TECHNIQUES OF LOCALIZATION IN THE THEORY OF ALGEBRAIC CYCLES

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Abstract. We extend the localization techniques developed by Bloch to simplicial spaces. As applications, we give an extension of Bloch's localization theorem for the higher Chow groups to schemes of finite type over a regular scheme of dimension at most one (including mixed characteristic) and, relying on a fundamental result of Friedlander-Suslin, we globalize the Bloch-Lichtenbaum spectral sequence to give a spectral sequence converging to the G-theory of a scheme X, of finite type over a regular scheme of dimension one, with E^1-term the motivic Borel-Moore homology.

1. Introduction

1.1. **Bloch's higher Chow groups.** We begin by recalling Bloch's definition of the higher Chow groups [1]. Fix a base field k. Let Δ^N_k denote the standard "algebraic N-simplex"

$$\Delta^N_k := \operatorname{Spec} k[t_0, \dots , t_N]/\sum_i t_i - 1,$$

let X be a quasi-projective scheme over k, and let Δ^*_X be the cosimplicial scheme

$$N \mapsto X \times_k \Delta^N_k.$$

A *face* of Δ^N_X is a subscheme defined by equations of the form $t_{i_1} = \ldots = t_{i_r} = 0$. Let $X_{(p,q)}$ be the set of dimension $q + p$ irreducible closed subschemes W of Δ^p_X such that W intersects each dimension r face F in dimension $\leq q + r$. We have Bloch's simplicial group

$$p \mapsto z_q(X, p),$$

with $z_q(X, p)$ the subgroup of the dimension $q + p$ cycles on $X \times \Delta^p$ generated by $X_{(p,q)}$. Denote the associated complex by $z_q(X, *)$. The *higher Chow groups* of X are defined by

$$\mathrm{CH}_q(X, p) := H_p(z_q(X, *)).$$

If X is locally equi-dimensional over k, we may label these complexes by codimension, and define

$$\mathrm{CH}^q(X, p) := H_p(z^q(X, *)),$$

where $z^q(X, p) = z_{d-p}(X, p)$ if X has dimension d over k.

These groups compute the motivic Borel-Moore homology of X and, for X smooth over k, the motivic cohomology of X by

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Theorem 1.2. *There is a natural isomorphism*

$$H_p^{\mathrm{B.M.}}(X, \mathbb{Z}(q)) \cong \mathrm{CH}_q(X, p-2q),$$

where $H_*^{\mathrm{B.M.}}$ *is the motivic Borel-Moore homology. Suppose* X *is smooth over* k*. There is a natural isomorphism*

$$H^p(X, \mathbb{Z}(q)) \cong \mathrm{CH}^q(X, 2q-p).$$

Here the motivic cohomology is that defined by the construction of [6], [8] or [12].

The categories of [12] and [8] are equivalent for k of characteristic zero [8, VI, Theorem 2.5.5]. In addition, motivic cohomology has many of the formal properties one expects, including Mayer-Vietoris, Chern classes and Chern character isomorphism from K-theory, and duality. Thus, one is somewhat justified in using Bloch's higher Chow groups as the definition of motivic cohomology for a smooth quasi-projective k-scheme.

Required for the above theorem is the fundamental localization result from [2].

Theorem 1.3 (Bloch)**.** *Let* $Z \subset X$ *be a closed subscheme,* U *the complement* $X \setminus Z$*. Then the sequence*

$$z_q(Z,*) \xrightarrow{i_{Z*}} z_q(X,*) \xrightarrow{j_U^*} z_q(U,*)$$

is a distinguished triangle, i.e., the quotient complex $z_q(U,*)/j_U^* z_q(X,*)$ *is acyclic.*

Remark 1.4. For $q < \dim X - 1$, the map j_U^* is not surjective, except for some trivial cases.

1.5. In this paper, we give an extension of the technique used to prove Theorem 1.3, which allows one to prove similar moving lemmas in a more general setting. Our main result (Theorem 1.9) is stated below.

Theorem 1.9 extends Bloch's results in two directions: it allows the base-ring A to be a semi-local PID instead of a field, and it gives a more precise formulation of the homological construction used to prove the acyclicity in Theorem 1.3. This will allow us to apply the moving lemma to certain interesting simplicial spaces.

1.6. **Applications.** We give two applications of our main result. The first is an extension of the localization result Theorem 1.3 to schemes of finite type over a regular scheme B of dimension at most one. For $f : X \to B$ a finite-type B-scheme, we have the set $X_{(p,q)}$ defined as above, where we use a certain dimension function instead of dimension over k; see §1.8 below for the precise definition. We let $z_q(X,p)$ denote the free abelian group on $X_{(p,q)}$, forming the simplicial abelian group $p \mapsto z_q(X,p)$, and the associated complex $z_q(X,*)$. The complexes $z_q(X,*)$ are covariant for proper morphisms, and contravariant for flat morphisms (with an appropriate shift in q). In particular, we have the complex of sheaves on B, $f_*z_q(X,*)$, associated to the presheaf $V \mapsto z_q(p^{-1}(V),*)$. These complexes of sheaves have the same functoriality as the complexes $z_q(-,*)$.

Here is our extension of Bloch's localization result:

Theorem 1.7. *Let* B *be a regular one-dimensional scheme. Let* $i : Z \to X$ *be a closed subscheme of a finite-type* B*-scheme* $f : X \to \operatorname{Spec} B$*, and let* $j : U \to X$ *be the complement. Then the (exact) sequence of sheaves on* B

$$0 \to (f \circ i)_* z_q(Z,*) \xrightarrow{i_*} f_* z_q(X,*) \xrightarrow{j^*} (f \circ j)_* z_q(U,*)$$

forms a distinguished triangle in the derived category; in other words, the quotient complex $(f \circ j)_* z_q(U, *)/j^*(f_* z_q(X, *))$ *is acyclic. If* B *is semi-local, then* $z_q(U, *)/j^* z_q(X, *)$ *is acyclic, hence the exact sequence of complexes*

$$0 \to z_q(Z, *) \xrightarrow{i_*} z_q(X, *) \xrightarrow{j^*} z_q(U, *)$$

forms a distinguished triangle.

If we set $\mathrm{CH}_q(X, p) := \mathbb{H}^{-p}(B_{\mathrm{Zar}}, f_* z_q(X, *)) =: H^{\mathrm{B.M.}}_{p-2q}(X, \mathbb{Z}(q))$, then Theorem 1.7 gives a long exact localization sequence for the higher Chow groups/motivic Borel-Moore homology. We also have the identity

$$\mathbb{H}^{-p}_{\mathrm{Zar}}(B, f_* z_q(X, *)) = H_p(z_q(X, *))$$

for B semi-local.

The second application is a globalization of the Bloch-Lichtenbaum spectral sequence [3]

$$E_2^{p,q} = H^p(F, \mathbb{Z}(-q/2)) \Longrightarrow K_{-p-q}(F),$$

F a field, to a spectral sequence (of homological type) for $X \to B$ of finite type, B a regular noetherian scheme of dimension at most one,

$$E^2_{p,q} = H^{\mathrm{B.M.}}_p(X, \mathbb{Z}(-q/2)) \Longrightarrow G_{p+q}(X).$$

The proof relies on the Bloch-Lichtenbaum sequence for field, plus the fundamental result of Friedlander-Suslin [5, Theorem 6.1], which gives a natural interpretation of the Bloch-Lichtenbaum sequence as coming from the "niveau" tower associated to the cosimplicial scheme Δ^*_X, in case $X = \operatorname{Spec} F$. For details, see §8. In [10], we examine various properties of this spectral sequence, including the construction of Adams operations, functoriality, multiplicative properties, and comparison with étale cohomology and étale K-theory. In [5], Friedlander and Suslin have given a gobalization of the Bloch-Lichtenbaum spectral sequence to schemes of finite type over a field by another method.

1.8. **Statement of results.** Before we state our main result, we introduce some notation.

Let B be a regular noetherian scheme of dimension at most one. For $f : X \to B$ an irreducible B-scheme of finite type, the *dimension* of X is defined as follows: Let $\eta \in B$ be the image of the generic point of X, X_η the fiber of X over η. If η is a closed point of B, then X is a scheme over the residue field $k(\eta)$, and we set $\dim X := \dim_{k(\eta)} X$. If η is not a closed point of B, we set $\dim X := \dim_{k(\eta)} X_\eta + 1$. If $X \to B$ is proper, then $\dim X$ is the Krull dimension of X, but in general $\dim X$ is only greater than or equal to the Krull dimension. If X is equi-dimensional over B, we write $\dim_B X$ for the dimension of X over B.

We let $\Delta^n = \operatorname{Spec}_B(\mathcal{O}_B[t_0, \dots, t_n]/\sum_i t_i - 1)$, giving the cosimplicial B-scheme Δ^*. We have for each B-scheme X the cosimplicial scheme $\Delta^*_X := X \times_B \Delta^*$, and for each (p, q) the set $X_{(p,q)}$ of irreducible closed subsets C of Δ^p_X of dimension $p + q$, such that, for each face F of Δ^p, we have

$$\dim(C \cap X \times F) \le \dim_B F + q.$$

If U is an open subscheme of X, we let $U^X_{(p,q)}$ be the subset of $U_{(p,q)}$ consisting of those irreducible closed subsets whose closure in Δ^p_X are in $X_{(p,q)}$.

We work in the additive category $\mathbb{Z}\mathbf{Sch}_B$, with the same objects as $\mathbf{Sch}_B$, where $\mathrm{Hom}_{\mathbb{Z}\mathbf{Sch}_B}(X,Y)$ is the free abelian group on $\mathrm{Hom}_{\mathbb{Z}\mathbf{Sch}_B}(X,Y)$ for X and Y connected, and disjoint union is the direct sum. By taking the usual alternating sum of the coboundary maps, the cosimplicial scheme Δ^* becomes an object of $\mathbf{C}^+(\mathbb{Z}\mathbf{Sch}_B)$, also denoted Δ^*. We also have the object $(\Delta^N;\partial\Delta^N)$ of $\mathbf{C}^b(\mathbb{Z}\mathbf{Sch}_B)$, defined as follows: In degree $-r$, $(\Delta^N;\partial\Delta^N)$ is the direct sum of the objects $\partial\Delta^N_I$, I a proper subset of $\{0,\ldots,N\}$ having r elements, where

$$\partial\Delta^N_I = \cap_{i\in I}(t_i = 0).$$

The differential in $(\Delta^N;\partial\Delta^N)$ is an alternating sum over the various inclusion maps. The appropriate alternating sum of identity maps defines the map of complexes (of degree $-N$)

$$\Psi_N : \Delta^* \to (\Delta^N, \partial\Delta^N).$$

For details on these constructions, we refer the reader to §2, §2.2 and §2.4.

The coordinates t_j, $j \notin I$, in the standard order, give a canonical isomorphism $\iota_I : \Delta^{M-|I|} \to \partial\Delta^M_I$. We define $X_{(I,q)}$ to be the set of irreducible closed subsets of $X \times \partial\Delta^M_I$ that correspond to elements of $X_{(M-|I|,q)}$ via $\mathrm{id}\times\iota_I$. For U open in X, we define $U^X_{(I,q)}$ similarly.

We can now state our extension of the main technical result of [2].

Theorem 1.9 (*cf.* [2], §3). *Let $B = \mathrm{Spec}\,A$, where A is a semi-local PID with infinite residue fields, and let U be an open subscheme of a B-scheme X of finite type over B. Let $\{C_{I,j}\}$ be a finite collection of irreducible closed subsets, $C_{I,j} \in U_{(I,q_j)}$, $I \subsetneq \{0,1,\ldots,N\}$. Then there is a degree $-N$ map of complexes*

$$\Psi : \Delta^* \to (\Delta^N;\partial\Delta^N),$$

and a homotopy H of Ψ with Ψ_N, with the following property: Write Ψ and H as sums with $\mathbb{Z}$-coefficients

$$\Psi = \sum_{\substack{s,\\ I\subsetneq\{0,\ldots,N\}}} n^s_I f^s_I; \quad H = \sum_{\substack{s,\\ I\subsetneq\{0,\ldots,N\}}} m^s_I g^s_I; \quad n^s_I, m^s_I \neq 0,$$

with

$$f^s_I : \Delta^{N-|I|} \to \partial\Delta^N_I; \quad g^s_I : \Delta^{N-|I|+1} \to \partial\Delta^N_I,$$

maps of B-schemes. Then

1. *Each component of $(\mathrm{id}\times f^s_I)^{-1}(C_{I,j})$ is in $U^X_{(N-|I|,q_j)}$ for each I, s and j.*
2. *Each component of $(\mathrm{id}\times g^s_I)^{-1}(C_{I,j})$ is in $U_{(N-|I|+1,q_j)}$ for each I, s and j.*
3. *If $C_{I,j}$ is in $U^X_{(I,q_j)}$, then each component of $(\mathrm{id}\times g^s_I)^{-1}(C_{I,j})$ is in $U^X_{(N-|I|+1,q_j)}$ for each s.*

We actually prove a somewhat finer result, Theorem 6.12, which is useful in case A has some finite residue fields. The proof of Theorem 1.9 and Theorem 6.12 uses our extension of the fundamental result [2, Theorem 2.1.2]:

Theorem 1.10 ([9, Theorem 1.3]). *Let B be a regular scheme of dimension at most one. Let X be a B-scheme of finite type, $S \to B$ a smooth B scheme with strict reduced relative normal crossing divisor ∂S, and $Z \subset X\times_B S$ a closed subscheme, not contained in $X\times\partial S$. Then there is an iterated blow-up of faces $p : S' \to S$ (see §3) such that $(\mathrm{id}\times p)^{-1}[Z]$ intersects $X\times F$ properly for all faces F of $\partial S'$.*

Here $(\mathrm{id}\times p)^{-1}[Z]$ is the proper transform of Z, $\partial S' := p^{-1}(\partial S)_{\mathrm{red}}$, and a "face" F of $\partial S'$ is a subscheme of S' of the form $\partial S'_{i_1}\cap\ldots\cap\partial S'_{i_s}$, where the $\partial S'_j$ are the irreducible components of the normal crossing divisor $\partial S'$. The divisor $\partial S'$ is refered to as the *distinguished divisor* on S'. A *vertex* of $\partial S'$ is a face of dimension zero over B.

A rough sketch of the proof of Theorem 1.9 is as follows: One first reduces to the case of quasi-projective X. We may assume, by adding in even more irreducible subschemes, that the subschemes $C_{I,j}$ for $|I|>0$ are contained in the intersection of the $C_{\emptyset,j}$ with faces of $U\times\Delta^N$. We replace the simplex Δ^N with the cube $\square^N=\mathbb{A}^N$, with $\partial\square^N$ the union of the divisors $x_i=0,1$, via a birational morphism $\pi:\Delta^N\to\square^N$ with $\partial\square^N=\pi^{-1}(\partial\Delta^N)_{\mathrm{red}}$. We then use Theorem 1.10 to form an iterated blow up of $\square^N$ along faces of $\partial\square^N$, forming the scheme $p:S_M\to\square^N$ with distinguished divisor $\partial S_M:=p^{-1}(\partial\square^N)_{\mathrm{red}}$, so that the proper transform of each $C_{\emptyset,j}$ to $U\times S_M$ has closure in $X\times S_M$ which intersectsw each face properly.

Choose a general A-point c of $\square^N\setminus\partial\square^N$; since $p:S_M\to\square^N$ is an isomorphism away from $\partial\square^N$, we may consider c as in S_M. One defines, for each vertex v of $(S_M,\partial S_M)$, a distinguished coordinate systems $t^v_1,\ldots,t^v_N$ on a neighborhood of v in S_M. We form the "little cube" with origin v by using the divisors $t^v_j=0$, $t^v_j=t^v_j(c)$. We paste these little cubes together, triangulate the little cubes into simplices, and compose with $\pi\circ p$. The result turns out to be a map of complexes

$$\Psi^c_p:\Delta^*\to(\Delta^N;\partial\Delta^N).$$

To construct the homotopy, take $\square^N\times\square^1$, and perform the same blow-up on $\square^N\times 1$ that we just used to construct S_M. The same division into cubes and then into simplices gives part of the desired homotopy; the rest comes from a comparison of $\square^N$ and Δ^N.

The paper is organized as follows: In §2, we define the "relative complex" $(X;D_1,\ldots,D_n)$ associated to a scheme X and a collection of closed subschemes $D_1,\ldots,D_n$. We also describe a method for constructing homotopies of maps into $(X;D_1,\ldots,D_n)$. In §3, we consider schemes constructed by a sequence of blow-ups of faces of a normal crossing divisor, and show how one constructs a distinguished coordinate atlas on such blow-ups. We look more closely at the iterated blow-ups of the "n-cube" in §4, which forms the heart of the paper. In §5, we show how to pass from the n-cube to the n-simplex, and we prove our main results in §6. We conclude with the applications to the localization problem for the higher Chow groups in §7 and the globalization of the Bloch-Lichtenbaum spectral sequence in §8.

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2. Complexes associated to schemes and subschemes

2.1. If $\mathcal{A}$ is an additive category, we have the differential graded category $\mathbf{C}(\mathcal{A})$ of complexes, where the Hom-complex $\mathrm{Hom}((A,d_A),(B,d_B))$ is the graded group whose element in degree n are given by sequences of maps in $\mathcal{A}$

$$f := (f^i : A^i \to B^{i+n}; i \in \mathbb{Z}),$$

with

$$df := (d_B^{i+n} \circ f^i + (-1)^{n-1} f^{i+1} \circ d_A^i : A^i \to B^{i+n+1}).$$

We call a map of degree n, $f : A \to B$, a *map of complexes* if $df = 0$.

As a matter of notation, if I is a finite subset of an ordered set $\mathcal{S}$, we write $I = i_1 < \ldots < i_r$ to indicate that $I = \{i_1, \ldots, i_r\}$ and $i_j < i_{j+1}$ for $j = 1, \ldots, r-1$.

2.2. **The relative complex.** Let B be a noetherian scheme, $\mathbf{Sch}_B$ the category of B-schemes, essentially of finite type over B. We form the additive category $\mathbb{Z}\mathbf{Sch}_B$ generated by connected B-schemes, i.e., disjoint union is direct sum, and $\mathrm{Hom}_{\mathbb{Z}\mathbf{Sch}_B}(X, Y)$ is the free abelian group on $\mathrm{Hom}_{\mathbf{Sch}_B}(X, Y)$ for X and Y connected and non-empty. In particular, the empty scheme is canonically isomorphic to zero. We work in the category of complexes $\mathbf{C}(\mathbb{Z}\mathbf{Sch}_B)$. Let X be a B-scheme, and D a closed subscheme. Form the complex $(X; D)$: $D \xrightarrow{i_D} X$, with X in degree 0. More generally, suppose we have closed subschemes $D_1, \ldots, D_N$ of X. For $I \subset \{1, \ldots, N\}$, let $D_I = \cap_{i \in I} D_i$. We have the N-dimenisonal complex which in multi-degree $J = (-j_1, \ldots, -j_n) \in \{0, -1\}^N$ is $D_{I(J)}$, where $I(J) := \{i \mid j_i = 1\}$, and zero otherwise, with all maps being the inclusions. For later use, we let

$$\iota_{I,j} : D_I \to D_{I \setminus \{j\}}$$

denote the inclusion, for $j \in I \subset \{1, \ldots, N\}$. We let $(X; D_1, \ldots, D_N)$ denote the total complex of this N-complex. Explicitly, the maps in the total complex are defined by summing over the maps

$$(-1)^{j-1} \iota_{i_1 < \ldots < i_r, i_j} : D_{i_1 < \ldots < i_r} \to D_{i_1 < \ldots < \hat{i_j} < \ldots < i_r}.$$

Example 2.3. We have the ordered set of closed subschemes of Δ^N,

$$\partial \Delta^N := \{(t_0 = 0), \ldots, (t_N = 0)\},$$

giving the complex $(\Delta^N; \partial \Delta^N)$. Using the coordinates $\{t_0, \ldots, t_N\} \setminus \{t_i \mid i \in I\}$ in the usual order gives a canonical isomorphism

$$\iota_I : \Delta^{N-|I|} \to \partial \Delta^N_I.$$

Let $Y := (\oplus_r Y^r, d)$ be an object of $\mathbf{C}(\mathbb{Z}\mathbf{Sch}_B)$. A degree d map

$$F : Y \to (X; D_1, \ldots, D_N)$$

decomposes into the sum

$$F = \sum_I F_I; \quad F_I : Y^{-|I|-d} \to D_I.$$

We call F_I the *I-component of F*.

2.4. **Triangulations.**

Definition 2.5. Let X be a B-scheme, $D_1, \ldots, D_N$ closed subschemes. We call $\mathrm{Hom}(\Delta^*, (X; D_1, \ldots, D_n))$ the *singular chain complex* of $(X; D_1, \ldots, D_n)$, which we write as $\mathrm{Sing}(X; D_1, \ldots, D_n)$.

We have the element $\Psi_N := \sum_i \Psi_N^i$ of $\mathrm{Sing}(\Delta^N;\partial\Delta^N)$, with Ψ_N^{N-r} being the direct sum of the maps

$$(-1)^{\sum_j i_j+Nr}\iota_{i_1<\ldots<i_r} : \Delta^{N-r} \to \partial\Delta^N_{i_1<\ldots<i_r}.$$

Ψ_N has degree $-N$ and $d\Psi_N = 0$.

For $M \leq N$, we identify Δ^M with the face $t_{M+1} = \ldots = t_N = 0$ of Δ^N via $\iota_{M+1<\ldots<N}$. This identifies the dimension i faces of Δ^M with a subset of the dimension i faces of Δ^N. Taking the sum of the maps

$$(\pm)\iota^{-1}_{M+1<\ldots<N} : \partial\Delta^N_{i_1<\ldots<i_{r+M-N}<M+1<\ldots<N} \to \partial\Delta^M_{i_1<\ldots<i_{r+M-N}},$$

where the sign $\pm$ is $(-1)^{(N-M)r+\binom{N-M+1}{2}}$, defines a map $\chi_{M,N}$, giving the commutative diagram of complexes

(2.1)
$$\begin{array}{ccc} (\Delta^N;\partial\Delta^N) & \xrightarrow{\chi_{M,N}} & (\Delta^M;\partial\Delta^M) \\ {\scriptstyle\Psi_N}\uparrow & \nearrow{\scriptstyle\Psi_M} & \\ \Delta^* & & \end{array}$$

Let **Ord** be the category with objects the ordered sets $[n] := \{0 < 1 < \ldots < n\}$, and morphisms order-preserving maps of sets. Suppose we have a functor

$$F : \mathbf{Ord}^{\mathrm{op}} \to \mathbf{C}_{\geq 0}(\mathbf{Ab}),$$

where $\mathbf{C}_{\geq 0}(\mathbf{Ab})$ is the category of homological complexes which are zero in negative degree. We may apply F to the complex Δ^* (identifying $[n]$ with Δ^n, and similarly for the morphisms), and take the total complex, forming the homological complex $F(\Delta^*)$; we may similarly form the complex $F(\Delta^N;\partial\Delta^N)$. This gives the map (of homological degree N)

$$\Psi_N^* : F(\Delta^N;\partial\Delta^N) \to F(\Delta^*) \tag{2.2}$$

and the commutative diagram

(2.3)
$$\begin{array}{ccc} F(\Delta^N;\partial\Delta^N) & \xleftarrow{\chi^*_{M,N}} & F(\Delta^M;\partial\Delta^M) \\ {\scriptstyle\Psi_N^*}\downarrow & \swarrow{\scriptstyle\Psi_M^*} & \\ F(\Delta^*) & & \end{array}$$

The following result is an elementary consequence of the Dold-Kan equivalence of the categories of simplicial abelian groups and chain complexes.

Lemma 2.6. *(i) For $M \leq N$, the map on homology*

$$\chi^*_{M,N} : H_{i-M}(F(\Delta^M;\partial\Delta^M)) \to H_{i-N}(F(\Delta^N;\partial\Delta^N))$$

is an isomorphism for $i < M$ and a surjection for $i = M$.

(ii) The map (2.2) *induces a homology isomorphism*

$$\Psi_N^* : H_{i-N}(F(\Delta^N;\partial\Delta^N)) \to H_i(F(\Delta^*))$$

for $i < N$.

Proof, taken from [8, Part II, Chap. III, Lemma 1.1.5(ii)]. We first prove (i); it suffices to consider the case $M = N-1$. For each n we have the spectral sequence

$$E^1_{p,q}(n) = H_{q-n}(F_p(\Delta^n;\partial\Delta^n)) \Longrightarrow H_{p+q-n}(F(\Delta^n;\partial\Delta^n)).$$

The map $\chi^*_{N-1,N}$ gives a map of spectral sequences $E(N-1) \to E(N)$; this reduces to the case of a functor $F : \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Ab}$.

Let $\partial^{\le i}\Delta^N$ be the subset $\{(t_j = 0);\ j = 0, \ldots, i\}$ of $\partial\Delta^N$. We have the term-wise split exact sequence of complexes

$$0 \to (\Delta^N;\partial^{\le N-1}\Delta^N) \xrightarrow{\mathrm{id}_*} (\Delta^N;\partial\Delta^N) \xrightarrow{\chi_{N-1,N}} (\Delta^{N-1};\partial\Delta^{N-1})[-1] \to 0.$$

Applying F, we have the term-wise exact sequence of complexes

(2.4)

$$0 \to F(\Delta^{N-1};\partial\Delta^{N-1})[1] \xrightarrow{\chi^*_{N-1,N}} F(\Delta^N;\partial\Delta^N) \xrightarrow{\mathrm{id}^*} F(\Delta^N;\partial^{\le N-1}\Delta^N) \to 0.$$

Thus, it suffices to show that $H_p(F(\Delta^N;\partial^{\le N-1}\Delta^N)) = 0$ for $p < 0$. We will show $H_p(F(\Delta^N;\partial^{\le i}\Delta^N)) = 0$ for $p < 0$ and $i < N$ by induction on i and N, the case $i = -1$ being evidently true.

The inclusion $\iota = \iota_i : \Delta^{N-1} \to \Delta^N$ induces the map

$$\iota^* : F(\Delta^N;\partial^{\le i-1}\Delta^N) \to F(\Delta^{N-1};\partial^{\le i-1}\Delta^{N-1}),$$

which identifies $F(\Delta^N;\partial^{\le i}\Delta^N)$ with $\mathrm{cone}(\iota^*)$. From the resulting long exact homology sequence and induction on N and i, it follows that $H_p(F(\Delta^N;\partial^{\le i}\Delta^N)) = 0$ for $p < -1$, and we have the exact sequence

$$\begin{aligned} 0 \to H_0(F(\Delta^N;\partial^{\le i}\Delta^N)) &\to H_0(F(\Delta^N;\partial^{\le i-1}\Delta^N)) \\ &\to H_0(F(\Delta^{N-1};\partial^{\le i-1}\Delta^{N-1})) \to H_{-1}(F(\Delta^N;\partial^{\le i}\Delta^N)) \to 0 \end{aligned}$$

The degeneracy $\sigma : \Delta^N \to \Delta^{N-1}$, $\sigma(t_0, \ldots, t_N) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_N)$, gives a splitting

$$\sigma^* : F(\Delta^{N-1};\partial^{\le i-1}\Delta^{N-1}) \to F(\Delta^N;\partial^{\le i-1}\Delta^N)$$

to ι^*. This shows that $H_{-1}(F(\Delta^N;\partial^{\le i}\Delta^N)) = 0$, and the induction goes through.

For (ii), we proceed by induction on N. Using (i) and the commutative diagram (2.3), we see that (2.2) induces a homology isomorphism $H_{i-N}(F(\Delta^N;\partial\Delta^N)) \to H_i(F(\Delta^*))$ for $i < N-1$. From (i) and the sequence (2.4), we have the exact sequence

(2.5)

$$H_0(F(\Delta^N;\partial^{\le N-1}\Delta^N)) \to H_0(F(\Delta^{N-1};\partial\Delta^{N-1})) \to H_{-1}(F(\Delta^N;\partial\Delta^N)) \to 0.$$

We have the normalized subcomplex $NF(\Delta^*)$ of $F(\Delta^*)$, with

$$NF(\Delta^*)_n = \cap_{i=0}^{n-1} \ker(\iota_i^* : F(\Delta^n) \to F(\Delta^{n-1})).$$

By the results of Dold-Kan (see e.g. [4]), the inclusion $NF(\Delta^*) \to F(\Delta^*)$ is a quasi-isomorphism. Clearly, the maps Ψ_N and Ψ_{N-1} induce isomorphisms

$$\Psi^*_N : H_0(F(\Delta^N;\partial^{\le N-1}\Delta^N)) \to NF(\Delta^*)_N$$

$$\Psi^*_{N-1} : H_0(F(\Delta^{N-1};\partial\Delta^{N-1})) \to \ker(d : NF(\Delta^*)_{N-1} \to NF(\Delta^*)_{N-2}).$$

Combining these with the exact sequence (2.5), we see that Ψ^*_N induces an isomorphism

$$H_{-1}(F(\Delta^N;\partial\Delta^N)) \to H_{N-1}(F(\Delta^*)),$$

completing the proof. □

Via Lemma 2.6, we may view the complexes $(\Delta^N;\partial\Delta^N)[-N]$ as giving approximations to Δ^*, by using the diagram (2.3) and Lemma 2.6 to give the identity

$$H_p(F(\Delta^*)) = \lim_{\substack{\to \\ N}} H_p(F((\Delta^N;\partial\Delta^N)[-N])). \tag{2.6}$$

Definition 2.7. An element Ψ of $\mathrm{Sing}(\Delta^N;\partial\Delta^N)$ which is homotopic to Ψ_N is called a *triangulation* of $(\Delta^N;\partial\Delta^N)$.

It follows directly from Lemma 2.6 that a triangulation Ψ of $(\Delta^N;\partial\Delta^N)$ induces a homology isomorphism

$$\Psi^* : H_{i-N}(F(\Delta^N;\partial\Delta^N)) \to H_i(F(\Delta^*))$$

in degrees $i < N$, for $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{C}_{\geq 0}(\mathbf{Ab})$ a functor, where $\mathcal{C}$ is a subcategory of $\mathbf{Sch}_B$ containing the maps in Δ^*, the maps used to construct Ψ and the maps in a choice of homotopy between Ψ and Ψ_N.

2.8. **Functorialities.** We describe various mapping properties of the complexes $(X;D_1,\ldots,D_N)$.

Suppose we have B-schemes X and Y, $D_1,\ldots,D_N$ closed subschemes of X, and $E_1,\ldots,E_M$ closed subschemes of Y, $f : X \to Y$ a morphism, and $\tau : \{1,\ldots,N\} \to \{1,\ldots,M\}$ a map with the property that

$$f(D_j) \subset E_{\tau(j)}. \tag{2.7}$$

Thus, for each $I \subset \{1,\ldots,N\}$, f induces the map

$$f_I : D_I \to E_{\tau(I)}.$$

For a subset $I = \{i_1 < \ldots < i_r\}$ of $\{1,\ldots,N\}$, define $\mathrm{sgn}(\tau, I)$ to be zero if $|\tau(I)| < |I|$, and the sign of the permutation which puts the sequence $(\tau(i_1),\ldots,\tau(i_r))$ in increasing order if $|\tau(I)| = |I|$. Let

$$(f,\tau)^I = \mathrm{sgn}(\tau, I) f_I : D_I \to E_{\tau(I)}$$

and let

$$(f,\tau) : (X;D_1,\ldots,D_N) \to (Y;E_1,\ldots,E_M)$$

be the sum of the $(f,\tau)^I$. It is easy to check that (f,τ) commutes with d, and thus defines a map of complexes of degree 0. The functoriality

$$(g,\eta)\circ(f,\tau) = (g\circ f, \eta\circ\tau)$$

follows directly from the definitions.

Lemma 2.9. *The map (f,τ) is independent of the choice of τ, up to homotopy.*

Proof. It suffices to consider the case of a second map τ' satisfying the condition (2.7), and differing from τ at a single element $i \in \{1,\ldots,N\}$; we may suppose $\tau(i) < \tau'(i)$. We have

$$f(D_i) \subset E_{\tau(i),\tau(i')}.$$

Let I be a subset of $\{1,\ldots,N\}$ containing i, and suppose that $\tau'(i)$ is not in $\tau(I)$. Let

$$\mathrm{sgn}(\tau,\tau',I) = (-1)^{j-1}\mathrm{sgn}(\tau, I)$$

if $\tau(i)$ is the jth element in the sequence with elements $\tau(I)\cup\tau'(I)$, written in increasing order. Note that, if $\tau'(i)$ is the kth element in the sequence with elements $\tau(I)\cup\tau'(I)$, written in increasing order, then we have

$$(-1)^{j-1}\mathrm{sgn}(\tau,I) = -(-1)^{k-1}\mathrm{sgn}(\tau',I). \tag{2.8}$$

If i is in I and $\tau'(i)$ is in $\tau(I)$, or if i is not in I, we set $\mathrm{sgn}(\tau,\tau',I)=0$.

Define the map $h_I: D_I \to E_{\tau(I)\cup\tau'(I)}$ to be the map induced by f, and let

$$h:(X;D_1,\ldots,D_N)\to(Y;E_1,\ldots,E_M)$$

be the sum of the maps $\mathrm{sgn}(\tau,\tau',I)h_I$. With aid of (2.8), one easily verifies that

$$d\circ h + h\circ d = (f,\tau') - (f,\tau).$$

□

Via the lemma, we may write f_* for the homotopy class of the map (f,τ).

2.10. **Homotopies.** Fix a B-scheme X, with closed subschemes $D_1,\ldots,D_N$. For $1\le i,j\le N$, we have the closed subscheme $D_{j,i} := D_j\cap D_i$ of D_j. In this section, we describe a method for converting a map of complexes

$$F: Y\to(X;D_1,\ldots,D_N)$$

into a pair of maps of complexes

$$F_{j,0},F_{j,1}: Y\to(X;D_1,\ldots,D_{j-1},D_{j,j+1},D_{j+2},\ldots,D_N),$$

together with an explicit homotopy $F_{j,h}$ between $F_{j,0}$ and $-F_{j,1}$.

We begin by defining the maps

$$p_j:(X;D_1,\ldots,D_N)\to(D_j;D_{1,j},\ldots,\widehat{D_{j,j}},\ldots,D_{j,N}),$$

$$q_{j,0}:(D_j;D_{1,j},\ldots,\widehat{D_{j,j}},\ldots,D_{j,N})\to (X;D_1,\ldots,D_{j-1},D_{j,j+1},D_{j+2},\ldots,D_N),$$

and

$$q_{j,1}:(D_{j+1};D_{1,j+1},\ldots,\widehat{D_{j+1,j+1}},\ldots,D_{j+1,N})\to (X;D_1,\ldots,D_{j-1},D_{j,j+1},D_{j+2},\ldots,D_N).$$

The map p_j, $j=1,\ldots,N$, is the sum of the maps

$$(-1)^{l-1}\mathrm{id}: D_{i_1<\ldots<i_l=j<\ldots<i_r}\to D_{j,i_1<\ldots<\hat{i}_l<\ldots<i_r};$$

p_j is the zero map on D_I if $j\notin I$. To define the map $q_{j,0}$, take an index $I\subset\{1,\ldots,\hat{j},\ldots,N\}$. If $j+1$ is in I, let $I'=I\cup\{j\}$, otherwise, let $I'=I$. We consider $D_{I'}$ as a summand of $(X;D_1,\ldots,D_{j-1},D_{j,j+1},D_{j+2},\ldots,D_N)$ by writing $D_{I'}$ as $\cap_{i\in I}D_i$ if $j+1\notin I$, and as $D_{j,j+1}\cap(\cap_{i\in I\setminus\{j+1\}}D_i)$ if $j+1\in I$. Let

$$q_{j,0,I}: D_{j,I}\to D_{I'}$$

be the inclusion, and set $q_{j,0}=\sum_I q_{j,0,I}$. The map $q_{j,1}$ is defined similarly, reversing the role of j and $j+1$.

Clearly p_j is a map of degree one, $q_{j,0}$ and $q_{j,1}$ are degree zero maps, and

$$dp_j = dq_{j,0} = dq_{j,1} = 0.$$

Let $H_{j,I} : D_I \to D_I$ be the map

$$H_{j,I} = \begin{cases} \mathrm{id} & \text{if } I \subset \{1, \dots, N\} \setminus \{j, j+1\}, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$H_j : (X; D_1, \dots, D_N) \to (X; D_1, \dots, D_{j-1}, D_{j,j+1}, D_{j+2}, \dots, D_N)$$

be the sum of the $H_{j,I}$.

We let $S(N)$ denote the set of subsets of $\{1, \dots, N\}$, $S(N)_{a\setminus b} \subset S(N)$ the set of I with $a \in I$, $b \notin I$, and $S(N)_{a,b} \subset S(N)$ the set of I with $\{a, b\} \subset I$.

Lemma 2.11. $dH_j = q_{j,0} \circ p_j + q_{j,1} \circ p_{j+1}$.

Proof. We prove the case $j = N-1$; the general case follows from this by reordering the D_i. We write H for H_{N-1}, q_{N-1} for $q_{N-1,0}$ and q_N for $q_{N-1,1}$.

Write the identity map on $(X; D_1, \dots, D_N)$ as a sum

$$\mathrm{id} = \mathrm{id}_{N-1} + \mathrm{id}_N + \mathrm{id}_{N-1,N} + \mathrm{id}_h,$$

where id_{N-1} (resp. id_N) is the sum of the identity maps on D_I with $N-1$ in I, and N not in I (resp. $N \in I$ and $N-1 \notin I$). $\mathrm{id}_{N-1,N}$ is the sum of the identity maps on D_I with $\{N-1, N\} \subset I$, and id_h is the sum of the remaining terms, i.e., the identity maps on those D_I with $I \subset \{1, \dots, N-2\}$.

We have

$$\begin{aligned} 0 &= d \circ \mathrm{id} - \mathrm{id} \circ d \\ &= (d \circ \mathrm{id}_{N-1} - \mathrm{id}_{N-1} \circ d) + (d \circ \mathrm{id}_N - \mathrm{id}_N \circ d) \\ &\quad + (d \circ \mathrm{id}_{N-1,N} - \mathrm{id}_{N-1,N} \circ d) + (d \circ \mathrm{id}_h - \mathrm{id}_h \circ d). \end{aligned} \tag{2.9}$$

In particular, for each index $I \subset \{1, \dots, N\}$, we have vanishing of the I-component $(d \circ \mathrm{id} - \mathrm{id} \circ d)_I$ of $d \circ \mathrm{id} - \mathrm{id} \circ d$. Taking the sum of the I-components over all I containing $N-1$ but not containing N, we arrive at the identity

$$(d \circ \mathrm{id}_{N-1} - \mathrm{id}_{N-1} \circ d) + \sum_{I \in S(N)_{N-1,N}} (-1)^{|I|-1} \iota_{I,N} = \sum_{I \in S(N)_{N-1\setminus N}} (-1)^{|I|-1} \iota_{I,N-1}.$$

Taking the sum of the I-components over all I containing N but not containing $N-1$, gives

$$(d \circ \mathrm{id}_N - \mathrm{id}_N \circ d) + \sum_{I \in S(N)_{N-1,N}} (-1)^{|I|-2} \iota_{I,N-1} = \sum_{I \in S(N)_{N\setminus N-1}} (-1)^{|I|-1} \iota_{I,N},$$

and taking the sum of the I-components over all I containing N and $N-1$ gives

$$(d \circ \mathrm{id}_{N-1,N} - \mathrm{id}_{N-1,N} \circ d) = \sum_{I \in S(N)_{N-1,N}} (-1)^{|I|-2} \iota_{I,N-1} + \sum_{I \in S(N)_{N-1,N}} (-1)^{|I|-1} \iota_{I,N}.$$

These together with (2.9) yield the identity

$$(d \circ \mathrm{id}_h - \mathrm{id}_h \circ d) + \sum_{I \in S(N)_{N-1\setminus N}} (-1)^{|I|-1} \iota_{I,N-1} + \sum_{I \in S(N)_{N\setminus N-1}} (-1)^{|I|-1} \iota_{I,N} = 0. \tag{2.10}$$

Let $\rho_I : D_I \to D_J$ be the map

$$\rho_I = \begin{cases} \mathrm{id} : D_I \to D_I & \text{if } I \subset \{1, \dots, N-2\}, \\ \iota_{I,N-1} : D_I \to D_{I \setminus \{N-1\}} & \text{if } I \in S(N)_{N-1 \setminus N}, \\ \iota_{I,N} : D_I \to D_{I \setminus \{N\}} & \text{if } I \in S(N)_{N \setminus N-1}, \\ 0 & \text{if } \{N-1, N\} \subset I. \end{cases}$$

Let

$$\rho : (X; D_1, \dots, D_N) \to (X; D_1, \dots, D_{N-2}, D_{N-1,N})$$

be the sum of the ρ_I. One computes directly that

$$\rho \circ (\sum_{I \in S(N)_{N-1 \setminus N}} (-1)^{|I|-1} \iota_{I,N-1} + \sum_{I \in S(N)_{N \setminus N-1}} (-1)^{|I|-1} \iota_{I,N}) = q_{N-1} \circ p_{N-1} + q_N \circ p_N,$$

and

$$\rho \circ \mathrm{id}_h = H.$$

In addition, since the I-component of $d \circ \mathrm{id}_h$ and id_h is zero if either $N-1$ or N is in I, we have

$$\rho \circ d \circ \mathrm{id}_h = d \circ \rho \circ \mathrm{id}_h.$$

Thus, applying ρ to (2.10) gives the desired identity. □

Remark 2.12. One could also give a "coordinate-free" proof of a weaker, but still usable result, by first restricting to the subcategory $\mathcal{C}$ of $\mathbf{Sch}_B$ with objects the D_I, and with morphisms $D_I \to D_J$ being the inclusion if $J \subset I$, and the empty set otherwise. Let Z be the cone of the map

$$(D_j; D_{1,j}, \dots, \widehat{D_{j,j}}, \dots, D_{j,N}) \oplus (D_{j+1}; D_{1,j+1}, \dots, \widehat{D_{j+1,j+1}}, \dots, D_{j+1,N}) \xrightarrow{q_{j,0}+q_{j,1}} (X; D_1, \dots, D_{j-1}, D_{j,j+1}, D_{j+2}, \dots, D_N)$$

and let

$$Z[-1] \xrightarrow{\eta} (D_j; D_{1,j}, \dots, \widehat{D_{j,j}}, \dots, D_{j,N}) \oplus (D_{j+1}; D_{1,j+1}, \dots, \widehat{D_{j+1,j+1}}, \dots, D_{j+1,N})$$

be the canonical map. One constructs an explict map $\tau : (X; D_1, \dots, D_N) \to Z[-1]$ in $\mathbf{C}^b(\mathbb{Z}\mathcal{C})$ with $(p_j, p_{j+1}) = \eta \circ \tau$. From this it follows that $q_{j,0} \circ p_j + q_{j,1} \circ p_{j+1}$ is homotopically trivial in $\mathbf{C}^b(\mathbb{Z}\mathcal{C})$. We can then conclude that there is a homotopy H which is a sum, with $\mathbb{Z}$-coefficients, of the $H_{j,I}$ described above, because the maps $H_{j,I}$ generate the group of degree zero maps from $(X; D_1, \dots, D_N)$ to $(X; D_1, \dots, D_{j-1}, D_{j,j+1}, D_{j+2}, \dots, D_N)$. Since this proof is essentially as long as the explicit version, and since it gives a somewhat weaker result, we have omitted the details.

If j is in I, write $I = i_1 < \ldots < i_l = j < \ldots < i_r$, and set $\mathrm{sgn}_j(I) = (-1)^{l-1}$. Suppose we have an object Y of $\mathbf{C}(\mathbb{Z}\mathbf{Sch}_B)$, and a degree n map of complexes

$$F : Y \to (X; D_1, \dots, D_N).$$

Form the maps

$$F_{j,0}, F_{j,1}, F_{j,h} : Y \to (X; D_1, \dots, D_{j-1}, D_{j,j+1}, D_{j+2}, \dots, , D_N)$$

by setting

$$
\begin{aligned}
(2.11)\qquad F_{j,0} &:= \sum_{I\in S(N)_{j\setminus j+1}} \mathrm{sgn}_j(I)\iota_{I,j}\circ F_I + \sum_{I\in S(N)_{j,j+1}} \mathrm{sgn}_j(I)F_I,\\
F_{j,1} &:= \sum_{I\in S(N)_{j+1\setminus j}} \mathrm{sgn}_{j+1}(I)\iota_{I,j+1}\circ F_I + \sum_{I\in S(N)_{j,j+1}} \mathrm{sgn}_{j+1}(I)F_I,\\
F_{j,h} &:= \sum_{I\subset\{1,\ldots,N\}\setminus\{j,j+1\}} F_I.
\end{aligned}
$$

Proposition 2.13. *$F_{j,0}$ and $F_{j,1}$ are degree n maps of complexes, and*

$$dF_{j,h} = F_{j,0} + F_{j,1}.$$

Proof. This follows directly from Lemma 2.11 and the identities

$$F_{j,0} = q_{j,0}\circ p_j\circ F,\ F_{j,1} = q_{j,1}\circ p_{j+1}\circ F,\ F_{j,h} = H_j\circ F.$$

□

3. Blowing up faces

3.1. We fix a noetherian base scheme B, which we assume to be irreducible. Let T be a B-scheme, smooth over B, ∂T a codimension one closed subscheme of T with irreducible components $\partial T_1,\ldots,\partial T_N$. We say that ∂T is a *strict reduced relative normal crossing divisor on T* if for each $I\subset\{1,\ldots,N\}$, the subscheme ∂T_I has pure codimension $|I|$ on T, and is smooth over B. If $D_1,\ldots,D_N$ are distinct codimension one reduced closed subschemes of T such that the union of the D_i is a strict reduced relative normal crossing divisor on T, we say that $D_1,\ldots,D_N$ *form a normal crossing divisor on T*.

If ∂T is a strict reduced relative normal crossing divisor on T, we will sometimes also denote the set of irreducible components of ∂T by ∂T, and we will often give a specific ordering to this set. The context will make the distinction clear.

Let ∂T be a strict reduced relative normal crossing divisor on T. A *face* of $(T;\partial T)$ is an irreducible component of some ∂T_I, a *vertex* is a face of dimension zero, and an edge is a face of dimension one (both over $\mathrm{Spec}\,B$). If the divisor ∂T is understood, we often refer to a face, vertex or edge of $(T;\partial T)$ as a face, vertex or edge of T.

It is easy to show that, if ∂T is a strict reduced relative normal crossing divisor on T, and if $p: T'\to T$ is the blow-up of T along a face F (of codimension at least two) of T, then $\partial T' := p^{-1}(\partial T)_{\mathrm{red}}$ is a strict reduced relative normal crossing divisor on T'. Let us call ∂T the *distinguished divisor on T*. We then define the distinguished divisor on $T_1 := T'$ to be ∂T_1. We may continue, blowing up a face of T_1 to form T_2 with its distinguished divisor, and so on. We call such a tower of blow-ups

$$T_M\to\ldots\to T_1\to T_0 := T$$

a sequence of blow-ups of faces, and the composition $T_M\to T$ an *iterated blow-up of faces.*

Let Y be a smooth B-scheme with strict reduced relative normal crossing divisor ∂Y. We let $\mathfrak{B}_Y$ be the full subcategory of $\mathbf{Sch}_Y$ with objects $p: X\to Y$ the iterated blow-ups of faces of Y. For each $p: X\to Y$ in $\mathfrak{B}_Y$, we have the distinguished divisor $\partial X := p^{-1}(\partial Y)_{\mathrm{red}}$.

Lemma 3.2. *Let T be a B-scheme with distinguished divisor ∂T, and let*

$$T_r \xrightarrow{p} T_{r-1} \to \ldots \to T_1 \to T_0 = T$$

a sequence of blow-ups of faces. Suppose that

1. *Each edge of T contains exactly two vertices.*
2. *Let l be an edge of T with vertices v_1 and v_2. There are irreducible components $D_1 \neq D_2$ of ∂T with $D_i \cap l = v_i$, $i = 1, 2$.*

Then (1) and (2) are true for T_r.

Proof. (Taken from [2, Lemma(1.3.2)]) We proceed by induction on r, reducing us to the case $r = 1$. If l is an edge of T_1, then $p(l)$ is either an edge or a vertex of T. Suppose $p(l)$ is an edge l', with vertices v' and w'. Replacing T with an open neighborhood of l in T, and changing notation, we may suppose that

$$l' = D_2 \cap \ldots \cap D_{n-1}; v' = l' \cap D_1, w' = l' \cap D_n,$$

with the D_i distinct irreducible components of ∂T.

Let F be the face of T we blow up to form T_1. If F contains l', we may suppose

$$F = D_2 \cap \ldots \cap D_s$$

for some s, $3 \leq s \leq n - 1$. Let E be the exceptional divisor of p, and let $[D_i]$ denote the proper transform of D_i to T_1. Then the irreducible components of the distinguished divisor of T_1 lying over a neighborhood of l' are E, $[D_1], \ldots, [D_n]$. Each edge mapping onto l' is thus of the form

$$l = E \cap \bigcap_{j \neq i, 2 \leq j \leq n-1} [D_j]$$

for some choice of i, $2 \leq i \leq n-1$, and the vertices of l are thus $l \cap [D_1]$ and $l \cap [D_n]$.

If F does not contain l', say $l' \cap F = v'$, then

$$l = [D_2] \cap \ldots \cap [D_{n-1}]$$

and l has vertices $l \cap E$, and $l \cap [D_n]$.

If $p(l)$ is a vertex $v = D_1 \cap \ldots \cap D_n$, then we have

$$l = E \cap \bigcap_{j \neq i, j \neq i', 1 \leq j \leq n} [D_j],$$

and l has vertices $l \cap [D_i]$, $l \cap [D_{i'}]$. □

Let $D_1, \ldots, D_N$ define a reduced strict normal crossing divisor ∂T on T, and let v be a vertex of T. Suppose that $D_{i_1}, \ldots, D_{i_n}$ are the divisors containing v. We call an n-tuple of regular functions $(f_1, \ldots, f_n)$ defined in a neighborhood U of v a *coordinate system adapted to ∂T at v* if the map

$$(f_1, \ldots, f_n) : U \to \mathbb{A}^n$$

is an open immersion, and if the divisor $D_{i_j} \cap U$ is given by $f_j = 0$, $j = 1, \ldots, n$.

Let us start with a B-scheme T with distinguished divisor ∂T, such that each vertex v of T has a neighborhood U_v with regular functions $f_1^v, \ldots, f_n^v$ giving a coordinate system adapted to ∂T at v. We assume in addition that, for each face F of T, we have

$$F \subset \cup_{v \in F} U_v;$$

in particular, the U_v cover T. Having fixed such a choice of the coordinate systems, we refer to the coordinate system $f^v := (f_1^v, \ldots, f_n^v)$, or any other coordinate system gotten by reordering the f_j^v, as *a distinguished coordinate system at* v. Let F be a face of T, and $p : T' \to T$ the blow-up of T along F with exceptional divisor E, giving the distinguished divisor $\partial T'$ on T'. If v is a vertex of T' with $p(v) = w$, we define the distinguished coordinate system at v' as follows: If F does not contain w, take $U_v = p^{-1}(U_w \setminus F)$, and $f^v = f^w \circ p$. If F does contain w, then there is a subset J of $\{1, \dots, n\}$ and an $i \in J$ such that F is defined on U_w by the equations $f_j^w = 0$, $j \in J$, and

$$v = E \cap \bigcap_{j \in J, j \neq i} [(f_j^w = 0)].$$

We let $U_v = p^{-1}(U_w) \setminus [(f_i^w = 0)]$, and

$$f_j^v = \begin{cases} p^*(f_j^w) & j \in \{1, \dots, n\}, j \notin J \setminus \{i\}, \\ p^*(f_j^w/f_i^w) & j \in J \setminus \{i\}. \end{cases}$$

We also allow a reordering of the f_j^v.

Thus, given a sequence of blow-ups of faces

$$T_M \to T_{M-1} \to \dots \to T_1 \to T_0 = T,$$

we have defined the distinguished coordinate systems f^v for each vertex v of T_M.

Example 3.3. Fix $n \geq 0$ and let $\square^n = \mathbb{A}^n$, with distinguished divisor $\partial \square^n$ having components $D_1, \dots, D_{2n}$,

$$D_i := \begin{cases} (t_i = 0); & i = 2k, \ k = 1, \dots, n, \\ (1 - t_i = 0); & i = 2k - 1, \ k = 1, \dots, n. \end{cases}$$

At each vertex $v = (\epsilon_1, \dots, \epsilon_n)$, $\epsilon_i \in \{0, 1\}$, we take as a distinguished coordinate system $t^v := (t_1^v, \dots, t_n^v)$, where

$$t_i^v := \begin{cases} t_i & \text{if } \epsilon_i = 0, \\ 1 - t_i & \text{if } \epsilon_i = 1. \end{cases}$$

4. Blowing up the n-cube

4.1. **Preliminaries.** As in §3, we fix an irreducible noetherian base scheme B. Let $S = \square^n$ with distinguished divisor $\partial \square^n$ and distinguished coordinate systems as in Example 3.3. We fix a sequence of blow-ups of faces of S, as in §3:

$$S_M \to \dots \to S_1 \to S,$$

we let $p : S_M \to S$ be the resulting morphism, and let ∂S_j denote the distinguished divisor on S_j.

Since S satifies the conditions (1) and (2) of Lemma 3.2, the same is true for each S_j.

For a vertex v of S_M, we call the divisors in ∂S_M which contain v the *coordinate divisors through* v. We let $\partial^v S_M$ denote the subset of ∂S_M consisting of those D with $v \notin D$, and U_M^v the open neighborhood $S_M \setminus \partial^v S_M$ of v.

Lemma 4.2. *Let* v *be a vertex of* S_M*,* $t^v = (t_1^v, \dots, t_n^v)$ *the corresponding distinguished coordinate system. Let* w *be the image of* v *in* S*. Then*

1. *There is a matrix* $(a_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ *with* $a_{ij} \geq 0$ *for all* i, j *such that*
$$t_i^w = \prod_j (t_j^v)^{a_{ij}}.$$
2. *The coordinate functions* t_i^v *are regular functions on* U_M^v*.*
3. *The morphism* $t^v : U_M^v \to \mathbb{A}^n$ *determined by* t^v *is an open immersion.*

Proof. All three statements are obviously true for $S_M = S$. Suppose (1) and (2) are true for S_{M-1}. Let u be the image of v in S_{M-1}, and let $F \subset S_{M-1}$ be the face we blow up to form S_M. If F does not contain u, then $S_M \to S_{M-1}$ is an isomorphism over U_{M-1}^u, and $t^v = t^u$, up to reordering, whence the result for S_M. Suppose F contains u, and let $U = S_{M-1} \setminus \partial^u S_{M-1}$. Then the coordinate system t^u defines an isomorphism of U with a Zariski open neighborhood of 0 in S. This reduces us to the case $M = 1$, $u = w = 0$. If F has dimension r, we have the isomorphism $(S, F) \cong (\mathbb{A}^{n-r}, 0) \times F$, which reduces us to the case $F = 0$. In this case, the blow-up is the closed subscheme of $\mathbb{P}^{n-1}_{\mathbb{A}^n}$ defined by the equations $X_j t_i = X_i t_j$, where we use homogeneous coordinates $X_1, \ldots, X_n$ for $\mathbb{P}^{n-1}$. The vertex v is given by a choice of some $i \in \{1, \ldots, n\}$, with $v = \cap_{j \neq i}(X_j = 0)$, the open neighborhood U_1^v is given by $X_i \neq 0$, and the coordinate system t^v is then
$$t_j^v = \begin{cases} t_j/t_i & \text{for } j \neq i, \\ t_i & \text{for } j = i, \end{cases}$$
up to reordering. Thus
$$t_j = \begin{cases} t_j^v t_i^v & \text{for } j \neq i, \\ t_i^v & \text{for } j = i, \end{cases}$$
proving (1). As the t_j^v are regular away from the proper transform of the divisor $t_i = 0$, (2) is proved as well. Clearly the coordinate system t^v gives an isomorphism $t^v : U_1^v \to \mathbb{A}^n$, proving (3). □

4.3. **Orientations.** We associate to each vertex of S_M an *orientation*, i.e., a sign. For this, fix an ordering of the divisors in ∂S_M:
$$\partial S_M = \{D_1, \ldots, D_N\}$$
compatible via p with the ordering of the components of the distinguished divisor of S given in Example 3.3. We let v_0 be the vertex $(0, \ldots, 0)$ of S.

We note that the scheme S with its distinguished divisor is the extension to B of the $\mathbb{Z}$-scheme $S_{\mathbb{Z}} := \square^n_{\mathbb{Z}}$, together with the distinguished divisor $\partial S_{\mathbb{Z}}$. It follows by an elementary induction that the same is true for each $(S_j, \partial S_j)$. In particular, to define a sign at each vertex of S_M, it suffices to make the definition in case $B = \operatorname{Spec} \mathbb{Z}$; we therefore assume in this section that $B = \operatorname{Spec} \mathbb{Z}$.

At each vertex v of S_M, we have the coordinate system $t^v := (t_1^v, \ldots, t_n^v)$. The closure of the divisor $t_i^v = 0$ is one of the D_j, say $D_{j(i)}$. We call t^v *ordered* if $j(1) < j(2) < \ldots < j(n)$. This condition determines the coordinate system t^v uniquely. We henceforth use only ordered coordinate systems, unless explicitly mentioned.

Let $U_j \subset S_j$ be the subset $S_j \setminus \partial S_j$; each U_j is isomorphic to $U := U_0$ via the map $S_j \to S$. Let $U(\mathbb{R})^+ \subset U(\mathbb{R})$ be the subset $\{(r_1, \ldots, r_n) \mid 0 < r_i < 1\}$, and let $U_M(\mathbb{R})^+$ be the inverse image of $U(\mathbb{R})^+$ via p.

If v is a vertex of S_M, then by Lemma 4.2, U_M is contained in the domain of definition of the coordinate mapping t^v, and t^v is a coordinate system at each point of U_M, which we identify with U via $p : S_M \to S$. Thus, the Jacobian determinant

$$J(v, w) := \frac{\partial t^v}{\partial t^w}$$

is a well-defined regular function on U_M for each pair of vertices v and w, even if we take one vertex from S_M and one from S. Since $U_M(\mathbb{R})^+$ is contractible, the sign of $J(v, w)$ is constant over $U_M(\mathbb{R})^+$.

Definition 4.4. Let v be a vertex of S_M. The *orientation* $\epsilon(v)$ is the sign of $J(v, v_0)$ on $U_M(\mathbb{R})^+$. We call two vertices v and w *adjacent* if there is an edge l of S_M with $v, w \in l$; we call l the *edge joining* v *and* w (*cf.* Lemma 3.2).

Let v and w be adjacent vertices, joined by an edge l. Since S_M satisfies the conditions (1) and (2) of Lemma 3.2, there is a unique divisor D among the D_j such that D contains v, but does not contain w. We call the coordinate t^v_p with $\overline{(t^v_p = 0)} = D$ the *coordinate for* l *at* v. Suppose that t^w_q is the coordinate for l at w. Let $g(v, w)$ be the permutation of $\{1, \ldots, n\}$ such that $g(p) = q$, and the closures of $t^v_j = 0$ and $t^w_{g(j)} = 0$ agree for $j \neq p$.

Lemma 4.5. *Let* v *and* w *be adjacent vertices of* S_M*,* l *the edge joining* v *and* w*. Then*

1. $\epsilon(v) = -\mathrm{sgn}(g(v, w))\epsilon(w)$.
2. *Let* t^v_p *be the coordinate for* l *at* v*, which we consider as a rational function on* l*. Then* $t^v_p(w)$ *is either* ∞ *or* 1.

Proof. We first prove (1). Let $g = g(v, w)$. It suffices to show that $-\mathrm{sgn}(g)$ is the sign of the Jacobian matrix $J(v, w)$ evaluated at some point of $U_M(\mathbb{R})^+$. For this, it is convenient to change coordinates in S as follows: Let $x_i = t_i/1 - t_i$. The affine line $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ is transformed to the affine line $\mathbb{P}^1 - \{-1\}$, and the region $0 \le t_i \le 1$ is transformed to the region $0 \le x_i \le \infty$. The new vertices on S are the points $u = (\epsilon_1, \ldots, \epsilon_n)$, with $\epsilon_i \in \{0, \infty\}$. We use for a distinguished coordinate system at u the coordinates $x^u = (x^u_1, \ldots, x^u_n)$, with

$$x^u_i = x_i^{\pm 1},$$

where the exponent is $+1$ if $\epsilon_i = 0$, -1 if $\epsilon_i = \infty$.

We have

$$\frac{dx_i}{dt_i} = \frac{1}{(1 - t_i)^2}, \quad \frac{d(x_i^{-1})}{d(1 - t_i)} = \frac{1}{t_i^2},$$

which are positive on $0 < t_i < 1$, so making these substitutions will not affect the sign of the various Jacobian determinants involved. We write the distinguished coordinate system at a vertex v with respect to these new coordinates as x^v. We let $\bar{S} = (\mathbb{P}^1)^n$, and let $\bar{S}_M \to \bar{S}$ be the extension of S_M gotten by blowing up the corresponding faces over $\bar{S}$.

Since the coordinates x_i are all regular at the "missing" point -1, the statement (2) of Lemma 4.2 remain valid for $\bar{S}_M$, i.e., each x^v_i is a regular function on $\bar{S}_M \setminus \partial^v \bar{S}_M$. It follows by Lemma 4.2(1) that there is a matrix $(a_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$x^w_i = \prod_j (x^v_j)^{a_{ij}}. \qquad (4.1)$$

Let l be the edge connecting v and w, x^v_p the coordinate for l at v and x^w_q the coordinate for l at w. Let $\bar{l}$ be the closure of l in $\bar{S}_M$. Then $\bar{l}$ is a $\mathbb{P}^1$; since v and w are the only vertices of $\bar{S}_M$ on $\bar{l}$ (*cf.* Lemma 3.2), it follows that x^v_p has a single zero with multiplicity one on $\bar{l}$ at p, hence x^v_p has its unique pole at w. Thus (4.1) implies that

$$x^w_q = (x^v_p)^{-1}. \tag{4.2}$$

By considering the divisors of the functions $x^w_{g(j)}$ and x^v_j, we see that

$$x^w_{g(j)} = x^v_j (x^v_p)^{a_{jp}}$$

for all $j \neq p$. From this and (4.2), we have

$$J(w, v) = -(x^v_p)^{-2}\mathrm{sgn}(g),$$

completing the proof of (1).

The statement (2) is clearly true for $S_M = S$; by induction, we may assume (2) for S_{M-1}. We have the morphism $p_M : S_M \to S_{M-1}$; let $v' = p_M(v)$, $w' = p_M(w)$ and $l' = p_M(l)$. If $v' = w'$, then, arguing as above, we have $t^v_p = (t^w_q)^{-1}$, as rational functions on l, hence $t^v_p(w) = \infty$. If $v' \neq w'$, then l' is the edge connecting v' and w'. Let $t^{v'}_{p'}$ be the coordinate for l' at v'. Then one can easily check that $p^*_M(t^{v'}_{p'}) = t^v_p$, so $t^v_p(w) = t^{v'}_{p'}(w')$, which by induction is either 1 or ∞. □

4.6. **The cubical complex.** For $j = 1, \ldots, n+1$, $\epsilon = 0, 1$, let

$$\iota_{j,\epsilon} : \square^n \to \square^{n+1}$$

be the inclusion with

$$\iota^*_{j,\epsilon} t_i = \begin{cases} t_i & \text{for } 1 \leq i < j, \\ t_{i-1} & \text{for } j < i \leq n+1, \\ \epsilon & \text{for } i = j. \end{cases}$$

Let

$$d^r_+, d^r_- : \square^r \to \square^{r+1}$$

be the signed sums

$$d^r_+ := \sum_{j=1}^{r+1} (-1)^j \iota_{j,0},$$
$$d^r_- := \sum_{j=1}^{r+1} (-1)^{j+1} \iota_{j,1}.$$

One easily checks that $d_+ \circ d_+ = 0 = d_- \circ d_-$, and that $d_+ \circ d_- = -d_- \circ d_+$, giving us the complexes $(\square^*, d_+)$, $(\square^*, d_-)$ and $(\square^*, d)$, with $d = d_+ + d_-$. We write $\square^*$ for $(\square^*, d)$.

We let $\square^r_0$ be the "semi-local scheme" of the vertices in $\square^r$, i.e., the limit of open subschemes gotten by removing closed subsets C with $C \cap v = \emptyset$ for all vertices v. $\square^r_0$ really is a semi-local scheme if B is semi-local. It is conceivable that $\square^r_0$ may not even be a scheme if B is not affine. In this case, we consider $\square^r_0$ as a limit object in the category of B-schemes.

The differential d^r restricts to the map

$$d^r : \square^r_0 \to \square^{r+1}_0,$$

giving the complexes $(\square_0^*, d_+)$, $(\square_0^*, d_-)$ and $\square_0^* := (\square_0^*, d)$ and the maps of complexes

$$(\square_0^*, d_+) \to (\square^*, d_+),\ (\square_0^*, d_-) \to (\square^*, d_-),\ \square_0^* \to \square^*.$$

4.7. **Little cubes for S_M.** In this section, we show how a sequence of blow-ups as in §3 leads to a "cubiculation" of S_M. We assume that the B-scheme $(\mathbb{A}^1 - \{0,1\})^n \to B$ admits a section, i.e., that there exists a regular function u on B such that u and $1-u$ are units. For our applications, we will assume that B is a semi-local scheme such that all residue fields are infinite, so the assumption on the existence of sections is fulfilled; in general, one can replace B with a suitable B-scheme $B' \to B$, make a base-extension to B', and change notation.

Let $c := (c_1, \ldots, c_n)$ be a section of $(\mathbb{A}^1 - \{0,1\})^n$ over B. Since $(\mathbb{A}^1 - \{0,1\})^n = \square^n \setminus \partial\square^n = U_M$, we may consider c as a section of $U_M \subset S_M$ over B. We let $t^{v,c} := (t_1^{v,c}, \ldots, t_n^{v,c})$ be the modified coordinate system

$$t_j^{v,c} := t_j^v / t_j^v(c);$$

via Lemma 4.2, we have the B-morphism $t^{v,c} : U_M^v \to \mathbb{A}^n$. By Lemma 4.2(3), $t^{v,c}$ is an open immersion, mapping v to the origin. It follows from Lemma 4.5(2) that the image $t^{v,c}(U_M^v)$ contains all the vertices of $\square^n = \mathbb{A}^n$, hence we have the morphism

$$\lambda^{v,c} : \square_0^n \to S_M$$

defined by inverting $t^{v,c}$ over $\square_0^n$, and then including U_M^v in S_M.

Let $\partial^+\square_0^n$ be the set of divisors $\{D_i := (t_i = 0) \subset \square_0^n \mid i = 1, \ldots, n\}$. The coordinate system

$$(t_1, \ldots, \widehat{t_{i_1}}, \ldots, \widehat{t_{i_r}}, \ldots, t_n)$$

on $D_{i_1,\ldots,i_r}$ defines the isomorphism

$$\iota_I : \square^{n-|I|} \to D_I.$$

Recall from §4.6 the complexes $(\square_0^*, d_+)$, $(\square_0^*, d_-)$ and $\square_0^* := (\square_0^*, d)$. We have the natural map of complexes (of degree $-n$)

$$\rho_n : (\square_0^*, d_+) \to (\square_0^n; \partial^+\square_0^n) \tag{4.3}$$

defined as the sum of the maps

$$(-1)^{\sum_j i_j + nr} \iota_{i_1<\ldots<i_r} : \square^{n-r} \to D_{i_1<\ldots<i_r}.$$

Let $\partial_v S_M$ denote the set of divisors in ∂S_M which contain v, in the same order as in ∂S_M. The ordered inclusion $\partial_v S_M \subset \partial S_M$ defines the map of complexes

$$\eta^v : (S_M; \partial_v S_M) \to (S_M; \partial S_M).$$

We define the map

$$\phi_+^{v,c} : (\square_0^*, d_+) \to (S_M; \partial S_M)$$

as the composition

$$(\square_0^*, d_+) \xrightarrow{\rho_n} (\square_0^n; \partial_+\square_0^n) \xrightarrow{\lambda_*^{v,c}} (S_M; \partial_v S_M) \xrightarrow{\eta^v} (S_M; \partial S_M).$$

Clearly $d\phi_+^{v,c} = 0$. We have the map

$$\phi^{v,c} : (\square_0^*, d) \to (S_M; \partial S_M)$$

with the same definition as $\phi_+^{v,c}$; $d\phi^{v,c}$ is not zero.

Let

$$\phi^c : (\square_0^*, d) \to (S_M; \partial S_M) \tag{4.4}$$

be the sum

$$\phi^c = \sum_v \epsilon(v)\phi^{v,c}.$$

Proposition 4.8. *$d\phi^c = 0$.*

For the proof, we require the following result:

Lemma 4.9. *Let v and w be adjacent vertices, connected by an edge l. Suppose that t_p^v is the coordinate for l at v and t_q^w is the coordinate for l at w. Then the diagram*

$$\begin{array}{ccccc} \square_0^n & \xrightarrow{\lambda^{v,c}} & S_M & \xleftarrow{\lambda^{w,c}} & \square_0^n \\ & \nwarrow{\scriptstyle \iota_{p,1}} & & \nearrow{\scriptstyle \iota_{q,1}} & \\ & & \square_0^{n-1} & & \end{array}$$

commutes.

Proof. (Following [2, Lemma (1.3.4)]) We proceed by induction on M, the case $S_M = S$ being obvious. Let v', w' and l' be the image of v, w and l, respectively, in S_{M-1}, and let $F \subset S_{M-1}$ be the face we blow up to form S_M. It suffices to prove the result for some choice of order on the set of components of the distinguished divisor.

Suppose at first that l' is an edge and $l' \subset F$. As in the proof of Lemma 3.2, we may assume we have components $D_1, \ldots, D_{n+1}$ of the distinguished divisor on S_{M-1} with

$$D_i = \overline{(t_i^v = 0)},\ D_{i+1} = \overline{(t_i^w = 0)};\quad i = 1, \ldots, n,$$

and with

$$l' = D_2 \cap \ldots \cap D_n,\ v' = l' \cap D_1,\ w' = l' \cap D_{n+1},\ F = D_s \cap \ldots \cap D_n.$$

We may also assume that

$$l = E \cap [D_2] \cap \ldots \cap [D_{n-1}],\ v = l \cap [D_1],\ w = l \cap [D_{n+1}],$$

where $[-]$ denotes proper transform. This yields the following coordinate changes:

$$t_j^v = \begin{cases} t_j^{v'} & j = 1, \ldots, s-1, n, \\ t_j^{v'}/t_n^{v'} & j = s, \ldots, n-1, \end{cases}$$

$$t_j^w = \begin{cases} t_j^{w'} & j = 1, \ldots, s-2, n-1, n, \\ t_j^{w'}/t_{n-1}^{w'} & j = s-1, \ldots, n-2, \end{cases}$$

so t_n^w is the coordinate for l at w, and t_1^v is the coordinate for l at v. The induction hypothesis implies that the divisors $\overline{t_1^{v'} = t_1^{v'}(c)}$ and $\overline{t_n^{w'} = t_n^{w'}(c)}$ agree, and that, on this common divisor, we have the identities,

$$t_j^{v'}/t_j^{v'}(c) = t_{j-1}^{w'}/t_{j-1}^{w'}(c);\ j = 2, \ldots, n.$$

Combining these with the coordinate changes described above gives the identity of divisors $\overline{t_1^v = t_1^v(c)}$ and $\overline{t_n^w = t_n^w(c)}$ and on this common divisor, the identities

$$t_j^v/t_j^v(c) = t_{j-1}^w/t_{j-1}^w(c);\ j = 2, \dots, n.$$

This implies the desired commmutativity.

Now suppose that l' an edge and $F \cap l'$ a vertex, say $F \cap l' = w'$. Then we may assume

$$l' = D_2 \cap \ldots \cap D_n,\ v' = l' \cap D_1,\ w' = l' \cap D_{n+1},\ F = D_s \cap \ldots \cap D_{n+1},$$

and

$$l = [D_2] \cap \ldots \cap [D_n],\ v = l \cap [D_1],\ w = l \cap E.$$

This gives the following coordinate changes:

$$t_j^v = t_j^{v'} \quad j = 1, \dots, n,$$
$$t_j^w = \begin{cases} t_j^{w'} & j = 1, \dots, s-2, n, \\ t_j^{w'}/t_n^{w'} & j = s-1, \dots, n-1, \end{cases}$$

with t_n^w the coordinate for l at w, and t_1^v the coordinate for l at v. The argument proceeds as above.

If l' is a vertex v', then we may assume

$$v' = D_1 \cap \ldots \cap D_n,\ F = D_s \cap \ldots \cap D_n,$$

and

$$l = [D_1] \cap \ldots \cap [D_{n-2}] \cap E,\ v = l \cap [D_{n-1}],\ w = l \cap [D_n].$$

This gives the following coordinate changes:

$$t_j^v = \begin{cases} t_j^{v'} & j = 1, \dots, s-1, n, \\ t_j^{v'}/t_n^{v'} & j = s, \dots, n-1, \end{cases}$$
$$t_j^w = \begin{cases} t_j^{v'} & j = 1, \dots, s-1, n-1, \\ t_j^{v'}/t_{n-1}^{v'} & j = s, \dots, n-2, n, \end{cases}$$

with t_{n-1}^v the coordinate for l at v and t_n^w the coordinate for l at w. One then argues as above to complete the proof. □

Proof of Proposition 4.8. It suffices to show that

$$\phi^c \circ d_- = 0.$$

To understand this equation, we first note that the terms of $\phi^c \circ d_-$ occur in pairs. Indeed, fix a vertex v and a dimension $n-r$. Suppose that the components of the distinguished divisor containing v are $D_{i_1}, \dots, D_{i_n}$, with $i_1 < \ldots < i_n$. The terms in $\phi^{v,c}$ involving $\square_0^{n-r}$ are indexed by the sequences $1 \le j_1 < \ldots < j_r \le n$, with $\square_0^{n-r}$ mapping into the face $D_{i_{j_1} < \ldots < i_{j_r}}$ by the composition

$$\lambda^{v,c} \circ \iota_{j_1 < \ldots < j_r}.$$

The corresponding terms in $\phi^{c,v} \circ d_-$ are then a signed sum over the maps

$$\lambda^{v,c} \circ \iota_{j_1 < \ldots < j_r} \circ \iota_{p,1} : \square_0^{n-r-1} \to S_M, \tag{4.5}$$

for $1 \leq p \leq n-r$. For each such choice of p, we have the edge l containing v defined as the intersection

$$l := D_{i_{j_1}<\ldots<i_{j_r}} \cap D_{i'_1<\ldots<\widehat{i'_p}<\ldots<i'_{n-r}}$$

where $i'_1 < \ldots < i'_{n-r}$ is the complement of $i_{j_1} < \ldots < i_{j_r}$ in $i_1 < \ldots < i_n$.

Conversely, given a vertex v of S_M, an index $1 \leq j_1 < \ldots < j_r \leq n$, and an edge l containing v, the above construction gives a uniquely determined map $\square_0^{n-r-1} \to S_M$ occurring in $\phi^c \circ d_-$. By Lemma 3.2, we may therefore pair the terms occuring in $\phi^c \circ d_-$ by fixing l and $j_1 < \ldots < j_r$, and taking the two terms corresponding to the two vertices on l.

Let w be the other vertex on l, and let D_s be the distinguished divisor with $w = l \cap D_s$. For some b we have $i_b < s < i_{b+1}$, where we set $i_{n+1} = \infty$. To fix ideas, we suppose that $i'_p < s$, the other case is gotten by reversing the role of v and w. Let a be the index with $i'_p = i_a$. Define the index $i^*_1 < \ldots < i^*_n$ by writing the set $\{i_1, \ldots, \widehat{i_a}, \ldots, i_n, s\}$ in increasing order.

Let $j^*_1 < \ldots < j^*_r$ be the index with

$$D_{i^*_{j^*_k}} = D_{i_{j_k}}; \quad k = 1, \ldots, r,$$

and let $i^{*\prime}_1 < \ldots < i^{*\prime}_{n-r}$ be the complement of $i^*_{j^*_1} < \ldots < i^*_{j^*_r}$ in $i^*_1 < \ldots < i^*_n$. Clearly $s = i^{*\prime}_q$ for some q.

By Lemma 4.9, the map

$$\lambda^{w,c} \circ \iota_{(j^*_1,\ldots,j^*_r)} \circ \iota_{q,1} : \square_0^{n-r-1} \to S_M \tag{4.6}$$

agrees with the map (4.5). Thus, we need only show that (4.5) and (4.6) occur in $\phi^c \circ d_-$ with opposite sign.

We have

$$j^*_k = \begin{cases} j_k & \text{if } i_{j_k} < i'_p \text{ or } s < i_{j_k}, \\ j_k - 1 & \text{otherwise.} \end{cases}$$

Thus

$$\sum_k j_k - \sum_k j^*_k + (q-p) = b - a,$$

since both sides are counting the number of indices in the set $\{i_1, \ldots, i_n, s\}$ which are strictly between i'_p and s. On the other hand, $g(v,w)$ is the permuation

$$g(v,w)(k) = \begin{cases} k & \text{if } k < a \text{ or } k > b, \\ k-1 & \text{if } a < k \leq b, \\ b & \text{if } k = a, \end{cases}$$

so

$$\mathrm{sgn}(g(v,w)) = (-1)^{b-a}.$$

The map (4.5) occurs with sign

$$\epsilon(v)(-1)^{\sum_k j_k + n(r+1)}(-1)^{p-1}$$

and the map (4.6) occurs with sign

$$\epsilon(w)(-1)^{\sum_k j^*_k + n(r+1)}(-1)^{q-1}.$$

By Lemma 4.5, we have

$$\epsilon(w) = (-1)^{b-a+1}\epsilon(v),$$

so (4.5) and (4.6) occur with opposite sign, as desired. □

4.10. **Little cubes for S.** We now complete the program of defining a "cubiculation" for $S = \square^n$. We fix a sequence of blow-ups of faces

$$S_M \to \ldots \to S,$$

with $p : S_M \to S$ the composition, and a section $c : B \to \square^n \setminus \partial\square^n = (\mathbb{A}^1 \setminus \{0,1\})^n$.

Lemma 4.11. *Let v be a vertex of S_M. Then the morphism*

$$p \circ \lambda^{v,c} : \square^n_0 \to S$$

extends (uniquely) to a morphism

$$\Lambda^{v,c} : \square^n \to S. \tag{4.7}$$

Proof. The assertion is local on B, so we may assume that B is affine, $B = \operatorname{Spec} A$.

Let $w = p(v)$, giving us the distinguished coordinate system t^w; we note that the functions t^w_j are regular on all of S and define a global coordinate system for S, i.e.,

$$A[t^w_1, \ldots , t^w_n] = A[t_1, \ldots , t_n].$$

By Lemma 4.2, there is a matrix $(b_{ij}) \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$t^w_i = \prod_j (t^v_j)^{b_{ij}}; \quad b_{ij} \geq 0.$$

Thus, the map

$$\Lambda^{v,c*} : A[t_1, \ldots , t_n] \to \Gamma(\square^n_0, \mathcal{O})$$

has image in the subring $\Gamma(\square^n, \mathcal{O}) \cong A[t^v_1, \ldots , t^v_n]$, completing the proof. □

Chose a map $\tau : \{1, \ldots , N\} \to \{1, \ldots , 2n\}$ such that $p((\partial S_M)_j) \subset (\partial S)_{\tau(j)}$, giving the map of complexes

$$p_* : (S_M; \partial S_M) \to (S; \partial S).$$

By Proposition 4.8 and Lemma 4.11, the map of complexes

$$p_* \circ \phi^c : \square^*_0 \to (S; \partial S)$$

extends canonically to the map of complexes

$$\Phi^c_p : \square^* \to (S; \partial S). \tag{4.8}$$

The map Φ^c_p is independent of the choice of τ, up to homotopy (*cf.* Lemma 2.9). As a special case, we may take $S_M = S$, giving us the map

$$\Phi^c_{\mathrm{id}} : \square^* \to (S; \partial S).$$

Proposition 4.12. *Let $p : S_M \to S$ be an iterated blow-up of faces. For each section $c' : B \to (\mathbb{A}^1 \setminus \{0,1\})^{n+1}$ of the form $c' = (c, c_{n+1})$, there is a homotopy $H_0(c')$ between Φ^c_p and Φ^c_{id}.*

Proof. Let $T = \square^{n+1}$, with distinguished divisor ∂T chosen as for S. We identify S with the face $t_{n+1} = 1$ of T via the inclusion ι. If $S_1 \to S$ is the blow-up of S along a face F, form the blow-up $T_1 \to T$ of T along $\iota(F)$. We have the canonical identification of S_1 with the proper transform of $\iota(S)$, so we may iterate, forming the sequence of blow-ups of faces

$$T_M \to \ldots \to T_0 := T$$

with T_i containing S_i as the proper transform of $\iota(S)$, and T_{i+1} being the blow-up of T_i along the image of the face of S_i we blow up to form S_{i+1}. The divisors in ∂T_M (except for the proper transforms of $t_{n+1} = 0$ and $t_{n+1} = 1$) are thus in 1-1 correspondence with the divisors of ∂S_M, the correspondence given by intersection with $S_M \subset T_M$. We have the identification of S with the face $t_{n+1} = 0$ of T_M.

We alter our conventions a bit by taking the proper transforms $[(t_{n+1} = 0)]$, $[(t_{n+1} = 1)]$ to be the last two divisors in ∂T_M. We order ∂T as before, and we order the exceptional divisors in ∂T_M to correspond with the given ordering of the exceptional divisors in S_M. With this ordering, we have

$$[(t_{n+1} = 0)] = (\partial T_M)_{N-1},\ [(t_{n+1} = 1)] = (\partial T_M)_N,$$

with $N = 2n + 2 + M$. We note that $[(t_{n+1} = 1)] \cap [(t_{n+1} = 0)] = \emptyset$, i.e., $(\partial T_M)_{N-1,N} = \emptyset$. In particular, if $(\partial T_M)_{i_1<\ldots<i_r}$ is non-empty, then at most one i_j is in $\{N-1, N\}$, and if one i_j is in $\{N-1, N\}$, then $j = r$.

Choose a section $c' : B \to T \setminus \partial T$ with $c'_i = c_i$, $i = 1, \ldots, n$. The above sequence of blow-ups gives us via Proposition 4.8 the map of complexes

$$\phi^{c'} : \square^*_0 \to (T_M; \partial T_M).$$

Let $\partial_- T_M = \partial T_M \setminus \{[(t_{n+1} = 1)], [(t_{n+1} = 0)]\}$, and let $\partial_- T = \partial T \setminus \{(t_{n+1} = 0), (t_{n+1} = 1)\}$. Following the construction of §2.11, we have the maps

$$\phi^c_0, \phi^c_1, \phi^{c'}_h : \square^*_0 \to (T_M; \partial_- T_M)$$

given by

$$\phi^c_0 = \sum_r \sum_{I=(i_1<\ldots<i_{r-1}<i_r=N-1)} (-1)^{r-1} \iota_{I,N-1} \circ F_I,$$
$$\phi^c_1 = \sum_r \sum_{I=(i_1<\ldots<i_{r-1}<i_r=N)} (-1)^{r-1} \iota_{I,N} \circ F_I,$$

and

$$\phi^{c'}_h = \sum_r \sum_{I=i_1<\ldots<i_r} F_I,$$

where this last sum is over indices with $i_r < N - 1$. As the notation suggests, ϕ^c_0 and ϕ^c_1 depend only on c. By Proposition 2.13, we have

$$d\phi^{c'}_h = \phi^c_0 + \phi^c_1;$$

both ϕ^c_0 and ϕ^c_1 are degree $-n$ maps of complexes.

Let $\tau' : \partial T_M \to \partial T$ be unique map extending τ which preserves the correspondence with the divisors on S_M, and sends $[(t_{n+1} = 1)]$ and $[(t_{n+1} = 0)]$ to $(t_{n+1} = 1)$ and $(t_{n+1} = 0)$, respectively. Then τ' maps $\partial_- T_M$ to $\partial_- T$. It follows from Lemma 4.11 and the definition of ϕ^c_0, ϕ^c_1 and $\phi^{c'}_h$ that $\tau'_*\circ\phi^c_0$, $\tau'_*\circ\phi^c_1$ and $\tau'_*\circ\phi^{c'}_h$ extend uniquely to maps

$$\Phi^c_0, \Phi^c_1, \Phi^{c'}_h : \square^* \to (T; \partial_- T),$$

with

$$d\Phi^{c'}_h = \Phi^c_0 + \Phi^c_1.$$

Both Φ^c_0 and Φ^c_1 are degree $-n$ maps of complexes.

The projection $\pi : T \to S$ of T on S gives the map of complexes (of degree 0)

$$\pi_* : (T; \partial_- T) \to (S; \partial S).$$

Via π, we may identify $[(t_{n+1} = 0)]$ with S; the rational map $T \to S_M$ defined by π extends uniquely to a morphism $\tilde{\pi} : T_M \to S_M$, giving an identification of $[(t_{n+1} = 1)]$ with S_M. Under these identifications, we have

$$\epsilon_S(\pi(v)) = \epsilon_{T_M}(v) \tag{4.9}$$

for a vertex v in $[(t_{n+1} = 0)]$, while

$$\epsilon_{S_M}(\tilde{\pi}(v)) = -\epsilon_{T_M}(v) \tag{4.10}$$

for a vertex v in $[(t_{n+1} = 1)]$. Indeed, it follows directly from the definitions that, for $v \in [(t_{n+1} = 0)]$, the ordered distinguished coordinate system t^v on T_M is given by

$$t^v = (t^{\pi(v)}_1, \ldots, t^{\pi(v)}_n, t_{n+1}),$$

where $(t^{\pi(v)}_1, \ldots, t^{\pi(v)}_n)$ is the ordered distinguished coordinate system on S at $\pi(v)$. Similary, for $v \in [(t_{n+1} = 1)]$, the ordered distinguished coordinate system t^v on T_M is given by

$$t^v = (t^{\tilde{\pi}(v)}_1, \ldots, t^{\tilde{\pi}(v)}_n, (1 - t_{n+1}) \prod_{j=1}^{n} (t^{\tilde{\pi}(v)}_j)^{a_j}),$$

where $(t^{\tilde{\pi}(v)}_1, \ldots, t^{\tilde{\pi}(v)}_n)$ is the ordered distinguished coordinate system on S_M at $\tilde{\pi}(v)$, and the a_j are integers. Taking (4.9) and (4.10) into account, the definition of ϕ^c_0 and ϕ^c_1 readily imply that

$$\pi_* \circ \Phi^c_0 = -\Phi^c_{\mathrm{id}},\ \pi_* \circ \Phi^c_1 = \Phi^c_p.$$

This gives us

$$d(\pi_* \circ \Phi^{c'}_h) = \Phi^c_p - \Phi^c_{\mathrm{id}};$$

taking $H_0(c') = \pi_* \circ \Phi^{c'}_h$ completes the proof. □

5. From cubes to simplices

The object in this section is to convert the map (4.8) into a triangulation of $(\Delta^N; \partial\Delta^N)$.

5.1. **The product $\Delta^n\times\square^m$.** As preparation, we extend the maps Φ^c_{id} to products of cubes and simplices. We make $\mathbb{Z}\mathbf{Sch}_B$ into a tensor category with $\otimes=\times_B$; this makes the category of bounded below complexes $\mathbf{C}^+(\mathbb{Z}\mathbf{Sch}_B)$ into a differential graded tensor category. Explicitly, if $A=(\oplus_iA^i,d^i_A:A^i\to A^{i+1})$ and $B=(\oplus_jB^j,d^j_B:B^j\to B^{j+1})$ are complexes, then $(A\otimes B)^n=\oplus_{i+j=n}A^i\otimes B^j$ with differential

$$d^n_{A\otimes B}=\sum_{i+j=n}d^i_A\otimes\mathrm{id}_{B^j}+(-1)^i\mathrm{id}_{A^i}\otimes d^j_B.$$

If $f=\sum_if^i:A^i\to C^{i+n}$, $g=\sum_jg^j:B^j\to D^{j+m}$ are graded maps of degrees n and m respectively, then $f\otimes g$ is the sum $\sum_{i,j}(-1)^{mi}f^i\otimes g^j$.

We note that the sum of the evident identity maps

$$D_{i_1,\ldots,i_r}\otimes E_{j_1,\ldots,j_s}=\cap_{l=1}^r(D_{i_l}\times Y)\cap\cap_{l=1}^s(X\times E_{j_l})$$

gives an isomorphism

$$(X;D_1,\ldots,D_n)\otimes(Y;E_1,\ldots,E_m)\\ \to(X\times_kY;D_1\times Y,\ldots,D_n\times Y,X\times E_1\ldots,X\times E_m).$$

We have the degree $-N$ map of complexes

$$\Phi_N:\square^*\to(\square^N;\partial\square^N)$$

defined by the sum of the maps

$$(-1)^{\sum_ji_j+\epsilon_j+Nr}\iota_{(i_1<\ldots<i_r),(\epsilon_1,\ldots,\epsilon_r)}:\square^{N-r}\to\partial_{(i_1<\ldots<i_r),(\epsilon_1,\ldots,\epsilon_r)}\square^N,$$

where $\iota_{(i_1<\ldots<i_r),(\epsilon_1,\ldots,\epsilon_r)}$ is the canonical identification of $\square^{n-r}$ with image the subscheme defined by $t_{i_j}=\epsilon_j$, $\epsilon_j\in\{0,1\}$.

We thus have the map of complexes

$$\Psi_n\otimes\Phi_m:\Delta^*\otimes\square^*\to(\Delta^n;\partial\Delta^n)\otimes(\square^m;\partial\square^m).$$

Recall that the vertices of Δ^n are the subschemes v_i given by $t_i=1$, $t_j=0$, $j\neq i$. Fix integers m, n and $M\geq m+1$, and a section $c:=(c_1,\ldots,c_M):B\to(\mathbb{A}^1\setminus\{0,1\})^M$. Let $c^m=(c_{M-m+1},\ldots,c_M)$, and let v be a vertex of $\square^m$. Taking the identity blow-up $\square^m\to\square^m$, we have the map $\Lambda^{v,c^m}:\square^m\to\square^m$. This gives us the affine-linear maps

$$\Lambda^{v_0,v}_{n,m}(c):\Delta^n\times\square^m\to\Delta^n\times\square^m,$$
$$\Lambda^{v_0,v}_{n,m}(c)(t_0,\ldots,t_n;x)=(t_0+(1-c_1)t_1,c_1t_1,t_2,\ldots,t_n;\Lambda^{v,c^m}(x)),$$

and

$$\Lambda^{v_1,v}_{n,m}(c):\Delta^n\times\square^m\to\Delta^n\times\square^m,$$
$$\Lambda^{v_1,v}_{n,m}(c)(t_0,\ldots,t_n;x)=((1-c_1)t_1,t_0+c_1t_1,t_2,\ldots,t_n;\Lambda^{v,c^m}(x)),$$

We set

$$\epsilon(v_0,v)=\epsilon(v);\quad\epsilon(v_1,v)=-\epsilon(v).$$

As in §4.7, if ∂T is the set of irreducible components of a strict reduced normal crossing divisor on a B-scheme T, and v is a vertex, we let $\partial_vT\subset\partial T$ be the subset consisting of those $D\in\partial T$ which contain v. For vertices $v\in\Delta^n$, $w\in\square^m$, the sum of the evident identity maps defines the (additive) projection

$$\pi_{v,w}:(\Delta^n;\partial\Delta^n)\otimes(\square^m;\partial\square^m)\to(\Delta^n;\partial_v\Delta^n)\otimes(\square^m;\partial_w\square^m),$$

and the inclusion of complexes

$$\iota_{v,w} : (\Delta^n; \partial_v\Delta^n) \otimes (\square^m; \partial_w\square^m) \to (\Delta^n; \partial\Delta^n) \otimes (\square^m; \partial\square^m).$$

The maps $\Lambda^{v_i,v}_{n,m}(c)$, $i = 0, 1$, give the maps of complexes

$$\Lambda^{v_i,v}_{n,m*}(c) : (\Delta^n; \partial_{v_0}\Delta^n) \otimes (\square^m; \partial_0\square^m) \to (\Delta^n; \partial_{v_i}\Delta^n) \otimes (\square^m; \partial_v\square^m).$$

We set $\partial(\Delta^n \times \square^m) := \partial\Delta^n \times \square^m + \Delta^n \times \partial\square^m$, and define

$$\Psi_n \times \Phi_m(c) : \Delta^* \otimes \square^* \to (\Delta^n \times \square^m; \partial(\Delta^n \times \square^m)) \tag{5.1}$$

by

$$\Psi_n \times \Phi_m(c) = (-1)^{mn} \sum_{v,i=0,1} \epsilon(v_i, v)\iota_{v_i,v} \circ \Lambda^{v_i,v}_{n,m*}(c) \circ \pi_{v_0,0} \circ (\Psi_n \otimes \Phi_m),$$

where the v in the sum runs over the vertices of $\square^m$.

Lemma 5.2. *$\Psi_n \times \Phi_m(c)$ is a map of complexes (of degree $-n - m$).*

Proof. The proof is similar to that of Proposition 4.8, but easier. We have the projections

$$\pi_v : (\Delta^n; \partial\Delta^n) \to (\Delta^n; \partial_v\Delta^n); \quad \pi_w : (\square^m; \partial\square^m) \to (\square^m; \partial_w\square^m),$$

and the inclusions

$$\iota_v : (\Delta^n; \partial_v\Delta^n) \to (\Delta^n; \partial\Delta^n); \quad \iota_w : (\square^m; \partial_w\square^m) \to (\square^m; \partial\square^m).$$

Define $\Lambda^{v_0}_n(c_1)$ and $\Lambda^{v_1}_n(c_1)$ by

$$\Lambda^{v_0}_n(c_1)(t_0, \dots, t_n) = (t_0 + (1 - c_1)t_1, c_1t_1, t_2, \dots, t_n);$$
$$\Lambda^{v_1}_n(c_1)(t_0, \dots, t_n) = ((1 - c_1)t_1, t_0 + c_1t_1, t_2, \dots, t_n).$$

These define the maps of complexes

$$\Lambda^{v_i}_{n*}(c_1) : (\Delta^n; \partial_{v_0}\Delta^n) \to (\Delta^n; \partial_{v_i}\Delta^n); \quad i = 0, 1.$$

Letting $\epsilon(v_0) = 1$, $\epsilon(v_1) = -1$ gives the identity

$$\Psi_n \times \Phi_m(c) = \pm\Big(\sum_{i=0,1} \epsilon(v_i)\iota_{v_i} \circ \Lambda^{v_i}_{n*}(c_1) \circ \pi_{v_0} \circ \Psi_n\Big) \otimes \Big(\sum_v \epsilon(v)\iota_v \circ \Lambda^{v,c^m}_* \circ \pi_0 \circ \Phi_m\Big).$$

The second term in the tensor product is just $\Phi^{c^m}_{\mathrm{id}}$, so it suffices to show that

$$\Psi_n(c_1) := \sum_{i=0,1} \epsilon(v_i)\iota_{v_i} \circ \Lambda^{v_i}_{n*}(c_1) \circ \pi_{v_0} \circ \Psi_n : \Delta^* \to (\Delta^n; \partial\Delta^n) \tag{5.2}$$

is a map of complexes.

To see this, let $\delta^m_i : \Delta^m \to \Delta^{m+1}$ be the inclusion

$$\delta^m_i(t_0, \dots, t_m) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_m).$$

Write the differential d in Δ^* as the sum $d = d_+ + d_-$, where $d^m_+ : \Delta^m \to \Delta^{m+1}$ is $\sum_{i=1}^{m+1}(-1)^i\delta^m_i$, and $d^m_- = \delta^m_0$. Note that (Δ^*, d_+) is a complex, and $\pi_{v_0} \circ \Psi_n : (\Delta^*, d_+) \to (\Delta^n; \partial_{v_0}\Delta^n)$ is a map of complexes. Thus $\Psi_n(c_1) : (\Delta^*, d_+) \to (\Delta^n; \partial\Delta^n)$ is a map of complexes, and we need only show that $\Psi_n(c_1) \circ d_- = 0$. This follows directly from the identity

$$\Lambda^{v_0}_n(c_1) \circ \delta^n_0 = \Lambda^{v_1}_n(c_1) \circ \delta^n_0.$$

□

We have the dominant birational morphism

$$p_N : \Delta^{N-1} \times \square^1 \to \Delta^N$$

defined by

$$p_N((t_0, \dots, t_{N-1}), x) = ((1-x)t_0, \dots, (1-x)t_{N-1}, x).$$

We let

$$\pi_N : \square^N \to \Delta^N \tag{5.3}$$

be the composition

$$\square^N \xrightarrow{p_1\times \mathrm{id}} \Delta^1 \times (\square^1)^{N-1} \xrightarrow{p_2\times \mathrm{id}} \Delta^2 \times \square^{N-2} \xrightarrow{p_3\times \mathrm{id}} \dots \xrightarrow{p_{N-1}\times \mathrm{id}} \Delta^{N-1} \times \square^1 \xrightarrow{p_N} \Delta^N \tag{5.4}$$

Explicitly,

$$\pi_N(x_1, \dots, x_N) = ((1-x_1)\cdot \ldots \cdot (1-x_N), x_1(1-x_2)(1-x_3)\cdot \ldots \cdot (1-x_N), x_2(1-x_3)(1-x_4)\cdot \ldots \cdot (1-x_N), \dots, x_{N-1}(1-x_N), x_N).$$

We note that π_N maps the face $x_i = 0$ of $\square^N$ birationally onto the face $t_i = 0$ of Δ^N, for $i = 1, \dots, N$; all the other faces $x_i = 1$ of $\square^N$ land in the face $t_0 = 0$. This gives us the well-defined map

$$\pi_{N*} : (\square^N; \partial\square^N) \to (\Delta^N; \partial\Delta^N).$$

Similarly, we have the well-defined maps

$$(p_n \times \mathrm{id})_* : (\Delta^{n-1} \times \square^{m+1}; \partial(\Delta^{n-1} \times \square^{m+1})) \to (\Delta^n \times \square^m; \partial(\Delta^n \times \square^m))$$

Proposition 5.3. *Let $c : B \to (\mathbb{A}^1 \setminus \{0,1\})^{n+m}$ be a section, giving the maps $\Psi_n \times \Phi_m(c)$ and $(p_n \times \mathrm{id})_* \circ (\Psi_{n-1} \times \Phi_{m+1}(c))$ from $\Delta^* \otimes \square^*$ to $(\Delta^n \times \square^m; \partial(\Delta^n \times \square^m))$. Suppose $n \geq 2$. Then there is a homotopy $H_{n,m}(c)$ of $\Psi_n \times \Phi_m(c)$ with $(p_n \times \mathrm{id})_* \circ (\Psi_{n-1} \times \Phi_{m+1}(c))$.*

Proof. We have the map of complexes

$$\Psi_n \times \Phi_{m+1}(c) : \Delta^* \otimes \square^* \to (\Delta^n \times \square^{m+1}; \partial(\Delta^n \times \square^{m+1})).$$

Let

$$\partial(\Delta^n \times \square^{m+1})_- = [\partial(\Delta^n \times \square^{m+1}) \setminus \{(\Delta^n \times (x_1 = 0), (t_n = 0) \times \square^{m+1}\}] \cup \{(t_n = 0) \times (x_1 = 0)\},$$

where we put $(t_n = 0) \times (x_1 = 0)$ in the spot vacated by $\Delta^n \times (x_1 = 0)$. Write $\tau_{v,i}$ for the map $\epsilon(v_i, v)\iota_{v_i,v} \circ \Lambda^{v_i,v}_{n,m+1*}(c) \circ \pi_{v_0,0}$, and ι_1 and ι_n for the maps $(\square^m, \partial\square^m) \to (\square^{m+1}, \partial\square^{m+1})$ and $(\Delta^{n-1}, \partial\Delta^{n-1}) \to (\Delta^n, \partial\Delta^n)$ induced by the respective inclusions $(x_1 = 0) \to \square^{m+1}$, $(t_n = 0) \to \Delta^n$.

We apply Proposition 2.13, where we take $j = n$. Adding in the signs which occur in the definition of the tensor product of maps of complexes, we have the homotopy $\Psi_n \times \Phi_{m+1}(c)_h$ between the maps

$$\Psi_n \times \Phi_{m+1}(c)_{n,0}, \Psi_n \times \Phi_{m+1}(c)_1 : \Delta^* \otimes \square^* \to (\Delta^n \times \square^{m+1}; \partial(\Delta^n \times \square^{m+1})_-),$$

with

$$(-1)^{(m+1)n}\Psi_n \times \Phi_{m+1}(c)_0 = \\ \sum_{v,i=0,1,I} (-1)^{(n-r)(m+1)+r-1}\Big[\sum_{J,j_1>1} (\iota_n \otimes \mathrm{id}) \circ \tau_{v,i} \circ (\Psi_{n,i_1<\ldots<i_{r-1}<n} \otimes \Phi_{m+1,j_1<\ldots<j_s}) \\ + \sum_J \tau_{v,i} \circ (\Psi_{n,i_1<\ldots<i_{r-1}<n} \otimes \Phi_{m+1,1<j_2<\ldots<j_s})\Big],$$

and

$$(-1)^{(m+1)n}\Psi_n \times \Phi_{m+1}(c)_1 = \\ \sum_{v,i=0,1,J} (-1)^{(n-r)(m+1)+r}\Big[\sum_{I,i_r<n} (\mathrm{id} \otimes \iota_1) \circ \tau_{v,i} \circ (\Psi_{n,i_1<\ldots<i_{r-1}<i_r} \otimes \Phi_{m+1,1<j_2<\ldots<j_s}) \\ + \sum_I \tau_{v,i} \circ \Psi_{n,i_1<\ldots<i_{r-1}<n} \otimes \Phi_{m+1,1<j_2<\ldots<j_s}\Big].$$

Here, the indices I and J in the sums are all indices $i_1 < \ldots < i_r$, $j_1 < \ldots < j_s$, with the various special conditions as indicated in the subscripts of Ψ_n or Φ_{m+1} or in the summations, i.e., sometimes $j_1 = 1$, $i_r = n$ or both, or $j_1 > 1$ or $i_r < n$. The v in the summation is over all vertices of $\square^{m+1}$

Let $\sigma : \Delta^{n+1} \to \Delta^n$ be the degeneracy map

$$\sigma(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_n + t_{n+1}).$$

The map

$$(\sigma \times \mathrm{id}) \circ (p_{n+1} \times \mathrm{id}) : \Delta^n \times \square^{m+1} \to \Delta^n \times \square^m$$

defines the map

$$(\Delta^n \times \square^{m+1}; \partial(\Delta^n \times \square^{m+1})_-) \xrightarrow{[(\sigma\times\mathrm{id})\circ(p_{n+1}\times\mathrm{id})]_*} (\Delta^n \times \square^m; \partial(\Delta^n \times \square^m)).$$

Noting that

$$\Psi_{n,i_1<\ldots<i_{r-1}<n} = (-1)^{r-1}\Psi_{n-1,i_1<\ldots<i_{r-1}},$$
$$\Phi_{m+1,1<j_2<\ldots<j_s} = (-1)^m \Phi_{m,j_2-1<\ldots<j_s-1},$$

we find

$$[(\sigma \times \mathrm{id}) \circ (p_{n+1} \times \mathrm{id})]_* \circ (\Psi_n \times \Phi_{m+1}(c)_0) \\ = (-1)^{m+1}(p_n \times \mathrm{id})_* \circ (\Psi_{n-1} \times \Phi_{m+1}(c)),$$

$$[(\sigma \times \mathrm{id}) \circ (p_{n+1} \times \mathrm{id})]_* \circ (\Psi_n \times \Phi_{m+1}(c)_1) = (-1)^m \Psi_n \times \Phi_m(c).$$

Thus, $(-1)^m[(\sigma \times \mathrm{id}) \circ (p_{n+1} \times \mathrm{id})]_* \circ (\Psi_n \times \Phi_{m+1}(c)_h)$ gives the desired homotopy between $\Psi_n \times \Phi_m(c)$ and $(p_n \times \mathrm{id})_* \circ (\Psi_{n-1} \times \Phi_{m+1}(c))$. □

5.4. **Triangulating $\square^*$.** We begin the conversion process by writing down the "standard" triangulation of $\square^*$. Let $[N]$ denote the ordered set $\{0 < \ldots < N\}$. If S and T are partially ordered sets, we have the partially ordered set $S \times T$ with $(a, b) \le (a', b')$ if and only if $a \le a'$ and $b \le b'$. For a map

$$g : [N] \to [1]^N,$$

we have the unique affine-linear map

$$L(g) : \Delta^N \to \square^N$$

with $L(g)(v_j) = g(j)$, where we have the obvious identitification of $[1]^N$ with the set of vertices of $\square^N$. If g is injective and order-preserving, we have the well-defined permutation $\sigma(g)$ of $\{1, \ldots, N\}$ which sends j to i if the ith coordinate of $g(j-1)$ is zero and the ith coordinate of $g(j)$ is one. We let $\mathrm{sgn}(g)$ be the sign of the permutation $\sigma(g)$. This gives us the map in $\mathbb{Z}\mathbf{Sch}_B$:

$$T_N := \sum_g \mathrm{sgn}(g) L(g) : \Delta^N \to \square^N,$$

where the sum is over injective order-preserving $g : [N] \to [1]^N$. One easily checks that the sum of the T_N gives a map of complexes

$$T : \Delta^* \to \square^*.$$

5.5. **Triangulating $\Delta^* \otimes \Delta^*$.** We have the partially ordered set $[m] \times [n]$. For an injective, order preserving map

$$g := (g_{[m]}, g_{[n]}) : [m+n] \to [m] \times [n],$$

we have the affine-linear map

$$L(g) : \Delta^{m+n} \to \Delta^m \times \Delta^n$$

with $L(g)(v_j) = (v_{g_{[m]}(j)}, v_{g_{[n]}(j)})$. We also have the well-defined permutation $\sigma(g)$ of $\{1, \ldots, n+m\}$ defined by sending j to $i \in \{1, \ldots, m\}$ if $g_{[m]}(j-1) = i-1$ and $g_{[m]}(j) = i$, and to $m+i \in \{m+1, \ldots, m+n\}$ if $g_{[n]}(j-1) = i-1$ and $g_{[n]}(j) = i$. We define $\mathrm{sgn}(g) := \mathrm{sgn}(\sigma(g))$, and let

$$T_{m,n} := \sum_g \mathrm{sgn}(g) L_g.$$

The sum of the $T_{m,n}$ defines the well-known *Eilenberg-Maclane map*

$$\delta : \Delta^* \to \Delta^* \otimes \Delta^*.$$

We have the following complement to Proposition 5.3:

Proposition 5.6. *For each section $c : B \to (\mathbb{A}^1 \setminus \{0,1\})^N$, there is a homotopy $H_N(c)$ between $\Psi_N \times \Phi_0(c) \circ (\mathrm{id} \otimes T) \circ \delta$ and Ψ_N.*

Proof. We have the identity $(\mathrm{id} \otimes \Phi_0) \circ (\mathrm{id} \otimes T) \circ \delta = \mathrm{id}_{\Delta^*}$, where we make the identification $Y \otimes \square^0 = Y \otimes B = Y$ for B-schemes Y. From this we have

$$\Psi_N \times \Phi_0(c) \circ (\mathrm{id} \otimes T) \circ \delta = [\iota_{v_0} \circ \Lambda^{v_0}_{N*}(c_1) - \iota_{v_1} \circ \Lambda^{v_1}_{N*}(c_1)] \circ \pi_{v_0} \circ \Psi_N.$$

Here we write Λ_N for $\Lambda_{N,0}$, etc.

Let $\sigma : \Delta^{N+1} \to \Delta^N$ be the degeneracy map

$$(t_0, \ldots, t_{N+1}) \mapsto (t_0 + t_1, t_2, \ldots, t_{N+1}).$$

Let $\rho : \Delta^{N+1} \to \Delta^{N+1}$ be the cyclic permuation

$$\rho(t_0, \ldots, t_{N+1}) = (t_1, \ldots, t_{N+1}, t_0).$$

Let $\hat{\Lambda}^{v_1}_{N+1}$ and $\hat{\Lambda}^{v_2}_{N+1}$ be the maps $\Delta^{N+1} \to \Delta^{N+1}$ defined by

$$\hat{\Lambda}^{v_1}_{N+1} = \rho^{-1} \circ \Lambda^{v_0}_{N+1} \circ \rho,$$
$$\hat{\Lambda}^{v_2}_{N+1} = \rho^{-1} \circ \Lambda^{v_1}_{N+1} \circ \rho.$$

We apply Proposition 2.13 to the map

$$\Psi'_{N+1}(c_1) := [\iota_{v_1} \circ \hat{\Lambda}^{v_1}_{N+1*}(c_1) - \iota_{v_2} \circ \hat{\Lambda}^{v_2}_{N+1*}(c_1)] \circ \pi_{v_0} \circ \Psi_{N+1},$$

with the selected divisors $D_1 := (t_0 = 0)$ and $D_2 := (t_1 = 0)$ of Δ^{N+1}. One easily computes that

$$\sigma_* \circ \Psi'_{N+1}(c_1)_0 = -[\iota_{v_0} \circ \Lambda^{v_0}_{N*}(c_1) - \iota_{v_1} \circ \Lambda^{v_1}_{N*}(c_1)] \circ \pi_{v_0} \circ \Psi_N,$$
$$\sigma_* \circ \Psi'_{N+1}(c_1)_1 = \Psi_N,$$

so $\sigma_* \circ \Psi'_{N+1}(c_1)_h$ gives the desired homotopy. □

5.7. **Triangulating** $(\Delta^N; \partial\Delta^N)$**.** Given an iterated blow-up of faces

$$p : S_M \to S = \square^N$$

and section $c : B \to (\mathbb{A}^1 \setminus \{0,1\})^N$, we have the map of complexes $\Phi^c_p : \square^* \to (\square^N; \partial\square^N)$. We have as well the map $\pi_N : \square^N \to \Delta^N$ (5.3) and the triangulation $T : \Delta^* \to \square^*$ of §5.4. Define the map of complexes

$$\Psi^c_p : \Delta^* \to (\Delta^N; \partial\Delta^N) \tag{5.5}$$

to be the composition $\pi_{N*} \circ \Phi^c \circ T$.

More generally, suppose we have a B-scheme $B' \to B$, and a B-morphism $c : B' \to (\mathbb{A}^1 \setminus \{0,1\})^n$. Writing $\tilde{c} : B' \to (\mathbb{A}^1_{B'} \setminus \{0,1\})^N$ for the corresponding section, we have the map $\Psi^{\tilde{c}}_p : \Delta^*_{B'} \to (\Delta^N_{B'}; \partial\Delta^N_{B'})$. We let

$$\Psi^c_p : \Delta^*_{B'} \to (\Delta^N; \partial\Delta^N)$$

be the composition of $\Psi^{\tilde{c}}_p$ with the map $\pi_{B'} : (\Delta^N_{B'}; \partial\Delta^N_{B'}) \to (\Delta^N; \partial\Delta^N)$ induced by the sum of the projections $F_{B'} \to F$, F a face of Δ^N. We define $\Phi^c_p : \square^*_{B'} \to (\square^N; \partial\square^N)$, $\Psi_n \times \Phi_m(c) : \Delta^* \otimes \square^*_{B'} \to (\Delta^n \times \square^m, \partial(\Delta^n \times \square^m)$, etc. similarly. We let $\Psi_{NB'} : \Delta^*_{B'} \to (\Delta^N; \partial\Delta^N)$ be the composition $\pi_{B'} \circ (\Psi_N \times_B B')$.

Theorem 5.8. *Let $p : S_M \to S$ be an iterated blow-up of faces, $B' \to B$ a B-scheme, and $c : B' \to (\mathbb{A}^1 \setminus \{0,1\})^N$ a B-morphism. For each B-morphism $c' : B' \to (\mathbb{A}^1 \setminus \{0,1\})^{N+1}$ of the form (c, c_{N+1}) there is a homotopy $H^{c'}_p$ between Ψ^c_p and $\Psi_{NB'}$.*

Proof. It clearly suffices to consider the case $B' = B$. We have the homotopy $H_0(c')$ of Φ^c_p with Φ^c_{id} given by Proposition 4.12, giving the homotopy $\pi_{N*} \circ H_0(c') \circ T$ of Ψ^c_p with Ψ^c_{id}.

Recall the tower (5.4)

$$\square^N \cong \Delta^1 \times \square^{N-1} \to \Delta^2 \times \square^{N-2} \to \ldots \to \Delta^{N-1} \times \square^1 \to \Delta^N.$$

Let $\pi_{N-m,m} : \Delta^{N-m} \times \square^m \to \Delta^N$, $m = 1, \ldots, N$, be the composition in this tower.

It is an elementary computation to see that

$$(\Psi_0 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes T) \circ \delta = T,$$

where we make the identification $\Delta^0 \otimes Y = B \otimes Y = Y$, for B-schemes Y. From this we see that

$$\pi_{N*} \circ (\Psi_0 \otimes \Phi_N(c)) \circ (\mathrm{id} \otimes T) \circ \delta = \Psi^c_{\mathrm{id}}.$$

The isomorphism $p_1 \times \mathrm{id} : \square^N \to \Delta^1 \times \square^{N-1}$ identifies Ψ^c_{id} with $\pi_{1,N-1*} \circ (\Psi_1 \otimes \Phi_{N-1}(c)) \circ (\mathrm{id} \otimes T) \circ \delta$

We have the homotopies $H_{N-m,m}(c)$ of Proposition 5.3; the sum

$$\sum_{m=0}^{N-2} \pi_{N-m,m*} \circ H_{N-m,m}(c) \circ (\mathrm{id} \times T) \circ \delta$$

thus gives a homotopy between Ψ^c_{id} and $\Psi_N \times \Phi_0(c) \circ (\mathrm{id} \otimes T) \circ \delta$.

We have the homotopy $H_N(c)$ between $\Psi_N \times \Phi_0(c) \circ (\mathrm{id} \otimes T) \circ \delta$ and Ψ_N given by Proposition 5.6. Thus, we may take

$$H_p^{c'} := \pi_{N*} \circ H_0(c') \circ T + \sum_{m=0}^{N-2} \pi_{N-m,m*} \circ H_{N-m,m}(c) \circ (\mathrm{id} \times T) \circ \delta + H_N(c).$$

□

6. Good position

In this section, we complete the proof of Theorem 1.9. The final step is to show that, for a suitable iterated blow-up of faces $p : S' \to S$, and a general choice of the auxiliary section $c' := (c, c_{N+1}) : B \to (\mathbb{A}^1 \setminus \{0,1\})^{N+1}$, the map Ψ^c_p and the homotopy $H_p^{c'}$ between Ψ^c_p and Ψ_N given by Theorem 5.8 satisfy the general position conditions required by Theorem 1.9. We suppose that B is an irreducible excellent noetherian scheme. Unless specified otherwise, all schemes will be reduced.

6.1. **Proper intersection.** Let Z be a B-scheme of finite type, T a smooth B-scheme with strict reduced relative normal crossing divisor ∂T, and $f : Z \to T$ a B-morphism, such that no generic point of Z lands in ∂T. We say that Z *intersects the faces of T properly* if, for each face F of T, we have

$$\mathrm{codim}_Z(f^{-1}(F)) \geq \mathrm{codim}_T(F).$$

Let $(T', \partial T')$ and $(T, \partial T)$ be smooth B-schemes with strict reduced relative normal crossing divisors, and $p : (T', \partial T') \to (T', \partial T')$ a B-morphism which induces an isomorphism $p : T' \setminus \partial T' \to T \setminus \partial T$. For a B-morphism $f : Z \to T$ we have the map $p^{-1} \circ f : Z \setminus f^{-1}(\partial T) \to T'$, inducing the section $\sigma : Z \setminus f^{-1}(\partial T) \to Z \times_T T'$. Let $p^{-1}[Z]$ be the closure of the image of σ, and $p^{-1}[f] : p^{-1}[Z] \to T'$ the morphism induced by p_2. We let $p^{-1}(Z)$ denote the reduced fiber product $(Z \times_T T')_{\mathrm{red}}$, and $p^{-1}(f) : p^{-1}(Z) \to T'$ the projection.

6.2. **Monomial morphisms.** A morphism $p : \mathbb{A}^n \to \mathbb{A}^m$ is *monomial* if there are integers $b_{ij} \geq 0$ such that $p^*(x_i) = \prod_{j=1}^n x_j^{b_{ij}}$ for $i = 1, \ldots, m$. For $n = m$, it is easy to show that a monomial morphism p is birational if and only if $\det(b_{ij}) = \pm 1$.

We have the category of pairs $(T, \partial T)$, where T is a smooth B-scheme, ∂T is a strict reduced relative normal crossing divisor, and a map of pairs $(T, \partial T) \to (T', \partial T')$ is a morphism of B-schemes $p : T \to T'$ such that p maps ∂T to $\partial T'$. We call such a map étale if p is étale and $p^{-1}(\partial T') = \partial T$. The étale topology on $\mathbf{Sch}_B$ induces a Grothendieck topology on the category of pairs, which we also call the étale topology.

Let $\partial \mathbb{A}^n \subset \mathbb{A}^n$ be the sum of the coordinate hyperplanes. Let $(X, \partial X)$, $(Y, \partial Y)$ be smooth B-schemes with strict reduced relative normal crossing divisors. A morphism $p : (X, \partial X) \to (Y, \partial Y)$ is a *locally birational monomial morphism* if p is locally isomorphic (in the étale topology of pairs) to a birational monomial morphism $(\mathbb{A}^n, \partial \mathbb{A}^n) \to (\mathbb{A}^n, \partial \mathbb{A}^n)$.

Let $p : \mathbb{A}^n \to \mathbb{A}^n$ be a birational monomial morphism, with corresponding matrix of exponents (b_{ij}). Let (a_{ij}) be the inverse to the matrix (b_{ij}), set $\rho_p(t)_j = (\prod_k t^{a_{jk}})$, and let ρ_p be the diagonal $\mathbb{G}_m^n$-action on $\mathbb{A}^n$ defined by

$$\rho_p((t_1, \ldots, t_n), (x_1, \ldots, x_n)) = (\rho_p(t)_1 x_1, \ldots, \rho_p(t)_n x_n).$$

For example, the fundamental action of $\mathbb{G}_m^n$ on $\mathbb{A}^n$,

$$(t_1, \dots, t_n) \cdot (x_1, \dots, x_n) = (t_1 x_1, \dots, t_n x_n).$$

is ρ_{id}. The map p is $\mathbb{G}_m^n$-equivariant, with $\mathbb{G}_m^n$ acting via ρ_p on the domain, and via the fundamental action on the range.

Lemma 6.3. *Let $p : (X, \partial X) \to (Y, \partial Y)$ be a locally birational monomial morphism of B-schemes, and let $f : Z \to Y$ be a finite type morphism of B-schemes, intersecting all faces of Y properly. Form the cartesian square*

$$\begin{array}{ccc} Z' & \xrightarrow{f'} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f]{} & Y. \end{array}$$

Then

1. *$f' : Z' \to X$ intersects all faces properly.*
2. *Suppose that $p : X \setminus \partial X \to Y \setminus \partial Y$ is an isomorphism. Then*

$$p^{-1}[Z] = Z'_{\mathrm{red}} = p^{-1}(Z).$$

Proof. We may assume that Z is irreducible. The second statement follows from the first. Indeed, since $Z' \to X$ intersects all faces properly, each generic point of Z' has image in $X \setminus \partial X$. If we assume that $X \setminus \partial X \to Y \setminus \partial Y$ is an isomorphism, this implies that each generic point of Z' is in $p^{-1}[Z]$, whence (2).

For the first statement we may assume that $X = Y = \mathbb{A}^n$, and that $p : (\mathbb{A}^n, \partial \mathbb{A}^n) \to (\mathbb{A}^n, \partial \mathbb{A}^n)$ is a birational monomial morphism.

Let F be a face of $\mathbb{A}^n$. The *open* face F^0 is the complement in F of all the faces of $\mathbb{A}^n$ properly contained in F. The following facts are easy to verify:

(a) Let F^0 be an open face in $\mathbb{A}^n$. There is a unique open face G^0 in $\mathbb{A}^n$ such that $p(F^0)$ is contained in G^0.

(b) Let G^0 be an open face in $\mathbb{A}^n$. Then G^0 is an orbit of $\mathbb{G}_m^n$ for the fundamental action of $\mathbb{G}_m^n$ on $\mathbb{A}^n$.

Since the map p is $\mathbb{G}_m^n$-equivariant (acting by ρ_p on the domain, and the fundamental action on the range), it follows from (a) and (b) that the restriction of p to the morphism

$$p : F^0 \to G^0$$

is surjective and $\mathbb{G}_m^n$-equivariant. From (b) it follows that a choice of a section $\sigma : B \to G^0$ determines an isomorphism of F^0 with $G^0 \times_B p^{-1}(\sigma(B))$, with p becoming the projection on G^0. If G is defined by the equations $X_j = 0$, $j \in J \subset \{1, \dots, n\}$, then we have the section σ with value $x_j = 0$ for $j \in J$, $x_j = 1$ for $j \notin J$. Thus, the morphism $p : F^0 \to G^0$ is flat.

As Z is of finite type over B, and B is excellent, Z has a well-defined finite Krull dimension. Since $\mathbb{A}^n$ is smooth over B, the map p is a birational l.c.i. morphism. This implies that each irreducible component of Z' has Krull dimension greater than or equal to the Krull dimension of Z. Since Z intersects all faces of $\mathbb{A}^n$ properly, the flatness of $p : F^0 \to G^0$ implies that each irreducible component of Z' which lies over ∂X has Krull dimension strictly less than the Krull dimension of Z. Thus Z' is irreducible, and the generic point of Z' maps to $X \setminus \partial X$. Using the flatness of the maps $p : F^0 \to G^0$, and the fact that $Z \to \mathbb{A}^n$ intersects all faces of $\mathbb{A}^n$

properly again, we see that $\mathrm{codim}_{Z'}(f'^{-1}(F)) \geq \mathrm{codim}_{\mathbb{A}^n}(F)$ for all faces F, i.e., that $f' : Z' \to \mathbb{A}^n$ intersects all faces properly. □

Example 6.4. Recall from §3 the category of iterated blow-ups $\mathfrak{B}_Y$ for a B-scheme Y with strict reduced normal crossing divisor ∂Y. Let $p_i : (X_i, \partial X_i) \to (Y, \partial Y)$, $i = 1, 2$, be in $\mathfrak{B}_Y$, such that $p_2 = p_1 \circ p$ for some Y-morphism $p : X_2 \to X_1$ (we say that X_2 *dominates* X_1). Then the induced morphism $p : (X_2, \partial X_2) \to (X_1, \partial X_1)$ is a locally birational monomial morphism. If $f : Z \to X_1$ intersects all faces properly, then by Lemma 6.3, so does $p^{-1}(f) : p^{-1}(Z) \to X_2$, and in addition, $p^{-1}(Z) = p^{-1}[Z]$.

6.5. For each vertex v of $S := \square^N$, we have the divisor $\partial_v \square^N$ consisting of those components of $\partial \square^N$ which contain v; the pair $(\square^N, \partial_v \square^N)$ is isomorphic to $(\mathbb{A}^N, \partial \mathbb{A}^N)$. Fix for each vertex v an iterated blow-up of faces of $(\square^N, \partial_v \square^N)$, $p_v : S(v) \to S$. By [9, Proposition 5.3], there is an iterated blow-up of faces of $(\square^N, \partial \square^N)$, $p : S_M \to S$, which dominates each $S(v)$; we let $q_v : S_M \to S(v)$ be the induced morphism.

If w is a vertex of $S(v)$, we have the open neighborhood U_w of w, being the complement of the union of components of $\partial S(v)$ which do not contain w, and the distinguished coordinate system t^w of regular functions on U_w. We let $\partial U_w = \partial S(v) \cap U_w$. By induction on the number of blow-ups used to construct p_v, and a direct computation of the distinguished coordinate system on a blow-up, as in the proof of Lemma 4.2, we see that the map $t^w : U_w \to \mathbb{A}^n$ is an isomorphism, and gives an isomorphism of pairs $(U_w, \partial U_w) \to (\mathbb{A}^N, \partial \mathbb{A}^N)$. Via this identification, we may speak of a monomial morphism $\mathbb{A}^n \to U_w$.

Let $\mathbb{A}^N_0$ be the "semi-local scheme" of the zero-section v_0 in $\mathbb{A}^N$, i.e., the limit of the open subschemes $\mathbb{A}^N \setminus C$, over closed subschemes C with $C \cap v_0 = \emptyset$. For a vertex u of S_M, we have the coordinate system t^u of regular functions on U_u, giving the morphism $t^w : U_u \to \mathbb{A}^N$ which is an isomorphism over a neighborhood of v_0; let

$$\lambda^u : \mathbb{A}^N_0 \to U_u$$

be the morphism induced by the inverse of t^w.

Lemma 6.6. *Let u be a vertex of S_M, let $v = p(u)$, and let $w = q_v(u)$. Then the composition $q_v \circ \lambda^u : \mathbb{A}^N_0 \to S(v)$ extends (uniquely) to a birational monomial morphism $\Lambda^u_w : \mathbb{A}^N \to U_w \subset S(v)$.*

Proof. It is clear that q_v maps the coordinate neighborhood $U_u \subset S_M$ into U_w, hence $q_v^*(t_j^w)$ is a regular function on U_u for each $j = 1, \ldots, n$.

The proof now proceeds essentially as in Lemma 4.11. We may assume that B is affine, $B = \mathrm{Spec}\, A$. Applying Lemma 4.2 to the birational maps p and p_v, there is a matrix $(b_{ij}) \in \mathrm{GL}_N(\mathbb{Z})$ such that

$$q_v^*(t_i^w) = \prod_{j=1}^N (t_j^u)^{b_{ij}}; \quad i = 1, \ldots, N.$$

Since the $q_v^*(t_j^w)$ are all regular on U_u, we have $\mathrm{div}(q_v^*(t_i^w)) \geq 0$ for each $i = 1, \ldots, N$, from which it follows that all the b_{ij} are non-negative. Thus, the map

$$(q_v \circ \lambda^u)^* : A[t_1^w, \ldots, t_N^w] \to \Gamma(\mathbb{A}^N_0, \mathcal{O})$$

has image in the subring $\Gamma(\mathbb{A}^N, \mathcal{O}) \cong A[t_1^u, \dots, t_N^u]$, giving the extension Λ_w^u. From the explicit formula

$$\Lambda_w^{u*}(t_j^w) = \prod_{j=1}^N (t_j^u)^{b_{ij}},$$

we see that Λ_w^u is a birational monomial morphism. □

6.7. **General position on $\mathbb{A}^N$.** We now assume that the base-scheme B is regular and has Krull dimension one. We call a morphism of B-schemes $B' \to (\mathbb{A}^1 \setminus \{0,1\})^n$ *allowable* if B' is a flat B-scheme.

Let $f : Z \to S$ be a B-morphism of finite type such that no generic point of Z lands in ∂S, let w be a vertex of S_M, and let $c : B' \to (\mathbb{A}^1 \setminus \{0,1\})^N$ be an allowable morphism of B-schemes.

Let $p : S_M \to S$ be an iterated blow-up of faces. We have the map (4.7)

$$\Lambda^{w,c} : \square^n_{B'} \to S_{B'};$$

we let $p^!_{w,c} Z_{B'}$ denote the proper transform $(\Lambda^{w,c})^{-1}[Z_{B'}]$, and $p^!_{w,c}(f) : p^!_{w,c} Z_{B'} \to \square^N_{B'}$ the induced morphism.

Recall the triangulation T_n of $\square^n$ given in §5.4. If $f : \Delta^n \to \square^n$ is one of the maps appearing in T_n, and if F is a face of Δ^n, we call the image $f(F)$ a *face of* $T\square^n$.

Let $B' \to B$ be a B-scheme, $d = (d_1, \dots, d_n) : B' \to \mathbb{G}_m^n$ a B-morphism, giving the B-isomorphism

$$i(d) : \square^n_{B'} \to \mathbb{A}^n_{B'}$$

with $i(d)^*(X_i) = d_i X_i$.

Lemma 6.8. *Let $f : Z \to \mathbb{A}^n$ be a B-morphism of finite type such that no generic point of Z lands in $\partial\mathbb{A}^n$. Suppose that Z intersects all faces of $(\mathbb{A}^n, \partial\mathbb{A}^n)$ properly. Then there is an open subscheme V_Z of $\mathbb{G}_m^n$, mapping onto B, such that, for all allowable B-morphisms $d : B' \to V_Z$, and for all faces F of $T\square^n$, the base-extension $f_{B'} : Z_{B'} \to \mathbb{A}^n_{B'}$ intersects $i(d)(F)$ properly, i.e.,*

$$\operatorname{codim}_Z\big(f_{B'}^{-1}(i(d)(F))\big) \geq \operatorname{codim}_{\mathbb{A}^n}(i(d)(F)).$$

Proof. It suffices to prove the lemma in case Z is irreducible. We have the map over $\mathbb{G}_m^n$, $(p_1, \rho) : \mathbb{G}_m^n \times \mathbb{A}^n \to \mathbb{G}_m^n \times \mathbb{A}^n$, with ρ the fundamental action ρ_{id}. Via the identification $\square^n = \mathbb{A}^n$, the map $i(d)$ is just the pull-back of (p_1, ρ) by $d : B' \to \mathbb{G}_m^n$. By induction on n, we need only consider faces F of $T\square^n$ which are not contained in $\partial\mathbb{A}^n$. Since the intersection of F with $\partial\mathbb{A}^n$ is again a face, it suffices to show that there is an open subscheme $j : V \to \mathbb{G}_m^n$ of $\mathbb{G}_m^n$, mapping onto B, such that open face $F^0 := F \setminus F \cap \partial\mathbb{A}^n$ has the property that $i(j)(F_0)$ intersects $f_V : Z_V \to \mathbb{A}^n_V$ properly.

We write $\mathbb{A}$ for $\mathbb{A}^n \setminus \partial\mathbb{A}^n$ and G for $\mathbb{G}_m^n$. The map $\rho : G \times_B F^0 \to \mathbb{A}$ is a locally trivial bundle over $\mathbb{A}$ with fiber F^0, thus, the same is true of the pull-back of ρ via f. In particular, $(G \times_B F^0) \times_{\mathbb{A}} Z$ is irreducible, of dimension $\dim Z + \dim_B F^0$. Consider the projection

$$\pi : (G \times_B F^0) \times_{\mathbb{A}} Z \to G.$$

If π is dominant, there is a codimension two subscheme C of G such that π is equi-dimensional over G. If π is not dominant, but $p : (G \times_B F^0) \times_{\mathbb{A}} Z \to B$ is dominant, let C be the closure of the image of π. If both π and p are not dominant, say the image of p is the closed point b of B, then there is a proper closed subset C of the fiber G_b such that $\pi : (G \times_B F^0) \times_{\mathbb{A}} Z \to G_b$ is equi-dimensional over $G_b \setminus C$. In each case, the closed subset C contains no fiber of the projection $G \to B$, hence the subscheme $V := G \setminus C$ maps onto B, and clearly the inclusion $j : V \to G$ has the property that $i(j)(F_0)$ intersects $f_V : Z_V \to \mathbb{A}^n_V$ properly. $\square$

We have a similar result for the triangulation of $\Delta^n \times \square^m$ given by composing the map $\Psi_n \times \Phi_m(c)$ of §5.1 with the triangulation $\mathrm{id} \times T_m$ and the Eilenberg-Maclane map δ of §5.5. We leave the proof of the following lemma to the reader; the proof is essentially the same for Lemma 6.8.

Lemma 6.9. *Let $f : Z \to \Delta^n \times \square^m$ a B-morphism which intersects all faces of $\Delta^n \times \square^m$ properly. Then there is an open subscheme V_Z of $(\mathbb{A}^1 \setminus \{0,1\})^{n+m}$, mapping onto B, such that, for all allowable B-morphisms $c : B' \to V_Z$, all faces F of Δ^{n+m}, and all maps $T(c) : \Delta^{n+m}_{B'} \to \Delta^n \times \square^m$ occuring in $\Psi_n \times \Phi_m(c) \circ (\mathrm{id} \times T_m) \circ \delta$, the map $f_{B'} : Z_{B'} \to \Delta^n \times \square^m_{B'}$ intersects $T(c)(F)$ properly.*

Now suppose that, as in §6.5, we have for each vertex v of $S = \square^N$, an iterated blow-up of faces $p_v : S(v) \to (S, \partial_v S)$, such that $p : S_M \to S$ factors through $S(v)$.

Proposition 6.10. *Let $f : Z \to S$ be a B-morphism of finite type such that no generic point of Z lands in ∂S. Suppose that $Z_v := p_v^{-1}[Z]$ intersects all faces of $S(v)$ properly, for all vertices v of S. Then there is a Zariski open subscheme V_Z of $(\mathbb{A}^1 \setminus \{0,1\})^N$, such that*

1. *The structure morphism $V_Z \to B$ is surjective.*
2. *For each allowable morphism of B-schemes $c : B' \to V_Z$, and each vertex u of S_M, the morphism $p^!_{u,c}(f) : p^!_{u,c} Z_{B'} \to \square^n_{B'}$ intersects all faces of $T\square^n_{B'}$ properly.*

In particular, if $f : Z \to S$ intersects all faces of S properly, then the above holds for all iterated blow-ups of faces $p : S_M \to S$.

Proof. Let u be a vertex of S_M, giving the coordinate system (t^u, U_u) around u, the vertex $v := p(u)$ of S, and the vertex $w := q_v(u)$ of $S(v)$. For a B-morphism $c : B' \to (\mathbb{A}^1 \setminus \{0,1\})^N$, we have the induced morphism $d := t^u(c) : B' \to \mathbb{G}_m^N$, and, via Lemma 6.6, the factorization of the map $\Lambda^{u,c}$ as

$$\Lambda^{u,c} = p_v \circ \Lambda^u_v \circ i(t^u(c)).$$

We let $Z_{v,u}$ be the fiber product $Z_v \times_{S(v)} \mathbb{A}^N$,

$$\begin{array}{ccc} Z_{v,u} & \xrightarrow{f_{v,u}} & \mathbb{A}^N \\ \downarrow & & \downarrow{\scriptstyle \Lambda^u_w} \\ Z_v & \xrightarrow[f_v]{} & S(v). \end{array}$$

By our assumption on the maps $Z_v \to S(v)$, together with Lemma 6.3, it follows that $f_{v,u} : Z_{v,u} \to \mathbb{A}^N$ intersects all faces of $\mathbb{A}^N$ properly. Let $V_{Z_{v,u}}$ be the open subscheme of $\mathbb{G}_m^N$ given by Lemma 6.8; by definition, if c is a morphism such that $t^u(c)$ lands in $V_{Z_{v,u}}$, then $p^!_{u,c}(f) : p^!_{u,c} Z_{B'} \to \square^N_{B'}$ intersects all faces of $T\square^N_{B'}$ properly.

Let V_Z be the intersection in $(\mathbb{A}^1\backslash\{0,1\})^N$ of the open subschemes $(\mathbb{A}^1\backslash\{0,1\})^N\cap(t^u)^{-1}(V_{Z_{p(u),u}})$, as u runs over all vertices of S_M. Since each t^u is dominant on each fiber over B, all the fibers of $V_Z\to B$ are non-empty. We have already seen that condition (2) holds for each morphism $c:B'\to V_Z$. This proves (1) and (2).

The remaining statement follows by taking all the maps $p_v:S(v)\to S$ to be the identity. $\square$

6.11. **Some reductions.** We now turn to the proof of Theorem 1.9. In this section, we make some preliminary reductions. We use the notations of §1 and Theorem 1.9. Let $B=\operatorname{Spec}A$, where A is a semi-local PID.

We will show the following sharper result:

Theorem 6.12. *Let $j:U\to X$ be an open subscheme of a finite type B-scheme X, and let $\{C_{I,j}\in U_{(I,q_j)}\}$ be a finite collection of irreducible closed subsets, as in the statement of Theorem 1.9. Then there is an iterated blow-up of faces $S'\to S$ such that, for each iterated blow-up of faces $p:S_M\to S$ which dominates S', there is an open subset $V(S_M)\subset(\mathbb{A}^1-\{0,1\})^{N+1}$ such that*

1. *The structure morphism $V(S_M)\to B$ is surjective.*
2. *For each flat B-scheme B' and each B-morphism $c':=(c,c_{N+1}):B'\to V(S_M)$, the map Ψ_p^c and the homotopy $H_p^{c'}$ of Ψ_p^c with $\Psi_{NB'}$ given by Theorem 5.8 satisfy the following analog of the conclusions of Theorem 1.9: Write Ψ and H as sums with $\mathbb{Z}$-coefficients*

$$\Psi=\sum_{\substack{s,\\ I\subsetneq\{0,\dots,N\}}}n_I^sf_I^s;\quad H=\sum_{\substack{s,\\ I\subsetneq\{0,\dots,N\}}}m_I^sg_I^s;\quad n_I^s,m_I^s\neq0,$$

with

$$f_I^s:\Delta_{B'}^{N-|I|}\to\partial\Delta_I^N;\quad g_I^s:\Delta_{B'}^{N-|I|+1}\to\partial\Delta_I^N,$$

maps of B-schemes. Then

(a) *Each component of $(\mathrm{id}\times f_I^s)^{-1}(C_{I,j})$ is in $(U_{B'})^X_{(N-|I|,q_j)}$ for each I, s and j.*

(b) *Each component of $(\mathrm{id}\times g_I^s)^{-1}(C_{I,j})$ is in $(U_{B'})_{(N-|I|+1,q_j)}$ for each I, s and j.*

(c) *If $C_{I,j}$ is in $U^X_{(I,q_j)}$, then each component of $(\mathrm{id}\times g_I^s)^{-1}(C_{I,j})$ is in $(U_{B'})^X_{(N-|I|+1,q_j)}$ for each s.*

We first reduce the proof of Theorem 6.12 to the case of affine X. Take X of finite type over B, let $X=\cup_{i=1}^nX_i$ be a finite affine cover, giving the open subscheme $U_i:=U\cap X_i$ of X_i. Assuming Theorem 6.12 for the affine schemes X_i, we have for each $i=1,\dots,n$ an iterated blow-up of faces $S_i'\to S$, satisfying Theorem 6.12 for $U_i\subset X_i$ and the collection of subsets $\{C_{I,j}\cap(U_i\times\partial\Delta_I^N)\}$. We denote the open subset corresponding to an S'' dominating S_i' by $V_i(S'')$. By [9, Proposition 5.3], there is an iterated blow-up of faces $S'\to S$ which dominates all the S_i. It then follows that, for each blow-up of faces $p:S_M\to S$ which dominates S', and each allowable B-morphism

$$c'=(c,c_{N+1}):B'\to V(S_M):=\cap_{i=1}^nV_i(S_M),$$

the map Ψ_p^c and the homotopy $H_p^{c'}$ of Ψ_p^c with $\Psi_{NB'}$ given by Theorem 5.8 satisfy, for each $i=1,\dots,n$, the conclusions of Theorem 6.12 for $U_i\subset X_i$ and the collection $\{C_{I,j}\cap(U_i\times\partial\Delta_I^N)\}$. Since the property of being in $U_{(s,q)}$ (resp. in $U^X_{(s,q)}$) is local over U (resp. local over X), the map Ψ_p^c and the homotopy $H_p^{c'}$ satisfy conclusions of Theorem 6.12 for $U \subset X$ and original collection $\{C_{I,j}\}$. Thus, it suffices to prove Theorem 6.12 for X affine.

We will make a further simplification; we first require the following elementary lemma:

Lemma 6.13. *Let Y be quasi-projective over B, $j : U \to Y$ a non-empty open subscheme, and C an element of $U_{(I,q)}$ for some $I \subset \{0, \ldots, N\}$, and some $q \geq -N$. Then there is an irreducible closed subset $\tilde{C}$ of Δ^N_U such that $\tilde{C}$ is in $U_{(\emptyset,q)}$, and C is an irreducible component of $\tilde{C} \cap U \times \partial\Delta^N_I$. If C is in $U^Y_{(I,q)}$, we may find a $\tilde{C}$ as above with $\tilde{C} \in U^Y_{(\emptyset,q)}$.*

Proof. If we can prove the result in the case of an element of $U_{(I,q)}$, the result follows for an element C of $U^Y_{(I,q)}$. Indeed the closure $\bar{C}$ in $Y \times \partial\Delta^N_I$ is in $Y_{(I,q)}$ by definition; if we have a $D \in Y_{(\emptyset,q)}$ with $\bar{C}$ an irreducible component of $D \cap Y \times \partial\Delta^N_I$, then taking $\tilde{C} = D \cap U \times \Delta^N$ gives the desired closed subset.

If Y' is an irreducible component of Y, then the inclusion $U' := U \cap Y' \to U$ induces maps $U'_{(I,q)} \to U_{I,q}$, $U'^{Y'}_{(I,q)} \to U^Y_{(I,q)}$, and similarly for $I = \emptyset$. Also, $U_{(I,q)}$ is the union of the $(U \cap Y')_{(I,q)}$, as Y' runs over the irreducible components of Y, and similarly for $U^Y_{(I,q)}$, $U_{(\emptyset,q)}$ and $U^Y_{(\emptyset,q)}$. Thus, we may assume that Y is irreducible.

In this case, since B has Krull dimension at most one, either Y is equi-dimensional over B, or Y maps to a closed point of B. We give the proof in the first case; the proof in the second case is essentially the same, and is left to the reader.

Suppose that C is in $U_{(I,q)}$. We fix an embedding of Y in a projective space $\mathbb{P}^n_B$. Since $\dim C = N - |I| + q$, C has pure codimension $r := \dim_B Y - q$ on $U \times \partial\Delta^N_I$, and intersects each subscheme $U \times \partial\Delta^N_J$ in codimension $\geq r$, for each face $I \subset J \subsetneq \{0, \ldots, N\}$. Let $\mathcal{I}$ be the ideal sheaf of C. For d large enough, the sheaf $\mathcal{I}(d)$ is generated by global sections. Thus, there are sections $s_1, \ldots, s_r$ of $\mathcal{I}(d)$ over $U \times \Delta^N$ such that the subscheme $\tilde{C}$ defined by $s_1, \ldots, s_r$ satisfies

1. $\tilde{C} \setminus C$ has pure codimension r on $U \times \Delta^N \setminus C$.
2. $(\tilde{C} \setminus C) \cap U \times \partial\Delta^N_J$ has pure codimension r on $U \times \partial\Delta^N_J \setminus C$ for all $J \subsetneq \{0, \ldots, N\}$.

Since C intersects all faces of $U \times \partial\Delta^N_I$ in codimension r, it follows that $\tilde{C}$ is in $U_{\emptyset,q}$. Since $\tilde{C}$ contains C, and each component of $\tilde{C} \cap U \times \partial\Delta^N_I$ has codimension r, it follows that C is a component of $\tilde{C} \cap U \times \partial\Delta^N_I$. □

We can now make our final reduction.

Proposition 6.14. *Suppose we can prove Theorem 6.12 for X affine, and for each finite collection of irreducible closed subsets $\{C_j \in U_{(\emptyset,q_j)}\}$. Then Theorem 6.12 is true in general.*

Proof. We have already seen that it suffices to prove Theorem 6.12 for X affine. Suppose then that X is affine, and that $\{C_{I,j} \in U_{(I,q_j)}\}$ is a finite set of irreducible closed subsets. By Lemma 6.13, we can find for each $C_{I,j}$ a $Z_{I,j} \in U_{\emptyset,q_j}$ such that $C_{I,j}$ is an irreducible component of $Z_{I,j} \cap \partial\Delta^N$, and if $C_{I,j}$ is in $U^X_{(I,q_j)}$, then we may take $Z_{I,j} \in U^X_{(\emptyset,q_j)}$.

Suppose we can find an iterated blow-up of faces $S' \to S$, and open subschemes $V(S_M)$ for each iterated blow-up $p : S_M \to S$ which dominates S', satisfying

Theorem 6.12 for the collection $\{Z_{I,j}\}$. Fix one such $p : S_M \to S$, and an allowable B-morphism $c' = (c, c_{N+1}) : B' \to V(S_M)$. Replacing B with B' and changing notation, we may assume that $B' = B$. Thus we have the map $\Psi_p^c : \Delta^* \to (\Delta^N, \partial\Delta^N)$, and the homotopy $H_p^{c'}$ of Ψ_p^c with Ψ_N.

Let $f : \Delta^* \to (Y; \partial Y)$ be a map of degree $-N$ in $\mathbf{C}(\mathbb{Z}\mathbf{Sch}_B)$. We say that f is *compatible with faces* if, for each component $f_I^s : \Delta^{N-|I|} \to \partial Y_I$ of f, there is a component $f_\emptyset^t : \Delta^N \to \partial Y_\emptyset = Y$, and a face F of Δ^N such that f_I^s composed with the inclusion $\partial Y_I \to Y$ factors as

$$\Delta^{N-|I|} \cong F \subset \Delta^N \xrightarrow{f_\emptyset^t} Y.$$

We make an analogous definition for maps $\square^* \to (Y; \partial Y)$, $\square_0^* \to (Y; \partial Y)$, or $\Delta^* \otimes \square^* \to (Y; \partial Y)$.

We claim that the maps Ψ_p^c and $H_p^{c'}$ are compatible with faces. Assuming this is the case, the fact that the $(f_\emptyset^t)^{-1}(Z_{I,j})$ intersects all faces of $U \times \Delta^N$ properly implies that $(f_I^s)^{-1}(C_{I,j})$ intersects all faces of $U \times \Delta^{N-|I|}$ properly. Similarly, if $(f_\emptyset^t)^{-1}(Z_{I,j})$ is in $U^X_{(N,q_j)}$, then $(f_I^s)^{-1}(C_{I,j})$ is in $U^X_{(N,q_j)}$.

We proceed to verify the above claim. Consider first the map (4.4) $\phi^c : \square_0^* \to (S_M, \partial S_M)$. It follows directly from the construction of ϕ^c that ϕ^c is compatible with faces. Since Φ_p^c is the unique extension of the map $p \circ \phi^c$, it follows that $\Phi_p^c : \square^* \to (S, \partial S)$ is compatible with faces. Since $\Psi_p^c = \pi_N \circ \Phi_p^c \circ T$, where $\pi_N : \square^N \to \Delta^N$ is the map (5.3) and $T : \Delta^* \to \square^*$ is the standard triangulation, the claim for Ψ_p^c is verified.

The proof for the homotopy $H_p^{c'}$ is similar. We use the notation from the proof of Theorem 5.8. The homotopy $H_p^{c'}$ is the sum

$$H_p^{c'} = \pi_{N*} \circ H_0(c') \circ T + \sum_{m=0}^{N-2} \pi_{N-m,m*} \circ H_{N-m,m}(c) \circ (\mathrm{id} \times T) \circ \delta + H_N(c).$$

Here $H_0(c')$ is the homotopy of Φ_p^c with Φ_{id}^c, constructed in the proof of Proposition 4.12, $H_{N-m,m}(c)$ is the homotopy of maps $\Delta^* \otimes \square^* \to (\Delta^{N-m} \times \square^m, \partial)$ constructed in Proposition 5.3, and $H_N(c)$ is the homotopy constructed in in Proposition 5.6. $T : \Delta^* \to \square^*$ is the triangulation constructed in §5.4, $\delta : \Delta^* \to \Delta^* \otimes \Delta^*$ is the Eilenberg-Maclane map, and the maps $\pi_{N-m,m}$ are the compositions in the tower (5.4). It clearly suffices to prove that the maps $H_0(c')$, $H_{N-m,m}(c)$, $m = 0, \ldots, N-1$, and $H_N(c)$ are compatible with faces.

For this, consider first the map $H_0(c')$. To construct $H_0(c')$, we started with the map $\Phi_q^{c'} : \square^* \to (T, \partial T)$, coming from a certain iterated blow-up of faces $q : T_M \to \square^{N+1}$. We took the projection $\pi : \square^{N+1} \to \square^N$ on the first N-factors, and then we applied the homotopy machine of §2.10 to the map $\pi_* \circ \Phi_q^{c'}$. It follows from the explicit form (2.11) of the homotopy $\Phi_h^{c'}$ corresponding to $\Phi_q^{c'}$ that the components of $H_0(c')$ and $\pi_* \circ \Phi_q^{c'}$ consisting of maps $\square^{N+1} \to \square^N$ are the same, and that each map $\square^{i+1} \to \square^i$ occurring in $H_0(c')$ occurs in $\pi_* \circ \Phi_q^{c'}$. Thus, since $\pi_* \circ \Phi_q^{c'}$ is compatible with faces, so is $H_0(c')$.

The maps $H_{N-m,m}(c)$, $m = 0, \ldots, N-2$, are constructed similarly (see the proof of Proposition 5.3) from tensor products of maps of the form $\Psi_{\mathrm{id}}^{c^1}$, $\Phi_{\mathrm{id}}^{c^2}$. As these latter maps are compatible with faces, the same argument as above shows that the maps $H_{N-m,m}(c)$ are compatible with faces as well. The proof for the homotopy $H_N(c)$ is similar. □

6.15. **Proof of Theorem 1.9.** We proceed to prove Theorem 6.12 for X affine, and for a collection of subsets $C_j \in U_{(\emptyset,q_j)}$, $j = 1,\ldots,s$, which will complete the proof of Theorem 1.9.

Let $\bar{C}_j$ denote the closure of C_j in $X \times \Delta^N$. We suppose that C_j is in $U^X_{(\emptyset,q_j)}$ for $j = 1,\ldots,r$; for these j, $\bar{C}_j$ is in $X_{(\emptyset,q_j)}$.

Recall the tower (5.4), built out of the maps $p_n \times \mathrm{id} : \Delta^{n-1} \times \square^{m+1} \to \Delta^n \times \square^m$, giving the compositions $\pi_N : \square^N \to \Delta^N$, and $\pi_{N-m,m} : \Delta^{N-m} \times \square^m \to \Delta^N$. We have the degeneracy morphisms $\sigma : \Delta^{n+1} \to \Delta^n$, $\sigma(t_0,\ldots,t_{n+1}) = (t_0,\ldots,t_{n-1},t_n + t_{n+1})$.

Lemma 6.16. *Let $f : Z \to \Delta^n \times \square^m$ be a morphism intersecting all faces properly. Then $(p_n \times \mathrm{id})^{-1}(f) : (p_n \times \mathrm{id})^{-1}(Z) \to \Delta^{n-1} \times \square^{m+1}$ intersects all faces of $\Delta^{n-1} \times \square^{m+1}$ properly, and $(\sigma \times \mathrm{id})^{-1}(f) : (\sigma \times \mathrm{id})^{-1}(Z) \to \Delta^{n+1} \times \square^m$ intersects all faces of $\Delta^{n+1} \times \square^m$ properly.*

Proof. The map $p_n \times \mathrm{id}$ is easily seen to be locally birational monomial, so the first assertion follows from Lemma 6.3. For the second assertion, let F be a face of $\Delta^{n+1} \times \square^m$. Then $(\sigma \times \mathrm{id})(F)$ is a face F' of $\Delta^n \times \square^m$, and the restriction of $\sigma \times \mathrm{id}$ to the open face F^0 factors as a flat map $F^0 \to F'$ followed by the inclusion of F' into $\Delta^n \times \square^m$. Since $f : Z \to \Delta^n \times \square^m$ intersects all faces properly, this implies that $(\sigma \times \mathrm{id})^{-1}(f) : (\sigma \times \mathrm{id})^{-1}(Z) \to \Delta^{n+1} \times \square^m$ intersects all faces of properly as well. □

Let $Z_j = \pi_N^{-1}(C_j)$, let $\bar{Z}_j = \pi_N^{-1}[\bar{Z}_j]$ and let

$$Z := \coprod_{j=1}^{s} Z_j; \quad \bar{Z} := \coprod_{j=1}^{s} \bar{Z}_j; \quad \bar{Z}_{\leq r} := \coprod_{j=1}^{r} \bar{Z}_j.$$

We let $f : Z \to \square^N$, $\bar{f} : \bar{Z} \to \square^N$ and $\bar{f}_{\leq r} : \bar{Z}_{\leq r} \to \square^N$ be the projections. It follows from Lemma 6.16 that f intersects all faces of $S := \square^N$ properly. From Lemma 6.3, we have $Z_j = \pi_N^{-1}[C_j]$ for $j = 1,\ldots,s$, and $\bar{Z}_j = \pi_N^{-1}(\bar{C}_j)$ for $j = 1,\ldots,r$.

By [9, Theorem 0.3], there is, for each vertex v of S, an iterated blow-up of faces of $(S, \partial_v S)$, $p_v : S(v) \to S$, such that $p_v^{-1}[\bar{f}] : p_v^{-1}[\bar{Z}] \to S(v)$ intersects all faces of $S(v)$ properly. Let $p : S_M \to S$ be an iterated blow-up of faces which dominates all the p_v. By Proposition 6.10(1), there is a Zariski open subscheme V_0 of $(\mathbb{A}^1 \setminus \{0,1\})^N$, faithfully flat over B, such that, for each allowable B-morphism $c : B' \to V_0$, and for each vertex u of S_M, the morphisms $p^!_{u,c}(f) : p^!_{u,c} Z_{B'} \to \square^N_{B'}$ and $p^!_{u,c}(\bar{f}) : p^!_{u,c} \bar{Z}_{B'} \to \square^N_{B'}$ intersects all faces of $T\square^N_{B'}$ properly. It follows from Proposition 6.10(2) and Lemma 6.3 that

$$p^!_{u,c} Z_{B'} = (\Lambda^{u,c})^{-1}(Z_{B'}), \tag{6.1}$$

$$p^!_{u,c} \bar{Z}_{\leq r B'} = (\Lambda^{u,c})^{-1}(\bar{Z}_{\leq r B'}). \tag{6.2}$$

Let $h_s : \square^N_{B'} \to \square^N$ be a component of Φ^c_p, and let $\overline{h_s^{-1}(Z_j)}$ denote the closure in $X \times \square^N_{B'}$ of $h_s^{-1}(Z_j)$. By the definition of the map Φ^c_p, $h_s = \Lambda^{u,c}$ for some vertex u of S_M. For this choice of u, it follows from (6.1) that

$$\overline{h_s^{-1}(Z_j)} \subset p^!_{u,c} \bar{Z}_{B'}.$$

Thus, the projection $\overline{h_s^{-1}(Z_j)} \to \square^N$ intersects all faces of $T\square^N$ properly.

From this it follows in turn that, if $f_s : \Delta^N_{B'} \to \Delta^N$ is a component of Ψ^c_p, then the irreducible components of $f_s^{-1}(C_j)$ are in $(U_{B'})^X_{N,q_j}$ for all j. Indeed, let $\overline{f_s^{-1}(C_j)}$ be the the closure of $f_s^{-1}(C_j)$ in $X \times \Delta^N_{B'}$. Each component f_s is of the form $\pi_N \circ h_s \circ t$ for some component $h_s : \square^N_{B'} \to \square^N$ of Φ^c_p, where $t : \Delta^N \to \square^N$ is a component of the triangulation T. Since $Z_j = \pi_N^{-1}(C_j)$, and t is an isomorphism of B-schemes, it follows that

$$\overline{f_s^{-1}(C_j)} = (\mathrm{id} \times t)^{-1}(\overline{h_s^{-1}(Z_j)}).$$

As $\overline{h_s^{-1}(Z_j)}$ intersects all the faces of $T\square^N$ properly, it follows that the closure $\overline{f_s^{-1}(C_j)}$ likewise intersects all faces of Δ^N properly, i.e., that the irreducible components of $f_s^{-1}(C_j)$ are in $(U_{B'})^X_{N,q_j}$.

This verifies the portion of Theorem 1.9 dealing with the map Ψ. We now turn to the homotopy H.

We write $H = H^{c'}_p$ as the sum (see the proof of Theorem 5.8)

$$H^{c'}_p = \pi_{N*} \circ H_0(c') \circ T + \sum_{m=0}^{N-2} \pi_{N-m,m*} \circ H_{N-m,m}(c) \circ (\mathrm{id} \times T) \circ \delta + H_N(c).$$

We first consider the term $\pi_N \circ H_0(c') \circ T$. The homotopy $H_0(c')$ is defined (see the proof of Proposition 4.12) by blowing-up $\square^{N+1} = \square^N \times \square^1$ along the locus $\square^N \times 1$ so as to form the sequence of blow-ups forming the map $p : S_M \to S$. This gives the iterated blow-up of faces $q : T_M \to T$ which is the identity over $\square^N \times 0$ and p over $\square^N \times 1$. A choice of a B-morphism $c' : B' \to (\mathbb{A}^1 \setminus \{0,1\})^{N+1}$ of the form (c, c_{N+1}) gives the map $\Phi^{c'}_q : \square^*_{B'} \to (T, \partial T)$. Feeding this map to the homotopy machine of §2.10 gives the homotopy Φ_h. We have the projection $\pi : T = \square^{N+1} \to S = \square^N$ on the first N-factors, and $H_0(c') = \pi_* \circ \Phi_h$.

Clearly $\pi^{-1}(Z) \to \square^{N+1}$ and $\pi^{-1}(\bar{Z}_{\leq r}) \to \square^{N+1}$ intersects all faces of $\square^{N+1}$ properly. Arguing as above, we have for all vertices u' of T_M

$$q^!_{u',c'}\pi^{-1}(Z)_{B'} = (\Lambda^{u',c'})^{-1}(\pi^{-1}(Z)_{B'}),$$
$$q^!_{u',c'}\pi^{-1}(\bar{Z}_{\leq r})_{B'} = (\Lambda^{u',c'})^{-1}(\pi^{-1}(\bar{Z}_{\leq r})_{B'}).$$

From this, it follows as above that, for each component $h_s : \square^{N+1}_{B'} \to \square^N$ of $\pi_* \circ \Phi^{c'}_q$, both $h_s^{-1}(Z) \to \square^{N+1}$ and $h_s^{-1}(\bar{Z}_{\leq r}) \to \square^{N+1}$ intersect all faces of $T\square^{N+1}$ properly.

From the explicit formula for $(\Phi^{c'}_q)_h$ given in (2.11), the components of $H_0(c')$ of the form $h' : \square^{N+1}_{B'} \to \square^N$ agree with those of $\pi_* \circ \Phi^{c'}_q$. The same argument as above shows that, for each component $h_s : \Delta^{N+1}_{B'} \to \Delta^N$ of $\pi_N \circ H_0(c') \circ T$, the irreducible components of $h_s^{-1}(C_j)$ are in $(U_{B'})_{(N,q_j)}$ for all $j = 1, \ldots, s$. Replacing U with X and C_j with $\bar{C}_j$, $j = 1, \ldots, r$, the same argument shows that $h_s^{-1}(\bar{C}_j)$ is in $(X_{B'})_{(N,q_j)}$ for all $j = 1, \ldots, r$, from which it follows that $h_s^{-1}(C_j)$ is in $(U_{B'})^X_{(N,q_j)}$ for all $j = 1, \ldots, r$.

We turn next to the terms $\pi_{N-m,m*} \circ H_{N-m,m}(c) \circ (\mathrm{id} \times T)$. We have the maps

$$\Psi_n \times \Phi_m(c) : \Delta^* \otimes \square^*_{B'} \to (\Delta^n \times \square^m, \partial).$$

In the proof of Lemma 5.2, we have noted that $\Psi_n \times \Phi_m(c)$ is defined by taking the tensor product of $\Psi_n(c_1)$ and $\Phi^{c^m}_{\mathrm{id}}$, and then identifying $(\Delta^n, \partial) \otimes (\square^m, \partial)$ with

$(\Delta^n\times\square^m, \partial(\Delta^n\times\square^m))$. The map

$$H_{n,m}(c):\Delta^*\otimes\square^*_{B'}\to(\Delta^n\times\square^m,\partial(\Delta^n\times\square^m))$$

gives a homotopy of $\Psi_n\times\Phi_m(c)$ with $(p_n\times\mathrm{id})_*\circ(\Psi_{n-1}\times\Phi_{m+1}(c))$, and is constructed by taking the homotopy $(\Psi_n\times\Phi_{m+1}(c))_h$ corresponding to $\Psi_n\times\Phi_{m+1}(c)$, and composing with $[(\sigma\times\mathrm{id})\circ(p_{n+1}\times\mathrm{id})]_*$ (up to sign). This gives the formula

$$\begin{aligned}&H_{N-m,m}(c)\circ(\mathrm{id}\times T)\circ\delta\\ &\qquad=\pm[(\sigma\times\mathrm{id})\circ(p_{N-m+1}\times\mathrm{id})]_*\circ(\Psi_{N-m}\times\Phi_{m+1}(c))_h\circ(\mathrm{id}\times T)\circ\delta.\end{aligned}$$

Let

$$\begin{aligned}Z^m_j&=(\pi_{N-m,m}\circ(\sigma\times\mathrm{id})\circ(p_{N-m+1}\times\mathrm{id}))^{-1}(C_j),\ j=1,\dots,s,\\ \bar Z^m_j&=(\pi_{N-m,m}\circ(\sigma\times\mathrm{id})\circ(p_{N-m+1}\times\mathrm{id}))^{-1}(\bar Z_j),\ j=1,\dots,r,\end{aligned}$$

and let

$$Z^m:=\coprod_{j=1}^s Z^m_j;\quad \bar Z^m_{\le r}:=\coprod_{j=1}^r\bar Z^m_j.$$

We let $f^m:Z^m\to\Delta^{N-m}\times\square^{m+1}$, $\bar f^m_{\le r}:\bar Z^m_{\le r}\to\Delta^{N-m}\times\square^{m+1}$ be the projections. It follows from Lemma 6.16 that f^m and $\bar f^m_{\le r}$ intersect all faces of $\Delta^{N-m}\times\square^{m+1}$ properly.

By Lemma 6.9, there is an open subscheme $V_{N-m,m}$ of $(\mathbb{A}^1\setminus\{0,1\})^N$, mapping onto B, such that for all allowable B-morphisms $c:B'\to V_{N-m,m}$, for each component $h:\Delta^{N+1}_{B'}\to\Delta^{N-m}\times\square^{m+1}$ of $\Psi_{N-m}\times\Phi_{m+1}(c)\circ(\mathrm{id}\times T)\circ\delta$, and for each face F of $\Delta^{N+1}_{B'}$, the maps f^m and $\bar f^m_{\le r}$ intersect $h(F)$ properly. Since each such component h induces an isomorphism of B'-schemes $\Delta^{N+1}_{B'}\to\Delta^{N-m}\times\square^{m+1}_{B'}$, this is the same as saying that $h^{-1}(Z^m)\to\Delta^{N+1}$ and $h^{-1}(\bar Z^m_{\le r})\to\Delta^{N+1}$ intersect all faces properly. Since $(\Psi_{N-m}\times\Phi_{m+1}(c))_h\circ(\mathrm{id}\times T)\circ\delta$ and $(\Psi_{N-m}\times\Phi_{m+1}(c))\circ(\mathrm{id}\times T)\circ\delta$ have the same components of this form, this implies that, for each component h_s of $H_{N-m,m}(c)\circ(\mathrm{id}\times T)\circ\delta$, $h_s^{-1}(C_j)$ is in $(U_{B'})_{(N+1,q_j)}$, $j=1,\dots,s$, and $h_s^{-1}(\bar C_j)$ is in $(X_{B'})_{(N+1,q_j)}$, $j=1,\dots,r$. Thus $h_s^{-1}(C_j)$ is in $(U_{B'})^X_{(N+1,q_j)}$, $j=1,\dots,r$, completing the discussion for the term $H_{N-m,m}(c)\circ(\mathrm{id}\times T)\circ\delta$.

The argument for $H_N(c)$ is similar. The homotopy $H_N(c)$ is gotten (up to sign) from the map $\Psi_{N+1}\times\Phi_0(c)$ by taking the associated homotopy $(\Psi_{N+1}\times\Phi_0(c))_h$, composing with σ_*, and then conjugating by the linear automorphism of Δ^{N+1} coming from a permutation of the vertices. For our purpose, we may ignore the conjugation by this automorphism, in which case the argument is the same as that for $H_{N-m,m}$. This gives us an open subscheme V_N of $(\mathbb{A}^1\setminus\{0,1\})^N$, faithfully flat over B, such that, for all allowable B-morphisms $c:B'\to V_N$, $h_s^{-1}(C_j)$ is in $(U_{B'})_{(N+1,q_j)}$, $j=1,\dots,s$, and $h_s^{-1}(C_j)$ is in $(U_{B'})^X_{(N+1,q_j)}$, $j=1,\dots,r$, for each component $h_s:\Delta^{N+1}_{B'}\to\Delta^N$ of $H_N(c)$.

Taking

$$V(S_M)=\big(V_0\cap\bigcap_{m=0}^{N-2}V_{N-m,m}\cap V_N\big)\times(\mathbb{A}^1\setminus\{0,1\})$$

completes the proof of Theorem 6.12 and Theorem 1.9.

7. Localization for motivic Borel-Moore homology

We can now prove our extension Theorem 1.7 of Bloch's localization theorem; we use the notations of §1. Let B be a regular noetherian scheme of Krull dimension at most one, $p : X \to B$ a finite type B scheme, and $j : U \to X$ an open subscheme. To prove Theorem 1.7, it suffices to show that, if B is semi-local, then the quotient complex $z_q(U, *)/j^*z_q(X, *)$ is acyclic.

Write $B = \operatorname{Spec} A$, with A a semi-local principal ideal domain.

Lemma 7.1. *Let V be an open subscheme of $\mathbb{A}^n_A$, mapping onto* $\operatorname{Spec} A$. *Then there exist finite étale extensions $A \to A_1$, $A \to A_2$, of relatively prime degree, with sections $\sigma_i : \operatorname{Spec} A_i \to V_{A_i}$.*

Proof. Let $A \to A'$ be an extension, with A' semi-local. Suppose that, for each closed point x of $\operatorname{Spec} A'$, there is a $k(x)$-valued point v_x of V. By the Chinese remainder theorem, there is a section $s : A' \to \mathbb{A}^n_{A'}$ with $s(x) = v_x$; since V is open, it follows that s has image in V. Thus, it suffices to find A_i which are étale and finite over A, of relatively prime degree, such that $V(k(x)) \neq \emptyset$ for each closed point x of $\operatorname{Spec} A_1$ and $\operatorname{Spec} A_2$.

Let y be a closed point of $\operatorname{Spec} A$. If $n = 1$, there is an integer $d_y \geq 1$ such that, for all separable field extensions $k(y) \to L$ of degree $\geq d_y$, $V(L) \neq \emptyset$; if $k(y)$ is infinite, we may and will take $d_y = 1$. The same holds for arbitrary n by an elementary induction. Let d be the maximum of the d_y, as y runs over the closed points of A, and take distinct primes $l_1, l_2 \geq d$. For each y, there is a monic separable polynomial $f_{y,i} \in k(y)[X]$ of degree l_i, such that each irreducible factor of $f_{y,i}$ has degree $\geq d_y$. Choose monic polynomials $f_i \in A[X]$ such that f_i reduces to $f_{i,y}$ at y. Letting $A_i = A[X]/(f_i)$ gives the desired extensions. □

If $i : A \to A'$ is a finite extension of semi-local principal ideal rings of degree d, then the composition

$$z_q(U, *)/j^*z_q(X, *) \xrightarrow{i^*} z_q(U_{A'}, *)/j^*z_q(X_{A'}, *) \xrightarrow{i_*} z_q(U_{A'}, *)/j^*z_q(X_{A'}, *)$$

is multiplication by degree d. Applying the lemma, we may assume throughout the subsequent constructions that each open $V \subset \mathbb{A}^1_A$ which maps onto $\operatorname{Spec} A$, admits a section.

Let $C \subset \sqcup_{j=0}^M U \times \Delta^j$ be a finite union of closed subsets $C_j \in U_{(p_j,q)}$, $0 \leq p_j \leq M$. We let $\mathbf{Ord}^{\leq N}$ be the full subcategory of the category $\mathbf{Ord}$ with objects $[p]$, $0 \leq p \leq N$. Since $\mathbf{Ord}^{\leq N}$ is a finite category, the union of the closed subsets $(\mathrm{id} \times g)^{-1}(C \cap U \times \Delta^p)$, as $g : \Delta^q \to \Delta^p$ runs over the morphisms in $\mathbf{Ord}^{\leq N}$, is again a finite union of the same form. We let $C^{\leq N}$ denote this completion of C with respect to the category $\mathbf{Ord}^{\leq N}$, and $C^{\leq N}_p \subset U \times \Delta^p$ the portion of $C^{\leq N}$ in $U \times \Delta^p$.

We let $z_q(U, p)_N(C)$ be the subgroup of $z_q(U, p)$ generated by the irreducible components of $C^{\leq N}_p$. For $p \leq N$, $z_q(U, p)$ is the direct limit of the $z_q(U, p)_N(C)$ as C runs over unions of closed subsets $C_j \in U_{(p_j,q)}$, as above. Similarly, $j^*z_q(X, p)$ is the direct limit of the $z_q(U, p)_N(C')$, as C' runs over unions of closed subsets $C'_j \in U^X_{(I_j,q)}$.

The association $p \mapsto z_q(U, p)_N(C)$ extends to an N-truncated simplicial subgroup $z_q(U, -)_N(C) : \mathbf{Ord}^{\leq N\ \mathrm{op}} \to \mathbf{Ab}$ of the N-truncated simplicial abelian group

$$z_q(U, -) := [p \mapsto z^q(U, p)] : \mathbf{Ord}^{\leq\ \mathrm{op}} \to \mathbf{Ab}.$$

We may apply $z^q(U,-)_N(C)$ and $z_q(U,-)$ to the complex $(\Delta^N, \partial\Delta^N)$, giving the subcomplex $z^q(U;(\Delta^N,\partial\Delta^N))(C)$ of the complex $z^q(U;(\Delta^N,\partial\Delta^N))$. We have as well the subcomplex $z_q(U,*)_N(C)$ of $z_q(U,*)$, associated to the N-truncated simplicial abelian group $z^q(U,-)_N(C)$.

Take $C_j \in U_{(N_j,q)}$, $j = 1, \ldots, s$, $C_i \in U^X_{(N_i,q)}$, $i = s+1, \ldots, r$, and let

$$C = \bigcup_{j=1}^{r} C_j,\ C' = \bigcup_{j=s+1}^{r} C_j.$$

The map Ψ_N gives the map of complexes

$$z_q(U,\Psi_N) : z^q(U;(\Delta^N,\partial\Delta^N))[-N] \to z_q(U,*),$$

which, by Lemma 2.6, is a homology isomorphism in degrees $< N$. Thus, the map

$$z_q(U,\Psi_N)(C) : z^q(U;(\Delta^N,\partial\Delta^N))(C)[-N] \to z_q(U,*)$$

is a homology isomorphism in degree $< N$ after taking the limit over C. Similarly, the map

$$z_q(U,\Psi_N)(C') : z^q(U;(\Delta^N,\partial\Delta^N))(C')[-N] \to j^*z_q(X,*)$$

is a homology isomorphism in degree $< N$ after taking the limit over C'.

From Theorem 6.12, we have the diagram

$$\begin{array}{ccc} z_q(U;(\Delta^N,\partial\Delta^N))(C')[-N] & \longrightarrow & z_q(U;(\Delta^N,\partial\Delta^N))(C) \\ \big\downarrow{\scriptstyle z_q(U,\Psi_N)(C')[-N]} & \overset{\Psi}{\swarrow} & \big\downarrow{\scriptstyle z_q(U,\Psi_N)(C)} \\ j^*z_q(X,*) & \xrightarrow[\iota]{} & z_q(U,*), \end{array} \tag{7.1}$$

where $\Psi = \Psi^c_p$ for suitable A-valued point c of $\square^N \setminus \partial\square^N$, and $p : S_M \to \square^N$ is an iterated blow-up of faces. The properties of the homotopy $H^{c'}_p$ imply that (7.1) is commutative up to homotopy. From the remarks in the previous paragraph, we see that the inclusion ι is a homology isomorphism in degrees $< N$. Since N was arbitrary, ι is a homology isomorphism, completing the proof of Theorem 1.7.

8. Atiyah-Hirzebruch spectral sequence

We now proceed to give an application to the construction of a spectral sequence from motivic cohomology to K-theory, globalizing the spectral sequence constructed by Bloch and Lichtenbaum in [3]. We let B be a regular scheme of Krull dimension at most one. In this section, unless specific mention to the contrary is made, "space" will mean "simplicial set"; we let $\mathcal{S}$ denote the category of simplicial sets.

8.1. **A variant of Theorem 6.12.** It will be useful to have a slightly different version of Theorem 6.12 where we replace various identities of dimensions with inequalities. Fortunately, this variant follows easily from Theorem 6.12.

Let B be a regular scheme of dimension at most one, and let $X \to B$ be a B-scheme of finite type. Let $X_{(p,\le q)}$ be the set of irreducible closed subsets W of $X \times \Delta^p$ such that, for each face F of Δ^p (including $F = \Delta^p$), and each irreducible component W' of $W \cap (X \times F)$, we have

$$\dim(W') \le q + \dim_B(F).$$

If U is an open subscheme of X, we let $U^X_{(p,\le q)}$ be the subset of $U_{(p,\le q)}$ consisting of those W whose closure in $X \times \Delta^p$ is in $X_{(p,\le q)}$.

Theorem 8.2. *Let $B = \operatorname{Spec} A$, where A is a semi-local PID, and let U be an open subscheme of a B-scheme X of finite type over B. Let $\{C_{I,j}\}$ be a finite collection of irreducible closed subsets, $C_{I,j} \in U_{(I,\leq q_j)}$, $I \subsetneq \{0,1,\dots,N\}$. Then there is an iterated blow-up of faces $S' \to S := \square^N$ such that, for each iterated blow-up of faces $p : S_M \to S$ which dominates S', there is an open subset $V(S_M) \subset (\mathbb{A}^1 - \{0,1\})^{N+1}$ such that*

1. *The structure map $V(S_M) \to B$ is surjective.*
2. *For each allowable B-morphism $c' := (c, c_{N+1}) : B' \to V(S_M)$, the map*
$$\Psi^c_p : \Delta^*_{B'} \to (\Delta^N; \partial\Delta^N)$$
and the homotopy $H^{c'}_p$ of Ψ^c_p with Ψ_N given by Theorem 5.8 satisfy the following: Write Ψ^c_p and $H^{c'}_p$ as sums with $\mathbb{Z}$-coefficients
$$\Psi^c_p = \sum_{\substack{s,\\ I\subsetneq\{0,\dots,N\}}} n^s_I f^s_I; \quad H^{c'}_p = \sum_{\substack{s,\\ I\subsetneq\{0,\dots,N\}}} m^s_I g^s_I; \quad n^s_I, m^s_I \neq 0,$$
with
$$f^s_I : \Delta^{N-|I|}_{B'} \to \partial\Delta^N_I; \quad g^s_I : \Delta^{N-|I|+1}_{B'} \to \partial\Delta^N_I,$$
maps of B-schemes. Then
   (a) *Each component of $(\mathrm{id} \times f^s_I)^{-1}(C_{I,j})$ is in $(U_{B'})^X_{(N-|I|,\leq q_j)}$ for each I, s and j.*
   (b) *Each component of $(\mathrm{id} \times g^s_I)^{-1}(C_{I,j})$ is in $(U_{B'})_{(N-|I|+1,\leq q_j)}$ for each I, s and j.*
   (c) *If $C_{I,j}$ is in $U^X_{(I,\leq q_j)}$, then each component of $(\mathrm{id} \times g^s_I)^{-1}(C_{I,j})$ is in $(U_{B'})^X_{(N-|I|+1,\leq q_j)}$ for each s.*

Proof. The arguments used in §6.11 reduce us to the case of X affine. The argument of Lemma 6.13 proves the following variant:

Lemma 8.3. *Let Y be quasi-projective over B, $j : U \to Y$ a non-empty open subscheme, and C an element of $U_{(I,\leq q)}$ for some $I \subset \{0,\dots,N\}$, and some $q \geq -N$. Then there is an irreducible closed subset $\tilde{C}$ of Δ^N_U such that $\tilde{C}$ is in $U_{(\emptyset,q)}$, and C is contained in $\tilde{C} \cap U \times \partial\Delta^N_I$. If C is in $U^Y_{(I,\leq q)}$, we may find a $\tilde{C}$ as above with $\tilde{C} \in U^Y_{(\emptyset,q)}$.*

Also, if we have a $C \in U_{(I,q)}$ (resp. $C \in U^X_{(I,q)}$), and an irreducible subset C' of C, then clearly C' is in $U_{(I,\leq q)}$ (resp. in $U^X_{(I,\leq q)}$). Thus, the argument of Proposition 6.14 reduces the proof of Theorem 8.2 to the case of a finite collection of irreducible closed subsets $\{C_j \in U_{(\emptyset,q_j)}\}$ (and also X affine, but we won't need this). The conclusion of Theorem 8.2 in this case follows from Theorem 6.12. ☐

8.4. **The G-theory spectral sequence.** Let X be a finite-type B-scheme. We have the exact category $\mathcal{M}_X$ of coherent sheaves on X, and the corresponding K-theory spectrum $G(X) := K(\mathcal{M}_X)$. The deloopings $\Omega^{-d}G(X)$ are given by Waldhausen's multiple Q-construction; in particular, the spectrum $G(X)$ has the natural structure of a spectrum of simplicial sets.

Remark 8.5. When we take the homotopy fiber or cofiber of a map of spectra, this will always be the homotopy fiber or cofiber in the category of spectra.

Remark 8.6. In what follows, we will be constructing various simplicial spaces, or simplicial spectra, by applying the Quillen/Waldhausen construction to simplicial exact categories. These in turn will arise from various *pseudo-functors* $F : \mathcal{C}^{\rm op} \to \mathbf{ECat}$, where $\mathcal{C}$ will be a small subcategory of the category of schemes, and $\mathbf{ECat}$ is the category of exact categories. Since F is not a functor, one runs into problems in applying the Quillen/Waldhausen construction. To avoid this, there is a standard method for changing the pseudo-functor F into a functor, namely, for each $X \in \mathcal{C}$, replace $F(X)$ with the category of tuples $(\mathcal{F}, \theta_g : g^*\mathcal{F} \to \mathcal{F}_g)$, with $\mathcal{F}$ an object of $F(X)$, $\mathcal{F}_g$ an object of $F(Y)$, for each morphism $g : Y \to X$ in $\mathcal{C}$, and θ_g an isomorphism (here we write $g^*\mathcal{F}$ for $F(g)(\mathcal{F})$). The morphisms are collections of maps for each component making the evident diagrams commute. For each $f : Y \to X$ in $\mathcal{C}$, define

$$f^*(\mathcal{F}, \theta_g : g^*\mathcal{F} \to \mathcal{F}_g) = (\mathcal{F}_f, \theta_{f\circ h} \circ \xi_{f,h} \circ h^*(\theta_f^{-1}) : h^*\mathcal{F}_f \to \mathcal{F}_{f\circ h}),$$

where $h : Z \to Y$ is a morphism in $\mathcal{C}$ and $\xi_{f,h} : h^*(f^*\mathcal{F}) \to (f\circ h)^*\mathcal{F}$ is the canonical isomorphism which is part of the data of the pseudo-functor F. By making this substitution, we may assume that our pull-back maps are functorial over any chosen small subcategory of the category of schemes; we will assume that we have done this, without further mention.

Let U be an open subscheme of $X \times \Delta^p$. We have the full subcategory $\mathcal{M}_U(\partial\Delta^p)$ of $\mathcal{M}_U$ with objects the coherent sheaves $\mathcal{F}$ such that $\mathrm{Tor}_q^{\mathcal{O}_U}(\mathcal{F}, \mathcal{O}_{U\cap(X\times F)}) = 0$ for all faces F of Δ^p and all $q > 0$. We write $\mathcal{M}_X(p)$ for $\mathcal{M}_{X\times\Delta^p}(\partial\Delta^p)$ and let $G(X,p)$ denote the K-theory spectrum $K(\mathcal{M}_X(p))$.

For a closed subset W of $X \times \Delta^p$, we have the spectrum with supports $G_W(X,p)$, defined as the homotopy fiber of the map of spectra

$$j^* : K(\mathcal{M}_X(p)) \to K(\mathcal{M}_U(\partial\Delta^p)),$$

where U is the complement $X \times \Delta^p \setminus W$ and $j : U \to X \times \Delta^p$ is the inclusion. We let $G_{(q)}(X,p)$ denote the direct limit of the $G_W(X,p)$, as W runs over finite unions of irreducible closed subsets $C \in X_{(p,\le q)}$.

Let $i : \Delta^q \to \Delta^p$ be a closed embedding in Δ^*, so i identifies Δ^q with a face of Δ^p, let $W \subset X \times \Delta^p$ be a finite union of $C_j \in X_{(p,\le q)}$, let U be the complement $X \times \Delta^p \setminus W$, and $V = i^{-1}(U)$. The vanishing Tor condition implies that the pull-back $i^* : \mathcal{M}_U(\partial\Delta^p) \to \mathcal{M}_V$ is exact, and has image in $\mathcal{M}_V(\partial\Delta^q)$. In particular, the assignment $p \mapsto \mathcal{M}_X(p)$ extends to a simplicial exact category $\mathcal{M}_X(-)$; we let $G(X,-) := K(\mathcal{M}_X(-))$ the corresponding simplicial spectrum. Similarly, the assignments $p \mapsto G_{(q)}(X,p)$ extend to simplicial spectra $G_{(q)}(X,-)$.

Lemma 8.7. *For $U \subset X \times \Delta^p$ open, the inclusion $\mathcal{M}_U(\partial\Delta^p) \to \mathcal{M}_U$ induces a weak equivalence $K(\mathcal{M}_U(\partial\Delta^p)) \to K(\mathcal{M}_U)$.*

Proof. We apply Quillen's resolution theorem [11, §4, Corollary 1] to the inclusion $\mathcal{M}_U(\partial\Delta^p) \to \mathcal{M}_U$; it thus suffices to show that each $\mathcal{F} \in \mathcal{M}_U$ admits a finite resolution by objects in $\mathcal{M}_U(\partial\Delta^p)$. Since each face is a complete intersection in U, it suffices to show that each $\mathcal{F} \in \mathcal{M}_U$ admits a surjection $\mathcal{G} \to \mathcal{F}$ with $\mathcal{G} \in \mathcal{M}_U(\partial\Delta^p)$. Since each $\mathcal{F} \in \mathcal{M}_U$ extends to a $\bar{\mathcal{F}} \in \mathcal{M}_{X\times\Delta^p}$, and since $j^* : \mathcal{M}_{X\times\Delta^p} \to \mathcal{M}_U$ maps $\mathcal{M}_X(p)$ to $\mathcal{M}_U(\partial\Delta^p)$, we may assume that $U = X \times \Delta^p$.

Let F be a face of Δ^p. Since F is flat over B, $X \times F$ is flat over X. Thus, if $\mathcal{F}_0$ is in $\mathcal{M}_X$, then $p_1^*\mathcal{F}_0$ is in $\mathcal{M}_X(p)$. Thus, it suffices to show that each $\mathcal{F} \in \mathcal{M}_{X\times\Delta^p}$ admits a surjection $p_1^*\mathcal{F}_0 \to \mathcal{F}$ for some $\mathcal{F}_0 \in \mathcal{M}_X$.

We identify Δ^p with $\mathbb{A}^p$. Let $i : \mathbb{A}^p \to \mathbb{P}^p$ be the standard inclusion, let $H = \mathbb{P}^p \setminus \mathbb{A}^p$, and let $\bar{p}_1 : X \times \mathbb{P}^p \to X$ the extension of $p_1 : X \times \mathbb{A}^p \to X$. Let $\mathcal{G}$ be a coherent sheaf on $X \times \mathbb{P}^p$ with $(\mathrm{id} \times i)^*\mathcal{G} \cong \mathcal{F}$. For all d, the sheaf $\bar{p}_{1*}\mathcal{G}(d)$ on X is a coherent $\mathcal{O}_X$-module, and for all d sufficiently large the natural map $\bar{p}_1^*\bar{p}_{1*}\mathcal{G}(d) \to \mathcal{G}(d)$ is surjective. The section $h : \mathcal{O}_{\mathbb{P}^p} \to \mathcal{O}_{\mathbb{P}^p}(d)$ with divisor $d \cdot H$ defines an isomorphism $(\mathrm{id} \times i)^*\mathcal{G} \to (\mathrm{id} \times i)^*\mathcal{G}(d)$, giving the surjection

$$p_1^*\bar{p}_{1*}\mathcal{G}(d) \to (\mathrm{id} \times i)^*\mathcal{G} \cong \mathcal{F},$$

completing the proof of the lemma. □

Proposition 8.8. *Let X be a B-scheme of finite type.*

1. *Let W be a closed subset of $X \times \Delta^p$. There is a natural weak equivalence $G(W) \sim G_W(X, p)$.*
2. *The projection $p_1 : X \times \Delta^p \to X$ induces a weak equivalence $p_1^* : G(X) \to G(X, p)$.*

Proof. Let $G_W(X \times \Delta^p)$ denote the homotopy fiber of the restriction map $j^* : G(X \times \Delta^p) \to G(X \times \Delta^p \setminus W)$. Quillen's localization theorem [11, §7, Proposition 3.1] gives us the natural weak equivalence $G(W) \to G_W(X \times \Delta^p)$. The natural map $G_W(X, p) \to G_W(X \times \Delta^p)$ is a weak equivalence by Lemma 8.7, proving (1).

The homotopy property for G-theory implies that the map $p_1^* : \mathcal{M}_X \to \mathcal{M}_{X \times \Delta^p}$ induces a weak equivalence $G(X) \to G(X \times \Delta^p)$. The natural map $G(X, p) \to G(X \times \Delta^p)$ is a weak equivalence by Lemma 8.7, proving (2). □

We let $\dim X$ denote the maximum of $\dim X_i$ over the irreducible components X_i of X. We note that $G_{(q)}(X, p) = G(X, p)$ for all $q \geq \dim X$. The evident maps

$$G_{(q-1)}(X, p) \to G_{(q)}(X, p)$$

give the tower of simplicial spectra

$$\ldots \to G_{(q-1)}(X, -) \to G_{(q)}(X, -) \to \ldots \to G_{(\dim X)}(X, -) = G(X, -). \tag{8.1}$$

By Proposition 8.8(2), the augmentation $X \to X \times \Delta^*$ gives a weak equivalence of $G(X, -)$ with $G(X)$. We let $G_{(q/q-1)}(X, -)$ denote the homotopy cofiber of the map $G_{(q-1)}(X, -) \to G_{(q)}(X, -)$ The tower (8.1) thus gives rise to a spectral sequence (of homological type)

$$E^1_{p,q} = \pi_{p+q}(G_{(p/p-1)}(X, -)) \Longrightarrow G_{p+q}(X). \tag{8.2}$$

Since the tower (8.2) is natural with respect to flat morphisms, replacing X with $U \times_B X$ for $U \subset B$ open forms a tower of presheaves of simplicial spectra on B_{Zar}. Jardine [7] has constructed a closed model structure on the category of presheaves of spectra; in particular, for a presheaf of simplicial spectra $\mathcal{G}$ on B_{Zar}, we have the homotopy groups $\pi_N(B; \mathcal{G})$ (also denoted $\mathbb{H}^{-N}(B; \mathcal{G})$). For a presheaf of Eilenberg-Maclane spectra, $\pi_N(B; \mathcal{G})$ agrees with the Zariski hypercohomology $\mathbb{H}^{-N}(B; \mathcal{G}^*)$, where $\mathcal{G}^*$ is the complex of presheaves of abelian groups corresponding to $\mathcal{G}$ via the Dold-Kan correspondence. For a B-scheme $f : X \to B$, presheafifying the tower (8.2) over B and taking the associated spectral sequence gives us the spectral sequence

$$E^1_{p,q} = \pi_{p+q}(B; f_*G_{(p/p-1)}(X, -)) \Longrightarrow G_{p+q}(X). \tag{8.3}$$

This spectral sequence converges to $G_{p+q}(X)$ since Quillen's localization theorem [11, §7, Proposition 3.1] implies that the natural map

$$\pi_N(K(\mathcal{M}_X)) \to \pi_N(B; f_*K(\mathcal{M}_X))$$

is an isomorphism.

Let $\pi_0 G_{(q/q-1)}(X, -)$ denote the simplicial abelian group

$$p \mapsto \pi_0(G_{(q/q-1)}(X, p)).$$

Taking the cycle-class of a coherent sheaf defines the map of simplicial abelian groups $\pi_0\mathrm{cl}_q : \pi_0 G_{(q/q-1)}(X, -) \to z_q(X, -)$; by replacing the simplicial abelian groups with the associated Eilenberg-Maclane spectra, we may consider $\pi_0\mathrm{cl}_q$ as a natural map of spectra. Since $\Omega^{-N}G_{(q/q-1)}(X, p)$ is N-connected for each N, we have the natural map of simplicial spectra $G_{(q/q-1)}(X, -) \to \pi_0 G_{(q/q-1)}(X, -)$. Composing this map with $\pi_0\mathrm{cl}_q$ gives us the natural map of simplicial spectra

$$\mathrm{cl}_q : G_{(q/q-1)}(X, -) \to z_q(X, -). \tag{8.4}$$

In case $X = \mathrm{Spec}\, F$ for a field F, Friedlander and Suslin [5, Theorem 6.1] have shown that this map is a weak equivalence. After the usual reindexing, the spectral sequence (8.2) thus becomes the (cohomological) spectral sequence of Atiyah-Hirzebruch type

$$E_2^{p,q} = H^p(F, \mathbb{Z}(-q/2)) \Longrightarrow G_{-p-q}(F) = K_{-p-q}(F). \tag{8.5}$$

The arguments of §7 used to prove Theorem 1.7 can be modified to show that the simplicial spectra $G_{(q)}(X, -)$ satisfy a *localization property*, namely, if $i : Z \to X$ is a closed subscheme of X with complement $j : U \to X$, then the sequence

$$G_{(q)}(Z, -) \xrightarrow{i_*} G_{(q)}(X, -) \xrightarrow{j^*} G_{(q)}(U, -) \tag{8.6}$$

is a homotopy fiber sequence, at least in case $B = \mathrm{Spec}\, A$, A a semi-local principal ideal ring. Assuming this is so, we are able to identify the E^1-terms in (8.3) via:

Proposition 8.9. *Let B be a regular scheme of Krull dimension at most one. Suppose that the sequence* (8.6) *is a homotopy fiber sequence for all pairs (X, Z), with X a scheme of finite type over a semi-local principal ideal ring, and $Z \subset X$ a closed subscheme. Then, for all finite-type B-schemes $f : X \to B$, the map* (8.4) *induces a weak equivalence of presheaves of simplicial spectra $f_*G_{(q)}(X, -) \to f_*z_q(X, -)$, and isomorphisms of the E^1-term in the spectral sequence* (8.3)

$$\begin{aligned} E^1_{p,q} = \pi_{p+q}(B; f_*G_{(p/p-1)}(X, -)) &\cong \pi_{p+q}(B; f_*z_p(X, -)) \\ &\cong \mathbb{H}_{p+q}(B; f_*z_p(X, *)) = \mathrm{CH}_p(X, p+q). \end{aligned}$$

In case $B = \mathrm{Spec}\, A$, A a semi-local principal ideal ring, the spectral sequences (8.2) *and* (8.3) *agree, with E^1-term*

$$\begin{aligned} E^1_{p,q} = \pi_{p+q}(G_{(p/p-1)}(X, -)) &\cong \pi_{p+q}(z_p(X, -)) \\ &\cong H_{p+q}(z_p(X, *)) = \mathrm{CH}_p(X, p+q). \end{aligned}$$

Proof. Since a map of presheaves of simplicial sets which is a stalk-wise weak equivalence induces an isomorphism on $\pi_n(B; -)$, it suffices to prove the proposition in case $B = \mathrm{Spec}\, A$, A a semi-local PID. Since we know that $\pi_n(z_p(X, -)) = \pi_n(B; f_*z_p(X, -))$ in this case (Theorem 1.7), we need only show that the E^1-term in the spectral sequence (8.2) is given by $\pi_{p+q}(z_p(X, -))$.

Let $G_{(q)}(X_{(r)}, -)$ be the direct limit of the spectra $G_{(q)}(Z, -)$, as Z runs over unions of irreducible closed subsets $C \subset X$ with $\dim C \leq r$. This gives us the map of simplicial spectra

$$G_{(q)}(X_{(r-1)}, -) \xrightarrow{i_r} G_{(q)}(X_{(r)}, -);$$

let $G_{(q)}(X_{(r/r-1)}, -)$ denote the cofiber of i_r.

Since the sequences (8.6) are homotopy fiber sequences for all (X, Z) by assumption, and since the operation of taking the K-theory spectrum is compatible with direct limits of exact categories [11, §2], we have the homotopy fiber sequence

$$G_{(q)}(X_{(\dim X-1)}, -) \xrightarrow{i_*} G_{(q)}(X, -) \xrightarrow{j^*} \coprod_{x \in X_{(\dim X)}} G_{(q)}(x, -),$$

natural in q. Since taking the homotopy cofiber (of spectra) preserves homotopy fiber sequences, we have the homotopy fiber sequence

$$G_{(q/q-1)}(X_{(\dim X-1)}, -) \xrightarrow{i_*} G_{(q/q-1)}(X, -) \xrightarrow{j^*} \coprod_{x \in X_{(\dim X)}} G_{(q/q-1)}(x, -).$$

Let $z_q(X_{(r)}, p)$ be the direct limit of the $z_q(Z, p)$, as Z runs over unions of irreducible closed subsets $C \subset X$ with $\dim C \leq r$. As A is a semi-local principal ideal ring, it follows from Theorem 1.7 and the Dold-Kan equivalence that

$$z_q(X_{(\dim X-1)}, -) \xrightarrow{i_*} z_q(X, -) \xrightarrow{j^*} \coprod_{x \in X_{(\dim X)}} z_q(x, -)$$

is a homotopy fiber sequence. The cycle map gives the commutative diagram

$$\begin{array}{ccccc} G_{(q/q-1)}(X_{(\dim X-1)}, -) & \xrightarrow{i_*} & G_{(q/q-1)}(X, -) & \xrightarrow{j^*} & \coprod_{x \in X_{(\dim X)}} G_{(q/q-1)}(x, -) \\ \downarrow{\scriptstyle \mathrm{cl}_q} & & \downarrow{\scriptstyle \mathrm{cl}_q} & & \downarrow{\scriptstyle \mathrm{cl}_q} \\ z_q(X_{(\dim X-1)}, -) & \xrightarrow[i_*]{} & z_q(X, -) & \xrightarrow[j^*]{} & \coprod_{x \in X_{(\dim X)}} z_q(x, -) \end{array}$$

By [5, Theorem 6.1], the cycle map $\mathrm{cl}_q : G_{(q/q-1)}(x, -) \to z_q(x, -)$ is a weak equivalence for all $x \in X$. By induction on $\dim X$, and the compatibility of the functors $G_{(q/q-1)}$ and z_q with direct limits, the cycle map

$$\mathrm{cl}_q : G_{(q/q-1)}(X_{(\dim X-1)}, -) \to z_q(X_{(\dim X-1)}, -)$$

is a weak equivalence. Thus $\mathrm{cl}_q : G_{(q/q-1)}(X, -) \to z_q(X, -)$ is a weak equivalence. □

8.10. After reindexing (8.3) to give an E^2-spectral sequence, we arrive at the homological spectral sequence

$$E^2_{p,q} = H^{\mathrm{B.M.}}_p(X, \mathbb{Z}(-q/2)) \Longrightarrow G_{p+q}(X), \quad (8.7)$$

reminiscent of the classical Atiyah-Hirzebruch spectral sequence from singular cohomology to topological K-theory. In case X is regular, the natural map $G_*(X) \to K_*(X)$ is an isomorphism. If X is irreducible, $\dim X = d$, we define $H^p(X, \mathbb{Z}(q)) := \mathrm{CH}_{d-q}(X, 2d-2q-p)$ and extend this definition to arbitrary regular X by taking the direct sum over the irreducible components. The sequence (8.7) then becomes the cohomological sequence

$$E_2^{p,q} = H^p(X, \mathbb{Z}(-q/2)) \Longrightarrow K_{-p-q}(X). \quad (8.8)$$

The above constructions give similar spectral sequences with finite coefficients as well. For instance, define the complex $z_q(X,*)/n$ as the cone of multiplication by n, $\times n : z_q(X,*) \to z_q(X,*)$, and set

$$H_p^{\mathrm{B.M.}}(X,\mathbb{Z}/n(q)) := \mathbb{H}^{2q-p}(B; f_*z_q(X,*)/n).$$

Replacing $\pi_s(-)$ with the mod n homotopy group $\pi_s(-;\mathbb{Z}/n)$ gives the spectral sequence

$$E^2_{p,q} = H_p^{\mathrm{B.M.}}(X,\mathbb{Z}/n(-q/2)) \Longrightarrow G_{p+q}(X;\mathbb{Z}/n). \tag{8.9}$$

The other spectral sequences discussed above have their mod n counterparts as well.

To complete the discussion, we now prove the localization property for spectra $G_{(q)}(X,-)$; we may take $B = \operatorname{Spec} A$, A a semi-local PID.

We let $j : U \to X$ be the inclusion of an open subscheme U of a finite-type B-scheme X. Let $G_{(q)}(U^X,p)$ be the spectrum constructed as $G_{(q)}(U,p)$, where we take supports in $W \subset U \times \Delta^p$ which is a union of finitely many $C \in U^X_{(p,\leq q)}$. We have the associated simplicial spectra $G_{(q)}(U^X,-)$, and the natural map

$$G_{(q)}(U^X,-) \to G_{(q)}(U,-) \tag{8.10}$$

Theorem 8.11. *The map* (8.10) *is a weak equivalence.*

Proof. As the spectra $G_{(q)}(U^X,-)$ and $G_{(q)}(U,-)$ are covariantly functorial for finite morphisms $X \to X'$, we may use the argument of §7 to show that we may assume that each open subscheme V of $\mathbb{A}^n_B$ which maps onto B admits a section.

Applying the Hurewicz isomorphism to the cofiber of (8.10), it suffices to show that (8.10) induces a homology isomorphism on all sufficently large deloopings. Since the homology group H_p of $\Omega^{-d}G_{(q)}(U^X,-)$ and $\Omega^{-d}G_{(q)}(U,-)$ are given by H_p of the truncated simplicial objects for all truncations $> p$, so it suffices to show that (8.10) induces a homology isomorphism in degrees $p < N$ on the associated N-truncated simplicial objects $\Omega^{-d}G_{(q)}(U^X,-)_N$ and $\Omega^{-d}G_{(q)}(U,-)_N$.

For a simplicial set T, we let $\operatorname{Sing}(T)$ denote the complex of integral simplicial chains, i.e., $\operatorname{Sing}(T)([p])$ is the free abelian group on $T([p])$. The functor $\operatorname{Sing}(-)$ thus transforms N-truncated simplicial spaces to N-truncated simplicial complexes (i.e. functors $\mathbf{Ord}^{\leq N\ \mathrm{op}} \to \mathbf{C}(\mathbf{Ab})$), and we have the natural quasi-isomorphism

$$\operatorname{Tot}(\operatorname{Sing}(T_*)) \to \operatorname{Sing}(|T_*|,\mathbb{Z}),$$

for an N-truncated simplicial space $T_* : \mathbf{Ord}^{\leq N\ \mathrm{op}} \to \mathcal{S}$, where $|T_*|$ is the geometric realization of T_* and $\operatorname{Sing}(|T_*|,\mathbb{Z})$ is the complex of singular chains. We therefore need only show that (8.10) induces a homology isomorphism in degrees $p < N$

$$\operatorname{Tot}(\operatorname{Sing}(\Omega^{-d}G_{(q)}(U^X,-))) \to \operatorname{Tot}(\operatorname{Sing}(\Omega^{-d}G_{(q)}(U,-))).$$

The proof is now essentially the same as the proof of Theorem 1.7. For a finite collection $C := \{C_j \in U_{(p_j,\leq q)}\}$, and an integer $N \geq 0$, we have the closed subset $C_p^{\leq N}$ of $U \times \Delta^p$, containing all the C_j with $p_j = p$, and functorial in p for morphisms $g : \Delta^q \to \Delta^p$ in $\Delta^{\leq N}$. $C_p^{\leq N}$ is a finite union of irreducible closed subsets in $U_{(p,\leq q)}$ for all p, and if in addition all C_j are in $U^X_{(p_j,\leq q)}$, then $C_p^{\leq N}$ is a finite union of irreducible closed subsets in $U^X_{(p,\leq q)}$ for all p.

Fix integers $N \geq 0$ and q, and take a finite collection $C := \{C_j \in U_{(p_j, \leq q)}\}$, $0 \leq p_j \leq N$. We may form the N-truncated simplicial spectrum $G_C(U, -)_N$ with

$$G_C(U,p)_N = G_{C_p^{\leq N}}(X \times \Delta^p).$$

We then have

$$H_p(\Omega^{-d} G_{(q)}(U, -)) = \varinjlim_{C} H_p(\Omega^{-d} G_C(U, -)_N)$$

for $p < N$, where C runs over finite collections $\{C_j \in U_{(p_j, \leq q)}\}$, $0 \leq p_j \leq N$. We have a similar description of $H_p(\Omega^{-d} G(U^X, -)_N)$ for $p < N$,

$$H_p(\Omega^{-d} G_{(q)}(U^X, -)) = \varinjlim_{C'} H_p(\Omega^{-d} G_C(U, -)_N)$$

where C' runs over finite collections $\{C'_j \in U^X_{(p_j, \leq q)}\}$, $0 \leq p_j \leq N$.

Take $C_j \in U_{(p_j, \leq q)}$, $j = 1, \ldots, s$, $C_j \in U^X_{(p_j, \leq q)}$, $j = s+1, \ldots, r$, with $0 \leq p_j \leq N$, and let

$$C = \bigcup_{j=1}^{r} C_j, \quad C' = \bigcup_{j=s+1}^{r} C_j.$$

This gives us the functors

$$\mathrm{Sing}(\Omega^{-d} G_C(U, -)_N) : \mathbf{Ord}^{\leq N \text{ op}} \to \mathbf{C}(\mathbf{Ab})$$
$$\mathrm{Sing}(\Omega^{-d} G_{C'}(U, -)_N) : \mathbf{Ord}^{\leq N \text{ op}} \to \mathbf{C}(\mathbf{Ab})$$

and the natural map

$$\mathrm{Sing}(\Omega^{-d} G_{C'}(U, -)_N) \to \mathrm{Sing}(\Omega^{-d} G_C(U, -)_N).$$

We may form the associated total complexes

$$\mathrm{Tot}(\mathrm{Sing}(\Omega^{-d} G_C(U, -)_N)),\ \mathrm{Tot}(\mathrm{Sing}(\Omega^{-d} G_{C'}(U, -)_N)).$$

We may also apply $\mathrm{Sing}(\Omega^{-d} G_C(U, -)_N)$ and $\mathrm{Sing}(\Omega^{-d} G_{C'}(U, -)_N)$ to the complex $(\Delta^N, \partial\Delta^N)$, forming the complexes

$$\mathrm{Sing}(\Omega^{-d} G_C(U; \Delta^N, \partial\Delta^N)),\ \mathrm{Sing}(\Omega^{-d} G_{C'}(U; \Delta^N, \partial\Delta^N)).$$

Taking the limit over C, C' gives the various complexes

$$\mathrm{Tot}(\mathrm{Sing}(\Omega^{-d} G_{(q)}(U, -)_N)),\ \mathrm{Tot}(\mathrm{Sing}(\Omega^{-d} G_{(q)}(U^X, -)_N))$$
$$\mathrm{Sing}(\Omega^{-d} G_{(q)}(U; \Delta^N, \partial\Delta^N)),\ \mathrm{Sing}(\Omega^{-d} G_{(q)}(U^X; \Delta^N, \partial\Delta^N)).$$

The map Ψ_N gives the map of complexes

$$\mathrm{Sing}(\Omega^{-d} G_{(q)}(U; \Delta^N, \partial\Delta^N))[-N] \xrightarrow{\mathrm{Sing}(G(\Psi_N))} \mathrm{Tot}(\mathrm{Sing}(\Omega^{-d} G_{(q)}(U, -)_N)),$$

which, by Lemma 2.6, is a homology isomorphism in degrees $< N$. Thus, the map

$$\mathrm{Sing}(\Omega^{-d} G_C(U; \Delta^N, \partial\Delta^N))[-N] \xrightarrow{\mathrm{Sing}(G(\Psi_N))(C)} \mathrm{Tot}(\mathrm{Sing}(\Omega^{-d} G_{(q)}(U, -)_N))$$

is a homology isomorphism in degree $< N$ after taking the limit over C. Similarly, the map

$$\mathrm{Sing}(\Omega^{-d} G_{C'}(U; \Delta^N, \partial\Delta^N))[-N] \xrightarrow{\mathrm{Sing}(G(\Psi_N))(C')} \mathrm{Tot}(\mathrm{Sing}(\Omega^{-d} G_{(q)}(U^X, -)_N))$$

is a homology isomorphism in degree $< N$ after taking the limit over C'.

From Theorem 8.2, we have the diagram

(8.11)

$$\begin{array}{ccc} \mathrm{Sing}(\Omega^{-d}G_{C'}(U;\Delta^N,\partial\Delta^N))[-N] & \longrightarrow & \mathrm{Sing}(\Omega^{-d}G_C(U;\Delta^N,\partial\Delta^N))[-N] \\ \Big\downarrow{\scriptstyle \mathrm{Sing}(G(\Psi_N))(C')} & \overset{\Psi}{\swarrow} & \Big\downarrow{\scriptstyle \mathrm{Sing}(G(\Psi_N))(C)} \\ \mathrm{Tot}(\mathrm{Sing}(\Omega^{-d}G_{(q)}(U^X,-)_N)) & \xrightarrow[\iota]{} & \mathrm{Tot}(\mathrm{Sing}(\Omega^{-d}G_{(q)}(U,-)_N)). \end{array}$$

As in the proof of Theorem 1.7 in §7, it follows from the properties of $\Psi := \Psi_p^c$ and $H := H_p^{c'}$ that (8.11) is commutative up to homotopy, and hence ι is a homology isomorphism in degrees $< N$. Since N was arbitrary, ι is a homology isomorphism, and hence (8.10) is a weak equivalence. □

It is now easy to show that (8.6) is a homotopy fiber sequence.

Corollary 8.12. *Let $B = \mathrm{Spec}\, A$, A a semi-local PID, and let X be a B-scheme of finite type. Let $j : U \to X$ be the inclusion of an open subscheme, $i : Z \to X$ of U the complement. Then the sequence*

$$G_{(q)}(Z,-) \xrightarrow{i_*} G_{(q)}(X,-) \xrightarrow{j^*} G_{(q)}(U,-)$$

is a homotopy fiber sequence for all integers q.

Proof. Let C be in $X_{(p,\leq q)}$. Then each irreducible component of $C \cap (Z \times \Delta^p)$ is in $Z_{(p,\leq q)}$. By definition, if C is in $U^X_{(p,\leq q)}$, then the closure of C in $X \times \Delta^p$ is in $X_{(p,\leq q)}$. From this, Proposition 8.8(1) and Quillen's localization theorem [11, §7, Proposition 3.1], we have the homotopy fiber sequence

$$G_{(q)}(Z,p) \xrightarrow{i_*} G_{(q)}(X,p) \xrightarrow{j^*} G_{(q)}(U^X,p)$$

for each p, giving the homotopy fiber sequence

$$G_{(q)}(Z,-) \xrightarrow{i_*} G_{(q)}(X,-) \xrightarrow{j^*} G_{(q)}(U^X,-).$$

By Theorem 8.11, the natural map $G_{(q)}(U^X,-) \to G_{(q)}(U,-)$ is a weak equivalence, giving the desired homotopy fiber sequence. □

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