The weight two K-theory of fields

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Introduction

The rational K-groups of a scheme X break up into a direct sum of the weight q eigenspaces $K_p(X)^{(q)}$ for the Adams operations ψ^k. The mod n K-groups $K_p(X, \mathbb{Z}/n)$ also break up in this way, after inverting sufficiently many primes (depending only on p and $\dim(X)$). In particular, if we consider the K-groups of a field F, we can split the gamma filtration on $K_p(F)$ and $K_p(F, \mathbb{Z}/n)$ by eigenspaces for the ψ^k after inverting $(p-1)!$. Additionally, for n prime to the characteristic of F, there are Chern class maps

$$c_{q,p}: gr^q_\gamma K_{2q-p}(F, \mathbb{Z}/n) \to H^p_{\acute{e}t}(F, \boldsymbol{\mu}_n^{\otimes q});$$

one version of the Quillen-Lichtenbaum conjectures is that the map $c_{q,p}$ should be an isomorphism after inverting “small” primes.

In this paper, we consider the case $q = 2$. We show that the maps $c_{q,p}$ are all isomorphisms, after inverting $(3-p)!$, for $p = 2, 1, 0, -1, \ldots$, and for n prime to the characteristic of F. In particular, $gr^2_\gamma K_p(F)/l = 0$ for all primes $l \geq p$, as long as $p \geq 4$ and l is prime to the characteristic. If F is a number field, this implies that $gr^2_\gamma K_p(F)[1/(p-1)!] = 0$ for $p \geq 4$. This fits into the conjecture of Soulé and Beilinson on the vanishing of the rational spaces $K_p(F)^{(q)}$ for arbitrary F, when $p \geq 2q$, and suggests the refined version:

Conjecture. *Let F be a field. Then $gr^q_\gamma K_p(F)[1/(p-1)!] = 0$ for $p \geq 2q$.*

The main point of our argument is to prove the analogue of Hilbert's Theorem 90 for the relative K_2 of regular semi-local rings containing a field. This was first shown for fields by Merkurjev and Suslin in [MS], reproved and moved to a central position in the theory by Suslin in [S], and was extended to the case of a semi-local PID containing a field in [MS2] and [L]. The proof here is essentially a modification of the arguments of [MS2], with some techniques borrowed from [L]. Aside from any intrinsic interest of the result, the argument points out the commonality among the various weight two components of the K-groups $K_p(F)$ for varying p. This suggests that a bi-graded motivic cohomology theory, in the sense of Lichtenbaum [Li] and Beilinson [B], is lurking in the background. In a different direction, this paper makes clear that the Galois symbol

$$\theta_q: K^M_q(F) \to H^q_{\acute{e}t}(F, \boldsymbol{\mu}_m^{\otimes q}).$$

plays a central rôle in our understanding of the Chern class maps $c_{q,p}$.

There are questions raised by this result regarding the inversion of small primes: is this an artifact of the proof, is it a reflection of the lack of degeneration of the Atiyah-Hirzebruch spectral sequence comparing étale cohomology and étale K-theory, or does it arise from an honest difference between the algebraic mod-n K-theory and étale K-theory of F? From the point of view of the conjectural motivic cohomology theory, our result may be an indication that K-theory should be computable in terms of motivic cohomology, not only rationally, but after inverting small primes. It would be interesting to see if the results of this paper could be carried over to the setting of Bloch's higher Chow groups [Bl2], where one would hopefully be able to avoid the inverting of small primes, and prove a purely integral result.

The paper is organized as follows: In §1, we give a description of the relative K_2 of a semi-local ring containing an infinite field in terms of Matsumoto-type symbols. As a preliminary result, we present an alternate proof of some results of Ellis [E], which describe the relative K_2 of a semi-local ring via ”relative Dennis-Stein symbols”; our results require somewhat different hypotheses than those of Ellis. In §2 we define norms for relative K-theory in the necessary generality. In §3 and §4 we collect some of the main results on the Quillen spectral sequence for relative K-theory, and show how the computations of this spectral sequence for projective spaces and affine spaces extend to the relative setting. In §5 we prove the main technical result of the paper, Hilbert's Theorem 90 for relative K_2, and in §6 we give applications of Hilbert's Theorem 90 to results of primary interest.

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§1. Relative K_2 revisited

We recall that the s-cube $<s>$ is the category associated to the partially ordered set of subsets of $\{1, 2, \ldots, s\}$, ordered by inclusion; we let

$$\rho_{I\subset J}: I \to J$$

denote the unique morphism for $I \subset J$. An s-cube in a category $\mathcal{C}$ is a functors $X: <s> \to \mathcal{C}$. This gives us, for example, the notion of an s-cube of sets, of topological spaces, of rings, etc. If R is a ring, $(I_1, \ldots, I_s)$ ideals, and $S \in <s>$, we let $I_S = \sum_{j\in S} I_j$, and define the s-cube of rings $(R; I_1, \ldots, I_s)_*$ by

$$(R; I_1, \ldots, I_s)(S) = R/I_S.$$

The *split s-cube* is the category $<s>^{sp}$ with the same objects as $<s>$, containing $<s>$ as a subcategory, with additional morphisms

$$\sigma_{I\subset J}: J \to I$$

for each $I \subset J \subset \{1, \ldots, s\}$. The morphisms $\sigma_{I\subset J}$ satisfy the relations

i) $\rho_{I\subset J} \circ \sigma_{I\subset J} = \mathrm{id}_J$,
ii) $\sigma_{I\subset J} \circ \sigma_{J\subset K} = \sigma_{I\subset K}$, for $I \subset J \subset K$,
iii) $\rho_{I\cap J\subset I} \circ \sigma_{I\cap J\subset J} = \sigma_{I\subset I\cup J} \circ \rho_{J\subset I\cup J}$;

in words, the category $<s>^{sp}$ is the category $<s>$, together with a system of compatible splittings of the morphisms $\rho_{I\subset J}$. A *split s-cube in a category* $\mathcal{C}$ is a functor $F: <s>^{sp} \to \mathcal{C}$; a splitting of an s-cube $F: <s> \to \mathcal{C}$ is an extension of F to a functor $F^{sp}: <s>^{sp} \to \mathcal{C}$.

Each s-cube $I \mapsto X_I$ in $\mathcal{C}$ can be viewed as a map of $(s-1)$-cubes in $\mathcal{C}$: $f: X^+ \to X^-$ by setting

$$X_I^+ = X_I \text{ for } I \subset \{1, \ldots, s-1\}$$
$$X_I^- = X_{I\cup\{s\}} \text{ for } I \subset \{1, \ldots, s-1\},$$

and defining $f_I: X_I^+ \to X_I^-$ to be the map $X(I \subset I \cup \{s\}): X_I \to X_{I\cup\{s\}}$. This defines a functor $res_{s/s-1}$ from $\mathcal{C}<s>$ to $\mathrm{Maps}(\mathcal{C}<s-1>)$.

Let $\mathbf{Top}^*$ denote the category of pointed topological spaces. Given a map $f_*: X_* \to Y_*$ of t-cubes of pointed spaces, we can form the t-cube of homotopy fibers $F(f)_*$, $F(f)_I = \mathrm{fiber}(f_I: X_I \to Y_I)$. If X_* is an s-cube of spaces, forming the homotopy fiber of $res_{s/s-1}(X_*)$ defines the functor

$$\mathrm{fib}_{s/s-1}: \mathbf{Top}^*<s> \to \mathbf{Top}^*<s-1>.$$

Iterating these functors defines the functor

$$\mathrm{fib}^s: \mathbf{Top}^*<s> \to \mathbf{Top}^*; \quad \mathrm{fib}^s = \mathrm{fib}_{1/0} \circ \ldots \circ \mathrm{fib}_{s/s-1}.$$

We call $\mathrm{fib}^n(X_*)$ the *iterated homotopy fiber* over the s-cube X_*.

Fix a functorial model $K(R)$ for $\Omega\mathrm{BQ}P_R$, and define the s-cube of pointed topological spaces

$$K(R; I_1, \ldots, I_s)_*$$

by

$$K(R; I_1, \ldots, I_s)(S) = K(R/I_S).$$

Taking the iterated homotopy fiber over this s-cube defines the topological space $K(R; I_1, \ldots, I_s)$; the homotopy groups of $K(R; I_1, \ldots, I_s)$ are by definition the relative K-groups $K_p(R; I_1, \ldots, I_s)$.

Let R be a semi-local ring with ideals $I_1, \ldots, I_s$. We let $D_{(R;I_1,\ldots,I_s)}$ denote the s-fold double of R with respect to $I_1, \ldots, I_s$; this is the subring of $\prod_{S \in <s>} R$ consisting of 2^s-tuples $\{r_S | S \in <s>\}$ with

$$r_S - r_{S\setminus k} \in I_k; \text{ for all } k \in S \in <s>.$$

We have the ideals D_k and D' of $D_{(R;I_1,\ldots,I_s)}$ defined by

$$D_k = \{r_S |\ r_S = 0 \text{ if } k \in S\};\ k = 1, \ldots, s$$

and

$$D' = \{r_S |\ r_S = 0 \text{ if } S = \emptyset\}.$$

We map $(D_{(R;I_1,\ldots,I_s)}, D_1, \ldots, D_s)$ to $(R; I_1, \ldots, I_s)$ with kernel D' by projection on the factor $S = \emptyset$.

We have given in [L2] a definition of the relative Milnor K-groups of R, $K_p^M(R; I_1, \ldots, I_s)$, in terms of the usual Milnor K-groups of $D_{(R;I_1,\ldots,I_s)}$. In this section, we will show how to describe $K_2^M(R; I_1, \ldots, I_s) = K_2(R; I_1, \ldots, I_s)$ in terms of R and the ideals $I_1, \ldots, I_s$ without the aid of the s-fold double. This will use the presentation in terms of Dennis-Stein symbols, and will give a partial generalization of the work of Keune [Ke], Loday [Lo], and Guin-Loday [G-L] on relative and doubly relative K_2.

Following Ellis [E], we proceed to define the relative Dennis-Stein groups $D(R; I_1, \ldots, I_s)$.

Let R be a ring, $I_1, \ldots, I_s$ ideals. Fix $S, S' \in <s>$ with $\{1, \ldots, s\} = S \cup S'$. This gives rise to the subset $(R^2; \{I_1, \ldots, I_s\}, S, S')$ of R^2:

$$(R^2; \{I_1, \ldots, I_s\}, S, S') = \bigcap_{j \in S} I_j \times \bigcap_{j \in S'} I_j$$

(the empty intersection is R). Let $(R^2; \{I_1, \ldots, I_s\}, S, S')^\times$ be the subset of $(R^2; \{I_1, \ldots, I_s\}, S, S')$ consisting of pairs (a, b) with $1 - ab$ invertible in R. Taking the union over all pairs (S, S') gives us the subsets

$$(R^2; \{I_1, \ldots, I_s\}) = \cup_{(S,S')}(R^2; \{I_1, \ldots, I_s\}, S, S')$$
$$(R^2; \{I_1, \ldots, I_s\})^\times = \cup_{(S,S')}(R^2; \{I_1, \ldots, I_s\}, S, S')^\times$$

of R^2. We make a similar definition of $(R^3; \{I_1, \ldots, I_s\}, S, S', S'')$ and $(R^3; \{I_1, \ldots, I_s\})$; let

$$(R^3; \{I_1, \ldots, I_s\}, S, S', S'')^\times = \{(a, b, c) \in (R^3; \{I_1, \ldots, I_s\}, S, S', S'') \,|\, 1 - abc \in R^\times\},$$

and

$$(R^3; \{I_1, \ldots, I_s\})^\times = \{(a, b, c) \in (R^3; \{I_1, \ldots, I_s\}) \,|\, 1 - abc \in R^\times\}.$$

Definition 1.1. *Let R be a ring, $I_1, \ldots, I_s$ ideals of R. Define $D(R; I_1, \ldots, I_s)$ to be the abelian group with generators $<a, b>$, for (a, b) in $(R^2; I_1, \ldots, I_s)^\times$, and relations*

a) $<a, b> + <b, a> = 0$,
b) $<a, c> + <b, c> = <a + b - abc, c>$,
c) $<ab, c> + <ac, b> = <a, bc>$, *for* $(a, b, c) \in (R^3; I_1, \ldots, I_s)^\times$.

□

A ring homomorphism $R \to T$ which sends I_k to J_k; $k = 1, \dots, s$ induces a homomorphism

$$D(R; I_1, \dots, I_s) \to D(T; J_1, \dots, J_s).$$

We will show (Theorem 1.4) that there is a natural isomrphism

$$D(R; I_1, \dots, I_s) \to K_2(R; I_1, \dots, I_s)$$

under certain circumstances. It is useful as well to have a slightly different description of $K_2(R; I_1, \dots, I_s)$, via so-called *generic* symbols. This approach in various settings has been considered by van der Kallen ([vdK], [vdK2]). Here we give a slight modification of the presentation he describes for $D_{gen}(R; J)$ and show that, under some conditions on the ideals $I_1, \dots, I_s$, this generalizes to give a group $D_{gen}(R; I_1, \dots, I_s)$, isomorphic to $K_2(R; I_1, \dots, I_s)$.

Definition 1.2. *Let R be a ring, $I_1, \dots, I_s$ ideals. We consider the following conditions on the ideals $I_1, \dots, I_s$:*

i) each ideal I_j is principal: $I_j = (t_j)$.
ii) for each subset S of $\{1, \dots, s\}$, the ideal $\bigcap_{j \in S} I_j$ is principal, and

$$\bigcap_{j \in S} I_j = \prod_{j \in S} t_j.$$

iii) R contains an infinite field.

An element x of $\cap_{j \in S} I_j$ is called generic *if $x = u \prod_{j \in S} t_j$ for some unit u in R. In particular, a generic element of R is simply a unit of R. If, in addition to (i)-(iii), for each $S \subset \{1 \dots, s\}$ the ring $R_S := R / \sum_{j \in S} I_j$ is regular, we say $I_1, \dots, I_s$ define a normal crossing divisor.* □

Let $(R^2; I_1, \dots, I_s)^\times_{gen}$ denote the subset of $(R^2; I_1, \dots, I_s)^\times$ consisting of pairs (a, b) of generic elements.

Definition 1.3. *Let R be a semi-local ring, $I_1, \dots, I_s$ ideals. Suppose $(R; I_1, \dots, I_s)$ satisfies the conditions (i)-(iii) of Definition 1.2. Let $D_{gen}(R; I_1, \dots, I_s)$ be the abelian group with generators*

$$<a, b>, \text{with} (a, b) \in (R^2; I_1, \dots, I_s)^\times_{gen} \cup (R^\times \times \{0\}) \cup (\{0\} \times R^\times),$$

and relations

a) $<a, b> + <b, a> = 0$,
b) $<a, c> + <b, c> = <a + b - abc, c>$,
c) $<ab, c> + <ac, b> = <a, bc>$,

when all symbols are defined. □

There is a natural homomorphism

$$D_{gen}(R; I_1, \dots, I_s) \to D(R; I_1, \dots, I_s)$$

sending $<a, b>$ to $<a, b>$ (and $<a, 0>$, $<0, a>$ to 0).

If I is an ideal in a ring R, and if there is a finite set of maximal ideals $m_1, \dots, m_n$ such that $I \cap m_1 \cap \dots \cap m_n$ is a radical ideal, we say that I is an *almost radical ideal*.

Lemma 1.1. *Let R be a ring, $I_1, \ldots, I_s$ ideals, $I = I_1 \cap \ldots \cap I_t$, $T = R/I$, J_j the image of I_j in T. Suppose $\cap_{j=1}^s I_j$ is an almost radical ideal. Then the sequence*

$$D(R; I, I_{t+1}, \ldots, I_s) \to D(R; I_1, \ldots, I_s) \to D(T; J_1, \ldots, J_s) \to 0$$

is exact. If $(R; I_1, \ldots, I_s)$ satisfy the condtions (i)-(iii) of Definition 1.2, then the sequence

$$D_{gen}(R; I, I_{t+1}, \ldots, I_s) \to D_{gen}(R; I_1, \ldots, I_s) \to D_{gen}(T; J_1, \ldots, J_s) \to 0$$

is exact.

Proof. For the first assertion, our assumption that $\cap_{j=1}^s I_j$ is an almost radical ideal yields the surjectivity of the maps

$$(R^2; \{I_1, \ldots, I_s\}, S, S')^\times \to (T^2; \{J_1, \ldots, J_s\}, S, S')^\times$$

and

$$(R^3; \{I_1, \ldots, I_s\}, S, S', S'')^\times \to (T^3; \{J_1, \ldots, J_s\}, S, S', S'')^\times$$

Thus, the map

$$D(R; I_1, \ldots, I_s) \to D(T; J_1, \ldots, J_s).$$

is surjective.

Since the symbols $<0, b>$ and $<b, 0>$ are zero (by Def. 1.1(b)), the composition

$$D(R; I, I_{t+1}, \ldots, I_s) \to D(R; I_1, \ldots, I_s) \to D(T; J_1, \ldots, J_s)$$

is zero. Since each relation defining the group $D(T; J_1, \ldots, J_s)$ can be lifted to a relation defining the group $D(R; I_1, \ldots, I_s)$, the kernel of the map

$$D(R; I_1, \ldots, I_s) \to D(T; J_1, \ldots, J_s)$$

is generated by symbols $<a, b> - <a', b'>$, with $a - a'$ and $b - b'$ in I. Given the elements $<a, b>$ and $<a', b'>$, we get two pairs of subsets of the set $\{1, \ldots, s\}$:

$$\{1, \ldots, s\} = S \cup S' = V \cup V'$$

with

$$a \in \bigcap_{j \in S} I_j; \ b \in \bigcap_{j \in S'} I_j;$$

$$a' \in \bigcap_{j \in V} I_j; \ b' \in \bigcap_{j \in V'} I_j.$$

Since $a' - a$ and $b' - b$ are in I, we can find a'' and b'' in R with

$$a'' \in \bigcap_{j \in V \cup S} I_j; \ b'' \in \bigcap_{j \in V' \cup S'} I_j;$$

and

$$a'' - a, \ a'' - a', \ b'' - b, \ b'' - b' \in I.$$

Replacing $<a,b> - <a',b'>$ with $<a,b> - <a'',b''> + <a'',b''> - <a',b'>$, and changing notation, we see that the kernel of $D(R;I_1,\ldots,I_s) \to D(T;J_1,\ldots,J_s)$ is generated by symbols $<a,b> - <a',b'>$ with $a'-a$, $b'-b$ in I; we can also assume that (a,b') is in $(R^2;I_1,\ldots,I_s)^\times$. But

$$<a,b> - <a,b'> = <a,(b-b')(1-ab')^{-1}>$$
$$<a,b'> - <a',b'> = <(a-a')(1-a'b')^{-1},b'>.$$

As both $<a,(b-b')(1-ab')-1>$ and $<(a-a')(1-a'b')-1,b'>$ are in the image of $D(R;I,I_{t+1},\ldots,I_s)$, this completes the proof of the exactness of the first sequence.

For the second, the surjectivity follows by arguing as above. For the exactness at $D_{gen}(R;I_1,\ldots,I_s)$, we first note that by choosing a'' and b'' generically, the kernel of the map

$$D_{gen}(R;I_1,\ldots,I_s) \to D_{gen}(T;J_1,\ldots,J_s)$$

is generated by differences $<a,b> - <a',b'>$ with $<a,b'>$ in $D_{gen}(R;I_1,\ldots,I_s)$. In addition, we have

$$<a,b> - <a,b'> = <a,(b-b')(1-ab')-1>$$

and

$$<a,b'> - <a',b'> = <(a-a')(1-a'b')-1,b'>,$$

which shows these elements are in the image of $D_{gen}(R;J,I_{t+1},\ldots,I_s)$. The proof is then exactly as above. □

Lemma 1.2. *Let R be a ring, $I,I_1,\ldots,I_s$ ideals, $T=R/I$, J_j the image of I_j in T. Suppose $\cap_{j=1}^s I_j$ is an almost radical ideal. Then*

i) the sequence

$$D(R;I,I_1,\ldots,I_s) \to D(R;I_1,\ldots,I_s) \to D(T;J_1,\ldots,J_s) \to 0$$

is exact. If $(R;I_1,\ldots,I_s)$ satisfies the conditions (i)-(iii) of Definition 1.2, then the sequence

$$D_{gen}(R;I,I_1,\ldots,I_s) \to D_{gen}(R;I_1,\ldots,I_s) \to D_{gen}(T;J_1,\ldots,J_s) \to 0$$

is exact.

ii) if the map of s-cubes of rings $(R;I_1,\ldots,I_s)_* \to (T;J_1,\ldots,J_s)_*$ is split, then the map

$$D(R;I,I_1,\ldots,I_s) \to D(R;I_1,\ldots,I_s)$$

is injective. If in addition $(R;I_1,\ldots,I_s)$ satisfies the conditions (i)-(iii) of Definition 1.2, then the map

$$D_{gen}(R;I,I_1,\ldots,I_s) \to D_{gen}(R;I_1,\ldots,I_s)$$

is injective.

Proof. The proof of (i) is essentially the same as the proof of Lemma 1.1, and is left to the reader.

To prove (ii), let $s\colon (T;J_1,\ldots,J_s)_* \to (R;I_1,\ldots,I_s)_*$ be a splitting of the natural map $p\colon (R;I_1,\ldots,I_s)_* \to (T;J_1,\ldots,J_s)_*$. For $r \in R$, let r' denote $s(p(r))$. As in the proof of Lemma 1.1 we have the identity

$$<a,b> - <a',b'> = <a,(b-b')(1-ab')^{-1}> + <(a-a')(1-a'b')^{-1},b'>$$

in $D(R; I_1, \ldots, I_s)$. Set

$$\phi(a,b) = <a, (b-b')(1-ab')^{-1}> + <(a-a')(1-a'b')^{-1}, b'>,$$

considered as an element of $D(R; I, I_1, \ldots, I_s)$. One checks directly, using the relations (a)-(c) of Definition 1.1, that sending $<a,b>$ to $\phi(a,b)$ gives a well-defined homomorphism

$$\phi\colon D(R; I_1, \ldots, I_s) \to D(R; I, I_1, \ldots, I_s).$$

If $<a,b>$ is in the image of $D(R; I, I_1, \ldots, I_s)$, then either a' or b' is zero, in which case one easily checks that $\phi(<a,b>) = <a,b> \in D(R; I, I_1, \ldots, I_s)$, giving us the desired splitting to the map $D(R; I, I_1, \ldots, I_s) \to D(R; I_1, \ldots, I_s)$. This proves the first assertion of (ii).

For the generic case, start with an element $<a,b>$ of $D_{gen}(R; I_1, \ldots, I_s)$. Choosing a'' and b'' generically, with $a - a''$ and $b - b''$ generic elements of I, and with $<a, b''>$ and $<a'', b''>$ in $D_{gen}(R; I_1, \ldots, I_s)$, we may assume that the identity

$$<a,b> - <a'',b''> = <a, (b-b'')(1-ab'')^{-1}> + <(a-a'')(1-a''b'')^{-1}, b''>$$

holds in $D_{gen}(R; I_1, \ldots, I_s)$, and that the symbols $<a, (b-b'')(1-ab'')^{-1}>$ and $<(a-a'')(1-a''b'')^{-1}, b''>$ are elements of $D_{gen}(R; I, I_1, \ldots, I_s)$. Similarly, we may assume that the identity

$$<a'',b''> - <a',b'> = <a'', (b''-b')(1-a''b')^{-1}> + <(a''-a')(1-a'b')^{-1}, b'>$$

holds in $D_{gen}(R; I_1, \ldots, I_s)$, and that the symbols $<a'', (b''-b')(1-a''b')^{-1}>$ and $<(a''-a')(1-a'b')^{-1}, b'>$ are elements of $D_{gen}(R; I, I_1, \ldots, I_s)$. Let $\phi_{(a'',b'')}(a,b)$ be the element

$$<a, (b-b'')(1-ab'')^{-1}> + <(a-a'')(1-a''b'')^{-1}, b''> + <a'', (b''-b')(1-a''b')^{-1}> + <(a''-a')(1-a'b')^{-1}, b'>$$

of $D_{gen}(R; I, I_1, \ldots, I_s)$. One checks that, if (a^*, b^*) is another such choice, chosen sufficiently generic with respect to a, b, a', b', a'' and b'', we have $\phi_{(a'',b'')}(a,b) = \phi_{(a^*,b^*)}(a,b)$. This then implies that $\phi_{(a'',b'')}(a,b)$ is independent of the choice of (a'',b''), subject to the above conditions; we denote this element of $D_{gen}(R; I, I_1, \ldots, I_s)$ by $\phi_{gen}(a,b)$. One checks as above that sending $<a,b>$ to $\phi_{gen}(a,b)$ gives a well-defined homomorphism

$$\phi_{gen}\colon D_{gen}(R; I_1, \ldots, I_s) \to D_{gen}(R; I, I_1, \ldots, I_s),$$

splitting the map $D_{gen}(R; I, I_1, \ldots, I_s) \to D_{gen}(R; I_1, \ldots, I_s)$. This completes the proof of (ii). □

Let R be a ring, a and b elements of R with $1 - ab$ a unit in R. Following [Ke], we let $<a,b>^*$ be the element of of the Steinberg group $\mathrm{St}(R)$,

$$<a,b>^* = x_{12}^{b(1-ab)^{-1}} x_{12}^{a} x_{21}^{-b} x_{12}^{-(1-ab)^{-1}} h_{12}((1-ab)^{-1}).$$

Sending $<a,b>$ to $<a,b>^*$ defines a natural homomorphism $Dsym\colon D(R) \to K_2(R)$, which is an isomorphism in case R is semi-local (see [Ke]). Now suppose we have a ring R and ideals $I_1, \ldots, I_s$. Suppose further that the s-cube of rings $(R; I_1, \ldots, I_s)_*$ is split. Then, as the reader will easily verify, $K_p(R; I_1, \ldots, I_s)$ is just the intersection of the kernels of the maps

$$K_p(R) \to K_p(R/I_k);\ k = 1, \ldots, s.$$

By Lemma 1.2(ii), $D(R; I_1, \ldots, I_s)$ is the intersection of the kernels of the maps

$$D(R) \to D(R/I_k);\ \ k = 1, \ldots, s;$$

Thus the natural map $D(R) \to K_2(R)$ induces a natural map

$$Dsym\colon D(R; I_1, \ldots, I_s) \to K_2(R; I_1, \ldots, I_s),$$

which is an isomorphism if R is semi-local.

Lemma 1.3. *Let R be a ring with ideals $I_1, \ldots, I_s$ such that $\cap_{k=1}^s I_k$ is an almost radical ideal. Then there is a natural homomorphism*

$$Dsym\colon D(R; I_1, \ldots, I_s) \to K_2(R; I_1, \ldots, I_s)$$

agreeing with the previously defined homomorphism $Dsym$ in the split case. If R is semi-local, then $Dsym$ is injective.

Proof. Let T be the s-fold double $D_{(R; I_1, \ldots, I_s)}$; in particular, the s-cube $(T; D_1, \ldots, D_s)_*$ is split. By Lemma 1.2, we have the exact sequence

$$D(T; D_1, \ldots, D_s, D') \to D(T; D_1, \ldots, D_s) \to D(R; I_1, \ldots, I_s) \to 0. \tag{1.1}$$

We also have the commutative ladder of exact relativization sequences

$$\begin{array}{ccccc} K_2(T; D_1, \ldots, D_s, D') & \to & K_2(T; D_1, \ldots, D_s) & \to & K_2(R; I_1, \ldots, I_s) \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to K_2(T; D') & \to & K_2(T) & \to & K_2(R) \to 0; \end{array}$$

the splitting $s\colon R \to T$ to the projection $p\colon T \to R$ gives the exactness of the bottom line. Additionally, the map $K_2(T; D_1, \ldots, D_s) \to K_2(T)$ is injective. Thus, the image H of $K_2(T; D_1, \ldots, D_s, D')$ in $K_2(T; D_1, \ldots, D_s)$ is the subgroup of $K_2(T; D_1, \ldots, D_s)$ generated elements of the form $\eta - s_*(p_*(\eta))$, with $\eta \in K_2(T; D_1, \ldots, D_s)$.

For x in T, let $x' = s(p(x))$. Using Lemma 1.2, and arguing as above, the image of $D(T; D_1, \ldots, D_s, D')$ in $D(T; D_1, \ldots, D_s)$ is the subgroup of $D(T; D_1, \ldots, D_s)$ generated elements of the form $<a, b> - <a', b'>$, with $<a, b>$ in $D(T; D_1, \ldots, D_s)$. This shows that $Dsym$ sends the image of $D(T; D_1, \ldots, D_s, D')$ in $D(T; D_1, \ldots, D_s)$ into H. This suffices to descend the map

$$Dsym\colon D(T; D_1, \ldots, D_s) \to K_2(T; D_1, \ldots, D_s)$$

to the desired map

$$Dsym\colon D(R; I_1, \ldots, I_s) \to K_2(R; I_1, \ldots, I_s).$$

If R is semi-local, then so is T, hence the map

$$Dsym\colon D(T; D_1, \ldots, D_s) \to K_2(T; D_1, \ldots, D_s)$$

is an isomorphism. From this, it follows that the image of $K_2(T; D_1, \ldots, D_s, D')$ in $K_2(T; D_1, \ldots, D_s)$ is generated by elements of the form $Dsym(<a, b> - <a', b'>)$, with $<a, b>$ in $D(T; D_1, \ldots, D_s)$. Thus the image of $D(T; D_1, \ldots, D_s, D')$ in $D(T; D_1, \ldots, D_s)$ is mapped by $Dsym$ onto the image of $K_2(T; D_1, \ldots, D_s, D')$ in $K_2(T; D_1, \ldots, D_s)$. The injectivity of $Dsym\colon D(R; I_1, \ldots, I_s) \to K_2(R; I_1, \ldots, I_s)$ follows directly from this. □

Theorem 1.4. *Let R be a semi-local ring, $I_1, \dots, I_s$ ideals defining a normal crossing divisor. Then*

$$Dsym\colon D(R; I_1, \dots, I_s) \to K_2(R; I_1, \dots, I_s)$$

is an isomorphism.

Proof. By Lemma 1.3, we need only show that $Dsym$ is surjective. We have the commutative diagram

$$\begin{array}{ccc} D(T; D_1, \dots, D_s) & \to & D(R; I_1, \dots, I_s) \\ Dsym \downarrow & & \downarrow Dsym \\ K_2(T; D_1, \dots, D_s) & \to & K_2(R; I_1, \dots, I_s), \end{array}$$

with $Dsym\colon D(T; D_1, \dots, D_s) \to K_2(T; D_1, \dots, D_s)$ an isomorphism. Thus, it suffices to show that the map $K_2(T; D_1, \dots, D_s) \to K_2(R; I_1, \dots, I_s)$ is surjective.

From ([L2], Theorem 4.5), there is a natural isomorphism

$$K_2^M(R; I_1, \dots, I_s) \to K_2(R; I_1, \dots, I_s);$$

we have as well the natural map □

$$K_2^M(T; D_1, \dots, D_s) \to K_2(T; D_1, \dots, D_s),$$

and a commutative diagram

$$\begin{array}{ccc} K_2^M(T) & \to & K_2(T) \\ \downarrow & & \downarrow \\ K_2^M(T; D_1, \dots, D_s) & \to & K_2(T; D_1, \dots, D_s) \\ \downarrow & & \downarrow \\ K_2^M(R; I_1, \dots, I_s) & \to & K_2(R; I_1, \dots, I_s). \end{array}$$

Since the map

$$K_2^M(T) \to K_2^M(R; I_1, \dots, I_s)$$

is surjective by the definition of $K_2^M(R; I_1, \dots, I_s)$, it follows that the natural map

$$K_2(T; D_1, \dots, D_s) \to K_2(R; I_1, \dots, I_s)$$

is surjective, as desired. This completes the proof. □

We now consider the groups $D_{gen}(R; I_1, \dots, I_s)$. Suppose $I_1 \cap \dots \cap I_s$ is an almost radical ideal, and $(R; I_1, \dots, I_s)$ satisfies (i)-(iii) of Def. 1.2. We have the natural map $D_{gen}(R; I_1, \dots, I_s) \to D(R; I_1, \dots, I_s)$; following this by the map $Dsym\colon D(R; I_1, \dots, I_s) \to K_2(R; I_1, \dots, I_s)$ gives a commutative triangle

$$\begin{array}{ccc} D_{gen}(R; I_1, \dots, I_s) & \longrightarrow & D(R; I_1, \dots, I_s) \\ Dsym_{gen} \searrow & & \nearrow Dsym \\ & K_2(R; I_1, \dots, I_s) & \end{array}$$

In case $s = 0$ or 1, van der Kallen ([vdK] and [vdK2]) has shown that the map

$$Dsym_{gen}: D_{gen}(R; I_1, \ldots, I_s) \to K_2(R; I_1, \ldots, I_s)$$

is an isomorphism (actually, he uses a different group, say $D_{gen}(R; I_1, \ldots, I_s)^*$, but there is an obvious surjective map

$$D_{gen}(R; I_1, \ldots, I_s)^* \to D_{gen}(R; I_1, \ldots, I_s),$$

which followed by the map

$$D_{gen}(R; I_1, \ldots, I_s) \to K_2(R; I_1, \ldots, I_s)$$

is an isomorphism).

Theorem 1.5. *Let R be a semi-local ring, $I_1, \ldots, I_s$ ideals defining a normal crossing divisor. Then the map*

$$D_{gen}(R; I_1, \ldots, I_s) \to K_2(R; I_1, \ldots, I_s)$$

is an isomorphism.

Proof. We need only show that the map $\phi: D_{gen}(R; I_1, \ldots, I_s) \to D(R; I_1, \ldots, I_s)$ is an isomorphism. To see that ϕ is surjective, let $<a, b>$ be in $D(R; I_1, \ldots, I_s)$; we may suppose that $a \in \cap_{k=1}^t I_k$ and $b \in \cap_{k=t+1}^s I_k$. Since R contains an infinite field, we may find a generic c in $\cap_{k=t+1}^s I_k$ such that $b + c(1 - ab)$ is a generic element of $\cap_{k=t+1}^s I_k$. Since

$$<a, b> + <a, c> = <a, b + c(1 - ab)>,$$

we may assume that b is a generic element of $\cap_{k=t+1}^s I_k$. Using Def. 1.1(a), we can repeat this argument for a, showing the desired surjectivity.

For the injectivity, we note that the s-fold double $T := (D_{(R;I_1,\ldots,I_s)}; D_1, \ldots, D_s)$ satisfies conditions (i), (ii) and (iii). Since all the relevant maps are split, we may apply Lemma 1.2 as above to conclude that $D_{gen}(T; D_1, \ldots, D_s)$ is the intersection of the kernels of the maps $D_{gen}(T) \to D_{gen}(T/D_k)$, for $k = 1, \ldots, s$. Using van der Kallen's result [vdK], the maps $D_{gen}(T) \to D(T)$ and $D_{gen}(T/D_k) \to D(T/D_k)$ are all isomorphisms, so the map

$$D_{gen}(T; D_1, \ldots, D_s) \to D(T; D_1, \ldots, D_s).$$

is an isomorphism as well.

We have the commutative ladder with exact rows

$$\begin{array}{ccccccc}
D_{gen}(D_{(R;I_1,\ldots,I_s)}, D_1, \ldots, D_s, D') & \to & D_{gen}(D_{(R;I_1,\ldots,I_s)}, D_1, \ldots, D_s) & \to & D_{gen}(R; I_1, \ldots, I_s) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
D(D_{(R;I_1,\ldots,I_s)}, D_1, \ldots, D_s, D') & \to & D(D_{(R;I_1,\ldots,I_s)}, D_1, \ldots, D_s) & \to & D(R; I_1, \ldots, I_s) & \to & 0
\end{array}$$

Since all the vertical arrows are surjective, and the middle arrow is an isomorphism, the right hand arrow is an isomorphism, as desired. □

Using this result, we can give a description of the relative K_2 for certain semi-local rings and ideals, which looks very much like the Matsumoto presentation for K_2 of fields. This construction goes back to some ideas of Bloch [Bl] and Weibel [W], for semi-local PID's. Here, we suppose that R is semi-local, containing an infinite field, that the ideals $I_1, \ldots, I_s$ satisfy the conditions (i)-(iii) above, and in addition

iv) R is a domain.

By assumption (ii) the intersection $I_1 \cap \ldots \cap I_s$ is principal, say

$$I_1 \cap \ldots \cap I_s = (t).$$

Let $Symb_2(R; I_1, \ldots, I_s)$ be the group defined by

$$Symb_2(R; I_1, \ldots, I_s) = (1 + (t))^\times \otimes_{\mathbb{Z}} R[t^{-1}]^\times / \mathcal{R}$$

where $\mathcal{R}$ is the subgroup generated by tensors of the form $f \otimes (1-f)$, with $f \in (1+(t))^\times$ and $1-f \in R[t^{-1}]^\times$. We write as per usual the equivalence class of a tensor $a \otimes b$ as the symbol $\{a, b\}$.

We map $(1 + (t))^\times \otimes_{\mathbb{Z}} R[t^{-1}]^\times$ to $D(R; I_1, \ldots, I_s)$ by

$$(1 - at) \otimes u\Pi t_i^{n_i} \to <u^{-1}at, u> + \Sigma n_i <at/t_i, t_i>,$$

where u is a unit in R and $a \in R$. Since R is a domain, the division t/t_i is well-defined. The bilinearity of this mapping is an easy consequence of the relations (b) and (c).

The relations $\mathcal{R}$ are generated by tensors of the form $(1 - u\Pi t_i^{n_i}) \otimes u\Pi t_i^{n_i}$, with each n_i positive, $i = 1, \ldots, s$, and u a unit in R. Repeated application of relation (c) shows that the image of such a tensor is

$$<\Pi t_i^{n_i}, u> + \sum_{j=1}^{s} n_j <u\Pi t_i^{n_i}/t_j, t_j> = <\Pi t_i^{n_i}, u> + \sum_{j=1}^{s} <u \prod_{i \neq j} t_i^{n_i}, t_j^{n_j}>.$$

Let $s_i = t_i^{n_i}$. Using relation (c), we have

$$<u \prod_{i=1}^{j-1} s_i, \prod_{i=j}^{s} s_i> = <u \prod_{i=1}^{j} s_i, \prod_{i=j+1}^{s} s_i> + <u \prod_{i \neq j} s_i, s_j>; \text{ for } j = 2, \ldots, s,$$

and

$$<u \prod_{i=2}^{s} s_i, s_1> = <u, \prod_{i=1}^{s} s_i> - <us_1, \prod_{i=2}^{s} s_i>.$$

Putting these together, we find

$$<u \prod_{i=2}^{s} s_i, s_1> = <u, \prod_{i=1}^{s} s_i> - \sum_{j=2}^{s} <u \prod_{i \neq j} s_i, s_j>,$$

hence

$$<\Pi t_i^{n_i}, u> + \sum_{j=1}^{s} n_j <ut_i^{n_i}/t_j, t_j> = <\Pi t_i^{n_i}, u> + <u, \Pi t_i^{n_i}> = 0.$$

Thus we get a homomorphism

$$\tau\colon Symb_2(R; I_1, \ldots, I_s) \to D(R; I_1, \ldots, I_s).$$

We define a homomorphism

$$\phi\colon D_{gen}(R; I_1, \ldots, I_s) \to Symb_2(R; I_1, \ldots, I_s)$$

by $\phi(<a,b>) = \{1-ab, b\}$. This is well-defined, as (a,b) generic implies $1-ab$ is in $(1+(t))^{\times}$, and a and b are units in $R[t^{-1}]$. In addition, $<b,a>$ goes to

$$\{1-ab, a\} = \{1-ab, b^{-1}\} = -\{1-ab, b\};$$

the relations (b) and (c) are checked similarly. As the tensor product $(1+(t))^{\times} \otimes_{\mathbb{Z}} R[t^{-1}]^{\times}$ is generated by tensors $(1-ut) \otimes vt_i$, where u and v are units in R, we see this latter map is surjective. We have the composition

$$Symb_2(R; I_1, \ldots, I_s) \to D(R; I_1, \ldots, I_s) \cong D_{gen}(R; I_1, \ldots, I_s) \to Symb_2(R; I_1, \ldots, I_s).$$

One sees by direct computation that $\tau(\phi(<a,b>)) = <a,b>$, for $<a,b> \in D_{gen}(R; I_1, \ldots, I_s)$, and that

$$\tau(\{1-ut, vt_i\}) = <v^{-1}ut, v> + <ut/t_i, t_i>,$$

which is a sum of generic symbols. As

$$\phi(<v^{-1}ut, v> + <ut/t_i, t_i>) = \{1-ut, vt_i\}$$

we find that ϕ and τ are isomorphisms. We have thus shown

Theorem 1.6. *Let R be a semi-local domain, $I_1, \ldots, I_s$ ideals defining a normal corssing divisor. Then there is a natural isomorphism $Symb_2(R; I_1, \ldots, I_s) \to K_2(R; I_1, \ldots, I_s)$.*

□

§2. Norms for relative K-theory

If R is a ring, $I_1, \ldots, I_s$ ideals and S a subset of $\{1, \ldots, s\}$, we let I_S denote the sum $\sum_{j \in S} I_j$ and I^S the intersection $\cap_{j \in S} I_j$. If A is a ring (not necessarily unital), let $\tilde{A}$ be the unital extension $\mathbb{Z} \oplus A$, and let $\Delta(A)$ denote the simplicial ring with

$$\Delta_n(A) = A[\tilde{A}[t_0, \ldots, t_n]/\Sigma t_i - 1];$$

The group $GL(A)$ is defined as the kernel of the natural homomorphism

$$GL(\tilde{A}) \to GL(\mathbb{Z}).$$

The Karoubi-Villamayor K-groups of A, $KV_*(A)$, are defined as the homotopy groups of the simplicial group $GL(\Delta(A))$ (see [K-V])

$$KV_p(A) = \pi_{p-1}(GL(\Delta(A))) = \pi_p(BGL(\Delta(A))); \; p \geq 1$$

Considering A as a constant simplicial ring, we get a functorial map $BGL(A) \to BGL(\Delta(A))$, which extends (uniquely up to homotopy) to a map $BGL(A)^+ \to BGL(\Delta(A))$. If we do this for $A = \mathbb{Z}$, and use the model $BGL(A) \cup_{BGL(\mathbb{Z})} BGL(\mathbb{Z})^+$ for $BGL(A)^+$, we obtain a functorial map $BGL(A)^+ \to BGL(\Delta(A))$ for unital A. This then induces functorial maps on homotopy fibers, and we get a natural map for the relative groups

$$K_p(A; I_1, \ldots, I_s) \to KV_p(A; I_1, \ldots, I_s).$$

On the other hand, if I is an ideal of A, there is a natural isomomorphism (see [W2]) $KV_*(I) \to KV_*(A; I)$. By induction, this give a natural isomorphism

$$KV_*(I) \to KV_*(A; I_1, \ldots, I_s); \; I = \cap I_j$$

Lemma 2.1. *Let A be a (unital) ring, $I_1, \ldots, I_s$ ideals. Suppose that, for each subset S of $\{1, \ldots, s\}$, the ring A/I_S is regular. Let $I = \cap I_j$. Then the map*

$$K_p(A; I_1, \ldots, I_s) \to KV_p(I)$$

is an isomorphism for $p \geq 1$.

Proof. Quillen has shown that the map $K_p(R) \to KV_p(R)$ is an isomorphism for regular unital rings R. The result is then a consequence of the long exact relativization sequence for K-theory and KV-theory, and the five lemma. □

From now on, all rings will be assumed unital; we will consider ideals as non-unital rings.

Lemma 2.2. *Let A be a ring, A' a finite extension of A, free as an A module, I an ideal of A. Let I' be an ideal of A' with*

$$\mathrm{rad}(I') \subset \mathrm{rad}(IA').$$

Then there is a natural norm homomorphism

$$Nm_{I'/I} \colon KV_*(I') \to KV_*(I)$$

satisfying

i) Given a tower $A \to A' \to A''$, $I \to I' \to I''$ of such rings and ideals, we have

$$Nm_{I'/I} \circ Nm_{I''/I'} = Nm_{I''/I}.$$

ii) The composition

$$KV_*(I) \to KV_*(I') \stackrel{Nm_{I'/I}}{\longrightarrow} KV_*(I)$$

is multiplication by $\dim_A(A')$.

iii) Suppose $I = I_1 \cap I_2$, $I' = I'_1 \cap I'_2$, where $I'_j \subset I_j$, and the pairs $(I_1 \subset A, I'_1 \subset A')$ and $(I/I_2 \subset A/I_2, I'/I'_2 \subset A'/I'_2)$ satisfy the hypotheses of the Lemma. Then we have a commutative ladder

$$\begin{array}{ccccccccc} \ldots \to KV_{p+1}(I'/I'_2) & \to & KV_p\,(I') & \to & KV_p(I'_1) & \to & KV_p(I'/I'_2) & \to \ldots \\ Nm_{(I'/I'_2)/(I/I_2)}\downarrow & & Nm_{I'/I}\downarrow & & Nm_{I'_1/I_1}\downarrow & & \downarrow Nm_{(I'/I'_2)/(I/I_2)} & \\ \ldots \to KV_{p+1}(I/I_2) & \to & KV_p\,(I) & \to & KV_p(I_1) & \to & KV_p(I/I_2) & \to \ldots \end{array}$$

Proof. We first suppose $I' \subset IA'$. Choosing a basis $e_1, \ldots, e_n$ for A' as an A-module defines a map of simplicial groups

$$\rho: GL(\Delta(I')) \to GL(\Delta(I))$$

which defines the norm homomorphism. If $f_1, \ldots, f_n$ is another choice of basis, let P be the change of basis matrix. Then there is an element $F(t)$ in the group $E(A) * <t>$ with

$$P\rho(T)P^{-1} = F(\rho(T)); \quad \text{for } T \in GL(A'[X_1, \ldots, X_n]),$$

and with

$$\bar{F}(t) = t,$$

where $\bar{F}(t)$ denotes the image of $F(t)$ under the homomorphism $E(A) \to id, t \to t$ (see, for example, Milnor [M]). Since we can find, for each element $E \in E(A)$, an element $E(X) \in E(A[X])$ with

$$E(0) = I, E(1) = E,$$

this shows that the maps

$$\rho: GL(\Delta(I')) \to GL(\Delta(I)),$$

and

$$P\rho P^{-1}: GL(\Delta(I')) \to GL(\Delta(I))$$

are homotopic, hence the norm homomorphism is independent of the choice of basis. Arguing as in the unital case, one shows that the norm homomorphism satisfies (i) and (ii), and is natural. The assertion (iii) follows from the naturality of Nm, and the naturality of the homotopy equivalences

$$GL(\Delta(I)) \to \mathrm{F}\big((GL(\Delta(I_1)) \to GL(\Delta(I_1/I))\big)$$

and

$$GL(\Delta(I')) \to \mathrm{F}\big(GL(\Delta(I'_1)) \to GL(\Delta(I'_1/I))\big),$$

where "F" denotes homotopy fiber.

In general, Weibel [W2] has shown the inclusion $I \to \mathrm{rad}(I)$ induces an isomorphism $KV_*(I) \to KV_*(\mathrm{rad}(I))$; using this isomorphism, we reduce to the case discussed above. □

I am indebted to C. Weibel for the proof of Lemma 2.2, replacing an earlier, incorrect argument.

Proposition 2.3. *Let $i: A \to A'$ be a finite extension of rings, I an ideal of A and I' an ideal of A'. we suppose*

a) A' is free over A.
b) $\mathrm{rad}(I') \subset \mathrm{rad}(IA')$.
c) we can write I and I' as intersections such that, for each subset S of $\{1, \ldots, n\}$ and S' of $\{1, \ldots, m\}$, the rings A/I_S and $A'/I'_{S'}$ are regular:

$$I = \cap_{j=1}^{n} I_j \quad \text{and} \quad I' = \cap_{j=1}^{n} I'_j$$

Then there is a natural norm homomorphism

$$Nm_{(A';I'_1,\ldots,I'_m)/(A;I_1,\ldots,I_n)}: K_p(A';I'_1,\ldots,I'_m) \to K_p(A';I'_1,\ldots,I_n), \quad \text{for } p \geq 1$$

such that

i) $i_* \circ Nm_{(A';I'_1,\ldots,I'_m)/(A;I_1,\ldots,I_n)} = \dim_A(A') \cdot id$.
ii) $Nm_{(A';I'_1,\ldots,I'_m)/(A;I_1,\ldots,I_n)} \circ Nm_{(A'';I''_1,\ldots,I''_k)/(A';I'_1,\ldots,I'_m)} = Nm_{(A'';I''_1,\ldots,I''_k)/(A;I_1,\ldots,I_n)}$
iii) The norm homomorphism commutes with the boundary maps in the relevant long exact relativization sequences.

Proof. The construction of the norm homomorphism satisfying properties (i), (ii) and (iii) follows from the previous two lemmas. □

We fix a regular semi-local ring R, principal ideals $I_1 = (t_1), \ldots, I_s = (t_s)$, and a prime l. We assume that R contains an infinite field k_0 containing a primitive l^{th} root of unity ζ. We additionally assume that $t_1, \ldots, t_s$ form a regular sequence in R (i.e., for each maximal ideal m of R, the t_j that are in m form a regular sequence in the local ring R_m), and that each ring R/I_S is smooth over k_0. For a subset S of $\{1, \ldots, s\}$, we let t^S denote the product:

$$t^S = \prod_{j \in S} t_j,$$

so $I^S = (t^S)$.

If we have a semi-local smooth k_0-algebra T, and an element $s \in T$, we can factor s as

$$s = u \prod_{k=1}^{n} s_k^{e_k}$$

with u a unit and the s_k prime. Let $J_k = (s_k)$. Suppose $s_1, \ldots, s_n$ forms a regular sequence in T, and the rings T/J_S are all smooth over k_0. Then we say that *the reduced divisor of s is a normal crossing divisor.* If all the e_k are 1, we say that*the divisor of s in T is a normal crossing divisor.*

Let a be a unit in T, and let α be an l^{th} root of a if $l \neq \mathrm{char}(k_0)$, or a root of $X^l - X - a$ if $l = \mathrm{char}(k_0)$. We let T_a denote the ring $T[\alpha]$. T_a is the a regular semi-local ring, étale over T; let σ_a, (or simply σ) be a generator of the cyclic group $\mathrm{Gal}(T_a/T)$. Let F denote the total quotient field of R.

Lemma 2.4. *The group $(1 + tR_a)^\times$ is generated by elements of the form*

$$1 + \sum_{i=0}^{l-1} x_i \alpha^i \ ; \quad x_i \in tR,$$

such that

i) *there is a finite extension $R \to S$ of degree prime to l so that the polynomial $P(T)$,*

$$P(T) = 1 + \sum_{i=0}^{l-1} x_i T^i$$

has a linear factor $yT - 1$ in $S[T]$, with $y \in (1 + \mathrm{rad}((t)S))^\times$.

ii) *S is smooth over k_0.*

iii) *the reduced divisor of t in S is a normal crossing divisor*

iv) *the divisor of y in S a normal crossing divisor.*

If $\mathrm{char}(k_0) = 0$, we can find an S satisfying (i)-(iv) such that $P(T)$ splits completely in S.

Proof. We first assume that $\mathrm{char}(k_0) = 0$. Let $a_0, \ldots, a_{l-1}$ be in R. Take units b_i and c_i in R, with

$$1 + \sum_{i=0}^{l-1} ta_i\alpha^i = (1 + \sum_{i=0}^{l-1} tb_i\alpha^i)(1 + \sum_{i=0}^{l-1} tc_i\alpha^i)$$

For a general choice of the b_i, the normalization of R in $F[T]/(1 + \sum_{i=0}^{l-1} Tb_i\alpha^i)$ is unramified over R outside the locus $t = 0$, and similarly for a general choice of c_i. Thus we may assume, without loss of generality, that the normalization S_1 of R in $F[T]/(1 + \sum_{i=0}^{l-1} Ta_i\alpha^i)$ is unramified over R outside the locus $t = 0$. Let R' be the extension of R gotten by adjoining all q^{th} roots of unity, and the q^{th} roots of $t_1, \ldots, t_s$ for $q = 1, \ldots, l-1$. Then R' is a finite, prime to l, Galois extension of R and satisfies (ii) and (iii) of the lemma. In addition, letting S' be the normalization of $S_1 \otimes_R R'$, S' is unramified over R' at all codimension one points of $\mathrm{Spec}(R')$, hence, by Zariski's theorem on the purity of the branch locus, S' is étale over R'. Thus, S' satisfies the conditions of the lemma, completing the proof in this case.

We now consider the case $\mathrm{char}(k_0) > 0$. Call an element

$$1 + \sum_{i=0}^{l-1} ta_i\alpha^i; \quad a_i \in R,$$

of $(1 + tR_a)^\times$ *good* if the polynomial

$$P(T) = 1 + \sum_{i=0}^{l-1} ta_i T^i$$

has a linear factor $yT - 1$ in an extension S of R satisfying the conditions of the lemma. We proceed by induction on s.

Let $n = l - 1$. As above, we may assume that $1 + ta_0$ and a_n are units in R. We note that the ring $S := R[X]/X^n P(X^{-1})$ is then finite over R. If $s = 1$, $R[X]/X^n P(X^{-1})$ is also smooth over k_0, and étale over $R[t^{-1}]$. Let y be the image of X in S; one sees easily that $t = uy^n$ for some unit $u \in S$. This shows that the extension $R \to S$ is totally ramified over (t), and that $\mathrm{rad}((t)S) = (y)$. Putting all this together proves the lemma in the case $s = 1$.

We now take $s > 1$. Arguing as in the characteristic zero case, we see that an element of the form $1 + tu_0 + tu_i\alpha^i$; $i \le n$, is good if u_0 and u_i are suitably general units in R. Let $w = \prod_{i=2}^s t_i$, and let $v = t_1$. For suitably general $a_0, \ldots, a_n$ in R, the normalization S_1 of R in $F[T]/(1 + \sum_{i=0}^n wv^i a_i T^i)$ is unramified outside the locus $w = 0$, and the normalization S_2 of R in $F[T]/(1 + \sum_{i=0}^n w^i va_i T^i)$ is similarly unramified outside the locus $v = 0$. Thus, by an induction on s, elements of the form $1 + \sum_{i=0}^n wv^i a_i\alpha^i$ or

$1+\sum_{i=0}^n w^i v a_i \alpha^i$ are products of good elements. On the other hand, for suitably general $a_0, \ldots, a_n$, we can find good elements $z_i = 1 + tr_i + tu_i\alpha^i$ such that

$$(1+\sum_{i=0}^n ta_i\alpha^i)\cdot z_1 \cdot \ldots \cdot z_n = 1 + \sum_{i=0}^n tb_i\alpha^i,$$

with b_i in the ideal $(v^{i-1}, w^{i-1})R$, for $i = 1, \ldots, n$. We can then write the element $1+\sum_{i=0}^n tb_i\alpha^i$ as a product

$$1+\sum_{i=0}^n tb_i\alpha^i = (1+\sum_{i=0}^n w^i va_i\alpha^i)(1+\sum_{i=0}^n wv^i c_i\alpha^i),$$

completing the proof. □

Let x be an element of a semi-local ring T, smooth over k_0, such that the divisor of x is a normal crossing divisor. Factor (x) into primes as $\Pi_{j=1}^s x_j$, and let $I_j = (x_j)$. We may apply Theorem 1.6 to get a natural isomorphism

$$Symb_2(T; I_1, \ldots, I_s) \to K_2(T; I_1, \ldots, I_s)).$$

Note that $Symb_2(T; I_1, \ldots, I_s)$ only depends on $(x)T$; to simplify the notation, we will occasionally write $Symb_2(T;(x)T)$ for $Symb_2(T; I_1, \ldots, I_s)$. If T is an extension of R, and a is an element of T such that $Symb_2(T_a; \mathrm{rad}((t)T_a))$ is defined, we define the subgroup $Symb_2(T_a; \mathrm{rad}((t)T_a))^*$ of $Symb_2(T_a; \mathrm{rad}((t)T_a))$ to be the subgroup generated by symbols $\{a, b\}$ with $a \in (1 + \mathrm{rad}((t)T))^\times$ or with b a unit in $T[t^{-1}]$.

Corollary 2.5. *Let T be a finite extension of R, smooth over k_0, such that the reduced divisor of t on T is a normal crossing divisor. Let u be an element of $Symb_2(T_a; \mathrm{rad}((t)T_a))$. Then there is a finite, prime to l extension $f: T \to T'$ such that*

- *i) the reduced divisor of t on T' is a normal crossing divisor.*
- *ii) $f^*(u)$ is in $Symb_2(T'_a; \mathrm{rad}((t)T'_a))^*$*
- *iii) T' smooth over k_0*

Proof. The proof is essentially the same as the proof of Lemma 1.4 of [MS2], with the aid of our Lemma 2.4 above. □

§3. Quillen spectral sequence for relative K-theory

In this section we extend some of the results of [L] for relative K-theory of schemes over semi-local PID's to the setting of schemes over arbitrary semi-local rings. Most of the results carry over with minor changes, so we will be a little sketchy. We fix a semi-local ring R, and principal ideals $I_1 = (t_1), \ldots, I_s = (t_s)$. Let $t = \prod_i t_i$. We assume that the t_j are relatively prime, and that the divisor of t is a normal crossing divisor.

Let $f\colon X \to \mathrm{Spec}(R)$ be an R-scheme. We denote the X_j the subscheme $f^{-1}(\mathrm{Spec}(R/I_j))$; similarly, if S is a subset of $\{1, \ldots, s\}$, we let X_S and X^S denote the subschemes $f^{-1}(\mathrm{Spec}(R/I_S))$ and $f^{-1}(\mathrm{Spec}(R/I^S))$, resp. We let $\underline{X}$ denote the union of the fibers of f over the closed points of $\mathrm{Spec}(R)$. We let $\mathcal{M}_X$ be the category of coherent sheaves on X; $\mathcal{M}_{X/R}$ will be the full subcategory of $\mathcal{M}_X$ consisting of sheaves with support contained in some subscheme of X which is flat over R. The subcategory of $\mathcal{M}_X$ consisting of sheaves supported in codimension d is written $\mathcal{M}_X^d$ and the quotient category $\mathcal{M}_X^d/\mathcal{M}_X^{d'}$ is denoted $\mathcal{M}_X^{d/d'}$. Similarly, the subcategory of $\mathcal{M}_{X/R}$ consisting of sheaves with support contained in a codimension d subscheme of X which is flat over R is denoted $\mathcal{M}_{X/R}^d$.

Suppose $f\colon X \to \mathrm{Spec}(R)$ is flat. We call a closed irreducible subset W of X *geometrically flat over* R W intersects each fiber of f properly, i.e., for each (not necessarily closed) point p of $\mathrm{Spec}R$, and each irreducible component W' of $W \cap f^{-1}(p)$, we have

$$\mathrm{codim}_X(W) = \mathrm{codim}_{f^{-1}(p)}(W').$$

In general, we will call a closed subset of X geometrically flat over R if each irreducible component is so. We sometimes omit the reference to R if the context makes our meaning clear, refering for example to a flat subscheme of X, rather than a subscheme of X, flat over R, or a geometrically flat subset of X, rather than a subset of X, geometrically flat over R.

Suppose X Cohen-Macaulay, and is flat and quasi-projective over R, and let W be geometrically flat over R, and of pure codimension d on X. Then there are codimension one closed subschemes containing W, $H_1, \ldots, H_d$ of X, such that each intersection

$$H_{i_1} \cap \ldots \cap H_{i_s}$$

is reduced, of codimension s on X, and is flat over R. In particular, W is contained in the flat, local complete intersection, codimension d subscheme $H_1 \cap \ldots \cap H_d$. From this it follows that, for X Cohen-Macaulay, flat and quasi-projective over R,

i) a closed subset Z of X is contained in a codimension d flat closed subscheme of X, if and only if Z is contained in a codimension d geometrically flat closed subset of X.

ii) if Z_1 and Z_2 are closed subsets, with $Z_i \subset Y_i$, and with Y_i a flat codimension d closed subscheme X, $i = 1, 2$, then $Z_1 \cup Z_2$ is contained in a codimension d flat closed subscheme of X.

iii) Suppose W is a codimension d closed subset of X, geometrically flat over R. If Z is a closed subset of $X \backslash W$, contained in a codimension d' flat closed subscheme of $X \backslash W$, with $d' \leq d$, then the closure $\bar{Z}$ of Z in X is contained in a a codimension d' flat closed subscheme of X.

We let $\mathrm{Fl}^d(X/R)$ denote the set of codimension d closed subsets of X which are geometrically flat over R, ordered by inclusion. $X^{(d)}$ will denote the set of codimension d points of X, and $(X/R)^{(d)}$ the set of codimension d points x with closure $\bar{x}$ geometrically flat over R.

Suppose X is Cohen-Macaulay, flat and quasi-projective over R. From (i)-(iii) above, it follows that the categories $\mathcal{M}_{X/R}^d$ are abelian for all d, and $\mathcal{M}_{X/R}^d$ is a Serre subcategory of $\mathcal{M}_{X/R}^{d'}$ for all $d' \leq d$. It follows from (i)-(iii) above that, for $W \in \mathrm{Fl}^d(X/R)$, and $d' \leq d$, the restriction map

$$\mathcal{M}_{X/R}^{d'} \to \mathcal{M}_{(X\backslash W)/R}^{d'}$$

is surjective on isomorphism classes of objects. Thus, the standard arguments show that the quotient category $\mathcal{M}^{d'}_{X/R}/\mathcal{M}^{d}_{X/R}$ is equivalent to to the direct limit category

$$\lim_{\substack{\to \\ W\in \mathrm{Fl}^d(X/R)}} \mathcal{M}^{d'}_{X\setminus W/R}.$$

We denote $\mathcal{M}^{d'}_{X/R}/\mathcal{M}^{d}_{X/R}$ by $\mathcal{M}^{d'/d}_{X/R}$.

The following properties are simple consequences of the work of Quillen [Q], and their proof is a straightforward modification of the proofs of the analogous results in §1 of [L]:

a) If X is flat over R, then $\mathcal{M}_{X/R} = \mathcal{M}_X$.

b) Suppose X is Cohen-Macaulay, flat and quasi-projective over R. Then we have E_1-spectral sequences $E(X)$ and $E(X/R)$:

$$E_1^{p,q}(X) = K_{-p-q}(\mathcal{M}_X^{p/p+1}) \Longrightarrow K_{-p-q}(\mathcal{M}_X)$$

$$E_1^{p,q}(X/R) = K_{-p-q}(\mathcal{M}_{X/R}^{p/p+1}) \Longrightarrow K_{-p-q}(\mathcal{M}_{X/R}) = K_{-p-q}(\mathcal{M}_X),$$

and a map of spectral sequences $E(X/R) \to E(X)$.

We write $G_p(X)^{d/d'}$ for $K_p(\mathcal{M}_X^{d/d'})$, and $G_p(X/R)^{d/d'}$ for $K_p(\mathcal{M}_{X/R}^{d/d'})$. Let $\mathcal{M}^d_{X:R}$ denote the full subcategory of $\mathcal{M}^d_{X/R}$ consisting of sheaves flat over R. If X is Cohen-Macaulay, and flat and quasi-projective over R, each sheaf $\mathcal{F}$ in $\mathcal{M}^d_{X/R}$ admits a surjection

$$\mathcal{O}_Y(-n)^m \to \mathcal{F},$$

where Y is a codimension d closed subscheme of X, flat over R. If in addition, R is regular, then $\mathcal{F}$ admits a finite resolution by sheaves in $\mathcal{M}^d_{X:R}$, hence we may apply Quillen's resolution theorem [Q] to show

c) If R is regular, and X is Cohen-Macaulay, and flat and quasi-projective over R, then the inclusion

$$\mathcal{M}^d_{X:R} \to \mathcal{M}^d_{X/R}$$

induces an isomorphism $K_p(\mathcal{M}^d_{X:R}) \to K_p(\mathcal{M}^d_{X/R})$.

Since, for $S \subset S'$, the restriction map $j^*_{S/S'}\colon \mathcal{M}^d_{X_S:R/I_S} \to \mathcal{M}^d_{X_{S'}:R/I_{S'}}$ is exact, we can apply the technique of [L], §1.1, to form the space $G(X/R; X_1, \ldots, X_s)^d$ as the iterated homotopy fiber over the s-dimensional cube of maps

$$\mathcal{M'}^d_{X_S:R/I_S} \to \mathcal{M'}^d_{X_{S'}:R/X_{S'}};\quad \text{for } S \subset S' \subset \{1, \ldots, s\}.$$

Here $\mathcal{M'}^d_{X_S:R/I_S}$ is the category consisting of objects $\mathcal{F}_{S''}$ of $\mathcal{M}^d_{X_{S''}:R/X_{S''}}$ for each $S'' \supset S$, together with a choice of isomorphism $f_{S''}\colon j^*_{S/S''}(\mathcal{F}_S) \to \mathcal{F}_{S''}$. This is a technical device to make the restriction maps $j^*_{S/S'}$ functorial.

We define the groups $G_p(X/R; X_1, \ldots, X_s)^d$ and $G_p(X/R; X_1, \ldots, X_s)^{d/d'}$ by

$$G_p(X/R; X_1, \ldots . X_s)^d = \pi_p(G(X/R; X_1, \ldots, X_s)^d)$$

$$G_p(X/R; X_1, \ldots, X_s)^{d/d'} = \varinjlim_{Z \in \mathrm{Fl}^{d'}} G_p((X \setminus Z)/R; X_1 \setminus Z, \ldots, X_s \setminus Z)^d.$$

For Z a closed subset of X, we have the full subcategory $\mathcal{M}^{Zd}_{X:R}$ of $\mathcal{M}^{d}_{X:R}$ with objects those sheaves supported in Z; replacing $\mathcal{M}^{d}_{X:R}$ with $\mathcal{M}^{Zd}_{X:R}$ throughout in the above construction defines the groups $G^Z_p(X/R; X_1, \ldots, X_s)^d$ and $G^Z_p(X/R; X_1, \ldots, X_s)^{d/d'}$. If Z is the support of a flat, quasi-projective Cohen-Macaulay subscheme W of X, of codimension r, we have the natural isomorphisms

$$G^Z_p(X/R; X_1, \ldots, X_s)^d \to G_p(W/R; X_1 \cap W, \ldots, X_s \cap W)^{d-r}$$
$$G^Z_p(X/R; X_1, \ldots, X_s)^{d/d'} \to G_p(W/R; X_1 \cap W, \ldots, X_s \cap W)^{d-r/d'-r}.$$

For Z in $\mathrm{Fl}^{d'}$, repeated applications of the Quetzalcoatl lemma, together with the isomorphism of (c), gives the long exact localization sequence

$$\ldots \to G^Z_p(X/R; X_1 \cap Z, \ldots, X_s \cap Z)^d \to G_p(X/R; X_1, \ldots, X_s)^d \to$$
$$G_p((X \setminus Z)/R; X_1 \setminus Z, \ldots, X_s \setminus Z)^d \to G^Z_{p-1}(X/R; X_1 \cap Z, \ldots, X_s \cap Z)^d \to \ldots$$

As in Quillen, passing to a limit over all $Z \in \mathrm{Fl}^{d'}$, and linking all the resulting long exact seguences together, we arrive at

d) Suppose that R is regular and X is Cohen-Macaulay, flat and quasi-projective over R. Then there is an E_1-spectral sequence

$$E_1^{p,q}(X/R; X_1, \ldots, X_s) = G_{-p-q}(X/R; X_1, \ldots, X_s)^{*p/p+1} \Longrightarrow G_{-p-q}(X/R; X_1, \ldots, X_s).$$

Here $G_n(X/R; X_1, \ldots, X_s)^{*p/p+1}$ is defined by

$$G_n(X/R; X_1, \ldots, X_s)^{*p/p+1}$$
$$= \begin{cases} G_n(X/R; X_1, \ldots, X_s)^{p/p+1} & \text{for } n > 0 \\ \mathrm{Im}[G_0(X/R; X_1, \ldots, X_s)^p \to G_0(X/R; X_1, \ldots, X_s)^{p/p+1}] & \text{for } n = 0 \end{cases}$$

We let $Z^{p,q}_{r+1}(X/R; X_1, \ldots, X_s)$ denote the cycles in the r^{th} step of the above spectral sequence:

$$Z^{p,q}_{r+1}(X/R; -) = \ker\big[d_r \colon E_r^{p,q}(X/R; -) \to E_r^{p+r,q-r+1}(X/R; -)\big].$$

As in the absolute case $s = 0$, we have

e) If X is smooth over R, and R/I_S is regular for each S, then the map

$$K_*(X/R; X_1, \ldots, X_s) \to G_*(X/R; X_1, \ldots, X_s)$$

is an isomorphism.

Quillen's computation of the K-theory of projective space goes through in the relative case as follows:

f) Suppose $p \colon X \to \mathrm{Spec}(R)$ is a projective space $\mathbb{P}^n$ over R and that R/I_S is regular for each S. Let $[\mathcal{O}(-i)]\cup$ denote the map $K_p(R; I_1, \ldots, I_s) \to K_p(X/R; X_1, \ldots, X_s)$ defined by

$$[\mathcal{O}(-i)] \cup (\eta) = [\mathcal{O}(-i)] \cup p^*(\eta).$$

Then

$$\bigoplus_{i=0}^{n-1}[\mathcal{O}(-i)]\cup:\bigoplus_{i=0}^{n-1}K_p(R;I_1,\ldots,I_s)\longrightarrow K_p(X/R;X_1,\ldots,X_s)$$

is an isomorphism.

The arguments of Sherman [Sh] go through as in ([L] §1.7) to show

g) Suppose X is a projective space over R, and R/I_S is regular for each S. Then the spectral sequence of (d) degenerates at E_2. In addition, let $\pi: L \to \mathrm{Spec}(R)$ be a projective space $\mathbb{P}^{n-p}$ over R, with linear inclusion $i: L \to X$ over R. Then

$$E_2^{p,q}(X/R;X_1,\ldots,X_s)\cong K_{-p-q}(R;I_1,\ldots,I_s),$$

and one can lift $E_2^{p,q}=gr^pG_{-p-q}(X/R;X_1,\ldots,X_s)$ to a subgroup of $G_{-p-q}(X/R;X_1,\ldots,X_s)$ via the composition

$$K_{-p-q}(R;I_1,\ldots,I_s)\xrightarrow{\pi^*}G_{-p-q}(L/R;L_1,\ldots,L_s)\xrightarrow{i_*}G_{-p-q}(X/R;X_1,\ldots,X_s).$$

If X is an affine line over R, then a closed subscheme W of X which is flat over R must either be X or a codimension one subscheme of X. Thus, if R/I_S is regular for each S, the spectral sequence of (d) also degenerates at E_2 and we have

$$E_2^{p,q}=0 \text{ if } p\neq 0;\quad E_2^{0,q}=K_{-q}(R;I_1,\ldots,I_s).$$

□

h) Suppose $f: X \to \mathrm{Spec}(R)$ is a projective space (resp. an affine space) over R, with R regular and semi-local, containing a field. Then the spectral sequence $E(X)$ degenerates at E_2.

Indeed, suppose X is a projective space over R. Then, for each point x of $\mathrm{Spec}(R)$, the spectral sequence $E(f^{-1}(x))$ degenerates at E_2, and the E_2-terms are given by

$$E_2^{p,q}(f^{-1}(x))=K_{-p-q}(k(x)).$$

We filter the E_1-complexes of the spectral sequence $E(X)$,

$$E_1^{*,q}(X): K_{-q}(k(X))\to\oplus_{x\in X^{(1)}}K_{-1-q}(k(x))\to\ldots\to\oplus_{x\in X^{(p)}}K_{-p-q}(k(x))\to\ldots$$

by setting

$$F^rE_1^{p,q}(X)=\oplus_{x\in X^{(p)},f(x)\in\mathrm{Spec}(R)^{(\geq r)}}K_{-p-q}(k(x)).$$

Then

$$\mathrm{gr}^rE_1^{*,q}(X)=\oplus_{y\in\mathrm{Spec}(R)^{(r)}}E_1^{*-r,q+r}(f^{-1}(y)),$$

thus

$$H^p(\mathrm{gr}^rE_1^{*,q}(X))=\oplus_{y\in\mathrm{Spec}(R)^{(r)}}K_{-p-q}(k(y)).$$

By Gersten's conjecture for the semi-local ring R, the E_1-complex of the spectral sequence

$${}'E_1^{p,r}=H^p(\mathrm{gr}^{r-p}E_1^{*,q}(X))\Rightarrow E_2^{p,q}(X).$$

arising from the filtration F^* has

$${}'E_2^{p,r}=\begin{cases}0 & \text{if } r\neq p\\ K_{-p-q}(R) & \text{if } r=p.\end{cases}$$

Thus, we have $E_2^{p,q}(X)=K_{-p-q}(R)$; as the E_2-terms thus compute the K-groups of X, the spectral sequence $E(X)$ degenerates at E_2, as claimed.

The proof for X an affine space is essentially the same. □

Let $f: X \to \mathrm{Spec}(R)$ be a flat, quasi-projective, Cohen-Macaulay R-scheme. Let $(X/R)^{'(p)}$ be the set of codimension p points x of X with closure $\bar{x}$ contained in a codimension one subscheme of X which is flat over R; define $(X_S/R)^{'(p)}$ similarly. For $p > 0$, we let $\mathcal{M}(X/R)^{'p}$ denote the category of coherent sheaves on X with support contained in some codimension p subset W of X, such that W is contained in a codimension 1 subscheme Y of X, with Y flat over R; set $\mathcal{M}(X/R)^{'0} = \mathcal{M}(X)$. Let $\mathcal{M}(X/R)^{'p/p+1}$ be the quotient category $\mathcal{M}(X/R)^{'p}/\mathcal{M}(X/R)^{'p+1}$. The sequence of abelian categories

$$\mathcal{M}(X) \supset \mathcal{M}(X/R)^1 \supset \mathcal{M}(X/R)^{'2} \supset \ldots \supset \mathcal{M}(X/R)^{'p} \supset \ldots$$

defines an E_1-spectral sequence $E(X/R)'$

$$E_1^{p,q}(X/R)' = K_{-q}(\mathcal{M}(X/R)^{'p/p+1}) \Rightarrow K_{-p-q}(X);$$

let $Z_r^{p,q}(X/R)'$ denote the cycles in $E_{r-1}^{p,q}(X/R)'$. The inclusions

$$\mathcal{M}(X/R)^p \to \mathcal{M}(X/R)^{'p} \to \mathcal{M}(X)^p$$

gives the maps of spectral sequences

$$E(X/R) \to E(X/R)' \to E(X).$$

i) Suppose that R is regular, semi-local and contains a field, and X is a projective space (resp. an affine space) over R. Then the spectral sequence $E(X/R)'$ degenerates at E_2.

Indeed, letting $R(X)$ denote the semi-local ring $\mathcal{O}_{X,\overline{X}}$, and $i: \mathrm{Spec}(R(X)) \to \mathrm{Spec}(X)$ the inclusion, the E_1-complex $E_1^{*,q}(X/R)'$ is the global sections of the sheaf sequence

$$i_*(\mathcal{K}_{-q}(R(X))) \to \oplus_{x\in(X/R)^{(1)}} i_{x*}(K_{-1-q}(k(x)) \to \ldots \to \oplus_{x\in(X/R)'^{(p)}} i_{x*}(K_{-p-q}(k(x)) \to \ldots .$$

If y is a point of X, the the stalk at y of the sheaf $R^s i_*(\mathcal{K}_{-q}(R(X)))$ is easily seen to be given as

$$R^s i_*(\mathcal{K}_{-q}(R(X)))_y = H^s(\mathrm{Spec}(\mathcal{O}_{X,\overline{X}}) \cap \mathrm{Spec}(\mathcal{O}_{X,y}), \mathcal{K}_q).$$

Collino's argument in [C], essentially a modification of Quillen's proof (in [Q]) of Gersten's conjecture, shows that

$$H^s(\mathrm{Spec}(\mathcal{O}_{X,\overline{X}}) \cap \mathrm{Spec}(\mathcal{O}_{X,y}), \mathcal{K}_q) = 0; \quad \text{for } s > 0.$$

In fact, let D be a codimension one subscheme of $Y := \mathrm{Spec}(\mathcal{O}_{X,\overline{X}}) \cap \mathrm{Spec}(\mathcal{O}_{X,y})$, W a codimension p subscheme of D and η an element of $G_q(W)$. We need to find a codimension p subscheme T of Y, containing W, such that the image of η in $G_p(T)$ is zero; Quillen's argument then gives the desired vanishing of higher cohomology. We may find an affine scheme $\bar{Y}$, smooth and of finite type over a field k, a codimension one subscheme $\bar{D}$ of $\bar{Y}$, codimension one subschemes E_1 and E_2 of $\bar{D}$, and a codimension p subscheme $\bar{W}$ of $\bar{D}$ such that

i) Y is a localization of $\bar{Y}$, $\bar{Y}$ is an open subscheme of X and $\bar{Y}$ contains y and each generic point of $\bar{X}$.

ii) $\bar{D} \cap Y = D$, $y \notin E_1$ and $E_2 \cap \bar{X} = \emptyset$.

iii) $\bar{W} \cap Y = W$, and there is an element $\bar{\eta}$ of $G_p(\bar{W} \setminus (E_1 \cup E_2))$ such that $\bar{\eta}$ restricted to W is η.

Furthermore, there is a smooth morphism $\pi\colon \bar{Y} \to \mathbb{A}^n_k$, with one dimensional fibers, such that the restriction of π to $\bar{D}$ is finite; in addition, we may assume that

$$y \notin \pi^{-1}(\pi(E_1)), \quad \bar{X} \cap \pi^{-1}(\pi(E_2)) = \emptyset.$$

Form the square

$$\begin{array}{ccc} \bar{D}' := \bar{D} \times_{\mathbb{A}^n} \bar{Y} & \to & \bar{Y} \\ \downarrow\uparrow s & & \downarrow \\ \bar{D} & \to & \mathbb{A}^n. \end{array}$$

After removing a closed subset of $\bar{Y}$, and changing notation, we may assume that the codimenison one subscheme $s(\bar{D})$ is principal. Quillen's argument proving Gersten's conjecture then shows that η goes to zero in $G_p(T)$, with $T = \pi^{-1}(\pi(\bar{W})) \cap Y$.

The derived functors $R^s i_*(\mathcal{K}_{-q}(R(X)))$, $s > 0$, therefore vanish, and thus we have the natural isomorphism

$$H^p(X, \mathcal{K}_{-q}) = E_2^{p,q}(X/R)'.$$

As we have the same result for the spectral sequence $E(X)$, the map $E(X/R)' \to E(X)$ is an isomorphism from E_2 on; by (h), we have the desired degeneration. $\square$

§4. Relative K-theory of projective spaces: some computations

In this section, we fix a regular semi-local ring R of Krull dimension d, distinct principal prime ideals $I_1 = (t_1), \ldots, I_s = (t_s)$, such that the product $t = \prod t_i$ has a normal crossing divisor on R. Let $\pi: X \to \operatorname{Spec}(R)$ be a projective space over $\operatorname{Spec}(R)$. We will extend some of the basic consequences of the Quillen spectral sequence for the K-theory of projective spaces and affine spaces to the relative setting.

Let $R(X)$ denote the semi-local ring $\mathcal{O}_{X,\overline{X}}$ of $\overline{X}$ in X, $I_j(X)$ the ideal $I_jR(X)$. Define $R_j(X_j)$, $I_k(X_j)$ (for $k \neq j$) similarly, by replacing R with R_j and X with X_j. Let $e = e(R; I_1, \ldots, I_s) := \min(s, d)$.

We recall from ([L2], §5), that the relative K-groups have a natural λ-ring structure, compatible with the long exact relativization sequences, and extending the λ-ring structure defined by Hiller [H] and Kratzer [K] for the absolute K-groups. In addition, if we assume that T is a ring with ideals $J_1, \ldots, J_s$ such that T/J_S is regular for all $S \subset \{1, \ldots, s\}$, this λ-ring structure on $K_p(T; J_1, \ldots, J_s)$ is a *special* λ-ring structure; in particular, there are natural Adams operations ψ^k defined on the $K_p(T; J_1, \ldots, J_s)$, preserving the γ-filtration, commuting with one another, and on $\mathrm{gr}^n_\gamma K_p(T; J_1, \ldots, J_s)$, ψ^k acts by multiplication by k^n.

Lemma 4.1. *For $p > 2$, the map*

$$\pi^*: \mathrm{gr}^2_\gamma K_p(R; I_1, \ldots, I_s)[1/(p+e-1)!] \to \mathrm{gr}^2_\gamma K_p(R(X); I_1(X), \ldots, I_s(X))[1/(p+e-1)!]$$

is an isomorphism.

Proof. We proceed by induction on d and s. By ([L2], Theorem 5.7) the γ-filtration on $K_n(R; I_1, \ldots, I_s)$ satisfies

i) $K_n(R; I_1, \ldots, I_s) = \mathrm{F}^2_\gamma K_n(R; I_1, \ldots, I_s)$
ii) $\mathrm{F}^{n+1}_\gamma K_n(R; I_1, \ldots, I_s) = 0.$

for $n \geq 2$. and similarly for $K_n(R(X); I_1(X), \ldots, I_s(X))$. Let l be a prime, $l > n-1$, $n \geq 2$. Take an integer k which gives a generator of $\mathbf{F}_l^\times$; let $\mathbb{Z}_{(l)}$ denote the localization of $\mathbb{Z}$ away from l. Then, by ([L2], Lemma 5.8) we can split the γ-filtration on $K_n(R; I_1, \ldots, I_s) \otimes_{\mathbb{Z}} \mathbb{Z}_{(l)}$ and $K_n(R(X); I_1(X), \ldots, I_s(X)) \otimes_{\mathbb{Z}} \mathbb{Z}_{(l)}$ into k^i-eigenspaces for ψ^k, $K_n(R; I_1, \ldots, I_s)^{(i)}_l$ and $K_n(R(X); I_1(X), \ldots, I_s(X))^{(i)}_l$:

$$K_n(R; I_1, \ldots, I_s) \otimes_{\mathbb{Z}} \mathbb{Z}_{(l)} = \bigoplus_{i=2}^{n} K_n(R; I_1, \ldots, I_s)^{(i)}_l;$$
$$K_n(R(X); I_1(X), \ldots, I_s(X)) \otimes_{\mathbb{Z}} \mathbb{Z}_{(l)} = \bigoplus_{i=2}^{n} K_n(R(X); I_1(X), \ldots, I_s(X))^{(i)}_l.$$

If U is an open subset of X, intersecting each closed fiber of π, then there is a section to $\pi_{|U}: U \to \operatorname{Spec}(R)$; thus, the map

$$\pi^*: K_n(R; I_1, \ldots, I_s) \to K_n(R(X); I_1(X), \ldots, I_s(X))$$

is injective, and hence the map

$$\pi^*: \mathrm{gr}^2_\gamma K_p(R; I_1, \ldots, I_s)[1/(p+e-1)!] \to \mathrm{gr}^2_\gamma K_p(R(X); I_1(X), \ldots, I_s(X))[1/(p+e-1)!]$$

is injective. We proceed to prove surjectivity.

We first consider the case $d = 0$. If R is a direct sum of fields

$$R = \bigoplus_{k=1}^{t} L_j,$$

we can order the L_j so that

$$K_p(R; I_1, \ldots, I_s) = \bigoplus_{k=1}^{r} K_p(L_j).$$

This reduces us to proving the result in case R is a field F. By Soulé [So], the exact sequence

$$0 \to K_p(F) \xrightarrow{\pi^*} K_p(F(X)) \xrightarrow{\delta} \oplus_{x \in X^1} K_{p-1}(F(x)) \to 0$$

breaks up into eigenspaces for ψ^k as

$$0 \to K_p(F)_l^{(2)} \xrightarrow{\pi^*} K_p(F(X))_l^{(2)} \xrightarrow{\delta} \oplus_{x \in X^1} K_{p-1}(F(x))_l^{(1)} \to 0$$

If $p > 2$, then $K_{p-1}(F(x))_l^{(1)} = 0$, proving our result for $d = 0$.

If S is a regular semi-local ring containing a field k, and if F is the quotient field of S, the localization sequence

$$0 \to K_p(S) \to K_p(F) \to \oplus_{x \in X^1} K_{p-1}(k(x))$$

breaks up into eigenspaces as

$$0 \to K_p(S)_l^{(2)} \to K_p(F)_l^{(2)} \to \oplus_{x \in X^1} K_{p-1}(k(x))_l^{(1)},$$

so the map $K_p(S)_l^{(2)} \to K_p(F)_l^{(2)}$ is an isomorphism if $p > 2$. This, together with the case $d = 0$, proves the lemma in case $s = 0$.

We now assume that $d > 0$ and $s > 0$, and we assume the result for all regular semi-local rings of dimension $d' < d$ and for all $s' < s$. By considering the inclusions of $\mathrm{Spec}(R_1)$ in $\mathrm{Spec}(R)$ and of X_1 in X, we arrive at the commutative ladder with exact columns:

(4.1)

$$\begin{array}{ccc}
K_{p+1}(R_1; I_2R_1, \ldots, I_sR_1)_l^{(2)} & \to & K_{p+1}(R_1(X); I_2(X_1), \ldots, I_s(X_1))_l^{(2)} \\
\downarrow & & \downarrow \\
K_p(R; I_1, \ldots, I_s)_l^{(2)} & \to & K_p(R(X); I_1(X), \ldots, I_s(X))_l^{(2)} \\
\downarrow & & \downarrow \\
K_p(R; I_2, \ldots, I_s)_l^{(2)} & \to & K_p(R(X); I_2(X), \ldots, I_s(X))_l^{(2)} \\
\downarrow & & \downarrow \\
K_p(R_1; I_2R_1, \ldots, I_sR_1)_l^{(2)} & \to & K_p(R_1(X_1); I_2(X_1), \ldots, I_s(X_1))_l^{(2)}
\end{array}$$

Here we require $p > 2$ and $l > p+e-1$. The first and last two horizontal arrows in (4.1) are isomorphisms by the induction hypothesis. The surjectivity follows from this and the five lemma, proving the lemma. □

Let $X_{1j} = X_1 \cap X_j$, for $j = 2, \ldots, s$. Recall from §2(d) the spectral sequences

$$E_1^{p,q}(X/R; \{X_j\}_{j=2}^s) = G^*_{-p-q}(X/R; \{X_j\}_{j=2}^s)^{p/p+1} \Longrightarrow K_{-p-q}(X; \{X_j\}_{j=2}^s)$$

and

$$E_1^{p,q}(X_1/R_1;\{X_{1j}\}_{j=2}^s) = G^*_{-p-q}(X_1/R_1;\{X_{1j}\}_{j=2}^s)^{p/p+1} \Longrightarrow K_{-p-q}(X_1;\{X_{1j}\}_{j=2}^s),$$

with cycles $Z_{r+1}^{p,q}$ in E_r . The inclusion of X_1 in X induces a map of spectral sequences

$$i^*\colon E(X/R;\{X_j\}_{j=2}^s) \to E(X_1/R_1;\{X_{1j}\}_{j=2}^s).$$

Lemma 4.2. *Suppose $s \geq 1$. Then*

i) the restriction map

$$i^*\colon Z_2^{1,-2}(X/R;\{X_j\}_{j=2}^s) \to Z_2^{1,-2}(X_1/R_1;\{X_{1j}\}_{j=2}^s)$$

is surjective.

ii) the restriction map

$$i^*\colon Z_2^{1,-3}(X/R;\{X_j\}_{j=2}^s)[1/e!] \to Z_2^{1,-3}(X_1/R_1;\{X_{1j}\}_{j=2}^s)[1/e!]$$

is surjective

iii) the maps

$$Z_2^{1,-1}(X/R;\{X_j\}_{j=1}^s) \to Z_2^{1,-1}(X)$$

and

$$Z_2^{1,-2}(X/R;\{X_j\}_{j=1}^s)[1/e!] \to Z_2^{1,-2}(X)[1/e!]$$

are injective.

If we replace X with an affine line over Spec(R), the statements analogous to (i)-(iii) remain true.

Proof. We first prove (ii). Let η be in $Z_2^{1,-3}(X_1/R_1,\{X_{1j}\}_{j=2}^s)$. We recall from §2(g) that the Quillen spectral sequence

$$E_1^{p,q} = G_{-p-q}(X_1/R_1;\{X_{1j}\}_{j=2}^s)^{*p/p+1} \Rightarrow K_{-p-q}(X_1;\{X_{1j}\}_{j=2}^s)$$

degenerates at E_2, and that we can identify the $E_2^{1,-3}$ term as

$$E_2^{1,-3} = i_{1*}q_1^*(K_2(R_1;\{I_jR_1\}_{j=2}^s)),$$

where $i_1\colon L_1 \to X_1$ is a hyperplane in X_1, with structure morphism $q_1\colon L_1 \to \mathrm{Spec}(R_1)$, $q_1 = \ pi \circ i_1$. Thus, there is an element ξ of $K_3(R_1(X);\{I_jR_1(X_1)\}_{j=2}^s)$ and an element ρ of $K_2(R_1;\{I_jR_1\}_{j=2}^s)$ with

$$\delta\xi = [i_{1*}q_1^*(\rho)] - \eta;$$

here $[-]$ denotes the image in $Z_2^{1,-3}(X_1/R_1;\{X_{1j}\}_{j=2}^s)$. Fix a prime $l \geq e+1 \geq 2$; we can write the image of ξ in $K_3(R_1(X_1);\{I_jR_1(X_1)\}_{j=2}^s)\otimes\mathbb{Z}_{(l)}$ as

$$\xi = \xi^{(2)} + \xi^{(3)}; \quad \xi^{(i)} \in K_3(R_1(X);\{I_jR_1(X_1)\}_{j=2}^s)_l^{(i)}.$$

By the previous lemma, $\delta\xi = \delta\xi^{(3)}$ in $K_2 \otimes \mathbb{Z}_{(l)}$. By ([L2], Theorem 5.7), $\xi^{(3)}$ is the image of some element ξ^M of $K_3^M(R_1(X_1); \{I_jR_1(X_1)\}_{j=2}^s) \otimes \mathbb{Z}_{(l)}$, under the map

$$K_3^M(R_1(X_1); \{I_jR_1(X_1)\}_{j=2}^s) \otimes \mathbb{Z}_{(l)} \to K_3(R_1(X_1); \{I_jR_1(X_1)\}_{j=2}^s) \otimes \mathbb{Z}_{(l)}.$$

Since $R_1(X_1)$ and $R(X)$ are semi-local, the map

$$K_3^M(R(X); \{R(X_1)\}_{j=2}^s) \to K_3^M(R_1(X_1); \{I_jR_1(X_1)\}_{j=2}^s)$$

is surjective, so we can lift $\xi^{(3)}$ to an element τ of $K_3(R(X); \{I_jR(X)\}_{j=2}^s) \otimes \mathbb{Z}_{(l)}$. The element $\delta\tau$ then lifts $[i_{1*}q_1^*(\rho)] - \eta$ to $G_2(X; \{X_j\}_{j=2}^s)^{1/2} \otimes \mathbb{Z}_{(l)}$. In addition, we can extend $i_1: L_1 \to X_1$ to a hyperplane $i: L \to X$ with structure morphism $q: L \to \mathrm{Spec}(R)$. Since

$$K_2(R_1; \{I_jR_1\}_{j=2}^s) = K_2^M(R_1; \{I_jR_1\}_{j=2}^s),$$

we can lift ρ to $\tilde{\rho} \in K_2(R; \{I_j\}_{j=2}^s)$, giving us the lifting $i_*q^*(\tilde{\rho}) - \delta\tau$ of η to $Z_2^{1,-3}(X/R; \{X_j\}_{j=2}^s) \otimes \mathbb{Z}_{(l)}$. This proves (ii).

The proof of (i) is similar, but easier; we replace the use of Lemma 4.1 with the isomorphism

$$K_p(R_1; \{I_jR_1\}_{j=2}^s) \cong K_p^M(R_1; \{I_jR_1\}_{j=2}^s); \quad p = 1, 2.$$

We now prove (iii).

Let η be in $Z_2^{1,-2}(X/R; \{X_j\}_{j=1}^s)[1/e!]$. By the degeneration of the spectral sequence $E(X/R; \{X_j\}_{j=1}^s)$ at E_2, we may lift η to an element of $G_1(X/R; \{X_j\}_{j=1}^s)^1[1/e!]$; let η' be the image of such a lifting in $G_1(X/R; \{X_j\}_{j=1}^s)^{1/3}[1/e!]$. Suppose η goes to zero in $Z_2^{1,-2}(X/R; \{X_j\}_{j=2}^s)[1/e!]$. Then the image η'' of η' in $G_1(X/R; \{X_j\}_{j=2}^s)^{1/3}[1/e!]$ lands in the subgroup

$$\mathrm{Im}\Big(G_1(X/R; \{X_j\}_{j=2}^s)^{2/3}[1/e!] \to G_1(X/R; \{X_j\}_{j=2}^s)^{1/3}[1/e!]\Big).$$

By the degeneration of the spectral sequence $E(X/R; \{X_j\}_{j=2}^s)$, this implies there is a codimension two linear subspace $i: L \to X$, with structure morphism $p: L \to \mathrm{Spec}(R)$, and an element ψ of $K_1(R, \{I_j\}_{j=2}^s)[1/e!]$ such that

$$\eta'' = i_*p^*(\psi) \quad \text{in } G_1(X/R; \{X_j\}_{j=2}^s)^{1/3}[1/e!].$$

This in turn implies that the image of $i_*p^*(\psi)$ in $G_1(X_1; \{X_{1j}\}_{j=2}^s)^{1/3}[1/e!]$ is zero. Using the degeneration of the spectral sequence $E(X_1/R_1; \{X_{1j}\}_{j=2}^s)$, this implies that the image of ψ in $K_1(R_1, \{I_{1j}\}_{j=2}^s)[1/e!]$ is also zero. Thus we can lift ψ to an element Ψ of $K_1(R, \{I_j\}_{j=1}^s)[1/e!]$. Replacing η' with $\eta' - i_*p^*(\Psi)$ and changing notation, we may assume that η' goes to zero in $G_1(X/R; \{X_j\}_{j=2}^s)^{1/3}[1/e!]$. Thus η' lies in the subgroup

$$\mathrm{Im}\Big(G_2(X_1/R_1; \{X_{1j}\}_{j=2}^s)^{1/3}[1/e!] \to G_1(X/R; \{X_j\}_{j=1}^s)^{1/3}[1/e!]\Big).$$

By (ii), this implies that η' lies in

$$\mathrm{Im}\Big(G_2(X_1/R_1; \{X_{1j}\}_{j=2}^s)^{2/3}[1/e!] \to G_1(X/R; \{X_j\}_{j=1}^s)^{1/3}[1/e!]\Big),$$

i.e., in

$$\mathrm{Im}\Big(G_1(X/R;\{X_j\}_{j=1}^s)^{2/3}[1/e!] \to G_1(X/R;\{X_j\}_{j=1}^s)^{1/3}[1/e!]\Big)$$

so $\eta = 0$. Thus, the map

$$Z_2^{1,-2}(X/R;\{X_j\}_{j=1}^s)[1/e!] \to Z_2^{1,-2}(X/R;\{X_j\}_{j=2}^s)[1/e!]$$

is injective; by induction, the map

$$Z_2^{1,-2}(X/R;\{X_j\}_{j=1}^s)[1/e!] \to Z_2^{1,-2}(X/R)[1/e!]$$

is injective. We now show that the map

$$Z_2^{1,-2}(X/R) \to Z_2^{1,-2}(X)$$

is injective, which will complete the proof of (iii).

We may assume that R is a domain; let F be the quotient field of R, and X_F the fiber of X over F. Let $X^{(1)*}$ be the set of codimension one points x of X whose closure $\bar{x}$ in X is *not* flat over R; $X^{(1)*}$ is also the set of codimension one points of $\mathrm{Spec}(R(X))$. We have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
K_2(R) & \to & K_2(R(X)) & \to & Z_2^{1,-2}(X/R) & \to & K_1(R) & \to 0 \\
|| & & \downarrow & & \downarrow & & || & & \\
K_2(R) & \to & K_2(F(X_F)) & \to & Z_2^{1,-2}(X) & \to & K_1(R) & \to 0 \\
 & & \downarrow & & & & & & \\
 & & \oplus_{x\in X^{(1)*}} k(x)^* & & & & & &
\end{array}$$

The degeneration of the spectral sequences $E(X/R)$ and $E(X)$ at E_2 gives the identifications

$$K_1(R) = E_2^{1,-2}(X/R), \quad K_2(R) = E_2^{0,-2}(X/R)$$
$$K_1(R) = E_2^{1,-2}(X), \quad K_2(R) = E_2^{0,-2}(X)$$

and implies that the two rows are exact. Since $R(X)$ is a regular semi-local ring containing a field, Gersten's conjecture implies that the column

$$0 \to K_2(R(X)) \to K_2(F(X_F)) \to \oplus_{x\in X^{(1)*}} k(x)^*$$

is exact as well. The injectivity of the map $Z_2^{1,-2}(X/R) \to Z_2^{1,-2}(X)$ then follows by a diagram chase.

The remainder of the proof of (iii) is a simpler version of the above argument.

If we replace X with an affine line over $\mathrm{Spec}(R)$, the Quillen spectral sequence still degenerates at E_2, with the E_2-terms $E_2^{p,q} = 0$ for $p \neq 0$. Thus, a simpler version of the above argument gives the analogous results. □

We define the map

$$d\colon K_2(R(X);\{I_j(X)\}_{j=1}^s) \to \bigoplus_{x\in X^{(1)}} k(x)^*$$

as the composition

$$K_2(R(X);\{I_j(X)\}_{j=1}^s) \xrightarrow{\delta} G_1(X/R;\{X_j\}_{j=1}^s)^{1/2} \;\to\; G_1(X)^{1/2} = \bigoplus_{x\in X^{(1)}} k(x)^*.$$

Lemma 4.3. *Suppose $\pi\colon X \to \mathrm{Spec}(R)$ is a projective space over R. Then the sequence*

$$0 \to K_2(R;\{I_j\}_{j=1}^s)[1/e!] \;\xrightarrow{\pi^*}\; K_2(R(X);\{I_j(X)\}_{j=1}^s)[1/e!] \;\xrightarrow{d}\; \bigoplus_{x\in X^{(1)}} k(x)^*[1/e!]$$

is exact.

Proof. As in the proof of Lemma 4.1, the map

$$\pi^*\colon K_2(R;\{I_j\}_{j=1}^s) \to K_2(R(X);\{I_j(X)\}_{j=1}^s)$$

is injective. Since the spectral sequence $E(X/R;\{X_j\}_{j=1}^s)$ degenerates at E_2, we have

$$\begin{aligned}\mathrm{Im}\Big(&K_2(X/R;\{X_j\}_{j=1}^s) \to K_2(R(X);\{I_j(X)\}_{j=1}^s)\Big)\\ &= \ker\Big(K_2(R(X);\{I_j(X)\}_{j=1}^s) \to Z_2^{1,-2}(X/R;\{X_j\}_{j=1}^s)\Big).\end{aligned}$$

By the previous lemma, the map

$$Z_2^{1,-2}(X/R;\{X_j\}_{j=1}^s)[1/e!] \to \bigoplus_{x\in X^{(1)}} k(x)^*[1/e!]$$

is injective, completing the proof. □

We let $\overline{K}_2(R(X);\{I_j(X)\}_{j=1}^s)$ denote the quotient $K_2(R(X);\{I_j(X)\}_{j=1}^s)/\pi^*K_2(R;\{I_j\}_{j=1}^s)$.

Lemma 4.4. *After inverting $(e+1)!$, the sequence*

$$0 \to \overline{K}_2(R(X);\{I_j(X)\}_{j=1}^s) \to \overline{K}_2(R(X);\{I_j(X)\}_{j=2}^s) \to \overline{K}_2(R_1(X_1);\{I_{1j}(X_1)\}_{j=2}^s) \to 0$$

is exact.

Proof. We have the commutative ladder with exact columns

$$\begin{array}{ccc} K_3(R_1;\{I_{1j}\}_{j=2}^s) & \to & K_3(R_1(X_1);\{I_{1j}(X_1)\}_{j=2}^s) \\ \downarrow & & \downarrow \\ K_2(R;\{I_j\}_{j=1}^s) & \to & K_2(R(X);\{I_j\}_{j=1}^s) \\ \downarrow & & \downarrow \\ K_2(R;\{I_j\}_{j=1}^s) & \to & K_2(R(X);\{I_j\}_{j=1}^s) \\ \downarrow & & \downarrow \\ K_2(R_1;\{I_{1j}\}_{j=2}^s) & \to & K_2(R_1(X_1);\{I_{1j}(X_1)\}_{j=2}^s) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

the surjectivity follows from Lemma 1.2 and Theorem 1.4. By Lemma 4.1, the map

$$K_3(R_1; \{I_{1j}\}_{j=2}^s)^{(2)} \to K_3(R_1(X_1); \{I_{1j}(X_1)\}_{j=2}^s)^{(2)}$$

is an isomorphism after inverting $(e+1)!$. Arguing as in Lemma 4.3, the maps

$$K_2(R; \{I_j\}_{j=1}^s) \to K_2(R(X); \{I_j(X)\}_{j=1}^s)$$
$$K_2(R; \{I_j\}_{j=1}^s) \to K_2(R(X); \{I_j(X)\}_{j=1}^s)$$
$$K_2(R_1; \{I_{1j}\}_{j=2}^s) \to K_2(R_1(X_1); \{I_{1j}(X_1)\}_{j=2}^s)$$

are injective. The lemma then follows by a diagram chase. □

Let $R \to R'$ be a smooth ring extension. Let $X' = X_{R'}$, $I'_j = I_j R'$, etc. Let F, F' denote the total quotient fields of R, R'.

Lemma 4.5. *Suppose the map $F \to F'$ is injective. Then the map*

$$\overline{K}_2(R(X); \{I_j(X)\}_{j=1}^s)[1/e!] \to \overline{K}_2(R'(X'); \{I'_j(X')\}_{j=1}^s)[1/e!]$$

is injective.

Proof. This follows immediately from Lemma 4.3. □

§5. Hilbert's Theorem 90 for relative K_2

Fix a prime number l; let k_0 be a field, R a semi-local smooth k_0-algebra, and t an element of R with normal crossing divisor on $\mathrm{Spec}(R)$. Let T be a semi-local R-algebra, smooth over k_0, such that the reduced divisor of t on $\mathrm{Spec}(T)$ is a normal crossing divisor. We recall that, for a a unit in T, T_a is the finite extension of T defined by

$$T_a = \begin{cases} T[X]/X^l - a & \text{if } l \neq \mathrm{char}(k_0) \\ T[X]/X^l - X - a & \text{if } l = \mathrm{char}(k_0). \end{cases}$$

Let $\{I(T)_j\}$ be the (finite) set of principal prime ideals containing $(t)T$. Let $V(T) = V(T, a)$ be the first homology of the complex

$$K_2(T_a; \{I(T_a)_j\}) \xrightarrow{1-\sigma} K_2(T_a; \{I(T_a)_j\}) \xrightarrow{Nm} K_2(T; \{I(T)_j\})$$

(with $K_2(T; \{I(T)_j\})$ in degree 0). The main result of this section is

Theorem 5.1 (Hilbert's Theorem 90 for relative K_2)**.** *Let a be in $R^\times$. If $l > \dim(R)+1$, then $V(R, a) = 0$.*

The proof proceeds in a series of steps. Let $d = \dim(R)$.

Step 1) Let $p: T \to T'$ be a finite extension of semi-local R-algebras, where T' is smooth over k_0 and the reduced divisor of t on $\mathrm{Spec}(T')$ is a normal crossing divisor. We have the functorial norm maps

$$Nm_{T'/T}: K_*(T'; \{I(T')_j\}) \to K_*(T; \{I(T)_j\})$$

$$Nm_{T'_a/T_a}: K_*(T'_a; \{I(T'_a)_j\}) \to K_*(T_a; \{I(T_a)_j\}),$$

giving rise to the commutative diagram

$$\begin{array}{ccccc} K_2(T'_a; \{I(T'_a)_j\}) & \xrightarrow{1-\sigma} & K_2(T'_a; \{I(T'_a)_j\}) & \xrightarrow{Nm} & K_2(T'; \{I(T')_j\}) \\ p_* \uparrow\downarrow Nm & & p_* \uparrow\downarrow Nm & & p_* \uparrow\downarrow Nm \\ K_2(T_a; \{I(T_a)_j\}) & \xrightarrow{1-\sigma} & K_2(T_a; \{I(T_a)_j\}) & \xrightarrow{Nm} & K_2(T; \{I(T)_j\}) \end{array}$$

This defines maps

$$p_*: V(T) \to V(T'); \quad Nm_{T'/T}: V(T') \to V(T),$$

with

$$Nm_{T'/T} \circ p_* = \deg(T'/T)\mathrm{id}.$$

Taking $T' = T_a$, and noting that $(T_a)_a$ is a product of copies of T_a, we see that $V(T_a) = 0$, hence $V(T)$ is an l-torsion group. Additionally, if $\deg(T'/T)$ is prime to l, then $V(T) \to V(T')$ is injective.

Replacing k_0 with an infinite, prime to l extension, and changing notation, we may assume that k_0 is an infinite field.

Step 2) For a semi-local ring T, let $\mathrm{Jac}(T)$ denote the Jacobson radical of T, and $\overline{T}$ the quotient ring $T/\mathrm{Jac}(T)$.

Let a and b be units of T. If $l \neq \mathrm{char}(k_0)$, fix a primitive l^{th} root of unity ζ, and define the Azumaya algebra $A(a, b)$ by

$$A(a, b) = T\{X, Y\}/(X^l - a, Y^l - b, YX - \zeta XY).$$

Let $\pi: X(a, b) \to \mathrm{Spec}(T)$ be the Brauer-Severi scheme associated to $A(a, b)$, let $\overline{X}$ denote the fiber of $X = X(a, b)$ over $\overline{T}$, and let $T(X)$ denote the semi-local ring of $\overline{X}$ in X.

Proposition 5.1. *Let a be a unit in T, b an element of $(1+\mathrm{Jac}(T))$; set $X = X(a,b)$. Suppose that $l > \dim(T)+1$. Then the map*

$$\pi_*\colon V(T,a) \to V(T(X),a)$$

is injective.

Proof. The extension $X_a := X \otimes_T T_a$ of X, to a T_a-scheme is a projective space over T_a, hence the map

$$K_2(T_a;\{I(T_a)_j\}) \to K_2(T_a(X_a);\{I(T_a(X_a))_j\})$$

is injective. We need to show that

$$\big[(1-\sigma)K_2(T_a(X_a);\{I(T_a(X_a))_j\})\big] \cap K_2(T_a;\{I(T_a)_j\}) = (1-\sigma)K_2(T_a;\{I(T_a)_j\})$$

For this, it suffices to show that the map

$$K_2(T_a(X_a);\{I(T_a(X_a)_j\})^\sigma \to \big[K_2(T_a(X_a);\{I(T_a(X_a))_j\})/K_2(T_a;\{I(T_a)_j\})\big]^\sigma$$

is surjective. We will actually prove that

$$K_2(T(X);\{I(T(X))_j\}) \to \big[K_2(T_a(X_a);\{I(T_a(X_a))_j\})/K_2(T_a;\{I(T_a)_j\})\big]^\sigma$$

is surjective.

Let F and F_a denote the quotient fields of T and T_a. From [MS2] (2.2.3), the map

$$K_2(F(X)) \to \big[K_2(F_a(X_a))/K_2(F_a)\big]^\sigma$$

is surjective. We now show that the map

$$K_2(T(X)) \to \big[K_2(T_a(X_a))/K_2(T_a)\big]^\sigma \tag{5.1}$$

is surjective. Indeed, we have the commutative ladder with exact rows

$$\begin{array}{ccccccccc} 0\to & K_2(T) & \to & K_2(F) & \to & \bigoplus_{x\in\mathrm{Spec}(T)^{(1)}} k(x)^* & \xrightarrow{\alpha} & \bigoplus_{x\in\mathrm{Spec}(T)^{(2)}} \mathbb{Z} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0\to & K_2(T(X)) & \to & K_2(F(X)) & \to & \bigoplus_{x\in\mathrm{Spec}(T(X))^{(1)}} k(x)^* & \xrightarrow{\beta} & \bigoplus_{x\in\mathrm{Spec}(T(X))^{(2)}} \mathbb{Z} & \to 0 \end{array} \tag{5.2}$$

Let $Z_1^{1,-2}(T)$ be the kernel of α and let $Z_1^{1,-2}(T(X))$ be the kernel of β. We have analogous ladder for T_a and $T_a(X_a)$; define $Z_1^{1,-2}(T_a)$ and $Z_1^{1,-2}(T_a(X_a))$ in a like manner. Take an element η in $Z_1^{1,-2}(T(X))$, and suppose that η goes to zero in $Z_1^{1,-2}(T_a(X_a))/Z_1^{1,-2}(T_a)$. Since the maps

$$\bigoplus_{x\in\mathrm{Spec}(T(X))^{(1)}} k(x)^* \to \bigoplus_{x\in\mathrm{Spec}(T_a(X_a))^{(1)}} k(x)^*,$$

and

$$\bigoplus_{x\in X^{(1)}} \mathbb{Z} \to \bigoplus_{x\in X_a^{(1)}} \mathbb{Z}$$

are injective, η must be in the subgroup

$$[\bigoplus_{y\in\mathrm{Spec}(T)^{(1)}} H^0(\pi^{-1}(y), \mathcal{O}^*_{\pi^{-1}(y)})] \cap Z_1^{1,-2}(T(X))$$

of $Z_1^{1,-2}(T(X))$. As $H^0(\pi^{-1}(y), \mathcal{O}^\times_{\pi^{-1}(y)}) = k(y)^*$, η is therefore in $Z_1^{1,-2}(T)$. Thus, the map

$$Z_1^{1,-2}(T(X))\Big/Z_1^{1,-2}(T) \to [Z_1^{1,-2}(T_a(X_a))\Big/Z_1^{1,-2}(T_a)]^\sigma$$

is injective. The surjectivity in (5.1) follows from this and a diagram chase on (5.2).

We now proceed by induction on c, the number of ideals in $\{I(T)_j\}$. Let $\hat{T} = T/I(T)_1$, $I(\hat{T})_j = (I(T)_1 + I(T)_j)/I(T)_1$, etc. We have the commutative ladder with exact rows

$$\begin{array}{ccccccc}
\cdots \to & K_2(T; \{I(T)_j\}_{j=1}^c) & \to & K_2(T; \{I(T)_j\}_{j=2}^c) & \to & K_2(\hat{T}; \{I(\hat{T})_j\}_{j=2}^c) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & K_2(T(X); \{I(T(X))_j\}_{j=1}^c) & \to & K_2(T(X); \{I(T(X))_j\}_{j=2}^c) & \to & K_2(\hat{T}(\hat{X}); \{I(\hat{T}(\hat{X}))_j\}_{j=2}^c) & \to 0;
\end{array}$$

the surjectivity following from Lemma 1.2 and Theorem 1.4. This gives us the exact sequence

$$\overline{K}_2(T(X); \{I(T(X))_j\}_{j=1}^c) \to \overline{K}_2(T(X); \{I(T(X))_j\}_{j=2}^c) \to \overline{K}_2(\hat{T}(\hat{X}); \{I(\hat{T}(\hat{X}))_j\}_{j=2}^c) \to 0.$$

Let $e = \dim(T)$. For the remainder of the proof, we invert $(e+1)!$. By Lemma 4.4, we have the exact sequence

$$0 \to \overline{K}_2(T_a(X_a); \{I_j(X_a)\}_{j=1}^s) \to \overline{K}_2(T_a(X_a); \{I_j(X_a)\}_{j=2}^s) \to \overline{K}_2(T_{a1}(X_{a1}); \{I_{1j}(X_{a1})\}_{j=2}^s) \to 0.$$

Taking σ-invariants gives us the commutative ladder

$$\begin{array}{ccccccc}
 & \overline{K}_2(T(X); \{I(T(X))_j\}_{j=1}^c) & \to & \overline{K}_2(T(X); \{I(T(X))_j\}_{j=2}^c) & \to & \overline{K}_2(\hat{T}(\hat{X}); \{I(\hat{T}(\hat{X}))_j\}_{j=2}^c) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \overline{K}_2(T_a(X_a); \{I_j(X_a)\}_{j=1}^s)^\sigma & \to & \overline{K}_2(T_a(X_a); \{I_j(X_a)\}_{j=2}^s)^\sigma & \to & \overline{K}_2(T_{a1}(X_{a1}); \{I_{1j}(X_{a1})\}_{j=2}^s)^\sigma.
\end{array}$$

By Lemma 4.5, the right hand vertical arrow is injective; by induction, the middle vertical arrow is surjective, hence the left-hand vertical arrow is surjective. This completes the proof. □

Step 3) We now make a computation of certain norms.

Lemma 5.2. *Let S be a regular semi-local k_0-algebra, $J_1, \ldots, J_s$ distinct principal prime ideals. Let $J = \cap J_k$, $J = (t)S$, and suppose that the divisor of t in S is a normal crossing divisor. Let a be a unit is S, σ a generator of $Gal(S_a/S)$. Let x be an element of $(1+J)$, w a unit in $S[1/t]$, z an element of $(1+JS_a)$ and y a unit in $S_a[1/t]$. This gives us the elements $\{z, w\}$ and $\{x, y\}$ in $K_2(S_a; J_1S_a, \ldots, J_sS_a)$. Then*

i) $Nm_{S_a/S}(\{z, w\}) = \{Nm_{S_a/S}(z), w\}$; *in* $K_2(S; J_1, \ldots, J_s)$
ii) $Nm_{S_a/S}(\{x, y\}) = \{x, Nm_{S_a/S}(y)\}$; *in* $K_2(S; J_1, \ldots, J_s)$.

Proof. We first prove (i). Let $J_i = (t_i)$. Since each unit of $S[1/t]$ is a product $u\prod {t_i}^{n_i}$, we may assume that $w = ut_i$ for some i, so that w divides t in S. Let $\hat{B}$ be the ring

$$\hat{B} = k_0[Z_0, \ldots, Z_{l-1}, W, V, T, A]/(WV - T),$$

and let $P \in B[X]$ be the polynomial $X^l - A$ in case $l \neq \mathrm{char}(k_0)$, or $X^l - X - A$ if $l = \mathrm{char}(k_0)$. Let $\hat{B}_A$ be the ring

$$\hat{B}_A = \hat{B}[A^*]/(P(A^*)).$$

$\hat{B}_A$ is a finite Galois extension of the ring $\hat{B}$; let N denote the norm from B_A down to B, and let

$$Z = 1 + T\sum_{i=0}^{l-1} Z^i A^{*i}; \quad X = N(1 + T\sum_{i=0}^{l-1} Z^i A^{*i}).$$

Let $\mathcal{S}$ be the set of elements $f \in \hat{B}$ such that

$$(f, WV) = \hat{B};$$

define $B = \mathcal{S}^{-1}\hat{B}$, $B_A = \mathcal{S}^{-1}\hat{B}_A$. We have isomorphisms

$$\hat{B} \cong k_0[Z_0, \ldots, Z_{l-1}, W, V, A]; \quad \hat{B}_A \cong k_0[Z_0, \ldots, Z_{l-1}, W, V, A^*],$$

so B and B_A are localizations of polynomial rings, and all the face maps in the squares of rings

$$\begin{array}{ccc} B & \to & B/(V) \\ \downarrow & & \downarrow \\ B/(W) & \to & B/(W,V) \end{array} \qquad \begin{array}{ccc} B_A & \to & B_A/(V) \\ \downarrow & & \downarrow \\ B_A/(W) & \to & B_A/(W,V) \end{array}$$

are split. Additionally,

$$(WV)B \subset \mathrm{Jac}(B); \quad (WV)B_A \subset \mathrm{Jac}(B_A),$$

so we can apply the results of §1. In particular, we have the elements

$$<(1-Z)/W, W> \in D(B_A; (W), (V)); \quad <(1-N(Z))/W, W> \in D(B; (W), (V)),$$

and the corresponding elements

$$Dsym(<(1-Z)/W, W>) \in K_2(B_A; (W), (V)); \quad Dsym(<(1-N(Z))/W, W>) \in K_2(B; (W)(V)).$$

Denote the elements $Dsym(<(1-Z)/W, W>)$, $Dsym(<(1-N(Z))/W, W>)$ by $\{Z, W\}$ and $\{N(Z), W\}$, respectively.

Let $\phi: B \to S$, $\phi_A: B_A \to S_a$ be the obvious homomophisms, with

$$\phi_A(A^*) = \alpha,\ \phi_A(Z) = z,\ \phi(W) = w,\ \phi(N(Z)) = Nm_{S_a/S}(z), \mathrm{etc.}$$

Let $v = \prod_{j \neq i} t_j$. ϕ and ϕ_A induce natural homomorphisms

$$\phi_A: K_2(B_A; (W), (V)) \to K_2(S_a; (t_i), (v)); \quad \phi: K_2(B; (W), (V)) \to K_2(S; (t_i), (v)).$$

If we compose with the natural maps

$$\tau: K_2(S; (t_i), (v)) \to K_2(S; J_1, \ldots, J_s); \quad \tau_a: K_2(S_a; (t_i), (v)) \to K_2(S_a; J_1 S_a, \ldots, J_s S_a)$$

one checks that

$$\tau_a \circ \phi_A(\{Z, W\}) = \{z, w\}; \quad \tau \circ \phi(\{N(Z), W\}) = \{Nm_{S_a/S}(z), w\}.$$

Thus, we need only check that

$$Nm_{B_A/B}(\{Z, W\}) = \{N(Z), W\}.$$

Let F_A and F denote the quotient fields of B_A and B. Since B_A and B are localizations of polynomial rings, and since the face maps in the above squares of rings are all split, the maps

$$K_2(B; (W), (V)) \to K_2(F); \quad K_2(B_A; (W), (V)) \to K_2(F_A)$$

are injective; thus we need only check that

$$Nm_{F_A/F}(\{Z, W\}) = \{Nm_{F_A/F}(Z), W\},$$

which follows directly from the projection formula.

The proof of (ii) is similar, and will be left to the reader. □

Proposition 5.3. *Let S be a regular semi-local k_0-algebra, $J_1, \ldots, J_s$ distinct principal prime ideals. Let $J = \cap J_k$, $J = (t)S$, and suppose that the divisor of t in S is a normal crossing divisor. Let a be a unit in S, z an element of $(1 + JS_a)$. Then $\{z, 1 - Nm_{S_a/S}(Z)\}$ is in $(1 - \sigma)K_2(S_a; J_1S_a, \ldots, J_sS_a)$*

Proof. The proof is essentially the same as the proof of Propositions 3.3 and 3.4 of [MS2], with the help of our Lemma 5.2. □

§6. Applications

We begin by computing the torsion in $K_2(R; I_1, \ldots, I_s)$. Fix a prime l and a primitive l^{th} root of unity ζ. Let R be a semi-local ring smooth over a field k_0, with principal ideals $I_1 = (t_1), \ldots, I_s = (t_s)$. We let $t = \prod t_i$, and assume that the divisor of t on R is a normal crossing divisor.

Theorem 6.1. *Suppose that $k_0 \supset \mu_l$. Then for $l > \dim(R) + 1$, the l-torsion subgroup ${}_lK_2(R; I_1, \ldots, I_s)$ of $K_2(R; I_1, \ldots, I_s)$ is generated by symbols $\{f, \zeta\}$, $f \in (1 + (t))^\times$. In particular, if $l = \mathrm{char}(k_0)$, then ${}_lK_2(R; I_1, \ldots, I_s) = 0$.*

Proof. The proof is essentially the same as in the case of fields. Let S be an indeterminate, and consider the extension $R[S]_S$ of $R[S]$,

$$R[S]_S = R[S, T]/F(T),$$

where $F(T) = T^l - S$ if $\mathrm{char}(k_0) \neq l$, $F(T) = T^l - T - S$ if $\mathrm{char}(k_0) = l$. Let $X = \mathrm{Spec}(R[S])$, $X_S = \mathrm{Spec}(R[S]_S)$, with structure morphisms

$$\pi: X \to \mathrm{Spec}(R),$$
$$\pi: X_S \to \mathrm{Spec}(R),$$

and morphism $f: X_S \to X$. We note that X and X_S are both affine lines over R; let $i: \mathrm{Spec}(R) \to X$, $i_S: \mathrm{Spec}(R) \to X_S$ be the zero sections defined by $S = 0$, $T = 0$, resp., and let σ be a generator of $\mathrm{Gal}(X_S/X)$.

Claim. $[Z_2^{1,-2}(X_S/R; \{(X_S)_k\}_{k=1}^s)]^\sigma = f^*[Z_2^{1,-2}(X/R; \{X_k\}_{k=1}^s)] + i_{S*}[K_1(R; \{I_k\}_{k=1}^s)]$

Proof of claim. Let $R_1 = R/I_1$, X_1, X_{1S} the fiber of X, X_S over $\mathrm{Spec}(R_1)$, $I_{1k} = (I_1 + I_k)/I_1$. By Lemma 4.2 we have the short exact sequences (after inverting $\dim(R)!$):

$$0 \to Z_2^{1,-2}(X_S/R; \{(X_S)_k\}_{k=1}^s) \to Z_2^{1,-2}(X_S/R; \{(X_S)_k\}_{k=2}^s) \to Z_2^{1,-2}(X_{1S}/R_1; \{(X_{1S})_k\}_{k=2}^s) \to 0$$
$$0 \to Z_2^{1,-2}(X/R; \{X_k\}_{k=1}^s) \to Z_2^{1,-2}(X/R; \{X_k\}_{k=2}^s) \to Z_2^{1,-2}(X_1/R_1; \{(X_1)_k\}_{k=2}^s) \to 0$$

In addition, we have the short exact sequence

$$0 \to K_1(R; \{I_k\}_{k=1}^s) \to K_1(R; \{I_k\}_{k=2}^s) \to K_1(R_1; \{I_{1k}\}_{k=2}^s) \to 0.$$

By Lemma 4.2(iii), the group $Z_2^{1,-2}(X_S/R; \{(X_S)_k\}_{k=1}^s)$ is a subgroup of $\oplus_{x \in X^1} k(x)^*$, and similarly for the groups $Z_2^{1,-2}(X_S/R; \{(X_S)_k\}_{k=1}^s)$, etc. above. From this it follows that the kernel of the map

$$i_{S_1*} - f_1^*: K_1(R_1; \{I_{1k}\}_{k=2}^s) \oplus Z_2^{1,-2}(X_1/R_1; \{(X_1)_k\}_{k=2}^s) \to Z_2^{1,-2}(X_{1S}/R_1; \{(X_{1S})_k\}_{k=2}^s)$$

is the subgroup $(\nu^l, i_*(\nu^{-1}))$, $\nu \in K_1(R_1; \{I_{1k}\}_{k=2}^s)$, and similarly for the map

$$i_{S*} - f^*: K_1(R; \{I_k\}_{k=2}^s) \oplus Z_2^{1,-2}(X/R; \{X_k\}_{k=2}^s) \to Z_2^{1,-2}(X_S/R; \{(X_S)_k\}_{k=2}^s).$$

Using the snake lemma and induction, this reduces us to the absolute case (s=0).

Let F^* denote the quotient field of R. We use the notations of §3, especially (b), (d), (h) and (i).

By §2(i), the spectral sequence $E(X/R)'$ degenerates at E_2. The $E_1^{1,*}$ complex for the spectral sequence $E(X/R)'$ is

$$K_2(R(X)) \to \oplus_{x \in (X/R)^{(1)}} k(x)^* \to \oplus_{x \in (X/R)'^{(2)}} \mathbb{Z}.$$

By the degeneration of $E(X/R)'$ at E_2, and the injectivity of the map

$$Z_2^{1,-2}(X/R) \to Z_2^{1,-2}(X)$$

(Lemma 4.2(iii)), the map

$$Z_2^{1,-2}(X/R) \to Z_2^{1,-2}(X/R)'$$

is an isomorphism; hence we have the exact sequence

$$0 \to Z_2^{1,-2}(X/R) \to \oplus_{x\in(X/R)^{(1)}} k(x)^* \to \oplus_{x\in(X/R)'^{(2)}} \mathbb{Z} \to 0$$

Similarly, we have the commutative ladder with exact rows

$$\begin{array}{ccccc} 0 \to Z_2^{1,-2}(X/R) & \to & \oplus_{x\in(X/R)^{(1)}} k(x)^* & \xrightarrow{\partial} & \oplus_{x\in(X/R)'^{(2)}} \mathbb{Z} \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to Z_2^{1,-2}(X_S/R) & \to & \oplus_{x\in(X_S/R)^{(1)}} k(x)^* & \xrightarrow{\partial} & \oplus_{x\in(X_S/R)'^{(2)}} \mathbb{Z} \to 0 \end{array}$$

Since the map $X_S \to X$ is étale away from the zero-section i_S, we have

$$[\oplus_{x\in(X_S/R)^{(1)}} k(x)^*]^\sigma = f^*(\oplus_{x\in(X/R)^{(1)}} k(x)^*) + i_{S*}(F^*).$$

Thus, if we have an element η in $Z_2^{1,-2}(X_S/R)^\sigma$, then we can write η as

$$\eta = f^*(\tau) + i_{S*}(\rho); \quad \tau \in \oplus_{x\in(X/R)^{(1)}} k(x)^*, \rho \in F^*.$$

Since η comes from $Z_2^{1,-2}(X_S/R)$, it follows that $\partial\rho \in \oplus_{x\in(X_S/R)'^{(2)}}\mathbb{Z}$ is divisible by l, thus we may write ρ as

$$\rho = u\nu^l$$

with $u \in R^*$. Thus, we have $\eta = f^*(\tau\nu) + i_{S*}(u)$; as f^* is injective on $\oplus_{x\in(X/R)'^{(2)}}\mathbb{Z}$, we have $\tau\nu \in Z_2^{1,-2}(X/R)$, as desired. This completes the proof of the claim. □

Let η be in ${}_lK_2(R; \{I_k\}_{k=1}^s)$. Then $Nm(\pi_S^*(\eta)) = \eta^l = 0$. By Hilbert's Theorem 90, there is an element τ of $K_2(R(X_S); \{I_kR(X_S)\}_{k=1}^s)$ with

$$\pi_S^*(\eta) = (1-\sigma)\tau.$$

Let $z \in Z_2^{1,-2}(X_S/R; \{(X_S)_k\}_{k=1}^s)$ be the boundary $\delta(\tau)$. Then $z^\sigma = z$; by the above claim, there is a y in $Z_2^{1,-2}(X/R; \{X_k\}_{k=1}^s)$ and w in $K_1(R; \{I_k\}_{k=1}^s)$ with $z = f^*(y) + i_{S*}(w)$. Since the spectral sequence $E((X/R; \{X_k\}_{k=1}^s)$ degenerates at E_2, we can find $\psi \in K_2(R(X); \{I_k(X)\}_{k=1}^s)$ with $\delta(\psi) = y$. Modifying τ by $f^*(\psi)$, we may assume that $\delta(\tau) = i_{S*}(w)$. Then

$$\delta(\tau\{1/w, S\}) = 0,$$

so $t = \{w, S\} \cdot \pi_S^*(\xi)$ for some ξ in $K_2(R; \{I_k\}_{k=1}^s)$. Then

$$\pi_S^*(\eta) = (1-\sigma)\{w, S\} = \pi_S^*(\{w, \zeta\}^{-1}).$$

Since π_S^* is injective, this completes the proof in case $l \neq \mathrm{char}(k_0)$. The case $l = \mathrm{char}(k_0)$ is easier, as in this case X_S is étale over X. □

We let $H^*(X, \mathcal{F})$ denote the continuous étale cohomology of Jannsen [J]. If $\mathcal{F}$ is a torsion sheaf, this is the usual étale cohomology. If Y is a closed subscheme of X, we define $H^*(X; Y, \mathcal{F})$ by

$$H^*(X; Y, \mathcal{F}) = H^*(X, \mathrm{Cone}(\mathcal{F} \to Ri_{Y*}i_Y^*(\mathcal{F}))[-1]),$$

and define a multi-relative version by iteration. This is discussed in detatil in ([L] §1.13), where we show that Gillet's theory of Chern classes goes through in the relative situation, with no essential changes. One also finds, by direct computation, that

$$H^1(R; \{I_k\}_{k=1}^s, \boldsymbol{\mu}_l) = (1 + (t))^\times \Big/ ((1 + (t))^\times)^l$$

where R, I_k and $(t) = \cap I_k$ are as above. From the above theorem, we get

Corollary 6.2. *Suppose that $l > \dim(R) + 1$. There is a surjective homomorphism*

$$symb\colon H^1(R; \{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2}) \to {}_lK_2(R; \{I_k\}_{k=1}^s).$$

Proof. If k_0 contains μ_l, define $symb$ as the composition

$$\begin{aligned} H^1(R; \{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2}) &= H^1(R; \{I_k\}_{k=1}^s, \boldsymbol{\mu}_l) \otimes \mu_l \\ &= (1 + (t))^\times \otimes \mu_l \\ &\to {}_lK_2(R; \{I_k\}_{k=1}^s), \end{aligned}$$

the last map sending $f \otimes \zeta$ to $\{f, \zeta\}$. In general, one adjoins ζ to k_0 and takes norms. □

Let $\bar{H}^1_{\acute{e}t}(R; \{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2})$ be the kernel of $symb$, and define $K_3(R; \{I_k\}_{k=1}^s)^{ind}$ by

$$K_3(R; \{I_k\}_{k=1}^s)^{ind} = K_3(R; \{I_k\}_{k=1}^s)/\mathrm{Im}[K_3^M(R; \{I_k\}_{k=1}^s)].$$

We proceed as in [MS2] to define a surjective homomorphism

$$\bar{H}^1_{\acute{e}t}(R; \{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2}) \to K_3(R; \{I_k\}_{k=1}^s)^{ind}/l.$$

We first suppose $\zeta \in k_0$. Let Λ be the subgroup of $(1 + (t))^\times/l$ corresponding to $\bar{H}^1_{\acute{e}t}(R; \{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2})$ under the isomorphism

$$\begin{gathered} (1 + (t))^\times/l \to H^1_{\acute{e}t}(R; \{I_k\}_{k=1}^s, \boldsymbol{\mu}_l) \otimes \mu_l = H^1_{\acute{e}t}(R; \{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2}) \\ f \mapsto f \otimes \zeta \end{gathered}$$

Consider the ring extension $R[X] \to R[T]$, $X \mapsto T^l$. Let $R(X)$ be the localization of $R[X]$ away from the set of prime ideals $\{mR[X] \mid m \text{ is a maximal ideal of } R\}$, and similarly define $R(T)$. Let S be the localization of $R[X]$ away from the set of maximal ideals $\{(m, X-1)R[X] \mid m \text{ is a maximal ideal of } R\}$, and S' the localization of $R[T]$ away from the set $\{(m, T^l - 1)R[T] \mid m \text{ is a maximal ideal of } R\}$. Let σ be the generator of $G = \mathrm{Gal}(S'/S)$; $\sigma(T) = \zeta T$.

Let $I = (1 - X)S$, $I' = (1 - X)S'$. Let L and M be the kernel and image, respectively, of the map

$$K_2(S'; I', \{I_kS'\}_{k=1}^s) \to K_2(S'; \{I_kS'\}_{k=1}^s).$$

We have the short exact sequence

$$0 \to L \to K_2(S'; I', \{I_k S'\}_{k=1}^s) \to M \to 0. \tag{6.1}$$

We note that the map $K_2(S; I, \{I_k S\}_{k=1}^s) \to K_2(S; \{I_k S\}_{k=1}^s)$ is injective. By Hilbert's Theorem 90 applied to the extension S'/S, we have $L \subset (1-\sigma)K_2(S'; I', \{I_k S'\}_{k=1}^s)$. Thus, the homology sequence from (6.1) becomes

$$H_1(G, K_2(S'; I', \{I_k S'\}_{k=1}^s)) \to H_1(G, M) \to L/(1-\sigma)L \to 0.$$

We have the commutative diagrams with exact rows

$$\begin{array}{ccccc}
K_2(S'; I', \{I_k S'\}_{k=1}^s) & \to & K_2(S'; \{I_k S'\}_{k=1}^s) & \to & \bigoplus_{i=0}^{l-1} K_2(R; \{I_k\}_{k=1}^s) \to 0 \\
Nm \downarrow & & Nm \downarrow & & \Sigma \downarrow \\
0 \to K_2(S; I, \{I_k S\}_{k=1}^s) & \to & K_2(S; \{I_k S\}_{k=1}^s) & \to & K_2(R; ; \{I_k\}_{k=1}^s) \to 0
\end{array}$$

$$\begin{array}{ccccc}
0 \to K_2(S'; \{I_k S'\}_{k=1}^s) & \to & K_2(R(T); \{I_k R(T)\}_{k=1}^s) & \to & \bigoplus_{i=0}^{l-1} K_1(R; \{I_k\}_{k=1}^s) \to 0 \\
Nm \downarrow & & Nm \downarrow & & \Pi \downarrow \\
0 \to K_2(S; \{I_k S\}_{k=1}^s) & \to & K_2(R(X); \{I_k R(X)\}_{k=1}^s) & \to & K_1(R; \{I_k\}_{k=1}^s) \to 0;
\end{array}$$

the first being relativization sequences, the second, localization sequences. Since the G-modules

$$\bigoplus_{i=0}^{l-1} K_2(R; \{I_k\}_{k=1}^s) \quad \text{and} \quad \bigoplus_{i=0}^{l-1} K_1(R; \{I_k\}_{k=1}^s)$$

are induced, we have

$$H_1(G, M) = H_1(G, K_2(R(T); \{I_k R(T)\}_{k=1}^s))$$

This defines an isomorphism

$$K_2(R(T); \{I_k R(T)\}_{k=1}^s)^\sigma \Big/ [Nm(K_2(R(T); \{I_k R(T)\}_{k=1}^s)) + K_2(S'; I', \{I_k S'\}_{k=1}^s)^\sigma] \to L/(1-\sigma)L. \tag{6.2}$$

Let u be in $K_2(R(T); \{I_k R(T)\}_{k=1}^s)^\sigma$. Arguing as in the proof of Theorem 6.1, there is an f in $(1+(t))^\times$ such that

$$u \equiv \{T, f\} \mod K_2(R(X); \{I_k R(X)\}_{k=1}^s).$$

The argument of [MS2] Lemma 4.2.3 then proves

Lemma 6.3. *Each element of $K_2(R(T); \{I_k R(T)\}_{k=1}^s)^\sigma$ is congruent mod $Nm(K_2(R(T); \{I_k R(T)\}_{k=1}^s)) + K_2(S'; I', \{I_k S'\}_{k=1}^s)$ to a symbol $\{T, f\}$, with $f \in \Lambda$.*

□

Lemma 6.4. *The map*

$$K_3(R;\{I_k\}_{k=1}^s)^{ind}[1/(\dim R+2)!] \to K_3(S';\{I_kS'\}_{k=1}^s)^{ind}[1/(\dim R+2)!]$$

is an isomorphism.

Proof. By ([L2], Theorem 5.7), we have

$$\begin{aligned} F_\gamma^3 K_3(R;\{I_k\}_{k=1}^s)[1/2] &= K_3^M(R;\{I_k\}_{k=1}^s)[1/2] \\ F_\gamma^3 K_3(S';\{I_kS'\}_{k=1}^s)[1/2] &= K_3^M(S';\{I_kS'\}_{k=1}^s)[1/2] \\ F_\gamma^3 K_3(R(T);\{I_kR(T)\}_{k=1}^s)[1/2] &= K_3^M(R(T);\{I_kR(T)\}_{k=1}^s)[1/2]. \end{aligned}$$

From Lemma 4.1, the map

$$K_3(R;\{I_k\}_{k=1}^s)^{ind}[1/(\dim(R)+2)!] \to K_3(R(T);\{I_kR(T)\}_{k=1}^s)^{ind}[1/(\dim(R)+2)!]$$

is an isomorphism; the result follows from this and localization. □

Collecting these results, we have

Proposition 6.5. *If $l > \dim(R)+2$, there is a surjective homomorphism*

$$\Lambda \to K_3(R;\{I_k\}_{k=1}^s)^{ind}/l.$$

Proof. We have the long exact relativization sequence

$$\cdots \to K_3(S';\{I_kS'\}_{k=1}^s) \to \bigoplus_{i=0}^{l-1} K_3(R;\{I_k\}_{k=1}^s) \to K_2(S';I',\{I_kS'\}_{k=1}^s) \to K_2(S';\{I_kS'\}_{k=1}^s) \to \cdots,$$

which, after inverting 2, gives the exact sequence

$$\cdots \to K_3(S';\{I_kS'\}_{k=1}^s)^{ind} \to \bigoplus_{i=0}^{l-1} K_3(R;\{I_k\}_{k=1}^s)^{ind} \to K_2(S';I',\{I_kS'\}_{k=1}^s) \to K_2(S';\{I_kS'\}_{k=1}^s) \to \cdots.$$

This, applying Lemma 6.4 and inverting $(\dim(R)+2)!$, gives the short exact sequence

$$0 \to K_3(R;\{I_k\}_{k=1}^s)^{ind} \to \bigoplus_{i=0}^{l-1} K_3(R;\{I_k\}_{k=1}^s)^{ind} \to L \to 0.$$

Taking G-homology gives the isomorphism

$$K_3(R;\{I_k\}_{k=1}^s)^{ind}/l \to L/(1-\sigma)L. \tag{6.3}$$

By Lemma 6.3, sending f to $\{T,f\}$, and using the isomorphism (6.2), defines a surjective map

$$\Lambda \to L/(1-\sigma)L;$$

following this with the isomorphism (6.3) completes the proof. □

Lemma 6.6. *Suppose that R is the localization of a ring of finite type over $\mathbb{Z}$. Then $\bar{H}^1_{\acute{e}t}(R;\{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2})$ is a finite group.*

Proof. From ([L] §1.13), there are Chern class maps

$$\begin{aligned} c_{2,2}\colon K_2(R;\{I_k\}_{k=1}^s) &\to H^2(R;\{I_k\}_{k=1}^s, \mathbb{Z}_l(2)) \\ c_{1,1}\colon K_1(R;\{I_k\}_{k=1}^s) &\to H^1(R;\{I_k\}_{k=1}^s, \mathbb{Z}_l(1))\,. \\ c_{1,1}\colon K_1(R) &\to H^1_{\acute{e}t}(R, \mathbb{Z}_l(1)) \end{aligned}$$

These Chern classes satisfy the product formula: if $a \in (1+(t)R)^\times$, and $b \in R^\times$, then

$$c_{2,2}(\{a,b\}) = c_{1,1}(a) \cup c_{1,1}(b).$$

This gives rise to the commutative diagram, with exact rows

$$\begin{array}{ccccccc} H^1(R;\{I_k\}_{k=1}^s, \mathbb{Z}_l(2)) \xrightarrow{\times l} H^1(R;\{I_k\}_{k=1}^s, \mathbb{Z}_l(2)) & \to & H^1_{\acute{e}t}(R;\{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2}) & \to & H^2(R;\{I_k\}_{k=1}^s, \mathbb{Z}_l(2)) \\ & & symb \downarrow & & c_{2,2} \uparrow & . \\ & & {}_lK_2(R;\{I_k\}_{k=1}^s) & \to & K_2(R;\{I_k\}_{k=1}^s) \end{array}$$

From this, we see that

$$\bar{H}^1_{\acute{e}t}(R;\{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2}) \subset H^1(R;\{I_k\}_{k=1}^s, \mathbb{Z}_l(2))/l.$$

The relativization sequence, together with the vanishing of $H^0(\bar{R}, \mathbb{Z}_l(2))$ for each quotient $\bar{R}$ of R, shows that $H^1(R;\{I_k\}_{k=1}^s, \mathbb{Z}_l(2))$ is a sub-$\mathbb{Z}_l$-module of $H^1(R, \mathbb{Z}_l(2))$. By Suslin [S]. this latter is isomorphic to $H^1(k, \mathbb{Z}_l(2))$, where k is the algebraic closure of the prime field in R. Since $H^1(k, \mathbb{Z}_l(2))$ is a finitely generated $\mathbb{Z}_l$-module, by Tate [T], $H^1(R;\{I_k\}_{k=1}^s, \mathbb{Z}_l(2))/l$ is a finite group, which proves the lemma. □

Theorem 6.7. *Let l be an odd prime, $(l, \mathrm{char}(k_0)) = 1$. Suppose R is the localization of an algebra of finite type over $\mathbb{Z}$.*

i) Suppose $l > \dim(R) + 2$. Then the Chern class

$$c_{2,1}\colon K_3(R;\{I_k\}_{k=1}^s; \mathbb{Z}/l^\nu)^{ind} \to H^1_{\acute{e}t}(R;\{I_k\}_{k=1}^s, \boldsymbol{\mu}_{l^\nu}^{\otimes 2})$$

is an isomorphism.

ii) Suppose $l > \dim(R) + 1$. Then the Chern class

$$c_{2,2}\colon K_2(R;\{I_k\}_{k=1}^s)/l^\nu \to H^2_{\acute{e}t}(R;\{I_k\}_{k=1}^s, \boldsymbol{\mu}_{l^\nu}^{\otimes 2})$$

is an isomorphism.

Proof. We may assume that k_0 contains μ_l, and that $\mathrm{Spec}(R)$ is connected. Let $\beta \in K_2(k_0, \mathbb{Z}/l)$ be the Bott element corresponding to a primitive root of unity ζ; giving the homomorphism $i\colon \mu_l \to K_2(R, \mathbb{Z}/l)$ splitting the surjection $K_2(R, \mathbb{Z}/l) \to \mu_l$. Since $\mathrm{Spec}(R)$ is connected, we have

$$H^0_{\acute{e}t}(R;\{I_k\}_{k=1}^s, \boldsymbol{\mu}_l^{\otimes 2}) = \begin{cases} \mu_l^{\otimes 2} & \text{if } s = 0 \\ 0 & \text{if } s > 0 \end{cases}$$

Additionally, we have the isomorphism

$$K_1(R;\{I_k\}_{k=1}^s)\otimes\mu_l\rightarrow H^1_{\acute{e}t}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2}).$$

Define

$$symb_1\colon H^1_{\acute{e}t}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})\rightarrow K_3(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l)$$

as the composition

$$H^1_{\acute{e}t}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})\rightarrow K_1(R;\{I_k\}_{k=1}^s)\otimes\mu_l\stackrel{id\otimes i}{\rightarrow}K_1(R;\{I_k\}_{k=1}^s)\otimes K_2(R,\mathbb{Z}/l)\stackrel{\cup}{\rightarrow}K_3(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l).$$

Define

$$symb_0\colon H^0_{\acute{e}t}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})\rightarrow K_4(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l)$$

to be the zero map if $s>0$; if $s=0$, define $symb_0$ as the composition

$$H^0_{\acute{e}t}(R,\boldsymbol{\mu}_l^{\otimes 2})\rightarrow\mu_l\otimes\mu_l\stackrel{i\otimes i}{\rightarrow}K_2(R,\mathbb{Z}/l)\otimes K_2(R,\mathbb{Z}/l)\stackrel{\cup}{\rightarrow}K_4(R,\mathbb{Z}/l).$$

The map

$$symb\colon H^1_{\acute{e}t}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})\rightarrow {}_lK_2(R;\{I_k\}_{k=1}^s)$$

is just the composition of $symb_1$ with the surjection

$$K_3(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l)\rightarrow {}_lK_2(R;\{I_k\}_{k=1}^s).$$

Form the ladder

$$\begin{array}{ccc}
H^1_{\acute{e}t}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2}) & \stackrel{symb}{\rightarrow} & {}_lK_2(R;\{I_k\}_{k=1}^s)\\
\downarrow & & \downarrow\\
H^2_{\acute{e}t}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_{l^\nu}^{\otimes 2}) & \stackrel{c_{2,2}}{\leftarrow} & K_2(R;\{I_k\}_{k=1}^s)/l^\nu\\
\downarrow & & \downarrow\\
H^2_{\acute{e}t}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_{l^{\nu+1}}^{\otimes 2}) & \stackrel{c_{2,2}}{\leftarrow} & K_2(R;\{I_k\}_{k=1}^s)/l^{\nu+1}\\
\downarrow & & \downarrow\\
H^2_{\acute{e}t}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2}) & \stackrel{c_{2,2}}{\leftarrow} & K_2(R;\{I_k\}_{k=1}^s)/l
\end{array}$$

by using the Bockstein sequence on the left, and the obvious map of groups on the right. Both vertical sequences are complexes, the one on the left is exact, and the one on the right is exact, except possibly at $K_2(R;\{I_k\}_{k=1}^s)/l^\nu$. The product formula shows the ladder is commutative. Using the Bockstein sequences for étale cohomology and K-theory with coefficients, we construct the commutative ladder with exact columns

$$\begin{array}{ccc}
H^0_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2}) & \overset{symb_0}{\rightarrow} & {}_lK_4(R;\{I_k\}_{k=1}^s) \\
\downarrow & & \downarrow \\
H^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_{l^\nu}^{\otimes 2}) & \overset{c_{2,2}}{\leftarrow} & K_3(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l^\nu) \\
\downarrow & & \downarrow \\
H^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_{l^{\nu+1}}^{\otimes 2}) & \overset{c_{2,2}}{\leftarrow} & K_3(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l^{\nu+1}) \\
\downarrow & & \downarrow \\
H^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2}) & \overset{c_{2,2}}{\leftarrow} & K_3(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l) \\
\downarrow & & \downarrow \\
H^2_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_{l^\nu}^{\otimes 2}) & \overset{c_{2,2}}{\leftarrow} & K_2(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l^\nu) \\
\downarrow & & \downarrow
\end{array}$$

Using these two diagrams reduces us to the case $\nu = 1$.

Restricting the map $symb_1$ to the subgroup $\bar{H}^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2}))$ of $H^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l)$ defines the map

$$\overline{symb}\colon \bar{H}^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2}) \rightarrow K_3(R;\{I_k\}_{k=1}^s)^{ind}/l.$$

The product formula for Chern classes shows that $c_{2,1} \circ symb_1$ is the identity; thus

$$c_{2,1}\colon K_3(R;\{I_k\}_{k=1}^s)^{ind}/l \rightarrow \bar{H}^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})$$

is surjective. As $\bar{H}^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})$ is finite by Lemma 6.6, Proposition 6.5 implies that

$$c_{2,1}\colon K_3(R;\{I_k\}_{k=1}^s)^{ind}/l \rightarrow \bar{H}^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})$$

is an isomorphism. By Corollary 6.2, this implies that

$$c_{2,1}\colon K_3(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l)^{ind} \rightarrow H^1_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})$$

is an isomorphism, proving (i).

For (ii), let $R_1 = R/I_1$, $I_{1k} = (I_k + I_1)/I_1$. We proceed by induction on s, the case $s = 0$ being a consequence of the theorem of Merkurjev and Suslin [MS] on the K_2 of fields, and a localization argument.

We have the commutative diagram

$$\begin{array}{ccc}
K_3(R;\{I_k\}_{k=2}^s,\mathbb{Z}/l) & \overset{c_{2,1}}{\rightarrow} & H^1_{ét}(R;\{I_k\}_{k=2}^s,\boldsymbol{\mu}_l^{\otimes 2}) \\
\alpha\downarrow & & \downarrow\beta \\
K_3(R_1;\{I_{1k}\}_{k=2}^s,\mathbb{Z}/l) & \overset{c_{2,1}}{\rightarrow} & H^1_{ét}(R_1;\{I_{1k}\}_{k=2}^s,\boldsymbol{\mu}_l^{\otimes 2}),
\end{array}$$

where α and β are the appropriate restriction maps. The map β is surjective since R is semi-local; by (i) the map

$$K_3(R_1;\{I_{1k}\}_{k=2}^s,\mathbb{Z}/l)\overset{c_{2,1}}{\rightarrow}H^1_{ét}(R_1;\{I_{1k}\}_{k=2}^s,\boldsymbol{\mu}_l^{\otimes 2})$$

is an isomorphism. Thus the cup product

$$K_1(R_1;\{I_{1k}\}_{k=2}^s,\mathbb{Z}/l)\otimes K_2(k_0;\mathbb{Z}/l)\to K_3(R_1;\{I_{1k}\}_{k=2}^s,\mathbb{Z}/l)$$

is surjective. This implies that α is surjective as well. By looking at the relativization sequence

$$K_3(R;\{I_k\}_{k=2}^s,\mathbb{Z}/l)\to K_3(R_1;\{I_{1k}\}_{k=2}^s,\mathbb{Z}/l)\to K_2(R;\{I_k\}_{k=1}^s,\mathbb{Z}/l)\xrightarrow{\gamma}K_2(R;\{I_k\}_{k=2}^s,\mathbb{Z}/l)\to$$

we see that γ is injective. Thus the map $K_2(R;\{I_k\}_{k=1}^s)/l\to K_2(R;\{I_k\}_{k=2}^s)/l$ is also injective.

We have the commutative ladder with exact columns

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
K_2(R;\{I_k\}_{k=1}^s)/l & \xrightarrow{c_{2,2}} & H^2_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})\\
\downarrow & & \downarrow\\
K_2(R;\{I_k\}_{k=2}^s)/l & \xrightarrow{c_{2,2}} & H^2_{ét}(R;\{I_k\}_{k=2}^s,\boldsymbol{\mu}_l^{\otimes 2})\\
\downarrow & & \downarrow\\
K_2(R_1;\{I_{1k}\}_{k=1}^s)/l & \xrightarrow{c_{2,2}} & H^2_{ét}(R;\{I_{1k}\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})\\
\downarrow & & \\
0 & &
\end{array}$$

By induction, the lower two Chern classes are isomorphisms, hence

$$c_{2,2}\colon K_2(R;\{I_k\}_{k=1}^s)/l\to H^2_{ét}(R;\{I_k\}_{k=1}^s,\boldsymbol{\mu}_l^{\otimes 2})$$

is an isomorphism, proving (ii). □

Let S_X^p be the "algebraic p-sphere" over a scheme X, i.e. the co-simplicial scheme with non-degenerate co-simplices

$$X \rightrightarrows \Delta^1_X \ \substack{\to\\ \to\\ \to}\ \Delta^2_X \ \substack{\to\\ \to\\ \to\\ \to}\ \cdots \ \substack{\to\\ \vdots\\ \to}\ \Delta^p_X$$

where Δ^k_X is the affine space of dimension k over X. If X is regular, there is an natural isomorphism

$$K_n(X)\to K_q(S_X^{n-q}),$$

and similarly for étale cohomology. In [L2], §6, this was exploited to study the Chern class maps

$$c_{q,p}: gr^q_\gamma K_{2q-p}(F)\to H^p_{ét}(F,\boldsymbol{\mu}_m^{\otimes q})$$

for fields F. In particular, Theorem 6.1 of [L2], combined with Theorem 6.7 yields

Theorem 6.8. *Let F be a field, l a prime, $n \geq 2$ and ν integers. Suppose $l \geq n$ and $(l, \mathrm{char}(F)) = 1$. Then the Chern class*

$$c_{2,4-n}: gr_\gamma^2 K_n(F, \mathbb{Z}/l^\nu) \to H_{\acute{e}t}^{4-n}(F, \boldsymbol{\mu}_{l^\nu}^{\otimes 2})$$

is an isomorphism. In particular, $gr_\gamma^2 K_n(F, \mathbb{Z}/l^\nu) = 0$ for $n > 4$

□

If $n > 4$ and $gr_\gamma^2 K_n(F)$ is a direct sum of finitely generated groups, this yields the vanishing of $gr_\gamma^2 K_n(F)[1/(n-1)!]$. We in fact have

Corollary 6.9. *Let F be a field, l a prime, $n \geq 2$ and ν integers. Suppose $l \geq n$ and $(l, \mathrm{char}(F)) = 1$. Then*

$$gr_\gamma^2 K_n(F)/l^\nu = 0 \text{ for } n \geq 4.$$

If F is a number field, then

$$gr_\gamma^2 K_n(F)[1/(n-1)!] = 0$$

for $n \geq 4$.

Proof. For $l \geq n$, we have the exact sequence

$$0 \to gr_\gamma^2 K_n(F)/l^\nu \to gr_\gamma^2 K_n(F; \mathbb{Z}/l^\nu) \to {}_l gr_\gamma^2 K_{n-1}(F) \to 0.$$

Theorem 6.8 thus implies $gr_\gamma^2 K_n(F)/l^\nu = 0$ for $n \geq 5$.

For $n = 4$, the l^ν-torsion in $K_3(F)^{ind}$ has been computed in [L] and in [MS2]; this computation shows that the composition

$$H_{\acute{e}t}^0(F, \boldsymbol{\mu}_{l^\nu}^{\otimes 2}) \stackrel{symb_0}{\to} K_4(F, \mathbb{Z}/l^\nu) \to {}_{l^\nu} gr_\gamma^2 K_3(F)$$

is an isomorphism. The product formula shows that the composition

$$H_{\acute{e}t}^0(F, \boldsymbol{\mu}_{l^\nu}^{\otimes 2}) \stackrel{symb_0}{\to} K_4(F, \mathbb{Z}/l^\nu) \stackrel{c_{2,0}}{\to} H_{\acute{e}t}^0(F, \boldsymbol{\mu}_{l^\nu}^{\otimes 2})$$

is the identity, hence, using Theorem 6.8, $gr_\gamma^2 K_4(F, \mathbb{Z}/l^\nu) = {}_{l^\nu} gr_\gamma^2 K_3(F)$. Thus $gr_\gamma^2 K_4(F)/l^\nu = 0$, as desired.

Now suppose F is a number field. Let $\mathcal{O}$ be the ring of integers of F. The K-groups of $\mathcal{O}$ are finitely generated, by Quilllen [Q2]. From Soulé [So2], we have the short exact sequence (after inverting 2):

$$0 \to K_n(\mathcal{O}) \to K_n(F) \to \bigoplus_{P \in \mathrm{MaxSpec}(\mathcal{O})} K_{n-1}(\mathcal{O}/P) \to 0$$

for $n \geq 2$. Since $K_{n-1}(\mathcal{O}/P)$ is a finite group by Quillen [Q3], $K_n(F)$ is a direct sum of finitely generated groups for $n \geq 2$. This, together with the first part of the corollary, completes the proof. □

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