mathbb{Q}$-vector spaces.

Theorem 1.4. *Suppose T satisfies the vanishing condition:*

$$Hom_T(\mathbb{Q}(r), \mathbb{Q}(s)[n]) = 0 \text{ for } r < s \text{ and } n \leq 0. \tag{1.5}$$

Then

i) $(T_{[a,b]}^{\leq 0}, T_{[a,b]}^{\geq 0})$ *is a non-degenerate t-structure on $T_{[a,b]}$ for all $a \leq b$.*

ii) The heart $A_{[a,b]}$ of $(T_{[a,b]}^{\leq 0}, T_{[a,b]}^{\geq 0})$ contains the objects $\mathbb{Q}(-c/2)$, $a \leq c \leq b$, and is generated as an abelian category by the $\mathbb{Q}(-c/2)$.

iii) Each object X of $A_{[a,b]}$ has a functorial filtration

$$gr_a^W(X) \subset W_{[a,a+1]}(X) \subset \ldots \subset W_{[a,b-1]}(X) \subset X,$$

with quotients $gr_c^W(X)$ in A_c, $a \leq c \leq b$. We call this filtration the weight filtration on X.

v) The functor

$$\bigoplus_{i=a}^{b} gr_i^W : A_{[a,b]} \to \bigoplus_{i=a}^{b} A_i$$

is a faithful exact tensor functor.

Proof. We first claim that $\mathrm{Hom}_T(T^{\leq 0}_{[a+1,b]}, T^{\geq 0}_c) = 0$ for $c \leq a \leq b$. Indeed, let X be in $T^{\geq 0}_{[a,b]}$. We have the exact triangle

$$gr^W_{a+1}(X) \to X \to W_{[a+2,b]}(X)$$

with $gr^W_a(X)$ in $T^{\leq 0}_a$ and $W_{[a+2,b]}(X)$ in $T^{\leq 0}_{[a+2,b]}$. By induction, $\mathrm{Hom}_T(W_{[a+2,b]}(X), T^{\geq 0}_c) = 0$; since $\mathrm{Hom}_T(\mathbb{Q}(-(a+1)/2)[n], \mathbb{Q}(-c/2)[m]) = 0$ for $n \geq m$ by the vanishing hypothesis, and since $gr^W_{a+1}(X)$ is in $T^{\leq 0}_{a+1}$, we have $\mathrm{Hom}_T(gr^W_{a+1}(X), T^{\geq 0}_c) = 0$. Thus $\mathrm{Hom}_T(T^{\leq 0}_{[a+1,b]}, T^{\geq -1}_c) = 0$ as claimed.

We now check Def. 1.2(iii). Let X be in $T_{[a,b]}$. We have the exact triangle

$$gr^W_a(X) \to X \to W_{[a+1,b]}(X) \to gr^W_a(X)[1]$$

and the exact triangle

$$gr^W_a(X)^- \to gr^W_a(X) \to gr^W_a(X)^+,$$

with $gr^W_a(X)^-$ in $T^{\leq 0}_a$ and $gr^W_a(X)^+$ in $T^{\geq 1}_a$ By induction, we have the exact triangle

$$W_{[a+1,b]}(X)^- \to W_{[a+1,b]}(X) \to W_{[a+1,b]}(X)^+,$$

with $W_{[a+1,b]}(X)^-$ in $T^{\leq 0}_{[a+1,b]}$ and $W_{[a+1,b]}(X)^+$ in $T^{\geq 1}_{[a+1,b]}$. Since $\mathrm{Hom}_T(W_{[a+1,b]}(X)^-, gr^W_a(X)^+[1]) = 0$, we may apply Lemma 1.1, giving us exact triangles

$$X^- \to W_{[a+1,b]}(X)^- \to gr^W_a(X)^-[1] \to X^-[1]$$

$$X^+ \to W_{[a+1,b]}(X)^+ \to gr^W_a(X)^+[1] \to X^+[1]$$

and

$$X^- \to X \to X^+ \to X^-[1]$$

Thus, X^- is in $T^{\leq 0}_{[a,b]}$ and X^+ is in $T^{\geq 1}_{[a,b]}$, verifying Def. 1.3(iii).

Def. 1.2(ii) follows from Def. 1.2(iii), and induction, beginning with the result proved in the first paragraph of this proof. Def. 1.2(i) follows directly from Definition 1.4: since $\mathbb{Q}(-a/2)[n]$ is in $T^{\geq 0}_{[a,b]}$ if $n \leq 0$ and is in $T^{\leq 0}_{[a,b]}$ if $n > 0$. The t-structure is non-degenerate since the induced t-structures on the categories T_c are all non-degenerate. This completes the proof of (i).

For (ii), we have already seen that each $\mathbb{Q}(-c/2)$ is in $A_{[a,b]}$. If X is in $A_{[a,b]}$, then the exact triangles (1.1) give rise to the functorial filtration

$$gr^W_b(X) \subset W_{[b-1,b]}(X) \subset \ldots \subset W_{[a+1,b]}(X) \subset X,$$

with quotients $gr^W_c(X)$, $a \leq c \leq b$. Since $gr^W_c(X)$ is in $A_{[c,c]}$, which is generated as an additive category by $\mathbb{Q}(-c/2)$, we see that the $\mathbb{Q}(-c/2)$ generate $A_{[a,b]}$. This proves (ii) and (iii).

To prove (iv), it suffices by Lemma 1.3 to show that $\oplus gr^W_c$ is faithful. It suffices by induction to show that the functor

$$W_{[a,b-1]} \oplus gr^W_b \colon A_{[a,b]} \to A_{[a,b-1]} \oplus A_b$$

is faithful. For this, we first note that $\mathrm{Hom}_{A_{[a,b]}}(X, Y) = 0$ for X in A_b, Y in $A_{[a,b-1]}$. Indeed, this follows from the vanishing hypothesis (1.5), together with the exact sequences of Hom's arising from the weight filtration on Y. Let $f: X \to Y$ be a map in $A_{[a,b]}$ and suppose $gr^W_b(f) = W_{[a,b-1]}(f) = 0$. Then we can factor f as a composition

$$X \to gr^W_b(X) \xrightarrow{\alpha} W_{[a,b-1]}(Y) \to Y;$$

since $\alpha = 0$, we have $f = 0$, completing the proof of (iv) and the theorem. □

Sending $\mathbb{Q}(i)$ to the one dimensional vector space over $\mathbb{Q}$ defines an equivalence of A_i with $\mathbf{V}_{\mathbb{Q}}$, so $\bigoplus_{i=a}^{b} gr_i^W$ defines an equivalence of $A_{[a,b]}$ with a subcategory of $\mathbf{GrV}_{\mathbb{Q}}$.

Let A be a full, admissible abelian subcategory of a triangulated category D. We will now examine the relationship between $\mathrm{Ext}^p_A(M,N)$ and $\mathrm{Hom}_D(M,N[p])$, for objects M and N of A. This has been carried out in [BBD]; we include the discussion here for the reader's convenience.

Each short exact sequence $0 \to M' \to M \to M'' \to 0$ in A extends uniquely to an exact triangle $M' \to M \to M'' \to M'[1]$ in D. Indeed, if we have two extensions, say

$$M' \to M \to M'' \xrightarrow{\alpha} M'[1]$$

and

$$M' \to M \to M'' \xrightarrow{\beta} M'[1],$$

the identity maps on M' and M extend to a map of triangles

$$\begin{array}{ccccccc} M' & \to & M & \to & M'' & \xrightarrow{\alpha} & M'[1] \\ || & & || & & g\downarrow & & || \\ M' & \to & M & \to & M'' & \xrightarrow{\beta} & M'[1], \end{array}$$

for some map g. Since A is a full subcategory, and the map $M \to M''$ is surjective, we have $g = \mathrm{id}$. Thus $\alpha = \beta$, as claimed. More generally, let

$$0 \to M_0 \to M_1 \to \ldots \to M_{n+1} \to 0$$

be a long exact sequence in A. Breaking this sequence up into a series of short exact sequences, we get a uniquely defined map $M_{n+1} \to M_0[n]$. Letting $\mathrm{Seq}^n_A(M,N)$ denote the set of long exact sequences as above, with $M_0 = N$ and $M_{n+1} = M$, we have defined a map

$$\hat{\phi}_n \colon \mathrm{Seq}^n_A(M,N) \to \mathrm{Hom}_D(M,N[n]).$$

Lemma 1.5. *Suppose we have a commutative ladder*

$$\begin{array}{cccccccc} 0\to & M_0 & \to & M_1 & \to & \ldots & \to & M_{n+1} & \to 0 \\ & f_0\downarrow & & f_1\downarrow & & & & f_{n+1}\downarrow & \\ 0\to & N_0 & \to & N_1 & \to & \ldots & \to & N_{n+1} & \to 0, \end{array}$$

with exact rows. Let M_* denote the first row, and N_* the second. Then

$$\hat{\phi}_n(N_*) \circ f_{n+1} = f_0[n] \circ \hat{\phi}_n(M_*).$$

Proof. Let $M = \mathrm{cok}(M_0 \to M_1)$ and $N = \mathrm{cok}(N_0 \to N_1)$. Then we have

$$\hat{\phi}_n(M_*) = \hat{\phi}_1(M_0 \to M_1 \to M)[n-1] \circ \hat{\phi}_{n-1}(M \to M_2 \to \ldots \to M_{n+1}),$$

and similarly for $\hat{\phi}_n(N_*)$. By induction, this reduces us to the case $n = 1$.

Each ladder

$$\begin{array}{ccccccccc} 0\to & M_0 & \to & M_1 & \to & M_2 & \to 0 \\ & f_0\downarrow & & f_1\downarrow & & f_2\downarrow & \\ 0\to & N_0 & \to & N_1 & \to & N_2 & \to 0 \end{array}$$

extends to the map of triangles

$$\begin{array}{ccccccccc} 0\to & M_0 & \to & M_1 & \to & M_2 & \overset{\hat\phi_1(M_*)}{\to} & M_0[1] \\ & f_0\downarrow & & f_1\downarrow & & g\downarrow & & f_0[1]\downarrow \\ 0\to & N_0 & \to & N_1 & \to & N_2 & \overset{\hat\phi_1(N_*)}{\to} & N_0[1], \end{array}$$

for some map g. Since the maps $M_1 \to M_2$ and $N_1 \to N_2$ are surjective, we have $g = f_2$, proving the case $n = 1$, and completing the proof. □

Proposition 1.6. *Let A be a full admissible abelian subcategory of a triangulated category D. Let M and N be in A. Then the map $\hat\phi_n$ descends to a homomorphism*

$$\phi_n\colon Ext^n_A(M,N) \to Hom_D(M,N[n]).$$

In addition, if A is closed under extensions in D, then

$$\phi_1\colon Ext^1_A(M,N) \to Hom_D(M,N[1]).$$

is an isomorphism, and

$$\phi_2\colon Ext^2_A(M,N) \to Hom_D(M,N[2])$$

is an injection. Here Ext is the Yoneda Ext.

Proof. Suppose we have a commutative ladder

$$\begin{array}{cccccccccc} 0\to & N & \to & M_1 & \to & \dots & \to & M_n & \to & M & \to 0 \\ & \mathrm{id}_N\downarrow & & f_1\downarrow & & & & f_n\downarrow & & \mathrm{id}_M\downarrow & \\ 0\to & N & \to & N_1 & \to & \dots & \to & N_n & \to & M & \to 0, \end{array}$$

with exact rows M_* and N_*. By Lemma 1.5, $\hat\phi_n(M_*) = \hat\phi_n(N_*)$. As $\mathrm{Ext}^n_A(M,N)$ is the quotient of the set $\mathrm{Seq}^n_A(M,N)$ by the relations given by commutative ladders as above, we have shown that $\hat\phi_n$ descends to the map ϕ_n, as claimed. The addition in $\mathrm{Ext}^n_A(M,N)$ is gotten by taking direct sums of sequences, pushing out by the sum $N \oplus N \to N$ and pulling back by the diagonal $M \to M \oplus M$. As $f + g : M \to N[n]$ is the composition

$$M \overset{\Delta}{\to} M \oplus M \overset{f\oplus g}{\to} N[n] \oplus N[n] \overset{\Sigma}{\to} N[n],$$

applying Lemma 1.5 shows that ϕ_n is a homomorphism.

To show ϕ_1 is an isomorphism, let $\alpha\colon M \to N[1]$ be a map in D. If

$$N \to E \to M \overset{\alpha}{\to} N[1]$$

$$N \to E' \to M \xrightarrow{\alpha} N[1]$$

are two triangles in D, then we have the map of triangles

$$\begin{array}{ccccccc} N & \to & E & \to & M & \xrightarrow{\alpha} & N[1] \\ \| & & \downarrow f & & \| & & \| \\ N & \to & E' & \to & M & \xrightarrow{\alpha} & N[1] \end{array} .$$

This shows that filling in $\alpha\colon M \to N[1]$ to a triangle $N \to E \to M \xrightarrow{\alpha} N[1]$, and taking the extension class of the short exact sequence $0 \to N \to E \to M \to 0$ in A defines an inverse to ϕ_1.

Finally, to show that ϕ_2 is injective, we first note that, if $N \subset B \subset X$ is a filtration in A, then the class of the exact sequence

$$0 \to N \to B \to X/N \to X/B \to 0$$

in $\mathrm{Ext}^2_A(X/B, N)$ is zero. Now suppose

$$M_* = 0 \to N \to M_1 \to M_2 \to M \to 0$$

is in $\mathrm{Seq}^2_A(M, N)$, with $\phi_2(M_*) = 0$. Let $C = \mathrm{cok}(N \to M_1)$. Then we have the sequence

$$M[-1] \xrightarrow{\alpha} C \xrightarrow{\beta} N[1],$$

with $\alpha = \phi_1(C \to M_2 \to M)[-1]$, $\beta = \phi_1(N \to M_1 \to C)$, and $\beta \circ \alpha = 0$. This gives the commutative square

$$\begin{array}{ccc} M[-1] & \xrightarrow{\alpha} & C \\ \downarrow & & \downarrow \beta \\ 0 & \to & N[1] \end{array} ,$$

which we fill in to a map of triangles

$$\begin{array}{ccccc} M[-1] & \xrightarrow{\alpha} & C & \to & M_2 \\ \downarrow & & \downarrow \beta & & \downarrow \gamma \\ 0 & \to & N[1] & = & N[1]. \end{array} \tag{1.6}$$

Filling in $\gamma\colon M_2 \to N[1]$ to a triangle

$$X \to M_2 \to N[1],$$

we have the short exact sequence in A

$$0 \to N \to X \to M_2 \to 0.$$

The octahedral axiom applied to the map of triangles (1.6) gives the triangle

$$M[-1] \to M_1 \to X,$$

giving the short exact sequence in A

$$0 \to M_1 \to X \to M \to 0,$$

and the commutative diagram

$$\begin{array}{ccccccc} 0 \to & N & \to & X & \to & M_2 \to 0 \\ & \downarrow & & || & & \downarrow \\ 0 \to & M_1 & \to & X & \to & M \to 0 \end{array}$$

This gives the filtration $N \subset M_1 \subset X$ trivializing the element of $\mathrm{Ext}^2(M, N)$ defined by M_*, which shows that ϕ_2 is injective. □

§2. Duality

Our object in this section is to show that there is a canonical duality on the heart of a triangulated category of Tate type, assuming the vanishing condition (1.5) of Theorem 1.4. We begin with some generalities on duality in tensor categories.

Let A be a small tensor category with unit 1. A *duality* D on A is a map $D: Obj(A) \to Obj(A)$, $D(X) = X^D$, together with maps

$$1 \stackrel{\delta_X}{\to} X \otimes X^D \stackrel{\epsilon_X}{\to} 1,$$

such that the composition

$$X \to 1 \otimes X \stackrel{\delta\otimes id}{\to} X \otimes X^D \otimes X \stackrel{id\otimes\tau}{\to} X \otimes X \otimes X^D \stackrel{id\otimes\epsilon}{\to} X \otimes 1 \to X$$

is the identity. We assume in addition that δ_1 and ϵ_1 are the isomorphisms given by the product $1 \otimes 1 \to 1$.

A duality D on A gives rise to homomorphisms

$$\delta_{(M,N,Z)}: \mathrm{Hom}_A(Z \otimes M, N) \to \mathrm{Hom}_A(Z, N \otimes M^D);\ \epsilon_{(M,N,Z)}: \mathrm{Hom}_A(M, Z \otimes N) \to \mathrm{Hom}_A(M \otimes N^D, Z)$$

by taking the compositions

$$\mathrm{Hom}_A(Z \otimes M, N) \to \mathrm{Hom}_A(Z \otimes M \otimes M^D, N \otimes M^D) \stackrel{id_Z\otimes\delta_M^*}{\longrightarrow} \mathrm{Hom}_A(Z, N \otimes M^D)$$

and

$$\mathrm{Hom}_A(M, Z \otimes N) \to \mathrm{Hom}_A(M \otimes N^D, Z \otimes N \otimes N^D) \stackrel{id_Z\otimes\epsilon_{N*}}{\longrightarrow} \mathrm{Hom}_A(M \otimes N^D, Z).$$

We let $D_{M,N,Z}: \mathrm{Hom}_A(Z\otimes M, N) \to \mathrm{Hom}_A(Z\otimes N^D, M^D)$ be the map $\epsilon_{Z,N,M^D} \circ \tau^*_{N,M^D} \circ \delta_{M,N,Z}$. We have as well the canonical map $M \to M^{D^D}$ given by $\delta_{M^D,1,M} \circ \epsilon_{M,M,1}(id_M)$. We call D a *perfect* duality if $\delta_{(M,N,Z)}$ and $\epsilon_{(M,N,Z)}$ are isomorphisms for all M, N and Z. The maps $D_{M,N,Z}$ and the map $M \to M^{D^D}$ are then isomorphisms for all M, N and Z.

If A has a perfect duality D, we can define an internal Hom on A by $\mathit{Hom}(M,N) = N \otimes M^D$. This gives A the structure of a rigid Tannakian category.

If A is a graded tensor category, a duality D is a *graded* duality on A if $M[n]^D = M^D[-n]$, and the maps $\delta_{M[n]}$ and $\epsilon_{M[n]}$ are the maps δ_M and ϵ_M composed with the isomorphism $M[n]\otimes M^D[-n] \cong M\otimes M^D$.

Let T be a triangulated category of Tate type, satisfying the vanishing conditions of Theorem 1.4, A the heart of $(T^{\leq 0}, T^{\geq 0})$. The graded category $gr(A) := \oplus_c A_c$ is equivalent to the category of finite dimensional (even) graded $\mathbb{Q}$-vector spaces; fixing an equivalence, we denote the object corresponding to a vector space V in degree $-2a$ (V_a) by $V \otimes \mathbb{Q}(a)$. Similarly, the category $\oplus_c T_c$ is equivalent to the bounded derived category $D^b\mathbf{GrV}_{\mathbb{Q}}$, we denote the object corresponding to $V[n]$ in degree $-2a$ by $V \otimes \mathbb{Q}(a)[n]$. For a $\mathbb{Q}$-vector space V, denote the dual by V^D; this gives the usual duality in $\mathbf{GrV}_{\mathbb{Q}}$ by $(V_a)^D = (V^D)_{-a}$. This defines the duality in $\oplus_c A_c$ by

$$(V \otimes \mathbb{Q}(a))^D = V^D \otimes \mathbb{Q}(-a),$$

where the maps

$$1 \to (V \otimes \mathbb{Q}(a)) \otimes (V \otimes \mathbb{Q}(a))^D \to 1$$

defining the duality isomorphisms

$$\mathrm{Hom}_{gr(A)}(V \otimes \mathbb{Q}(a), W \otimes \mathbb{Q}(a)) \to \mathrm{Hom}_{gr(A)}(1, (W \otimes \mathbb{Q}(a)) \otimes (V \otimes \mathbb{Q}(a))^D)$$

and

$$\mathrm{Hom}_{gr(A)}(V \otimes \mathbb{Q}(a), W \otimes \mathbb{Q}(a)) \to \mathrm{Hom}_{gr(A)}((W \otimes \mathbb{Q}(a))^D \otimes (V \otimes \mathbb{Q}(a)), 1)$$

are given by the duality maps

$$\mathbb{Q} \to V \otimes V^D \to \mathbb{Q}$$

combined with the isomorphism

$$(V \otimes \mathbb{Q}(a)) \otimes (V^D \otimes \mathbb{Q}(-a)) \to (V \otimes V^D) \otimes (\mathbb{Q}(a) \otimes \mathbb{Q}(-a)) \to (V \otimes V^D) \otimes \mathbb{Q}(0)$$

and the isomorphism

$$\mathbb{Q} \otimes \mathbb{Q}(0) \to \mathbb{Q}(0) \to 1.$$

This extends in the obvious way to a graded duality on $\oplus_c T_c$. We let $A[*]$ denote the full graded subcategory of T generated by A

Proposition 2.1. *There is a perfect graded duality $D: A[*] \to A[*]$ on $A[*]$ such that the functor*

$$\oplus_a gr_a: A \to gr(A)$$

is compatible with the duality functors. The duality D is unique up to natural isomorphism.

Proof. We may assume by induction that we have defined a perfect duality D on the full additive subcategory $A(n)$ of A generated by objects those X in A having a weight filtration of length less than n; the induction starts by defining D on $A(1)$ via the isomorphism $A(1) \to gr(A)$, and using the duality we have defined above on $gr(A)$. D extends canonically to a duality on the graded category $A(n)[*]$ generated by $A(n)$. We may also assume that the maps $\delta_{M,N,Z}$ are isomorphisms for all M in $A(n)[*]$, and for all N and Z in $A[*]$. Similarly, we may assume that the maps $\epsilon_{M,N,Z}$ are isomorphisms for all N in $A(n)[*]$, and for all M and Z in $A[*]$; in particular, D defines a perfect duality on $A(n)[*]$.

Let X be in $A(n+1)$, and let a be the minimum integer for which $W_{\leq a}(X)$ is not zero. We have the exact sequence

$$0 \to gr_a(X) \to X \to W^{>a}(X) \to 0; \tag{2.1}$$

clearly $W^{>a}(X)$ is in $A(n)$. Thus, we have maps

$$1 \xrightarrow{\delta_{W^{>a}(X)}} W^{>a}(X) \otimes W^{>a}(X)^D \xrightarrow{\epsilon_{W^{>a}(X)}} 1$$

$$1 \xrightarrow{\delta_{gr_a(X)}} gr_a(X) \otimes gr_a(X)^D \xrightarrow{\epsilon_{gr_a(X)}} 1$$

giving the duality isomorphisms on the appropriate Hom groups.

The exact sequence (2.1) defines the map $\alpha: W^{>a}(X) \to gr_a(X)[1]$, which in turn defines the map $\alpha^D: gr_a(X)^D[-1] \to W^{>a}(X)^D$. Let X^D be an object of A fitting into the exact triangle

$$W^{>a}(X)^D \to X^D \to gr_a(X)^D \xrightarrow{\alpha^D[1]} W^{>a}(X)^D[1].$$

The exact sequence

$$0 \to W^{>a}(X)^D \to X^D \to gr_a(X)^D \to 0$$

is uniquely isomorphic to the sequence

$$0 \to W_{<-a}(X^D) \to X^D \to W^{\geq -a}(X) \to 0,$$

hence X^D is determined by X up to unique isomorphism.

We have the filtration

$$W_{<0}(X \otimes X^D) \subset W_{\leq 0}(X \otimes X^D) \subset X \otimes X^D,$$

and isomorphisms

$$W_{<0}(X \otimes X^D) \cong gr_a^W(X) \otimes W^{>a}(X)^D; W_{\leq 0}(X \otimes X^D) \cong gr_a^W(X) \otimes X^D + X \otimes W^{>a}(X)^D$$

and

$$gr_0^W(X \otimes X^D) \cong gr_a^W(X) \otimes (gr_a^W(X))^D \oplus W^{>a}(X) \otimes W^{>a}(X)^D.$$

In addition, via these isomorphisms, we can identify the map γ in the triangle

$$W_{<0}(X \otimes X^D) \to W_{\leq 0}(X \otimes X^D) \to gr_0^W(X \otimes X^D) \xrightarrow{\gamma} W_{<0}(X \otimes X^D)[1]$$

with the map

$$1 \otimes \alpha^D[1] + \omega \circ \alpha \otimes 1 \colon gr_a^W(X) \otimes (gr_a^W(X))^D \oplus W^{>a}(X) \otimes W^{>a}(X)^D \to gr_a^W(X) \otimes W^{>a}(X)^D[1],$$

where $\omega \colon gr_a^W(X)[1] \otimes W^{>a}(X)^D \to gr_a^W(X) \otimes W^{>a}(X)^D[1]$ is the canonical isomorphism.

One computes directly that

$$(1 \otimes \alpha^D[1]) \circ \delta_{gr_a(X)} = -(\omega \circ \alpha \otimes 1) \circ \delta_{W^{>a}(X)},$$

the minus sign coming from the shift by 1. Thus the map

$$\delta_{gr_a(X)} \oplus \delta_{W^{>a}(X)} \colon 1 \to gr_0(X \otimes X^D)$$

lifts to a map $\delta \colon 1 \to W_{\leq 0}(X \otimes X^D)$; since $\mathrm{Hom}_A(1, W_{<0}(A)) = 0$, the lifting is unique. This defines the map

$$\delta_X \colon 1 \to X \otimes X^D.$$

The constuction of $\epsilon_X \colon X \otimes X^D \to 1$ is similar, and is left to the reader.

The above construction is clearly compatible, via the functor $gr \colon A \to gr(A)$, with the perfect dualtity on $gr(A)$. As gr is faithful, this implies that we have extended D to a duality on $A(n+1)$, which then extends canonically to a duality on $A(n+1)[*]$. If M is in $A(n+1)$, we have the short exact sequences

(2.2)
$$0 \to gr_a^W(M) \to M \to W^{>a}(M) \to 0$$
$$0 \to W^{>a}(M)^D \to M^D \to gr_a^W(M)^D \to 0$$

where a is the minimal integer such that $gr_a^W(M) \neq 0$. Using the long exact sequences of Hom's coming from the appropriate short exact sequences, and using induction, we find that the maps $\delta_{N,M,Z}$ and $\epsilon_{N,M,Z}$ are isomorphisms for all N and Z in $A[*]$. Thus, D defines a perfect duality on $A(n+1)[*]$. The induction therefore goes through, completing the proof. □

§3. The triangulated category of Tate motives

In this section we recall some aspects of the construction of the motivic triangulated category **DM** given in [L], and describe the triangulated Tate category **DTM** as a subcategory of **DM**.

Fix a base field k, and let $\mathbf{Sch}_k$ denote the category of smooth, quasi-projective schemes over k. Let $\mathbf{PSch}_k$ denote the category of pairs (X, F), where X is in $\mathbf{Sch}_k$ and F is a closed subset of X; a map of pairs $f\colon (X, F) \to (Y, G)$ is a map $f\colon X \to Y$ with $f^{-1}(G) \subset F$. The category $\mathbf{DM}_k$ is a triangulated tensor category, equipped with a functor

$$\mathbb{Z}^{mot}\colon \mathbf{PSch}_k^{op} \times \mathbb{Z} \to \mathbf{DM}_k.$$

We write $\mathbb{Z}^{mot}_{X,F}(n)$ for $\mathbb{Z}^{mot}((X, F), n)$, and for a morphism p in $\mathbf{PSch}_k$, we denote $\mathbb{Z}^{mot}(p)$ by p^*. We denote $\mathbb{Z}^{mot}_{X,X}(n)$ by $\mathbb{Z}^{mot}_X(n)$, and $\mathbb{Z}^{mot}_{Spec(k)}(n)$ by $\mathbb{Z}(n)$; $\mathbb{Z}(0)$ is the unit for the tensor structure on $\mathbf{DM}_k$.

The functor $\mathbb{Z}^{mot}$ satisfies the axioms for a twisted duality theory (see Gillet [G]) in the following sense:

a) (Homotopy) Let $p\colon \mathbb{A}^1_X \to X$ be the projection. Then

$$p^*\colon \mathbb{Z}^{mot}_{X,F}(n) \to \mathbb{Z}^{mot}_{\mathbb{A}^1_X, \mathbb{A}^1_F}(n)$$

is an isomorphism for every n.

b) (Localization) Let $F \subset G$ be closed subsets of some X in $\mathbf{Sch}_k$, let $j\colon U \to X$ be the inclusion of $X \backslash F$, let $H = G \backslash F$, and let $i_*\colon \mathbb{Z}^{mot}_{X,F}(n) \to \mathbb{Z}^{mot}_{X,G}(n)$ be the map induced by the identity on X. Then the sequence

$$\mathbb{Z}^{mot}_{X,F}(n) \xrightarrow{i_*} \mathbb{Z}^{mot}_{X,F}(n) \xrightarrow{j^*} \mathbb{Z}^{mot}_{U,H}(n)$$

extends canonically to an exact triangle

$$\mathbb{Z}^{mot}_{X,G}(n) \xrightarrow{i_*} \mathbb{Z}^{mot}_{X,F}(n) \xrightarrow{j^*} \mathbb{Z}^{mot}_{U,H}(n) \to \mathbb{Z}^{mot}_{X,G}(n)[1].$$

c) (Künneth formula) There are functorial exterior products

$$\Box : \mathbb{Z}^{mot}_{X,F}(n) \otimes \mathbb{Z}^{mot}_{Y,G}(m) \to \mathbb{Z}^{mot}_{X\times Y, F\times G}(n+m)$$

which are isomorphisms. This defines the cup product

$$\cup\colon \mathbb{Z}^{mot}_{X,F}(n) \otimes \mathbb{Z}^{mot}_{X,G}(m) \to \mathbb{Z}^{mot}_{X,F\cap G}(n+m)$$

by $\cup = \Delta^* \circ \Box$.

d) (Poincaré duality) Let $i\colon Z \to X$ be a closed embedding of pure codimension d in $\mathbf{Sch}_k$, F a closed subset of Z. There is an isomorphism

$$i_*\colon \mathbb{Z}^{mot}_{Z,F}(n) \to \mathbb{Z}^{mot}_{X,F}(n+d)[2d].$$

e) (cycle classes) For (X, F) in $\mathbf{PSch}_k$, let $\mathcal{Z}^d_F(X)$ denote the group of codimension d cycles on X, supported on F, $Ch^d_F(X)$ the quotient group of cycles modulo rational equivalence (on F). There is a homomorphism

$$cl\colon Ch^d_F(X) \to \mathrm{Hom}_{\mathbf{DM}}(\mathbb{Z}(0), \mathbb{Z}^{mot}_{Z,F}(d)[2d]).$$

f) (projective bundle formula) Let $p: P_X \to X$ be a $\mathbb{P}^n$-bundle over X in $\mathbf{Sch}_k$, F a closed subset of X, and $P_F = p^{-1}(F)$. Let $\xi = cl(\mathcal{O}(1))$. Then the maps

$$(\xi^i \cup ?) \circ p^*: \mathbb{Z}^{mot}_{X,F}(q-i)[-2i] \to \mathbb{Z}^{mot}_{P_X,P_F}(q)$$

define an isomorphism

$$\bigoplus_{i=0}^{n} \mathbb{Z}^{mot}_{X,F}(q-i)[-2i] \to \mathbb{Z}^{mot}_{P_X,P_F}(q).$$

g) (projective pushforward) For a $\mathbb{P}^n$-bundle as in (f), let

$$p_*: \mathbb{Z}^{mot}_{P_X,P_F}(q) \to \mathbb{Z}^{mot}_{X,F}(q-n)[-2n]$$

be the inverse of the isomorphism of (f), followed by projection on the factor $\mathbb{Z}^{mot}_{X,F}(q-n)[-2n]$. Let $f: Y \to X$ be a projective map in $\mathbf{Sch}_k$, G a closed subset of Y and F a closed subset of X containing $f(G)$. Factor f as $p \circ i$, with $i: Y \to P$ a closed embedding, and $p: P \to X$ a projective bundle. Let $d = \dim(Y) - \dim(X)$. Then the composition

$$p_* \circ i_*: \mathbb{Z}^{mot}_{Y,G}(q) \to \mathbb{Z}^{mot}_{X,F}(q-d)[-2d]$$

is independent of the factorization of f. Defining f_* as this composition, we have $f_* \circ g_* = (f \circ g)_*$.

h) (projection formula) Let $f: Y \to X$ be projective. Then

$$f_* \circ (f^*(?) \cup (?)) = (?) \cup f_*(?)$$

as maps from $\mathbb{Z}^{mot}_{X,F}(n) \otimes \mathbb{Z}^{mot}_{Y,G}(m)$ to $\mathbb{Z}^{mot}_{X,F\cap f(G)}(n+m-d)[-2d]$. In addition, the cycle class map cl is compatible with pullback and pushforward. □

For a sub-ring A of $\mathbb{C}$, we let $\mathbf{DM}_k \otimes A$ denote the category with the same objects as $\mathbf{DM}_k$, with $\mathrm{Hom}_{\mathbf{DM}_k \otimes A}(X,Y) = \mathrm{Hom}_{\mathbf{DM}_k}(X,Y) \otimes A$. If A is flat over $\mathbb{Z}$, this defines a triangulated tensor category over A, satisfying the properties (a)-(h) above. We denote the object $\mathbb{Z}^{mot}_{X,F}(n)$, considered as an object of $\mathbf{DM}_k \otimes A$, by $A^{mot}_{X,F}(n)$.

We define the motivic cohomology groups, $H^p_{mot}(X,F,\mathbb{Z}(q))$, by

$$H^p_{mot}(X,F,\mathbb{Z}(q)) = \mathrm{Hom}_{\mathbf{DM}_k}(\mathbb{Z}(0), \mathbb{Z}^{mot}_{X,F}(q)[p]).$$

For A flat over $\mathbb{Z}$ as above, set $H^p_{mot}(X,F,A(q)) := H^p_{mot}(X,F,\mathbb{Z}(q)) \otimes A = \mathrm{Hom}_{\mathbf{DM}_k \otimes A}(A(0), A^{mot}_{X,F}(q)[p])$.

The category $\mathbf{DM}_k$ has a universal mapping property for cohomology theories which are constructed in a certain way, which we won't spell out here. In particular, the theories of singular and étale cohomology, as well as Beilinson's absolute Hodge cohomology, admit realization functors from the category $\mathbf{DM}_k$. These in turn give rise to functorial maps

$$Re_{\mathcal{B},\sigma}: H^p_{mot}(X,F,\mathbb{Z}(q)) \to H^p_{\mathcal{B}}(X^\sigma(\mathbb{C}), F^\sigma(\mathbb{C}), \mathbb{Z}(q))$$

$$Re_{ét}: H^p_{mot}(X,F,\mathbb{Z}(q)) \to H^p_{ét}(X,F,\mathbb{Z}_l(q))$$

and

$$Re_{\mathcal{H},\sigma}\colon H^p_{mot}(X,F,\mathbb{Z}(q)) \to H^p_{\mathcal{H}}(X^\sigma(\mathbb{C}),F^\sigma(\mathbb{C}),\mathbb{Z}(q)).$$

Here σ is an embedding of k into $\mathbb{C}$ and $H^p_{\mathcal{B}}$, $H^p_{\acute{e}t}$ and $H^p_{\mathcal{H}}$ denote the Betti, étale and Hodge cohomology, respectively. We can define the mod-n theory $H^p_{mot}(X,F,\mathbb{Z}/n(q))$ by defining the object $\mathbb{Z}^{mot}_{X,F}(q)/n$, fitting into an exact triangle

$$\mathbb{Z}^{mot}_{X,F}(q)/n \to \mathbb{Z}^{mot}_{X,F}(q) \xrightarrow{\times n} \mathbb{Z}^{mot}_{X,F}(q),$$

and setting

$$H^p_{mot}(X,F,\mathbb{Z}/n(q)) = \mathrm{Hom}_{\mathbf{DM}_k}(\mathbb{Z}(0),\mathbb{Z}^{mot}_{X,F}(q)/n[p]).$$

The realization functors $Re_{\mathcal{B},\sigma}$ and $Re_{\acute{e}t}$ extend to realization functors

$$Re_{\mathcal{B},\sigma}\colon H^p_{mot}(X,F,\mathbb{Z}/n(q)) \to H^p_{\mathcal{B}}(X^\sigma(\mathbb{C}),F^\sigma(\mathbb{C}),\mathbb{Z}/n(q))$$

$$Re_{\acute{e}t}\colon H^p_{mot}(X,F,\mathbb{Z}/n(q)) \to H^p_{\acute{e}t}(X,F,\mathbb{Z}/n(q)),$$

compatible with the appropriate Bockstein sequences.

Let $Ch^q(X,n)$ denote Bloch's higher Chow group of codimension q cycles on the "algebraic n-sphere" S^n_X over X (see [B] for details). For a closed subset F of X, let $Ch^q(X,F,n)$ denote the higher Chow group of codimension q cycles on S^n_X, with support on S^n_F. Combining the cycle class map cl with the homotopy property (a), we arrive at the homomorphism

$$cl_{q,p}\colon Ch^q(X,F,2q-p) \to H^p_{mot}(X,F,\mathbb{Z}(q)).$$

Theorem 1(Theorem 5.2 of [L])**.** *Let A be a sub-ring of $\mathbb{C}$, flat over $\mathbb{Z}$, such that Bloch's higher Chow groups $Ch^q(?,n)\otimes A$ satisfy the localization property. Then the map*

$$cl_{q,p}\colon Ch^q(X,F,2q-p)\otimes A \to H^p_{mot}(X,F,A(q))$$

is an isomorphism. □

Presumably, Bloch's higher Chow groups, (or some suitable modification) satisfy the localization property over $\mathbb{Z}$, but this is not at present known. Since the higher Chow group $Ch^q(X,F,p)\otimes\mathbb{Q}$ agrees with the weight q portion of $G_p(F)\otimes\mathbb{Q}$, it follows that Bloch's higher Chow groups $Ch^q(?,n)\otimes\mathbb{Q}$ satisfy the localization property. Thus we have

Corollary 3.1. *The map*

$$cl_{q,p}\colon Ch^q(X,2q-p)\otimes\mathbb{Q} \to H^p_{mot}(X,\mathbb{Q}(q))$$

defines an isomorphism of $K_{2q-p}(X)^{(q)}$ with $H^p_{mot}(X,\mathbb{Q}(q))$. □

Definition 3.1. Let $\mathbf{DTM}_k$, the triangulated category of *Tate motives*, be the strictly full subcategory of $\mathbf{DM}_k\otimes\mathbb{Q}$ generated by the objects $\mathbb{Q}(n)$ for $n\in\mathbb{Z}$. □

§4. Tate motives and the vanishing conjectures

In this section, we apply the results of §1 and §2 to the category $\mathbf{DTM}_k$.

By Corollary 3.1, we have the isomorphism

$$K_{2q-p}(k)^{(q)} \to \mathrm{Hom}_{\mathbf{DTM}_k}(\mathbb{Q}(n), \mathbb{Q}(n+q)[p]). \tag{4.1}$$

Since $K_{2q-p}(k)^{(q)} = 0$ for $q < 0$, $K_p(k)^{(0)} = 0$ for $p \neq 0$ and $K_0(k)^{(0)} = \mathbb{Q}$, we have

Theorem 4.1. *The category $\mathbf{DTM}_k$ is a $\mathbb{Q}$-triangulated tensor category of Tate type.*

We recall the strong version of the vanishing conjectures of Soulé and Beilinson:

Conjecture. *$K_{2q-p}(k)^{(q)} = 0$ if $p \leq 0$ and $q > 0$.*

Define the full subcategories $\mathbf{DTM}_k^{\geq 0}$ and $\mathbf{DTM}_k^{\leq 0}$ of $\mathbf{DTM}_k$ by

$$X \text{ is in } \mathbf{DTM}_k^{\geq 0} \text{ if and only if } gr_a^W(X) \cong \oplus_{n\leq 0}\mathbb{Q}(-a/2)^{m_n}[n] \text{ for all } a$$

$$X \text{ is in } \mathbf{DTM}_k^{\leq 0} \text{ if and only if } gr_a^W(X) \cong \oplus_{n\geq 0}\mathbb{Q}(-a/2)^{m_n}[n] \text{ for all } a.$$

Theorem 4.2. *Suppose the field k satisfies the vanishing conjecture of Soulé and Beilinson. Then*

i) $(\mathbf{DTM}_k^{\leq 0}, \mathbf{DTM}_k^{\geq 0})$ is a t-structure on $\mathbf{DTM}_k$, with heart $\mathbf{TM}_k$ generated by the objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.

ii) Composing the functor gr_i^W with the equivalence $\mathbf{TM}_{k,i} \to \mathbf{V}_\mathbb{Q}$ gives a faithful exact tensor functor

$$\bigoplus_{i=a}^{b} gr_i^W : \mathbf{TM}_k \to \mathbf{GrV}_\mathbb{Q}$$

iii) There is a perfect duality on $\mathbf{TM}_k$, making $\bigoplus_{i=a}^{b} gr_i^W$ into a Tannakian functor.

iv) For each p, there is a natural map

$$\phi_p : Ext^p_{\mathbf{TM}_k}(M, N) \to Hom_{\mathbf{DTM}_k}(M, N[p]).$$

ϕ_1 is an isomorphism, and ϕ_2 is injective.

Proof. The first three assertions follow from Theorem 1.4, the isomorphism (4.1) and Proposition 2.1. Item (iv) follows from Prop. 1.6. □

Take $M = \mathbb{Q}(0)$, $N = \mathbb{Q}(q)$ in Thm. 4.2(iv). Composing the map ϕ_p with the isomorphism (4.1)

$$\mathrm{Hom}_{\mathbf{DTM}_k}(\mathbb{Q}(0), \mathbb{Q}(q)[p]) \to K_{2q-p}(k)^{(q)},$$

we arrive at the homomorphism

$$\tau_{q,p} : \mathrm{Ext}^p_{\mathbf{TM}_k}(\mathbb{Q}(0), \mathbb{Q}(q)) \to K_{2q-p}(k)^{(q)};$$

$\tau_{q,1}$ is an isomorphism, and $\tau_{q,2}$ is injective.

Let k be a number field. It follows from Borel's computation (see [Bo]) of the rational K-groups of number fields that, for $q > 0$, $K_{2q}(k) \otimes \mathbb{Q} = 0$ and $K_{2q-1}(k) \otimes \mathbb{Q} = K_{2q-p}(k)^{(q)}$.

Corollary 4.3. *Let k be a number field. Then ($\mathbf{DTM}_k^{\leq 0}$, $\mathbf{DTM}_k^{\geq 0}$) is a t-structure on $\mathbf{DTM}_k$, with heart $\mathbf{TM}_k$ generated by the objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$. The functors gr_i^W give an equivalence of $\mathbf{TM}_k$ with a tensor subcategory of $\mathbf{GrV}_{\mathbb{Q}}$. In addition, for M and N in $\mathbf{TM}_k$ the maps*

$$\phi_p \colon Ext^p_{\mathbf{TM}_k}(M,N) \to Hom_{\mathbf{DTM}_k}(M,N[p]).$$

are isomorphisms for all p (both sides are zero for $p > 1$). In particular, the maps

$$\tau_{q,p} \colon Ext^p_{\mathbf{TM}_k}(\mathbb{Q}(0),\mathbb{Q}(q)) \to K_{2q-p}(k)^{(q)}$$

are isomorphisms for all p and q.

Proof. We have $\mathrm{Hom}_{\mathbf{DTM}_k}(\mathbb{Q}(0),\mathbb{Q}(q)[p]) = K_{2q-p}(k)^{(q)}$. Since $K_{2q-p}(k)^{(q)} = 0$ if $q \neq 0$ and $p \neq 1$, we may apply Theorem 4.2 to prove the first two assertions. Also, we have

$$\mathrm{Hom}_{\mathbf{DTM}_k}(\mathbb{Q}(0),\mathbb{Q}(q)[p]) = 0$$

for $q \neq 0$ and $p \neq 1$. By Theorem 4.2(iv), this implies that $\mathrm{Ext}^2_{\mathbf{TM}_k}(\mathbb{Q}(a),\mathbb{Q}(b)) = 0$ for all a and b. Since each object in $\mathbf{TM}_k$ has its weight filtration, with quotients direct sums of the $\mathbb{Q}(a)$ for varying a, this implies that $\mathrm{Ext}^2_{\mathbf{TM}_k}(M,N) = 0$ for all M and N in $\mathbf{TM}_k$. This in turn implies that $\mathrm{Ext}^p_{\mathbf{TM}_k}(M,N) = 0$ for all M and N in $\mathbf{TM}_k$, and for all $p \geq 2$. A similar argument shows that $\mathrm{Hom}_{\mathbf{DTM}_k}(M,N[p]) = 0$ for all all M and N in $\mathbf{TM}_k$, and for all $p \geq 2$. Since ϕ_1 is an isomorphism, the proof is complete. ∎

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