BLOWING UP MONOMIAL IDEALS

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Abstract. Let k be a field. Spivakovsky's theorem on the solution of Hironaka's polyhedral game has been extended by Bloch to show that a morphism $f: Z \to S$ of finite type k-schemes can be put in good position with respect to a normal crossing divisor ∂S on S by taking the proper transform with respect to an iterated blowing up of faces of ∂S. We extend these results to schemes of finite type over a regular scheme of dimension one, including the case of mixed characteristic.

0. Introduction

In this paper, we consider a version of Hironaka's polyhedral game for positive codimension two cycles in $\mathbb{A}^n_{\mathcal{O}}$, where $\mathcal{O}$ is a discrete valuation ring (including the mixed characteristic case). Spivakovsky's solution of Hironaka's game [5] has been used by Bloch [3] in his proof of the localization property of the higher Chow groups for schemes over a field; the results proved here are used in [4] to give an extension of the localization property to schemes in mixed characteristic.

We begin by describing the algebro-geometric version of Hironaka's polyhedral game for codimension one cycles in $\mathbb{A}^n_k$, k a field, and then the codimension two version.

Fix a ring R. We use coordinates $X_1, \ldots, X_n$ for $\mathbb{A}^n_R$. For a non-empty subset J of $\{1, \ldots, n\}$, and an element $i \in J$, we have the map

$$\mu_{J,i} : \mathbb{A}^n_R \to \mathbb{A}^n_R$$

with

$$\mu^*_{J,i}(X_j) = \begin{cases} X_j & \text{for } j \notin J \setminus \{i\}, \\ X_i X_j & \text{for } j \in J \setminus \{i\}. \end{cases}$$

Let $F_R(J)$ be the subscheme of $\mathbb{A}^n_R$ defined by the equations $X_j = 0$, $j \in J$; we have

$$\mu^{-1}_{J,i}(F_R(J)) = F_R(\{i\}),$$

and $\mu_{J,i} : \mathbb{A}^n_R \setminus F_R(\{i\}) \to \mathbb{A}^n_R \setminus F_R(J)$ is an isomorphism. If the context makes the meaning clear, we write E_R for $F_R(\{i\})$

Let $Z > 0$ be a positive codimension one cycle in $\mathbb{A}^n_k$. We call a non-empty subset J of $\{1, \ldots, n\}$ *allowable* for Z if $\mathrm{supp}(Z)$ contains $F_k(J)$. This is the cases if and only if $\mu^*_{J,i}(Z) - E_k \geq 0$.

We have the following game with two players A and B. The game starts with a codimension one cycle $Z > 0$ in $\mathbb{A}^n_k$, with $0_k \in \mathrm{supp}(Z)$. A moves first, chosing

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an allowable subset J of $\{1,\dots,n\}$. B moves by chosing an element i of J, and forming the non-negative cycle $Z' := \mu_{J,i}^*(Z) - E_k$. If $\mathrm{supp}(Z')$ does not contain 0_k, A wins at this point, if 0_k is in $\mathrm{supp}(Z')$, the game continues, with the cycle $Z' > 0$ replacing the cycle Z.

Theorem 0.1 (Spivakovsky/Bloch)**.** *For each codimension one cycle $Z > 0$ in $\mathbb{A}^n_k$, there is a strategy for the player A to win after finitely many moves.*

We now describe the codimension two version of this game, which we call the codimension two blow-up game. Let $\mathcal{O}$ be a discrete valuation ring with residue field k and quotient field K. Let $Z > 0$ be a codimension two cycle in $\mathbb{A}^n_{\mathcal{O}}$. We say that Z is *generically in good position* if $\mathrm{supp}(Z_K) \cap F$ has codimension two in F (or is empty) for all F of the form $F_K(J)$, $J \subset \{1,\dots,n\}$. A non-empty subset J of $\{1,\dots,n\}$ is allowable for J if $\mathrm{supp}(Z) \supset F_k(J)$. This is the case if and only if

$$\mu_{J,i}^*(Z) - E_k \geq 0.$$

Here, the multiplicities in the pull-back $\mu_{J,i}^*(Z)$ are defined using the alternating sum of lengths of Tors introduced by Serre [7] (this is not in general the same as taking the cycle associated to the scheme-theoretic pull-back), and the above inequality is a consequence of the positivity part of Serre's intersection multiplicity conjecture for local rings smooth over a DVR.

The game starts with the choice of a codimension two cycle $Z > 0$ in $\mathbb{A}^n_{\mathcal{O}}$, generically in good position, with $0_k \in \mathrm{supp}(Z)$. Player A moves by choosing a subset J of $\{1,\dots,n\}$, allowable for Z. B moves by choosing an element i of J, and forming the cycle $Z' := \mu_{J,i}^*(Z) - E_k$. If 0_k is not in $\mathrm{supp}(Z')$, A wins; if 0_k is in $\mathrm{supp}(Z')$, the game continues with Z' replacing Z. Our main theorem is

Theorem 0.2. *For each codimension two cycle $Z > 0$ in $\mathbb{A}^n_{\mathcal{O}}$, generically in good position, A has a winning stategy for the blow-up game for Z.*

The idea of the proof is as follows: Spivakovsky proves his result by translating the codimension one game into a purely combinatorial game, replacing the cycle Z with the *Newton polygon* of the defining equation of Z. The various conditions and transformations in the game are then translated into conditions and transformations on the Newton polygon. Spivakovsky then considers this purely combinatorial game, with the starting point being a so-called positively convex polyhedron with integral vertices, and develops a winning strategy in this setting. Actually, the game Spivakovsky considers is a bit weaker than the one we need; the extra steps to win the desired game were supplied by Bloch. See §5.6 for details.

Our method of proof is to define the Newton polygon of a positive codimension two cycle Z in $\mathbb{A}^n_{\mathcal{O}}$, generically in good position, and to show that this polygon (which is positively convex by definition) has integral vertices, and behaves under the transformation $Z \mapsto \mu_{J,i}^*(Z) - E_k$ the same way as in the codimension one case. Spivakovsky's winning strategy for the polyhedral game then gives a winning strategy for the geometric game in codimension two.

Our main application of Theorem 0.2 is a generalization of Bloch's result on "moving cycles by blowing-up", as described in [3], especially Theorem 2.1.2 of [3]. Let B be the spectrum of a Dedekind domain, $S \to B$ a smooth B-scheme, and ∂S a reduced strict relative normal crossing divisor on S, i.e., ∂S is a reduced closed subscheme of B of pure codimension one, and if we write ∂S as a union of irreducible components,

$$\partial S = \cup_{i=1}^{t} \partial S_i,$$

then, for each subset I of $\{1, \dots, t\}$, the closed subscheme $\partial S_I := \cap_{j \in I} \partial S_j$ is smooth over B, and of pure codimension $|I|$ in S. A *face* of S is a subscheme of the form ∂S_I. If $p_1 : S_1 \to S$ is the blow up a face of S, then S_1 is a smooth B-scheme, and the subscheme $\partial S_1 := p^{-1}(\partial S)_{\mathrm{red}}$ is a reduced strict relative normal crossing divisor on S_1, so we may blow up a face of S_1, and so on, forming a *sequence of blow-ups of faces*

$$S_M \to \dots \to S_1 \to S_0 = S.$$

The induced map $p : S_M \to S$ is called an *iterated blow-up of faces*.

Now let $f : Z \to S$ be a morphism of finite type, and $p : S' \to S$ an iterated blow-up of faces. Suppose that, for each generic point η of Z, $f(\eta)$ is not in ∂S. Since p is an isomorphism over $S \setminus \partial S$, the projection $Z \times_S S' \to Z$ has a canonical section σ over the dense open subscheme $f^{-1}(S \setminus \partial S)$. We let $p^{-1}[Z]$ denote the closure of the image of σ in $Z \times_S S'$, and $p^{-1}[f] : p^{-1}[Z] \to S'$ the induced morphism; we call $p^{-1}[Z]$ the *strict transform* of Z, and $p^{-1}[f]$ the strict transform of f. We say that a morphism $p : Z \to S$ *intersects all faces of* S *properly* if $\mathrm{codim}_Z(f^{-1}(\partial S_I)) \geq |I|$ for all indices I. Using Theorem 0.2 and the arguments of [3], we show:

Theorem 0.3. *Let $f : Z \to S$ be a morphism of finite type. Then there exists an iterated blow-up of faces $p : S' \to S$ such that $p^{-1}[Z] \to S'$ intersects all faces of S' properly.*

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1. Notions from polyhedral geometry

We include this section to fix notation, and for the reader's convenience. This material is mostly taken from [6] and [5].

A *polyhedron* P in $\mathbb{R}^n$ is a subset defined by a matrix inequality of the form

$$A \cdot x \geq b,$$

with $A \in M_{r \times n}(\mathbb{R})$, $x = (x_1, \dots, x_n)$, and $b \in \mathbb{R}^r$. If $b = 0$, we call the resulting polyhedron a *polyhedral cone*. We call a polyhedron *rational* if we may take A in $M_{r \times n}(\mathbb{Q})$ and $b \in \mathbb{Q}^r$.

For $\omega \in \mathbb{R}^n$, let l_ω be the linear function on $\mathbb{R}^n$ defined by dot product: $l_\omega(x) = \omega \cdot x$. Let P be a polyhedron. The ω-face of P is the subset

$$\{u \in P \mid l_\omega(u) \leq l_\omega(v) \text{ for all } v \in P\}.$$

We denote the ω-face of P by $F_P(\omega)$. Each subset F of P which is an ω-face for some ω is called a *face* of P. A *vertex* of P is a zero-dimensional face. A face of a polyhedron is obviously also a polyhedron.

Let F be a face of a polyhedron P. The *open face* F^0 is the subset of F gotten by removing all proper faces $F' \subset F$ from F. F^0 is relatively open in F. If P is a rational polyhedron (resp. a polyhedral cone), we call P^0 a rational relatively open polyhedron (resp. a relatively open polyhedral cone).

We set

$$\mathbb{R}^n_+ = \{(r_1, \dots , r_n) \in \mathbb{R}^n \mid r_i \geq 0;\ i = 1, \dots , n\};$$

if S is a subset of $\mathbb{R}^n$, we write S_+ for $S \cap \mathbb{R}^n_+$. A subset S of $\mathbb{R}^n_+$ is called *positively convex* if S contains the translate $s + \mathbb{R}^n_+$ of $\mathbb{R}^n_+$ for each $s \in S$.

A *polyhedral complex* Δ in $\mathbb{R}^n$ is a finite collection of polyhedra in $\mathbb{R}^n$ such that

1. if $P \in \Delta$ and F is a face of P, then $F \in \Delta$.
2. if $P_1, P_2 \in \Delta$, then $P_1 \cap P_2$ is a face of P_1 and P_2.

A *fan* is a polyhedral complex $\mathcal{F}$ such that each polyhedron in $\mathcal{F}$ is a polyhedral cone. A fan $\mathcal{F}$ in $\mathbb{R}^n_+$ is called *complete* if $\mathbb{R}^n_+$ is the union of the polyhedra in $\mathcal{F}$. We have the relation of *containment* among polyhedral complexes, namely, $\Delta_1 \subset \Delta_2$ if each polyhedron in Δ_1 is a union of polyhedra in Δ_2.

Let Δ be a polyhedral complex, P a polyhedron in Δ. The *closed star neighborhood* of P in Δ, $\mathcal{C}_\Delta(P)$, is the union of all $P' \in \Delta$ with $P' \supset P$. The *open star neighborhood* of P^0, $\mathcal{U}_\Delta(P^0)$, is the union of all relatively open polyhedral P'^0 with $P' \supset P$. $\mathcal{U}_\Delta(P^0)$ is the interior (relative to Δ) of $\mathcal{C}_\Delta(P)$. If p is a point of Δ, we let $\mathcal{C}_\Delta(p) = \mathcal{C}_\Delta(P)$, $\mathcal{U}_\Delta(p) = \mathcal{U}_\Delta(P^0)$, where P is the smallest polyhedron in Δ containing p

Let $\mathcal{F}$ be a complete fan in $\mathbb{R}^n_+$. A continuous function

$$m : \mathbb{R}^n_+ \to \mathbb{R}_+$$

is called *piecewise linear* with respect to $\mathcal{F}$ if the restriction of m to each polyhedral cone $C \in \mathcal{F}$ is linear, i.e., there is a linear function $L : \mathbb{R}^n \to R$ such that L and m agree on C. If $m : \mathbb{R}^n_+ \to \mathbb{R}_+$ is piecewise linear, then the set

$$P(m) := \cap_{\omega \in \mathbb{R}^n_+} \{x \in \mathbb{R}^n_+ \mid l_\omega(x) \geq m(x)\}$$

is a positively convex polyhedron in $\mathbb{R}^n_+$.

A function $m : \mathbb{R}^n_+ \to \mathbb{R}$ is called *convex* if

$$m(x \cdot p + (1 - x) \cdot q) \leq x \cdot m(p) + (1 - x) \cdot m(q)$$

for all $x \in \mathbb{R}$, $0 \leq x \leq 1$, and all $p, q \in \mathbb{R}^n_+$. If m is continuous, then convexity is a local property; if m is piecewise linear with respect to some complete flag, then $-m$ is convex if and only if, for each $x \in \mathbb{R}^n_+$, there is a linear function L on $\mathbb{R}^n$ such that

$$m(x) = L(x) \text{ and } m(y) \leq L(y) \text{ for all } y \in \mathbb{R}^n_+.$$

Let $P \subset \mathbb{R}^n_+$ be a positively convex polyhedron. For each $\omega \in \mathbb{R}^n_+$, let

$$\mathcal{N}_P(\omega) = \{\eta \in \mathbb{R}^n_+ \mid F_P(\eta) = F_P(\omega)\}$$

Clearly the closure $\overline{\mathcal{N}_P(\omega)}$ is the set of all η with $F_P(\eta) \supset F_P(\omega)$. Each $\overline{\mathcal{N}_P(\omega)}$ is a convex polyhedral cone in $\mathbb{R}^n_+$, and the collection of the $\overline{\mathcal{N}_P(\omega)}$ forms a complete fan in $\mathbb{R}^n_+$, the *normal fan* of P, which we denote by $\mathcal{N}_P$.

More generally, if $P_1, \dots , P_r$ are positively convex polyhedra in $\mathbb{R}^n_+$, let

$$\mathcal{N}_{P_1,\dots,P_r}(\omega) = \{\eta \in \mathbb{R}^n_+ \mid F_{P_i}(\eta) = F_{P_i}(\omega) \text{ for } i = 1, \dots , r\}$$

Then $\overline{\mathcal{N}_{P_1,\dots,P_r}(\omega)}$ is a convex polyhedral cone in $\mathbb{R}^n_+$, and the collection of the $\overline{\mathcal{N}_{P_1,\dots,P_r}(\omega)}$ form the complete fan $\mathcal{N}_{P_1,\dots,P_r}$.

Let P be a positively convex polyhedron in $\mathbb{R}^n_+$. For each $\omega \in \mathbb{R}^n_+$, the function l_ω has a minimum on P, namely, on the ω-face of P; let $\min_P(\omega)$ be this value. The function $\min_P$ is piecewise linear with respect to $\mathcal{N}_P$, $-\min_P$ is convex, and

$$P = \cap_{\omega \in \mathbb{R}^n_+} \{x \in \mathbb{R}^n_+ \mid l_\omega(x) \geq \min_P(\omega)\}.$$

$\min_P$ is called the *characteristic function* of the positively convex polyhedron P. There is a converse to this identity, namely:

Lemma 1.1. *Let $\mathcal{F}$ be a complete fan in $\mathbb{R}^n_+$, $m : \mathbb{R}^n_+ \to \mathbb{R}_+$ a continuous function which is piecewise linear with respect to $\mathcal{F}$. Let P be the positively convex polyhedron in $\mathbb{R}^n_+$,*

$$P = \cap_{\omega \in \mathbb{R}^n_+} \{x \in \mathbb{R}^n_+ \mid l_\omega(x) \geq m(\omega)\}.$$

Suppose that $-m$ is convex. Then $m = \min_P$; in particular, for each $\omega \in \mathbb{R}^n_+$, there is a point $p \in P$ such that $l_\omega(p) = m(p)$. In addition, $\mathcal{F} \supset \mathcal{N}_P$.

Proof. Take $\omega \in \mathbb{R}^n_+$. Since m is piecewise linear with respect to $\mathcal{F}$ and $-m$ is convex, there is a linear function L on $\mathbb{R}^n_+$ such that

$$L(\omega) = m(\omega),\ L(\eta) \geq m(\eta) \text{ for all } \eta \in \mathbb{R}^n_+.$$

Since L is linear and non-negative on $\mathbb{R}^n_+$, the intersection of the hyperplanes $l_\eta(x) = L(\eta)$ contains a point p of $\mathbb{R}^n_+$. Indeed, we have

$$L(r_1, \dots, r_n) = \sum_i a_i r_i$$

for some $a_i \in \mathbb{R}$. We have $a_i \geq 0$ since $L \geq 0$ on $\mathbb{R}^n_+$. Then $p = (a_1, \dots, a_n)$ is a point in the intersection of the hyperplanes $l_\eta(x) = L(\eta)$. Clearly $l_\omega(p) = m(\omega)$. Since $L(\eta) \geq m(\eta)$ for all $\eta \in \mathbb{R}^n_+$, p is in P, so $l_\omega(p) \geq \min_P(\omega)$. As $\min_P(\omega) \geq m(\omega)$, we have $\min_P(\omega) = m(\omega)$, as desired.

The containment $\mathcal{F} \supset \mathcal{N}_P$ follows from the characterization of the normal polyhedron $\overline{\mathcal{N}_P(\omega)}$ as the largest polyhedral cone in $\mathbb{R}^n_+$ which contains ω in its interior, and on which $\min_P$ is linear. □

2. Gröbner bases

We review some notions related to Gröbner bases for ideals in polynomial rings over a noetherian ring. As a general reference, we refer the reader to [6]. Many of the arguments here are adapted directly from [6], where the treatment is given for Gröbner bases in $k[X_1, \dots, X_n]$, k a field. Some aspects of the theory for a noetherian ring have been treated by Adams and Loustaunou in [1]; we recall the basic concepts in the following section.

2.1. **Definitions.** Let R be a noetherian commutative ring. For $I = (i_1, \dots, i_n) \in \mathbb{N}^n$, we have the element $X^I := X_1^{i_1} \cdot \ldots \cdot X_n^{i_n}$ of $R[X_1, \dots, X_n]$. An element of $R[X_1, \dots, X_n]$ of the form rX^I, $r \in R$, is called a *monomial*, an ideal $\mathcal{I} \subset R[X_1, \dots, X_n]$ is called a *monomial ideal* if $\mathcal{I}$ is generated as an ideal by monomials.

It is evident that an ideal $\mathcal{I}$ is a monomial ideal if and only if $\mathcal{I}$ is generated as an additive group by the monomials in $\mathcal{I}$. Also, if $\mathcal{J}$ is a monomial ideal, and $f = \sum_I r_I X^I$ is in $\mathcal{J}$, then $r_I X^I$ is in $\mathcal{J}$ for each I. Similarly, if monomials $r_j X^{I_j}$ generate $\mathcal{J}$ as an ideal, and if rX^I is a monomial in $\mathcal{J}$, then

$$rX^I = \sum_j s_j X^{J_j} r_j X^{I_j}$$

for certain monomials $s_1X^{J_1}, \dots, s_NX^{J_N}$ in $R[X_1, \dots, X_n]$.

Let $<$ be an *additive well-ordering* of $\mathbb{N}^n$, i.e., $<$ is a total order, each non-empty subset of $\mathbb{N}^n$ has a minimal element, and $a < b$ implies $a + c < b + c$ for all $c \in \mathbb{N}^n$. It follows that 0 is the unique minimal element of $(\mathbb{N}^n, <)$.

Definition 2.2. For $f \in R[X_1, \dots, X_n]$, write $f = \sum_I r_IX^I$. If $f \neq 0$, let the *leading term* of f, $\mathrm{LT}_<(f)$, be the element r_IX^I, with I the maximal element in $\{J \mid r_J \neq 0\}$ with respect to $<$. Set $\mathrm{LT}_<(0) = 0$. If S is a subset of $R[X_1, \dots, X_n]$, we let $\mathrm{LT}_<(S)$ be the subgroup of $R[X_1, \dots, X_n]$ generated by the element $\mathrm{LT}_<(f)$ with $f \in S$.

Lemma 2.3. *If $\mathcal{I}$ is an ideal in $R[X_1, \dots, X_n]$, then $\mathrm{LT}_<(\mathcal{I})$ is a monomial ideal. In addition, if a monomial rX^I is in $\mathrm{LT}_<(\mathcal{I})$, then $rX^I = \mathrm{LT}_<(f)$ for some $f \in \mathcal{I}$.*

Proof. Fix an I in $\mathbb{N}^n$. To prove both statements, it suffices to show that

$$\{r \mid rX^I = \mathrm{LT}_<(f) \text{ for some } f \in \mathcal{I}\} \cup \{0\}$$

is an ideal in R. If $rX^I = \mathrm{LT}_<(f)$, $sX^I = \mathrm{LT}_<(g)$, and $r + s \neq 0$, then $(r+s)X^I = \mathrm{LT}_<(f + g)$. Similarly, if a is in R and $ar \neq 0$, then $arX^I = \mathrm{LT}_<(af)$. □

Definition 2.4. Let $\mathcal{I}$ be an ideal in $R[X_1, \dots, X_n]$. A *Gröbner basis* of $\mathcal{I}$ with respect to $<$ is a finite subset $\{f_1, \dots, f_N\}$ of I such that $\mathrm{LT}_<(f_1), \dots, \mathrm{LT}_<(f_N)$ generates $\mathrm{LT}_<(\mathcal{I})$ as an ideal.

Proposition 2.5. *Every ideal of $R[X_1, \dots, X_n]$ has a Gröbner basis.*

Proof. Let $\mathcal{I} \subset R[X_1, \dots, X_n]$ be an ideal, $\mathcal{J} = \mathrm{LT}_<(\mathcal{I})$. Since $R[X_1, \dots, X_n]$ is noetherian, $\mathcal{J}$ has finite generating set as an ideal. Since $\mathcal{J}$ is a monomial ideal, $\mathcal{J}$ is generated as an ideal by finitely many monomials in $\mathcal{J}$. By Lemma 2.3, this shows that $\mathcal{J}$ is generated by finitely many elements of the form $\mathrm{LT}_<(f)$, $f \in \mathcal{L}$, which proves the proposition. □

Proposition 2.6. *Let $\mathcal{I} \subset R[X_1, \dots, X_n]$ be an ideal. Then a Gröbner basis of $\mathcal{I}$ generates $\mathcal{I}$ as an ideal.*

Proof. Let $f_1, \dots, f_N$ be a Gröbner basis of $\mathcal{I}$, and let g be in $\mathcal{I}$. Write $\mathrm{LT}_<(g) = rX^I$. We show by induction on I (with respect to the well-ordering $<$) that g is in the ideal generated by $f_1, \dots, f_N$. If $\mathrm{LT}_<(g) = 0$, then this is clear. If $\mathrm{LT}_<(g) \neq 0$, then there are elements $r_1, \dots, r_N$ of R and $I_j \in \mathbb{N}^n$ such that

$$\mathrm{LT}_<(g) = \sum_{j=1}^{N} r_jX^{I_j}\mathrm{LT}_<(f_j).$$

Thus, if $g - \sum_{j=1}^N r_jX^{I_j}f_j \neq 0$, then $\mathrm{LT}_<(g - \sum_{j=1}^N r_jX^{I_j}f_j) = sX^J$, with $J < I$, and the induction goes through. □

2.7. **Universal Gröbner bases.** Let $\mathcal{I}$ be an ideal in $R[X_1, \dots, X_n]$. A *universal* Gröbner basis of $\mathcal{I}$ is a finite set of elements $f_1, \dots, f_N$ of $\mathcal{I}$ that is a Gröbner basis of $\mathcal{I}$ with respect to *every* additive well-ordering $<$ of $\mathbb{N}^n$.

Proposition 2.8. *Let R be a commutative Noetherian ring. Then every ideal in $R[X_1, \dots, X_n]$ has a universal Gröbner basis.*

We first prove the following lemma:

Lemma 2.9. *Let $\mathcal{I}$ be an ideal in $R[X_1,\dots,X_n]$, $f_1,\dots,f_r$ elements of $\mathcal{I}$, and $<$ an additive well-order on $\mathbb{N}^n$. Suppose there is a $g\in\mathcal{I}$ with*

$$\mathrm{LT}_<(g)\notin(\mathrm{LT}_<(f_1),\dots,\mathrm{LT}_<(f_r)).$$

Then there is an $h\in\mathcal{I}$, $h\neq 0$, such that no monomial occuring in h is in $(\mathrm{LT}_<(f_1),\dots,\mathrm{LT}_<(f_r))$.

Proof. Write $g=\sum_I r_IX^I$, and take I to be maximal such that r_IX^I is in $(\mathrm{LT}_<(f_1),\dots,\mathrm{LT}_<(f_r))$. It suffices to find a $g'\in\mathcal{I}$, $g'=\sum_J s_JX^J\neq 0$ such that, either no s_JX^J is in $(\mathrm{LT}_<(f_1),\dots,\mathrm{LT}_<(f_r))$, or, if J is the maximal index such that s_JX^J is in $(\mathrm{LT}_<(f_1),\dots,\mathrm{LT}_<(f_r))$, then $J<I$.

For this, write

$$r_IX^I=\sum_j r_jX^{I_j}\mathrm{LT}_<(f_j).$$

We may assume that, if $\mathrm{LT}_<(f_j)=a_jX^{A_j}$, then $I=I_j+A_j$ for all j with $r_j\neq 0$. Then $r_IX^I=\mathrm{LT}_<(\sum_j r_jX^{I_j}f_j)$. Thus, if $K>I$, the monomial s_KX^K occurs in $g':=g-\sum_j r_jX^{I_j}f_j$ if and only if s_KX^K occurs in g; in particular, no monomial s_KX^K occuring in g' is in $(\mathrm{LT}_<(f_1),\dots,\mathrm{LT}_<(f_r))$ if $K>I$. Clearly no monomial of the form s_IX^I occurs in g', so g' satisfies the required condition. □

Proof of Proposition 2.8. Let $\mathcal{I}\subset R[X_1,\dots,X_n]$ be an ideal, and suppose that $\mathcal{I}$ has no universal Gröbner basis. Choose for each leading term ideal $\mathrm{LT}_<(\mathcal{I})$ an additive well-order $<$ giving the ideal; this gives an identification of the set of leading term ideals $\mathrm{LT}_<(\mathcal{I})$ with a subset $\mathcal{M}$ of the set of additive well-orderings of $\mathbb{N}^n$.

It follows from Proposition 2.5 that the set of leading term ideals,

$$\{\mathrm{LT}_<(\mathcal{I})\mid\ <\in\mathcal{M}\}$$

is infinite, for if $\mathcal{M}$ were finite, the union of the Gröbner bases for each of the finitely many ideals $\mathrm{LT}_<(\mathcal{I})$ would give a universal Gröbner basis for $\mathcal{I}$.

Take $f_1\neq 0$ in $\mathcal{I}$. Since f_1 is a finite sum of monomials, $\{\mathrm{LT}_<(f_1)\mid\ <\in\mathcal{M}\}$ is a finite set, hence there is a monomial m_1 appearing in f_1 such that there are infinitely many $<\in\mathcal{M}$ with $\mathrm{LT}_<(f_1)=m_1$, and with $\mathrm{LT}_<(\mathcal{I})\neq(m_1)$. Let $\mathcal{M}_1$ be the set of such $<$.

Take a $<\in\mathcal{M}_1$. There is an $f\in\mathcal{I}$ such that $\mathrm{LT}_<(f)$ is not in (m_1). By the above lemma, we may find an $f_2\in\mathcal{I}$ such that no monomial which occurs in f_2 is in (m_1). Since f_2 has only finitely many monomials, there is a monomial m_2 occuring in f_2, and an infinite subset $\mathcal{M}_2$ of $\mathcal{M}_1$ such that, for each $<\in\mathcal{M}_2$, we have $m_2=\mathrm{LT}_<(f_2)$ and $(m_1,m_2)\neq\mathrm{LT}_<(\mathcal{I})$.

Suppose then we have elements $f_1,\dots,f_s$ of $\mathcal{I}$, monomials m_j occuring in f_j, and an infinite subset $\mathcal{M}_s$ of $\mathcal{M}$ such that for each $<\in\mathcal{M}_s$, we have

$$m_j=\mathrm{LT}_<(f_j);\quad j=1,\dots,s,$$

and $\mathrm{LT}_<(\mathcal{I})$ strictly contains $(m_1,\dots,m_s)$. In addition, we suppose that $m_{j+1}\notin(m_1,\dots,m_j)$ for $j=1,\dots,s-1$. We repeat the above argument to find an $f_{s+1}\in\mathcal{I}$ such that no monomial occuring in f_{s+1} is in $(m_1,\dots,m_s)$. There is similarly a monomial m_{s+1} occuring in f_{s+1} such that $m_{s+1}=\mathrm{LT}_<(f_{s+1})$ and $(m_1,\dots,m_{s+1})\neq\mathrm{LT}_<(\mathcal{I})$ for infinitely many $<\in\mathcal{M}_s$. We let $\mathcal{M}_{s+1}$ be this infinite subset, and the induction goes through. This gives us the ascending chain of ideals

$$(m_1) \subsetneq (m_1, m_2) \subsetneq \ldots \subsetneq (m_1, \ldots, m_s) \subsetneq \ldots,$$

contrary to the Noetherian hypothesis on R. □

Theorem 2.10. *Let R be a commutative Noetherian ring, $\mathcal{I} \subset R[X_1, \ldots, X_n]$ an ideal. Then the set of leading term ideals $\mathrm{LT}_<(\mathcal{I})$, as $<$ runs over all additive well-orderings of $\mathbb{N}^n$, is finite.*

Proof. Let $f_1, \ldots, f_N$ be a universal Gröbner basis for $\mathcal{I}$ (Proposition 2.8), and $<$ an additive well-ordering of $\mathbb{N}^n$. By definition, $\mathrm{LT}_<(\mathcal{I}) = (\mathrm{LT}_<(f_1), \ldots, \mathrm{LT}_<(f_N))$, but as each f_i is a sum of finitely many monomials, there are only finitely many such ideals. □

2.11. **Weight vectors.** We call an element of $\mathbb{R}_+^n$ a *weight vector* for $\mathbb{A}^n$, or simply a weight vector. We have the standard dot product on $\mathbb{R}^n$,

$$(v_1, \ldots, v_n) \cdot (w_1, \ldots, w_n) = \sum_j v_j w_j.$$

If ω is a weight vector, and $<$ is an additive well-ordering on $\mathbb{N}^n$, we form the additive well-ordering $<_\omega$ defined by

$$I <_\omega J \iff \begin{cases} \omega \cdot I < \omega \cdot J \\ \quad\text{or} \\ \omega \cdot I = \omega \cdot J \text{ and } I < J \end{cases}$$

For $f = \sum_I r_I X^I \neq 0$ in $R[X_1, \ldots, X_n]$, ω a weight vector, we let

$$\deg_\omega(f) = \max\{\omega \cdot I \mid r_I \neq 0\},$$

and we set

$$\mathrm{in}_\omega(f) = \sum_{\substack{I \\ \omega \cdot I = \deg_\omega(f)}} r_I X^I.$$

We set $\mathrm{in}_\omega(0) = 0$ and $\deg_\omega(0) = -\infty$. We call $\mathrm{in}_\omega(f)$ the *initial form* of f for ω.

We say $f \in R[X_1, \ldots, X_n]$ is ω-homogeneous, of ω-degree d, if $\mathrm{in}_\omega(f) = f$, and if $d = \deg_\omega(f)$. Clearly $f = \sum_I r_I X^I$ is ω-homogeneous of degree d if and only if $\omega \cdot I = d$ for all I with $r_I \neq 0$. An ideal $\mathcal{J}$ of $R[X_1, \ldots, X_n]$ is called ω-homogeneous if $\mathcal{J}$ is generated by ω-homogeneous elements.

The elementary properties of homogeneous polynomials and ideals carry over without change to the ω-homogeneous case, with the exception that the subgroup of $\mathbb{R}_+$ of possible ω-degrees is not a discrete subgroup unless ω is in $\mathbb{R}_+\mathbb{N}^n$. For example:

(2.1)

1. Each $f \in R[X_1, \dots, X_n]$ is uniquely a finite sum
$$f = \sum_d f_d,$$
with f_d ω-homogeneous of degree d; we call f_d the ω-homogeneous component of f of ω-degree d.
2. $\deg_\omega(fg) = \deg_\omega(f) + \deg_\omega(g)$ if f and g are ω-homogeneous and $fg \neq 0$.
3. Suppose that R is an integral domain. If $f = gh$, then f is ω-homogeneous if and only if both g and h are ω-homogeneous.
4. An ideal $\mathcal{J}$ is ω-homogeneous if and only if $\mathcal{J} = \oplus_d \mathcal{J}_d$, where $\mathcal{J}_d$ is the subgroup of $\mathcal{J}$ of elements of ω-degree d (together with 0).

In addition, we have the following evident but useful formula

$$\mathrm{LT}_<(\mathrm{in}_\omega(f)) = \mathrm{LT}_{<_\omega}(\mathrm{in}_\omega(f)) = \mathrm{LT}_{<_\omega}(f). \tag{2.2}$$

For $S \subset R[X_1, \dots, X_n]$, we let $\mathrm{in}_\omega(S)$ be the subgroup of $R[X_1, \dots, X_n]$ generated by $\{\mathrm{in}_\omega(f) \mid f \in S\}$.

Lemma 2.12. *If $\mathcal{I} \subset R[X_1, \dots, X_n]$ is an ideal, then $\mathrm{in}_\omega(\mathcal{I})$ is a ω-homogeneous ideal. In addition, each ω-homogeneous element of $\mathrm{in}_\omega(\mathcal{I})$ is of the form $\mathrm{in}_\omega(f)$ for some $f \in \mathcal{I}$.*

Proof. For $f \in R[X_1, \dots, X_n]$, $r \in R$, $I \in \mathbb{N}^n$, we have
$$\mathrm{in}_\omega(rX^I f) = rX^I \mathrm{in}_\omega(f)$$
if $rX^I \mathrm{in}_\omega(f) \neq 0$. For $g \in R[X_1, \dots, X_n]$ with $\mathrm{in}_\omega(g)$ having the same ω-degree as $\mathrm{in}_\omega(f)$, we have
$$\mathrm{in}_\omega(f + g) = \mathrm{in}_\omega(f) + \mathrm{in}_\omega(g)$$
if $\mathrm{in}_\omega(f) + \mathrm{in}_\omega(g) \neq 0$. This suffices to prove the lemma. □

Proposition 2.13. *Let $f_1, \dots, f_r$ be a universal Gröbner basis for an ideal $\mathcal{I} \subset R[X_1, \dots, X_n]$. Then for each weight vector ω, we have*
$$(\mathrm{in}_\omega(f_1), \dots, \mathrm{in}_\omega(f_r)) = \mathrm{in}_\omega(\mathcal{I}).$$

Proof. Since $f_1, \dots, f_r$ is a universal Gröbner basis, $\mathrm{LT}_{<_\omega}(f_1), \dots, \mathrm{LT}_{<_\omega}(f_r)$ generates the leading term ideal $\mathrm{LT}_{<_\omega}(\mathcal{I})$. It follows from (2.2) that $\mathrm{LT}_{<_\omega}(\mathcal{I}) = \mathrm{LT}_{<_\omega}(\mathrm{in}_\omega(\mathcal{I}))$, and that $\mathrm{in}_\omega(f_1), \dots, \mathrm{in}_\omega(f_r)$ is a Gröbner basis for $\mathrm{in}_\omega(\mathcal{I})$. The proposition follows from this and Proposition 2.6. □

Theorem 2.14. *Let $\mathcal{I} \subset R[X_1, \dots, X_n]$ be an ideal. Then*

1. *The set of ideals of the form $\mathrm{in}_\omega(\mathcal{I})$, ω a weight vector, is finite.*
2. *For each weight vector ω, the set of weight vectors η with $\mathrm{in}_\eta(\mathcal{I}) = \mathrm{in}_\omega(\mathcal{I})$ contains a relatively open rational polyhedral cone C with $\omega \in C$.*

Proof. Let $f_1, \dots, f_s$ be a universal Gröbner basis for $\mathcal{I}$. For each f_i, there are only finitely many polynomials of the form $\mathrm{in}_\omega(f_i)$. By Proposition 2.13, the finite set of ideals $\{(\mathrm{in}_\omega(f_1), \dots, \mathrm{in}_\omega(f_s))\}$, as ω runs over all weight vectors, is equal to the set of ideals of the form $\mathrm{in}_\omega(\mathcal{I})$, proving (1).

For (2), it is clear that, for fixed $f = \sum_I r_I X^I \in R[X_1, \dots, X_n]$ and fixed weight vector ω, the set of weight vectors η such that $\mathrm{in}_\omega(f) = \mathrm{in}_\eta(f)$ forms an a relatively open rational polyhedral cone containing ω. Applying this remark to the generators

$$\mathrm{in}_\omega(f_1), \dots, \mathrm{in}_\omega(f_s)$$

of the ideal $\mathrm{in}_\omega(\mathcal{I})$, and noting that a finite intersection of a relatively open rational polyhedral cones containing ω is again a relatively open rational polyhedral cone, proves (2). □

Corollary 2.15. *Let $\mathcal{I} \subset R[X_1, \dots, X_n]$ be an ideal, ω a weight vector. Then for every $\epsilon > 0$, there is a weight vector η such that*

1. $\mathrm{in}_\omega(\mathcal{I}) = \mathrm{in}_\eta(\mathcal{I})$.
2. *η is in $\mathbb{Q}^n_+ \subset \mathbb{R}^n_+$.*
3. *$|\eta - \omega| < \epsilon$, where $| - |$ is the standard absolute value on $\mathbb{R}^n$.*

Proof. This follows from Theorem 2.14(2) and the fact that the $\mathbb{Q}$-points of a relatively open rational polyhedral cone are everywhere dense. □

2.16. **Homogeneous and non-homogeneous ideals.** For our applications, we are really interested in the ideals generated by the terms of *smallest* ω-degree in f, rather than the terms of maximal ω-degree $\mathrm{in}_\omega(f)$. One can go from the one to the other by the process of homogenization and dehomogenization, as we now explain.

Let ω be a weight vector, $f \neq 0$ an element of $R[X_1, \dots, X_n]$. Write f as a finite sum of its ω-homogeneous components,

$$f = \sum_d f_d, \ \deg_\omega(f_d) = d.$$

We let $\mathrm{ld}_\omega(f) = f_{d_0}$, where d_0 is the minimum among the ω-degrees d which occur. Similarly, if $\mathcal{I} \subset R[X_1, \dots, X_n]$ is an ideal, we let $\mathrm{ld}_\omega(\mathcal{I})$ be the ω-homogeneous ideal generated by the $\mathrm{ld}_\omega(f)$, for $f \in \mathcal{I}$.

For $f = \sum_I r_I X^I \in R[X_1, \dots, X_n]$ of (usual) degree d, let $f^h \in R[X_0, \dots, X_n]$ be the corresponding homogeneous polynomial of degree d,

$$f^h = \sum_I r_I X_0^{d-|I|} X^I.$$

If $F \in R[X_0, \dots, X_n]$ is homogeneous, we let $F^a \in k[X_1, \dots, X_n]$ be the polynomial $F(1, X_1, \dots, X_n)$. Clearly we have $f = (f^h)^a$, and $F = X_0^s (F^a)^h$, where $s = \deg(F) - \deg(F^a)$.

Applying these operations to ideals gives us the operations

$$\mathcal{I} \mapsto \mathcal{I}^h, \ \mathcal{J} \mapsto \mathcal{J}^a,$$

sending an ideal $\mathcal{I}$ in $R[X_1, \dots, X_n]$ to the homogeneous ideal $\mathcal{I}^h$, and sending the homogeneous ideal $\mathcal{J}$ in $R[X_0, \dots, X_n]$ to the ideal $\mathcal{J}^a$ in $R[X_1, \dots, X_n]$. We have

$$(\mathcal{I}^h)^a = \mathcal{I}; \quad \mathcal{J} \subset (\mathcal{J}^a)^h,$$

with the latter inclusion an equality if and only if $R[X_0, \dots, X_n]/\mathcal{J}$ is X_0-torsion free.

If $\omega = (\omega_0, \dots, \omega_n) \in \mathbb{R}_+^{n+1}$ is a weight vector, set

$$\omega^a := (\omega_0 - \omega_1, \dots, \omega_0 - \omega_n).$$

We call ω *non-negative* if ω^a is a weight vector, i.e., if $\omega_0 \geq \omega_i$ for $i = 1, \dots, n$. Clearly every weight vector η in $\mathbb{R}_+^n$ can be written in the form $\eta = \omega^a$ for ω a non-negative weight vector. Converslev, if $\eta = (\eta_1, \dots, \eta_n)$ is a weight vector, choose an i with η_i maximal, and let η^h be the weight vector

$$(\eta_i, \eta_i - \eta_1, \dots, \eta_i - \eta_n).$$

Clearly $(\eta^h)^a = \eta$ and $(\omega^a)^h = \omega - \omega_j(1, \dots, 1)$, where ω_j is the minimum among the components of ω.

If $X^I \in R[X_0, \dots, X_n]$ is a monomial of degree d, and $\omega = (\omega_0, \dots, \omega_n)$ is a non-negative weight vector, we have

$$\deg_\omega(X^I) = d\omega_0 - \deg_{\omega^a}((X^I)^a). \tag{2.3}$$

Lemma 2.17. *Let ω be a non-negative weight vector, $\eta = \omega^a$, $\mathcal{I} \subset R[X_1, \dots, X_n]$ an ideal. Then*

$$\mathrm{ld}_\eta(\mathcal{I}) = (\mathrm{in}_\omega(\mathcal{I}^h))^a.$$

Proof. It follows directly from (2.3) that, for $f \in R[X_1, \dots, X_n]$, we have

$$\mathrm{ld}_\eta(f) = (\mathrm{in}_\omega(f^h))^a.$$

The lemma is an immediate consequence of this identity. □

Theorem 2.18. *Let $\mathcal{I} \subset R[X_1, \dots, X_n]$ be an ideal. Then*

1. *There are elements $f_1, \dots, f_s$ of $\mathcal{I}$ such that*
$$\mathrm{ld}_\omega(\mathcal{I}) = (\mathrm{ld}_\omega(f_1), \dots, \mathrm{ld}_\omega(f_s))$$
for all weight vectors ω. In particular, the set of ideals $\{\mathrm{ld}_\omega(\mathcal{I}) \mid \omega \in \mathbb{R}_+^n\}$, is finite.
2. *For each weight vector η, the set of weight vectors τ with $\mathrm{ld}_\tau(\mathcal{I}) = \mathrm{ld}_\eta(\mathcal{I})$ contains a relatively open rational polyhedral cone C with $\eta \in C$.*
3. *Given a weight vector η and an $\epsilon > 0$, there is a weight vector τ such that*
   (a) $\mathrm{ld}_\eta(\mathcal{I}) = \mathrm{ld}_\tau(\mathcal{I})$.
   (b) *τ is in $\mathbb{Q}_+^n \subset \mathbb{R}_+^n$.*
   (c) $|\eta - \tau| < \epsilon$.

Proof. Take a universal Gröbner basis $F_1, \dots, F_s$ for $\mathcal{I}^h$, and let $f_i = F_i^a$. Then Lemma 2.17 together with Proposition 2.13 proves (1). The arguments of Theorem 2.14 and Corollary 2.15, together with (1), prove (2) and (3). □

2.19. **Torus actions.** With the aid of Theorem 2.18, we can give a description of the lowest degree ideals $\mathrm{ld}_\omega(\mathcal{I})$ in terms of a limit of $\mathcal{I}$ under a $\mathbb{G}_m$-action.

For $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}_+^n$, we have the algebraic action of the torus $\mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$ on $\mathbb{A}_R^n$:

$$\rho_\eta : \mathbb{G}_m \times \mathbb{A}^n \to \mathbb{A}^n$$
$$\rho_\eta(t, x_1, \dots, x_n) = (t^{\eta_1} x_1, \dots, t^{\eta_n} x_n).$$

Let

$$\iota_0 : \mathbb{A}^n \to \mathbb{A}^1 \times \mathbb{A}^n$$

be the inclusion $\iota_0(x) = (0, x)$. For a subscheme Y of $\mathbb{A}^n_R$, we have the subscheme $\rho_\eta^{-1}(Y)$ of $\mathbb{G}_m \times \mathbb{A}^n_R$; let $\overline{\rho_\eta^{-1}(Y)}$ denote the closure of $\rho_\eta^{-1}(Y)$ in $\mathbb{A}^1 \times \mathbb{A}^n_R$. We define the subscheme Y_η of $\mathbb{A}^n_R$ by

$$Y_\eta := \iota_0^{-1}(\overline{\rho_\eta^{-1}(Y)}).$$

Lemma 2.20. *If Y has defining ideal $\mathcal{J}$, then Y_η has defining ideal $\mathrm{ld}_\eta(\mathcal{J})$.*

Proof. Take f in $\mathcal{J}$, and write f as a sum of its η-homogeneous components,

$$f = \sum_{d=0}^{\deg_\eta(f)} f_d.$$

Then

$$\rho^*(f) = \sum_{d=0}^{\deg_\eta(f)} t^d f_d.$$

Thus, if g is in the ideal of $\overline{\rho_\eta^{-1}(Y)}$, then we can write the restriction of g to $\mathbb{G}_m \times \mathbb{A}^n$ as

$$j^* g = \sum_{j=-N}^{M} t^j \sum_d t^d f_d^j,$$

for suitable $f^j \in \mathcal{J}$. Since g extends to a regular function on $\mathbb{A}^1 \times \mathbb{A}^n_R$, we have

$$\sum_{i=0}^{j} f_i^{j-i-N} = 0$$

for $j = 0, \ldots, N-1$. Since η-homogeneous polynomials of different η-degree are R-independent, we must have $f_i^j = 0$ for $i + j < 0$. Thus $g = \sum_{i+j\geq 0} t^{i+j} f_i^j$ and $\deg_\eta(\mathrm{ld}_\eta(f^j)) \geq -j$ for $j \leq 0$, whence

$$\iota_0^* g = \sum_{j=-N}^{0} \epsilon_j \mathrm{ld}_\eta(f^j); \quad \epsilon_j = \begin{cases} 1 & \text{if } \deg_\eta(\mathrm{ld}_\eta(f^j)) = -j \\ 0 & \text{if } \deg_\eta(\mathrm{ld}_\eta(f^j)) > -j. \end{cases}$$

Therefore, the defining ideal of Y_η is contained in $\mathrm{ld}_\eta(\mathcal{J})$. On the other hand, if f is in $R[X_1, \ldots, X_n]$ with $\deg_\eta(\mathrm{ld}_\eta(f)) = d$, then $t^{-d}\rho_\eta^*(f)$ extends to a regular function on $\mathbb{A}^1 \times \mathbb{A}^n$ with

$$\iota_0^*(t^{-d}\rho_\eta^*(f)) = \mathrm{ld}_\eta(f),$$

giving the other containment. □

As an application, we have

Proposition 2.21. *Let $\mathcal{J} \subset R[X_1, \ldots, X_n]$ be an ideal, ω a weight vector. Let $\mathcal{P} \supset \mathcal{J}$ be a minimal prime ideal containing $\mathcal{J}$. If $\mathcal{J}$ is ω-homogeneous, then $\mathcal{P}$ is ω-homogeneous.*

Proof. By Theorem 2.18, there are $f_1, \ldots, f_s \in \mathcal{J}$ such that

$$\mathrm{ld}_\eta(\mathcal{J}) = (\mathrm{ld}_\eta(f_1), \ldots, \mathrm{ld}_\eta(f_s))$$

for all weight vectors η; since $\mathcal{J}$ is ω-homogeneous, we may assume that the f_i are also ω-homogeneous. Let $\mathcal{H} \subset \mathbb{R}_+^n$ be the set of η such that each f_i is η-homogeneous. $\mathcal{H}$ is clearly defined by finitely many linear equations with $\mathbb{Z}$ coefficients, hence $\mathcal{H} \cap \mathbb{Q}^n$ is everywhere dense in $\mathcal{H}$.

Suppose we can show that $\mathcal{P}$ is η-homogeneous for all $\eta \in \mathcal{H} \cap \mathbb{Q}^n$. Let $g_1, \ldots, g_r \in \mathcal{P}$ be elements such that $\mathrm{ld}_\eta(\mathcal{P}) = (\mathrm{ld}_\eta(g_1), \ldots, \mathrm{ld}_\eta(g_s))$ for all weight vectors η; we may also assume that $\mathcal{P} = (g_1, \ldots, g_s)$. If $\mathcal{P}$ is η-homogeneous for some η, then all the η-homogeneous components of each g_i are in $\mathcal{P}$. Since each g_i involves only finitely many monomials, we may assume that each g_i is η-homogeneous for all $\eta \in \mathcal{H} \cap \mathbb{Q}^n$. From this it follows that each g_i is η-homogeneous for all η in the $\mathbb{R}$-linear span of $\mathcal{H} \cap \mathbb{Q}^n$, i.e., for all of $\mathcal{H}$, in particular for ω, so $\mathcal{P}$ is ω-homogeneous.

We have therefore reduced to the case $\omega \in \mathbb{Q}_+^n$; scaling by the denominators in ω, we may assume $\omega \in \mathbb{Z}_+^n$. Let $Y = \mathrm{Spec}\,(R[X_1, \ldots, X_n]/\mathcal{J})$, $W = \mathrm{Spec}\,(R[X_1, \ldots, X_n]/\mathcal{P})$. Since $\mathcal{J}$ is ω-homogeneous, we have $\rho_\omega^{-1}(Y) = \mathbb{G}_m \times Y$, from which it follows that $\rho_\omega^{-1}(W) = \mathbb{G}_m \times W$. By Lemma 2.20, this implies $\mathrm{ld}_\omega \mathcal{P} = \mathcal{P}$, hence $\mathcal{P}$ is ω-homogeneous. □

3. Newton Polygons

3.1. **Newton polygons for polynomials.** Let $f = \sum_I r_I X^I \in R[X_1, \ldots, X_n]$ be a non-zero polynomial. The *Newton polygon* of f, $\mathrm{Np}(f)$, is the convex hull of the subset of $\mathbb{R}_+^n$

$$\bigcup_{\substack{I \\ r_I \neq 0}} I + \mathbb{R}_+^n,$$

where $I + \mathbb{R}_+^n$ denotes the translation of $\mathbb{R}_+^n$ by the element I, using the usual vector addition in $\mathbb{R}^n$. $\mathrm{Np}(f)$ is a positively convex polyhedron in $\mathbb{R}_+^n$.

One can give an alternative definition of $\mathrm{Np}(f)$ using the notion of ω-degree as follows:

Lemma 3.2. *For $f \in R[X_1, \ldots, X_n]$, $f \neq 0$, $\mathrm{Np}(f)$ is the intersection of the half-spaces*

$$l_\omega \geq \deg_\omega(\mathrm{ld}_\omega(f)),$$

for $\omega \in \mathbb{R}_+^n$ a weight vector.

We have the following elementary but useful result:

Lemma 3.3. *Take $f \in R[X_1, \ldots, X_n]$, $f \neq 0$. Then the function m_f,*

$$m_f(\omega) := \deg_\omega(\mathrm{ld}_\omega(f)) : \mathbb{R}_+^n \to \mathbb{R}_+$$

is the characteristic function of $\mathrm{Np}(f)$; in particular, m_f is piecewise linear with respect to the complete fan $\mathcal{N}_{\mathrm{Np}(f)}$, and $-m_f$ is convex.

Proof. This follows from the identity (*cf.* §1)

$$m_f(\omega) = \min_{x \in \mathrm{Np}(f)} l_\omega(x),$$

i.e., for each $\omega \in \mathbb{R}_+^n$, $\deg_\omega(\mathrm{ld}_\omega(f))$ is the minimum of l_ω on $\mathrm{Np}(f)$. □

3.4. **Multiplicities for hypersurfaces.** Let k be a field, Z a positive codimension one cycle in $\mathbb{A}^n_k$, i.e., $Z = \sum_{i=1}^N n_i Z_i$, where the Z_i are reduced, irreducible codimension one subschemes of $\mathbb{A}^n_k$, and the n_i are positive integers. Each such Z can be written as the divisor of some $f \in k[X_1, \ldots, X_n]$,

$$Z = \mathrm{div}(f);$$

f is uniquely determined by Z, up to multiplication by an element of $k \setminus \{0\}$. We call such an f a *defining equation* for Z.

Let ω be a weight vector for $k[X_1, \ldots, X_n]$. Define the multiplicity $\mathrm{mult}^{(1)}_\omega(Z)$ by

$$\mathrm{mult}^{(1)}_\omega(Z) := \deg_\omega(\mathrm{ld}_\omega(f)), \tag{3.1}$$

where f is a defining equation for Z. We extend the definition of $\mathrm{mult}^{(1)}_\omega$ to the zero cycle by setting $\mathrm{mult}^{(1)}_\omega(0) = 0$.

Lemma 3.5. *Let Z_1, Z_2 be positive codimension one cycles in $\mathbb{A}^n_k$, ω a weight vector. Then*

$$\mathrm{mult}^{(1)}_\omega(Z_1) + \mathrm{mult}^{(1)}_\omega(Z_2) = \mathrm{mult}^{(1)}_\omega(Z_1 + Z_2)$$

Proof. Let $Z = Z_1 + Z_2$, and let f_i be a defining equation for Z_i, $i = 1, 2$. Then $f_1 f_2$ is a defining equation for Z. Since k is an integral domain, we have

$$\mathrm{ld}_\omega(f_1 f_2) = \mathrm{ld}_\omega(f_1)\mathrm{ld}_\omega(f_2),$$

hence

$$\deg_\omega(\mathrm{ld}_\omega(f_1 f_2)) = \deg_\omega(\mathrm{ld}_\omega(f_1)) + \deg_\omega(\mathrm{ld}_\omega(f_2)).$$

□

Remark 3.6. Via Lemma 3.2, we may make the definition of the Newton polygon of a positive codimension one cycle Z in $\mathbb{A}^n_k$, $\mathrm{Np}(Z)$, as the intersection of the half-spaces

$$l_\omega \geq \mathrm{mult}^{(1)}_\omega(Z)$$

as ω runs over all weight vectors. If h is the defining equation for Z, we have

$$\mathrm{Np}(h) = \mathrm{Np}(Z).$$

3.7. Let $\mathcal{O}$ be a DVR with parameter π, quotient field K and residue field k. Using the formula of Remark 3.6 as a point of departure, we proceed to define the Newton polygon for certain codimension two cycles in $\mathbb{A}^n_{\mathcal{O}} := \mathrm{Spec}\, \mathcal{O}[X_1, \ldots, X_n]$ by first defining multiplicities for codimension two cycles.

Let ω be a weight vector. The *ring of constants for* ω, $R[X_1, \ldots, X_n]_\omega$, is the subring of $R[X_1, \ldots, X_n]$ consisting of those f with $\deg_\omega(f) = 0$. Explicitly, if $\omega = (\omega_1, \ldots, \omega_n)$, and $\mathcal{Z}(\omega) = \{i \mid \omega_i = 0\}$, then $R[X_1, \ldots, X_n]_\omega$ is the polynomial subring generated by the X_i with $i \in \mathcal{Z}(\omega)$.

Let $\mathcal{S}_\omega \subset \mathcal{O}[X_1, \ldots, X_n]$ be the subset $\mathcal{O}[X_1, \ldots, X_n]_\omega \setminus \pi[X_1, \ldots, X_n]$, i.e., the set of $f \in \mathcal{O}[X_1, \ldots, X_n]_\omega$ which are not divisible by π; $\mathcal{S}_\omega$ is multiplicatively closed by the Gauss lemma. The localization $\mathcal{O}(\omega) := \mathcal{S}_\omega^{-1}\mathcal{O}[X_1, \ldots, X_n]_\omega$ is a DVR containing $\mathcal{O}$, with fraction field $K(\omega)$ (resp. residue field $k(\omega)$) the quotient field of $K[X_1, \ldots, X_n]_\omega$ (resp. $k[X_1, \ldots, X_n]_\omega$). The localization $\mathcal{S}_\omega^{-1}\mathcal{O}[X_1, \ldots, X_n]$ is a polynomial algebra over $\mathcal{O}(\omega)$, with generators those X_i with $\deg_\omega(X_i) > 0$.

For $R = \mathcal{O}, K, k$, we let $F_R(\omega)$ is the subscheme of $\mathbb{A}^n_R$ defined by the ideal $(\{X_i \mid \deg_\omega(X_i) = 0\})$. We call $F_k(\omega)$ the *center* of the weight vector ω.

3.8. **Multiplicities for codimension two cycles.** We write 0_R for the subscheme of $\mathbb{A}^n_R$ defined by the ideal $(X_1, \ldots, X_n)$. A *face* of $\mathbb{A}^n_R$ is a linear subscheme of $\mathbb{A}^n_R$ defined by equations of the form $X_{i_1} = \ldots = X_{i_r} = 0$.

Let Z be a codimension two closed subscheme of $\mathbb{A}^n_\mathcal{O}$. We say that Z is *generically in good position* if for each face F_K of $\mathbb{A}^n_K$, $Z_K \cap F_K$ has codimension ≥ 2 in F_K. If Z is a codimension two cycle in $\mathbb{A}^n_\mathcal{O}$, we say that Z is generically in good position if the support of Z is generically in good position, or if $Z = 0$.

Let ω be a weight vector, Z a non-negative codimension two cycle in $\mathbb{A}^n_\mathcal{O}$, generically in good position. We want to define a real number $\mathrm{mult}^{(2)}_\omega(Z)$ with the following properties:

(3.2)

1. $\mathrm{mult}^{(2)}_\omega(Z_1 + Z_2) = \mathrm{mult}^{(2)}_\omega(Z_1) + \mathrm{mult}^{(2)}_\omega(Z_2)$.
2. $\mathrm{mult}^{(2)}_\omega(Z) \geq 0$; $\mathrm{mult}^{(2)}_\omega(Z) > 0$ if and only if $\mathrm{supp}(Z) \supset F_k(\omega)$.
3. Let $Z \geq 0$ be a codimension one cycle in $\mathbb{A}^n_k$, and let $i : \mathbb{A}^n_k \to \mathbb{A}^n_\mathcal{O}$ be the inclusion. Then $\mathrm{mult}^{(2)}_\omega(i_*Z) = \mathrm{mult}^{(1)}_\omega(Z)$.

We first need some preliminary results.

For a subscheme Y of $\mathbb{A}^n_\mathcal{O}$ with defining ideal $\mathcal{J}$, we let $\mathrm{ld}_\omega(Y)$ be the subscheme with defining ideal $\mathrm{ld}_\omega(\mathcal{J})$.

Lemma 3.9. *Let Y be a codimension two subscheme of $\mathbb{A}^n_\mathcal{O}$ with $O_K \notin Y_K$, $\omega \in \mathbb{R}^n_+$ with $F_k(\omega) = 0_k$. Then* $\mathrm{supp}(\mathrm{ld}_\omega(Y)) \subset \mathbb{A}^n_k$. *If $0_k \notin \mathrm{supp}(Y)$, then* $\mathrm{ld}_\omega(Y) = \emptyset$.

Proof. Let $\mathcal{J}$ be the defining ideal for Y. Since $Y_K \cap 0_K = \emptyset$, there is an element f of $\mathcal{J}$ with non-zero constant term $f_0 \in \mathcal{O}$. Since $\deg_\omega(X_i) > 0$ for all i by assumption, we must have

$$\mathrm{ld}_\omega(f) = f_0 \neq 0,$$

hence $\mathrm{ld}_\omega(\mathcal{J}) \cap \mathcal{O} \neq \{0\}$. Thus, $\mathrm{ld}_\omega(\mathcal{J})$ contains the ideal $\pi^s[X_1, \ldots, X_n]$ for some $s \geq 0$, hence $\mathrm{supp}(\mathrm{ld}_\omega(Y)) \subset \mathbb{A}^n_k$.

Similarly, if $0_k \notin \mathrm{supp}(Y)$, then there is an element $g \in \mathcal{J}$ with constant term g_0 having non-zero residue in k, i.e., g_0 is a unit in $\mathcal{O}$. As above, we have $g_0 \in \mathrm{ld}_\omega(\mathcal{J})$, so $\mathrm{ld}_\omega(Y) = \emptyset$. □

Lemma 3.10. *Let Y be a codimension two subscheme of $\mathbb{A}^n_\mathcal{O}$ with $O_K \notin Y_K$, $\omega \in \mathbb{R}^n_+$ with $F_k(\omega) = 0_k$. If 0_k is in* $\mathrm{supp}(Y)$, *then* $\mathrm{supp}(\mathrm{ld}_\omega(Y))$ *is a pure codimension one subset of $\mathbb{A}^n_k$ containing 0_k.*

Proof. We use the constructions of §2.19. By Theorem 2.18(2), we may assume that ω is in $\mathbb{Z}^n_+$. By Lemma 2.20, we have

$$\mathrm{ld}_\omega(Y) = Y_\omega = \iota_0^{-1}(\overline{\rho_\omega^{-1}(Y)}).$$

We can factor ρ_ω as the composition

$$\mathbb{G}_m \times \mathbb{A}^n \xrightarrow{(\mathrm{id}, \rho_\omega)} \mathbb{G}_m \times \mathbb{A}^n \xrightarrow{p_2} \mathbb{A}^n;$$

$(\mathrm{id}, \rho_\omega)$ is an automorphism of $\mathbb{G}_m \times \mathbb{A}^n_{\mathcal{O}}$ over $\mathbb{G}_m$. It follows that $\rho_\omega^{-1}(Y)$ has codimension two in $\mathbb{G}_m \times \mathbb{A}^n_{\mathcal{O}}$. Thus $\overline{\rho_\omega^{-1}(Y)}$ is codimension two in $\mathbb{A}^1 \times \mathbb{A}^n_{\mathcal{O}}$ and is Tor-independent with respect to $0 \subset \mathbb{A}^1$, i.e.,

$$\mathrm{Tor}^i_{\mathcal{O}_{\mathbb{A}^1}}(\mathcal{O}_{\overline{\rho_\omega^{-1}(Y)}}, \mathcal{O}_0) = 0; \quad i > 0.$$

Thus Y_ω has pure codimension two in $\mathbb{A}^n_{\mathcal{O}}$. If Y contains 0_k, then $\rho_\omega^{-1}(Y)$ contains $\mathbb{G}_m \times 0_k$, hence Y_ω contains 0_k.

Since $\mathrm{ld}_\omega(Y)$ is supported in $\mathbb{A}^n_k$ by Lemma 3.9, it follows that $\mathrm{supp}(\mathrm{ld}_\omega(Y))$ has pure codimension one in $\mathbb{A}^n_k$, and contains 0_k. □

We are now ready to define $\mathrm{mult}^{(2)}_\omega(W)$. Suppose first that W contains 0_k, and that $F_k(\omega) = 0_k$. Let $|\mathrm{ld}_\omega(W)| \geq 0$ be the cycle associated to the subscheme $\mathrm{ld}_\omega(W)$. By Lemma 3.10, $|\mathrm{ld}_\omega(W)|$ is a pure codimension two cycle in $\mathbb{A}^n_{\mathcal{O}}$, with support in $\mathbb{A}^n_k$, so we may consider $|\mathrm{ld}_\omega(W)|$ as a codimension one cycle in $\mathbb{A}^n_k$. Define

$$\mathrm{mult}^{(2)}_\omega(1 \cdot W) := \mathrm{mult}^{(1)}_\omega(|\mathrm{ld}_\omega(W)|). \tag{3.3}$$

If W does not contain 0_k, we set $\mathrm{mult}_\omega(1 \cdot W) = 0$.

For a general ω, we have the canonical isomorphism

$$\mathcal{S}_\omega^{-1}\mathcal{O}[X_1, \ldots, X_n] \cong \mathcal{O}(\omega)[X_1, \ldots, X_{n'}],$$

where $n - n'$ is the transcendence dimension of $K(\omega)$ over K. Let $j_\omega : \mathbb{A}^{n'}_{\mathcal{O}(\omega)} \to \mathbb{A}^n_{\mathcal{O}}$ be the corresponding inclusion, Then on $\mathbb{A}^{n'}_{\mathcal{O}(\omega)}$, ω has center $0_{k(\omega)}$, so the multiplicity $\mathrm{mult}^{(2)}_\omega(j_\omega^*(1 \cdot W))$ is defined. We set

$$\mathrm{mult}^{(2)}_\omega(1 \cdot W) := \mathrm{mult}^{(2)}_\omega(j_\omega^*(1 \cdot W)).$$

For a general codimension two cycle $Z = \sum_i n_i Z_i \geq 0$, generically in good position, we set

$$\mathrm{mult}^{(2)}_\omega(Z) := \sum_i n_i \mathrm{mult}^{(2)}_\omega(Z_i).$$

We now proceed to verify the properties (3.2).

Lemma 3.11. *The multiplicity defined in* (3.3) *satisfies the conditions (1)-(3) of* (3.2).

Proof. The property (1) is satisfied by construction. For (2), we may assume (after localization with respect to $\mathcal{S}_\omega$ and changing notation) that the center $F_k(\omega)$ is 0_k. By the additivity (1), we may assume that $Z = 1 \cdot W$, with W irreducible. By the definition (3.3), $\mathrm{mult}^{(2)}_\omega(1 \cdot W) \geq 0$. If $0_k \notin W$, then, by Lemma 3.9, $\mathrm{mult}^{(2)}_\omega(1 \cdot W) = 0$. If 0_k is in W, then by Lemma 3.10, the cycle $|\mathrm{ld}_\omega(W)|$ is non-zero. Let h be a defining equation for $|\mathrm{ld}_\omega(W)|$. By Proposition 2.21, each irreducible component of $\mathrm{supp}(|\mathrm{ld}_\omega(W)|)$ is ω-homogeneous, hence h is ω-homogeneous. If $\deg_\omega(h) = 0$, then clearly h is in k, which is impossible. Thus $\mathrm{mult}^{(2)}_\omega(1 \cdot W) = \deg_\omega(h) > 0$.

To prove (3), we may assume that $Z = 1 \cdot W$, with W a reduced, irreducible codimension one closed subscheme of $\mathbb{A}^n_k$. As in the proof of (2), we may assume that $F_k(\omega) = 0_k$. Let h be the defining equation for Z. The defining ideal for W is (π, h), where π is the parameter for $\mathcal{O}$. It is easy to see that (π, h) is a universal Gröbner basis for (π, h), so the defining ideal for $\mathrm{ld}_\omega(W)$ is $(\pi, \mathrm{ld}_\omega(h))$.

Thus $|\mathrm{ld}_\omega(W)|$, considered as a codimension one cycle in $\mathbb{A}^n_k$, has defining equation $\mathrm{ld}_\omega(h)$, and

$$\begin{aligned}\mathrm{mult}^{(2)}_\omega(i_*Z) &= \mathrm{mult}^{(1)}_\omega(|\mathrm{ld}_\omega(W)|)\\ &= \deg_\omega(\mathrm{ld}_\omega(\mathrm{ld}_\omega(h)))\\ &= \deg_\omega(\mathrm{ld}_\omega(h))\\ &= \mathrm{mult}^{(1)}_\omega(Z).\end{aligned}$$

□

Remark 3.12. Suppose that ω is in $\mathbb{Z}^n_+$, and Z is a codimension two cycle in $\mathbb{A}^n_{\mathcal{O}}$, generically in good position. Then $\mathrm{mult}^{(2)}_\omega(Z)$ is an integer. Indeed, $\mathrm{mult}^{(2)}_\omega(Z) = \mathrm{mult}^{(1)}_{\omega'}(Z')$ for some non-negative codimension one cycle Z' in $\mathbb{A}^{n'}_{k'}$, where $n' \leq n$, k' is an extension field of k, and ω' is gotten from $\omega = (\omega_1, \dots, \omega_n)$ by deleting the ω_i which are zero. Since $\deg_\eta(h)$ is an integer for all $h \in k'[X_1, \dots, X_r]$ and all $\eta \in \mathbb{Z}^r_+$, it follows that $\mathrm{mult}^{(1)}_{\omega'}(Z')$ is an integer.

With the multiplicities defined in above, we mimic the formula of Remark 3.6 to define the Newton polygon of a codimension two cycle.

Definition 3.13. Let $Z \geq 0$ be a codimension two cycle in $\mathbb{A}^n_{\mathcal{O}}$, generically in good position. The *Newton polygon* of Z, $\mathrm{Np}(Z)$, is defined by

$$\mathrm{Np}(Z) := \cap_{\omega \in \mathbb{R}^n_+}\{x \mid l_\omega(x) \geq \mathrm{mult}^{(2)}_\omega(Z)\}.$$

4. Properties of the Newton Polygon

We proceed to examine $\mathrm{Np}(Z)$. In the main result of this section, Theorem 4.8, we show that $\mathrm{Np}(Z)$ has integral vertices and that the function $\omega \mapsto \mathrm{mult}^{(2)}_\omega(Z)$ is the characteristic function of $\mathrm{Np}(Z)$.

4.1. Suppose at first that $Z = 1 \cdot W$, where W is a pure codimension two reduced and irreducible subscheme of $\mathbb{A}^n_{\mathcal{O}}$, generically in good position. Let $\mathcal{I}$ be the defining ideal of W. By Theorem 2.18(1), there are elements $f_1, \dots, f_s \in \mathcal{I}$ such that

$$\mathrm{ld}_\omega(\mathcal{I}) = (\mathrm{ld}_\omega(f_1), \dots, \mathrm{ld}_\omega(f_s)) \tag{4.1}$$

for all $\omega \in \mathbb{R}^n_+$, and there are only finitely many such ideals. Thus, enlarging the set $f_1, \dots, f_s$ if necessary, we may assume that

$$\mathrm{ld}_\eta(\mathrm{ld}_\omega(\mathcal{I})) = (\mathrm{ld}_\eta(\mathrm{ld}_\omega(f_1)), \dots, \mathrm{ld}_\eta(\mathrm{ld}_\omega(f_s))) \tag{4.2}$$

for all $\omega, \eta \in \mathbb{R}^n_+$.

Let $\mathcal{N}$ be the complete fan in $\mathbb{R}^n_+$, $\mathcal{N} := \mathcal{N}_{\mathrm{Np}(f_1), \dots, \mathrm{Np}(f_s)}$.

Lemma 4.2. *Let P be a polyhedral cone in $\mathcal{N}$. Then the function*

$$\omega \mapsto \mathrm{mult}^{(2)}_\omega(W)$$

is linear on the relatively open cone P^0.

Proof. Since P^0 is open, all $\omega \in P^0$ have the same center $F_k(\omega)$; after localizing and changing notation, we may assume that $F_k(\omega) = 0_k$ for all $\omega \in P^0$. By (4.1) and the definition of $\mathcal{N}$ we have

$$\mathrm{ld}_{\omega_1}(W) = \mathrm{ld}_{\omega_2}(W)$$

for all $\omega_1, \omega_2 \in P^0$. In particular, the cycle $|\mathrm{ld}_\omega(W)|$ is independent of the choice of $\omega \in P^0$. Let h be a defining equation for the cycle $|\mathrm{ld}_\omega(W)|$.

Since each $\mathrm{Np}(f_j)$ has integral vertices, the cone P is clearly rational, hence the rational points of P^0 are everywhere dense in P^0. On the other hand, if ω is in $\mathbb{Z}^n \cap P^0$, Proposition 2.20 implies that the scheme $\mathrm{ld}_\omega(W)$ is ω-homogeneous; as $\mathrm{ld}_{t\omega}(W) = \mathrm{ld}_\omega(W)$, $\mathrm{ld}_\omega(W)$ is ω-homogeneous for all rational points ω of P^0. Thus the cycle $|\mathrm{ld}_\omega(W)|$ has defining equation h which is ω-homogeneous for all $\omega \in P^0 \cap \mathbb{Q}^n$, hence h is ω-homogeneous for all $\omega \in P^0$. Therefore, if we write $h = \sum_I r_I X^I$, and choose some I with $r_I \neq 0$, we have

$$\begin{aligned}\mathrm{mult}^{(2)}_\omega(W) &= \deg_\omega(\mathrm{ld}_\omega(h)) \\ &= \deg_\omega(h) \\ &= \omega \cdot I\end{aligned}$$

for all $\omega \in P^0$. □

Lemma 4.3. *Take $\omega \in \mathbb{R}^n_+$. Then for all η in the open star neighborhood $\mathcal{U}_\mathcal{N}(\omega)$, we have*

$$\mathrm{ld}_\eta(\mathrm{ld}_\omega(W)) = \mathrm{ld}_\eta(W).$$

Proof. This is the same as showing the identity of ideals

$$\mathrm{ld}_\eta(\mathrm{ld}_\omega(\mathcal{I})) = \mathrm{ld}_\eta(\mathcal{I})$$

for all $\eta \in \mathcal{U}_\mathcal{N}(\omega)$. By (4.1) and (4.2), it suffices to show that

$$\mathrm{ld}_\eta(\mathrm{ld}_\omega(f_i)) = \mathrm{ld}_\eta(f_i),\ i = 1, \dots, s. \tag{4.3}$$

If P is a polyhedron in $\mathbb{R}^n_+$, and ω is in $\mathbb{R}^n_+$, we have the ω-face $F_P(\omega)$ of P. The smallest polyhedron of $\mathcal{N}_P$ containing ω is $\overline{\mathcal{N}_P(\omega)}$, and for all $\eta \in \overline{\mathcal{N}_P(\omega)}$ we have $F_P(\eta) \supset F_P(\omega)$, with $F_P(\eta) = F_P(\omega)$ if and only if $\eta \in \mathcal{N}_P(\omega) = \overline{\mathcal{N}_P(\omega)}^0$. From this it follows easily that

$$F_P(\eta) \subset F_P(\omega)$$

for all $\eta \in \mathcal{U}_{\mathcal{N}_P}(\omega)$.

If we now take $P = \mathrm{Np}(f)$ for some $f \in \mathcal{O}[X_1, \dots, X_n]$, $f = \sum_I r_I X^I$, and use the fact that

$$\mathrm{ld}_\eta(f) = \sum_{I \in F_{\mathrm{Np}(f)}(\eta)} r_I X^I,$$

we see that

$$\mathrm{ld}_\eta(f) = \mathrm{ld}_\eta(\mathrm{ld}_\omega(f))$$

whenever $F_{\mathrm{Np}(f)}(\eta) \subset F_{\mathrm{Np}(f)}(\omega)$, in particular,for $\eta \in \mathcal{U}_{\mathcal{N}_{\mathrm{Np}(f)}}(\omega)$.

Since $\mathcal{U}_\mathcal{N}(\omega) \subset \mathcal{U}_{\mathcal{N}_{\mathrm{Np}(f_i)}}(\omega)$ for each i, the identity (4.3) follows, completing the proof. □

Recall that for $\omega \in \mathbb{R}^n_+$ we have the residue field $k(\omega) \supset k$ of the DVR $\mathcal{O}(\omega)$; the localization $\mathcal{S}_\omega^{-1}\mathcal{O}[X_1, \dots, X_n]$ is canonically isomorphic to $\mathcal{O}(\omega)[X_1, \dots, X_{n'}]$, where $n - n'$ is the transcendence degree of $k(\omega)$ over k. We have the corresponding inclusion $j_\omega : \mathbb{A}^{n'}_{\mathcal{O}(\omega)} \to \mathbb{A}^n_\mathcal{O}$. In particular, we have the cycle $j^*_\omega|\mathrm{ld}_\omega(W)|$ on $\mathbb{A}^{n'}_{\mathcal{O}(\omega)}$, which we consider as a codimension one cycle in $\mathbb{A}^{n'}_{k(\omega)}$.

Lemma 4.4. *Let ω be in $\mathbb{R}^n_+$, and let $h \in k[X_1, \dots, X_n]$ be an element whose image in $k(\omega)[X_1, \dots, X_{n'}]$ is a defining equation for the codimension one cycle $j^*_\omega|\mathrm{ld}_\omega(W)|$ in $\mathbb{A}^{n'}_{k(\omega)}$. Then for all $\eta \in \mathcal{U}_\mathcal{N}(\omega)$ with center $F_k(\eta) = F_k(\omega)$, the image of $\mathrm{ld}_\eta(h)$ in $k(\omega)[X_1, \dots, X_{n'}]$ is a defining equation for $j^*_\omega|\mathrm{ld}_\eta(W)|$.*

Proof. Since the Newton polygons $\mathrm{Np}(f_i)$ have all integral vertices, each polyhedral cone in $\mathcal{N}$ is rational, and similarly for $\mathcal{N}_{\mathrm{Np}(h)}$. By (4.1), $\mathrm{ld}_\eta(W)$ depends only on the smallest polyhedral cone in $\mathcal{N}$ containing η; similarly, $\mathrm{ld}_\eta(h)$ depends only on the smallest polyhedral cone in $\mathcal{N}_{\mathrm{Np}(h)}$ containing η. Thus, it suffices to prove the lemma for η in $\mathbb{Q}^n_+$. Since scaling η by $t \in \mathbb{R}$, $t > 0$, does not affect $\mathrm{ld}_\eta(W)$ or $\mathrm{ld}_\eta(h)$, we may assume that η is in $\mathbb{Z}^n_+$. By passing to the localization $\mathcal{S}^{-1}_\omega\mathcal{O}[X_1, \dots, X_n]$ of $\mathcal{O}[X_1, \dots, X_n]$ and changing notation, we may assume that the center of η and ω is 0_k, i.e., $\eta = (\eta_1, \dots, \eta_n)$ with all $\eta_i > 0$, and similarly for ω.

We recall the $\mathbb{G}_m$-action corresponding to η from §2.19,

$$\rho_\eta : \mathbb{G}_m \times \mathbb{A}^n \to \mathbb{A}^n,$$

and the codimension two subscheme $\overline{\rho^{-1}_\eta(\mathrm{ld}_\omega(W))}$ of $\mathbb{A}^1 \times \mathbb{A}^n_\mathcal{O}$, which is Tor-independent with respect to $0 \subset \mathbb{A}^1$. We have the fiber $\mathrm{ld}_\omega(W)_\eta$ of $\overline{\rho^{-1}_\eta(\mathrm{ld}_\omega(W))}$ over $0 \in \mathbb{A}^1$; by Lemma 2.20 we have

$$\mathrm{ld}_\omega(W)_\eta = \mathrm{ld}_\eta(\mathrm{ld}_\omega(W)).$$

Since $\overline{\rho^{-1}_\eta(\mathrm{ld}_\omega(W))}$ and $0 \subset \mathbb{A}^1$ are Tor-independent, we have the identity of cycles

$$|\mathrm{ld}_\eta(\mathrm{ld}_\omega(W))| = \iota^*_0(|\overline{\rho^{-1}_\eta(\mathrm{ld}_\omega(W))}|),$$

where $\iota_0 : \mathbb{A}^n_\mathcal{O} \to \mathbb{A}^1 \times \mathbb{A}^n_\mathcal{O}$ is the inclusion $\iota(x) = (0, x)$.

We apply the same construction to the subscheme H of $\mathbb{A}^n_k$ with ideal (h), giving

$$|\mathrm{ld}_\eta(H)| = \iota^*_0(|\overline{\rho^{-1}_\eta(H)}|).$$

Since ρ_η is flat, we have the identity of cycles in $\mathbb{G}_m \times \mathbb{A}^n_\mathcal{O}$

$$|\rho^{-1}_\eta(H)| = \rho^*_\eta(|\mathrm{ld}_\omega(W)|) = |\rho^{-1}_\eta(\mathrm{ld}_\omega(W))|.$$

Taking the closure, this gives the identity of cycles in $\mathbb{A}^1 \times \mathbb{A}^n_\mathcal{O}$

$$|\overline{\rho^{-1}_\eta(H)}| = |\overline{\rho^{-1}_\eta(\mathrm{ld}_\omega(W))}|,$$

whence the identity of cycles in $\mathbb{A}^n_k$

$$|\mathrm{ld}_\eta(H)| = |\mathrm{ld}_\eta(\mathrm{ld}_\omega(W))|.$$

Since H is defined by the principal ideal (h), the former of these two cycles has defining equation $\mathrm{ld}_\eta(h)$. The latter cycle is equal to $|\mathrm{ld}_\eta(W)|$ by Lemma 4.3, which completes the proof. □

The analog of the above result in case ω and η have different centers is a bit more subtle, and we will content ourselves with a special case. We first prove

Lemma 4.5. *Suppose that Y is a codimension two subscheme of $\mathbb{A}^n_\mathcal{O}$, generically in good position, and take $\omega \in \mathbb{R}^n_+$. Then $\mathrm{ld}_\omega(Y)$ has pure codimension two in $\mathbb{A}^n_\mathcal{O}$ and $\mathrm{ld}_\omega(Y)$ is generically in good position.*

Proof. We may suppose without loss of generality that $\omega_i > 0$ for $i \le r$, and $\omega_i = 0$ for $i > r$. Let $F \subset \mathbb{A}^n_{\mathcal{O}}$ be the subscheme defined by $X_1 = \ldots, X_r = 0$.

Arguing as in the proof of Lemma 4.4, we see that each irreducible component of $\mathrm{ld}_\omega(Y)$ has codimension two in $\mathbb{A}^n_{\mathcal{O}}$.

We have the inclusion $i : F \to \mathbb{A}^n_{\mathcal{O}}$, split by the projection $p : \mathbb{A}^n_{\mathcal{O}} \to F$,

$$p(x_1, \ldots, x_n) = (0, \ldots, 0, x_{r+1}, \ldots, x_n).$$

Let T be an irreducible component of $p^{-1}i^{-1}(Y)$ which is not contained in $\mathbb{A}^n_k$. Then, since Y is generically in good position, T has codimension two in $\mathbb{A}^n_{\mathcal{O}}$ and T is generically in good position. It therefore suffices to show that $\mathrm{supp}(\mathrm{ld}_\omega(Y))$ is contained in $p^{-1}i^{-1}(Y)$.

For $f \in \mathcal{O}[X_1, \ldots, X_n]$, it follows directly from the conditions we have imposed on ω that

$$i^*f = i^*\mathrm{ld}_\omega(f),$$

hence

$$i^{-1}(Y) = i^{-1}(\mathrm{ld}_\omega(Y)).$$

On the other hand, if f is ω-homogeneous, and $i^*f \neq 0$, then f involves only the variables $X_{r+1}, \ldots, X_n$. Thus

$$p^{-1}(i^{-1}(\mathrm{ld}_\omega(Y))) \supset \mathrm{ld}_\omega(Y),$$

which completes the proof. □

We introduce some notation. Take $\omega \in \mathbb{R}^n_+$, and assume that $\mathcal{Z}(\omega) = \{i \mid \omega_i = 0\}$ is non-empty. We have the residue field $k(\omega)$ of the DVR $\mathcal{O}(\omega)$, and the codimension one cycle $|\mathrm{ld}_\omega(W)|$ in $\mathbb{A}^{n'}_{k(\omega)}$. Let $j_{\omega*}j^*_\omega|\mathrm{ld}_\omega(W)|$ be the codimension one cycle in $\mathbb{A}^n_k$ with

$$j^*_\omega j_{\omega*}j^*_\omega|\mathrm{ld}_\omega(W)| = j^*_\omega|\mathrm{ld}_\omega(W)|$$
$$\mathrm{supp}(j_{\omega*}j^*_\omega|\mathrm{ld}_\omega(W)|) = \overline{j^*_\omega|\mathrm{ld}_\omega(W)|}.$$

We call $\omega, \omega' \in \mathbb{R}^n_+$ *complementary* if $\mathcal{Z}(\omega) \cup \mathcal{Z}(\omega') = \{1, \ldots, n\}$.

Lemma 4.6. *Let $\omega, \omega' \in \mathbb{R}^n_+$ be complementary, let $\eta = \omega + \omega'$, and suppose that η is in $\mathcal{U}_\mathcal{N}(\omega)$. Let $h \in k[X_1, \ldots, X_n]$ be a defining equation for $j_{\omega*}j^*_\omega|\mathrm{ld}_\omega(W)|$, and let $j^*_\omega\mathrm{ld}_\eta(h)$ be the image of $\mathrm{ld}_\eta(h)$ in $k(\omega)[X_1, \ldots, X_{n'}]$. Then*

1. *The cycle $j^*_\omega|\mathrm{ld}_\eta(W)|$ in $\mathbb{A}^{n'}_{\mathcal{O}(\omega)}$ is supported in $\mathbb{A}^{n'}_{k(\omega)}$.*
2. *$j^*_\omega\mathrm{ld}_\eta(h)$ is a defining equation for $j^*_\omega|\mathrm{ld}_\eta(W)|$, considered as a codimension one cycle in $\mathbb{A}^{n'}_{k(\omega)}$.*

Proof. By Lemma 4.3, we have $\mathrm{ld}_\eta(W) = \mathrm{ld}_\eta(\mathrm{ld}_\omega(W))$. By Lemma 3.9, $j^*_\eta|\mathrm{ld}_\eta(W)|$ is supported in $\mathbb{A}^{n''}_{k(\eta)}$. Since ω and ω' are complementary, we have $\mathcal{O}(\eta) \subset \mathcal{O}(\omega)$, whence (1).

For (2), we may assume as in the proof of Lemma 4.4 that ω and η are in $\mathbb{Z}^n_+$. We may write the cycle $|\mathrm{ld}_\omega(W)|$ as a sum

$$|\mathrm{ld}_\omega(W)| = j_{\omega*}j^*_\omega|\mathrm{ld}_\omega(W)| + Z,$$

with $Z \ge 0$ and $j^*_\omega Z = 0$. By Proposition 2.21, each irreducible component of Z has ω-homogeneous defining ideal; by Lemma 4.5, each irreducible component of $\mathrm{supp}(Z)$ has pure codimension two in $\mathbb{A}^n_{\mathcal{O}}$ and is generically in good position.

Let $\rho_\eta : \mathbb{G}_m \times \mathbb{A}^n_{\mathcal{O}} \to \mathbb{A}^n_{\mathcal{O}}$ be the representation corresponding to η, and let $Y \subset \mathbb{A}^n_k$ be the locally principal subscheme with defining equation h. By Lemma 4.3 $\mathrm{ld}_\eta(h)$ is a defining equation for Y_η; arguing as in Lemma 4.4, and using the notation of this proof, we see that

$$|\mathrm{ld}_\eta(W)| = |Y_\eta| + \iota_0^*(\overline{\rho_\eta^*(Z)}).$$

Thus, it suffices to show that $j_\omega^* \iota_0^*(\overline{\rho_\eta^*(Z)}) = 0$. Replacing W with an irreducible component of $\mathrm{supp}(Z)$, we need to show that, for W ω-homogeneous and generically in good position, $j_\omega^*|W| = 0$ implies that $j_\omega^*|\mathrm{ld}_\eta(W)| = 0$.

So, suppose that $j_\omega^*|W| = 0$. By Lemma 3.9 (after suitable localization), this is the same as requiring that W does not contain the center $F_k(\omega)$. To fix ideas, we suppose that $\omega_i = 0$ for $i \le r$, and $\omega'_i = 0$ for $i > r$; without loss of generality we may suppose that $\omega'_i > 0$ for $i \le r$. For $I = (i_1, \dots, i_n)$, let $I_{\le r} = (i_1, \dots, i_r, 0, \dots, 0)$, and $I_{>r} = (0, \dots, 0, i_{r+1}, \dots, i_n)$. Let $i_* : k[X_1, \dots, X_n] \to k[X_1, \dots, X_r]$ be the homomorphism

$$i_*(X_i) = \begin{cases} X_i & \text{for } 1 \le i \le r \\ 0 & \text{for } r < i \le n. \end{cases}$$

Similarly, we let $i_*\omega' = (\omega'_1, \dots, \omega'_r)$.

Let $\mathcal{I}$ be the ideal defining W. The ideal $\mathcal{I}$ is ω-homogeneous; let $f_1, \dots, f_s$ be ω-homogeneous generators for $\mathcal{I}$ such that $\mathrm{ld}_\eta(f_1), \dots, \mathrm{ld}_\eta(f_s)$ generate $\mathrm{ld}_\eta(W)$.

Let $f \in k[X_1, \dots, X_n]$ be ω-homogeneous of ω-degree d, and write

$$f = \sum_I r_I X^{I_{\le r}} X^{I_{>r}}.$$

Then $d = \omega \cdot I_{>r}$ for all I with $r_I \ne 0$, and we have

$$\mathrm{ld}_{\omega'}(f) = \mathrm{ld}_\eta(f),\ \mathrm{ld}_{i_*\omega'}(i_*f) = i_*\mathrm{ld}_\eta(f).$$

From this, it follows that $\mathrm{ld}_\eta(f)$ is also ω-homogeneous, and that $j_\omega^*(\mathrm{ld}_{i_*\omega'}(i_*\mathcal{I}))$ is the unit ideal in $k(\omega)$ if and only if $j_\omega^*\mathrm{ld}_\eta(\mathcal{I})$ is the unit ideal in $k(\omega)[X_1, \dots, X_{n'}]$.

Since W does not contain $F_k(\omega)$ and is generically in good position, the pull-back $i^*(\mathrm{ld}_\omega(W))$ has pure codimension two and is generically in good position in $\mathbb{A}^r_{\mathcal{O}}$. By by Lemma 4.5, this implies that $\mathrm{ld}_{i_*\omega'}(i_*\mathcal{I})$ has pure codimension two in $\mathbb{A}^r_{\mathcal{O}}$, hence the restriction of $\mathrm{ld}_{i_*\omega'}(i_*\mathcal{I})$ to the generic point of $\mathbb{A}^r_k$ is empty. Thus $j_\omega^*(\mathrm{ld}_{i_*\omega'}(i_*\mathcal{I}))$ is the unit ideal in $k(\omega)$, completing the proof. □

Proposition 4.7. *Let W be a codimension two reduced irreducible closed subscheme of $\mathbb{A}^n_{\mathcal{O}}$, generically in good position, and let ω be a weight vector.*

1. *Let h be a defining equation for $j_{\omega*}j_\omega^*|\mathrm{ld}_\omega(W)|$. Then for all $\eta \in \mathcal{U}_{\mathcal{N}}(\omega)$ with $F_k(\eta) = F_k(\omega)$, we have*

$$\mathrm{mult}^{(2)}_\eta(W) = \deg_\eta(\mathrm{ld}_\eta(h)).$$

2. *Let ω' be complementary to ω. For $t \in \mathbb{R}_+$, let $\eta(t) = \omega + t \cdot \omega'$. Then*

$$\mathrm{mult}^{(2)}_\omega(W) = \lim_{t \to 0^+} \mathrm{mult}^{(2)}_{\eta(t)}(W).$$

Proof. To prove (1), we may assume $F_k(\omega) = 0_k$. It follows from the definition of $\mathrm{mult}^{(2)}_\eta(W)$ that we have

$$\mathrm{mult}^{(2)}_\eta(W) = \deg_\eta(\mathrm{ld}_\eta(g)),$$

where g is a defining equation for the cycle $|\mathrm{ld}_\eta(W)|$ in $\mathbb{A}^n_k$. For $\eta \in \mathcal{U}_\mathcal{N}(\omega)$, we may take $g = \mathrm{ld}_\eta(h)$, by Lemma 4.4. Since $\mathrm{ld}_\eta(\mathrm{ld}_\eta(h)) = \mathrm{ld}_\eta(h)$, (1) is proved.

For (2), let $h \in k[X_1, \dots, X_n]$ be a defining equation for $j_{\omega*}j^*_\omega|\mathrm{ld}_\omega(W)|$. By Lemma 4.6, $j^*_\omega \mathrm{ld}_{\eta(t)}(h)$ is a defining equation for $j^*_\omega|\mathrm{ld}_{\eta(t)}(W)|$ for all $t \geq 0$ sufficiently small. It follows from Proposition 2.21 that h is ω-homogeneous; clearly we have

$$\deg_\omega(j^*_\omega(h)) = \deg_\omega(h),$$
$$\deg_{\eta(t)}(j^*_{\eta(t)}(\mathrm{ld}_{\eta(t)}h)) = \deg_{\eta(t)}(\mathrm{ld}_{\eta(t)}h).$$

Since h is ω-homogeneous, we have

$$\lim_{t\to 0^+} \deg_{\eta(t)}(\mathrm{ld}_{\eta(t)}h) = \deg_\omega(h).$$

On the other hand, since there are only finitely many ideals of the form $\mathrm{ld}_{\eta(t)}(W)$, the cycle $|\mathrm{ld}_{\eta(t)}(W)|$ is independent of the choice of t for all sufficiently small $t > 0$. Thus there is a codimension one cycle A supported in $\mathbb{A}^n_k$ such that

$$j^*_\omega(A) = 0$$
$$j^*_{\eta(t)}|\mathrm{ld}_{\eta(t)}(W)| = j^*_{\eta(t)}(A + j_{\omega*}j^*_\omega|\mathrm{ld}_\omega(W)|)$$

for all $0 < t < \epsilon$. In particular, if g is a defining equation for A, we have $\deg_\omega(g) = 0$, hence

$$\mathrm{mult}^{(2)}_{\eta(t)}(W) = \deg_\omega(\mathrm{ld}_{\eta(t)}(h)) + t\deg_{\omega'}(\mathrm{ld}_{\omega'}(g)).$$

Putting this all together gives

$$\begin{aligned}\mathrm{mult}^{(2)}_\omega(W) &= \deg_\omega(j^*_\omega(h))\\ &= \deg_\omega(h)\\ &= \lim_{t\to 0^+} \deg_\omega(\mathrm{ld}_{\eta(t)}h)\\ &= \lim_{t\to 0^+} \mathrm{mult}^{(2)}_{\eta(t)}(W).\end{aligned}$$

□

The main foundation of our theory is the following theorem:

Theorem 4.8. *Let $Z > 0$ be a codimension two cycle in $\mathbb{A}^n_\mathcal{O}$, generically in good position. Then*

1. $\mathrm{Np}(Z)$ *is a positively convex polyhedron in* $\mathbb{R}^n_+$ *with integer vertices.*
2. $\mathrm{mult}^{(2)}_\omega(Z)$ *is the characteristic function of* $\mathrm{Np}(Z)$*.*

Proof. The additivity (3.3)(1) of the defining function $\mathrm{mult}^{(2)}_-(Z)$ reduces us to the case of a prime cycle. Indeed, the additivity of $\mathrm{mult}^{(2)}_-(Z)$ in Z implies that $\mathrm{Np}(Z_1 + Z_2)$ is the *Minkowski sum* of $\mathrm{Np}(Z_1)$ and $\mathrm{Np}(Z_2)$, i.e., the convex hull of the set of sums $\{x_1 + x_2 \mid x_i \in \mathrm{Np}(Z_i)\}$. The properties (1) and (2) for $\mathrm{Np}(Z_1 + Z_2)$ then follow easily from (1) and (2) for $\mathrm{Np}(Z_1)$ and $\mathrm{Np}(Z_2)$.

Thus we may assume that $Z = 1 \cdot W$, with W reduced and irreducible. Take generators $f_1, \dots, f_s$ for the defining ideal of W, satisfying the conditions (4.1) and (4.2), and let $\mathcal{N}$ be the complete fan $\mathcal{N}_{\mathrm{Np}(f_1),\dots,\mathrm{Np}(f_s)}$. From Lemma 3.3 and Proposition 4.7, the function $-\mathrm{mult}^{(2)}_-(W)$ is continuous and convex on $\mathbb{R}^n_+$. By

Lemma 4.2 together with the continuity we have just proved, $-\mathrm{mult}^{(2)}_{-}(W)$ is piecewise linear with respect to the complete fan $\mathcal{N}$.

By Lemma 1.1, this implies that $\mathrm{mult}^{(2)}_{\omega}(W)$ is the characteristic function of $\mathrm{Np}(W)$. From this, it follows that the vertices of $\mathrm{Np}(W)$ are all of the form $F_{\mathrm{Np}(W)}(\omega)$, where ω is an interior point on a polyhedral cone $P \in \mathcal{N}$, with $\dim P = n$.

Take such a polyhedral cone P and an $\omega \in P^0$. Let h be a defining equation for $|\mathrm{ld}_\omega(W)|$. Via Proposition 4.7, we have

$$F_{\mathrm{Np}(W)}(\omega) = F_{\mathrm{Np}(h)}(\omega).$$

Since $\mathrm{Np}(h)$ has integral vertices, this implies that the vertex $F_{\mathrm{Np}(W)}(\omega)$ of $\mathrm{Np}(W)$ is integral. $\square$

For a subset J of $\{1, \dots, n\}$, we let $\omega_J = ((\omega_{J,1}, \dots, \omega_{J,n})$ be the weight vector with

$$\omega_{J,j} = \begin{cases} 1 & \text{for } j \in J \\ 0 & \text{for } j \neq J. \end{cases}$$

Corollary 4.9. *Let $Z \geq 0$ be a codimension two cycle in $\mathbb{A}^n_{\mathcal{O}}$, generically in good position, and let J be a subset of* $\{1, \dots, n\}$. *Then* $\mathrm{supp}(Z) \supset F_k(\omega_J)$ *if and only if*

$$l_{\omega_J}(p) \geq 1$$

for all $p \in \mathrm{Np}(Z)$.

Proof. If $\mathrm{supp}(Z) \supset F_k(\omega_J)$, then, by (3.2)(3), we have $\mathrm{mult}^{(2)}_{\omega_J}(Z) > 0$. Since $\mathrm{mult}^{(2)}_{\omega_J}(Z)$ is an integer (see Remark 3.12), it follows that $\mathrm{mult}^{(2)}_{\omega_J}(Z) \geq 1$; since $l_\omega(p) \geq \mathrm{mult}^{(2)}_{\omega}(Z)$ for all $p \in \mathrm{Np}(Z)$ and all ω by the defining equations for $\mathrm{Np}(Z)$, we have $l_{\omega_J}(p) \geq 1$ for all $p \in \mathrm{Np}(Z)$.

Conversely, suppose that $l_{\omega_J}(p) \geq 1$ for all $p \in \mathrm{Np}(Z)$. Then the minimum of l_{ω_J} is at least one; by Theorem 4.8(2), this shows that $\mathrm{mult}^{(2)}_{\omega_J}(Z) \geq 1$. By (3.2)(3), this implies that $\mathrm{supp}(Z) \supset F_k(\omega_J)$. $\square$

5. Blowing up faces

5.1. **Affine blow-up.** Let J be subset of $\{1, \dots, n\}$, $i \in J$. We let $F(J)$ be the face of $\mathbb{A}^n_R$ defined by the ideal $(\{X_j \mid j \in J\})$. We let

$$\mu_{J,i} : \mathbb{A}^n_R \to \mathbb{A}^n_R$$

be the morphism with

$$\mu^*_{J,i}(X_j) = \begin{cases} X_j & \text{for } j \notin J \setminus \{i\} \\ X_j X_i & \text{for } j \in J \setminus \{i\}. \end{cases}$$

We have the blow-up $\mu_J : \mathbb{A}^n_{R,J} \to \mathbb{A}^n_R$ of $\mathbb{A}^n_R$ along $F(J)$, which is naturally the closed subscheme of $\mathbb{A}^n_R \times \mathbb{P}^{|J|-1}$ defined by equations $X_j T_{j'} - X_{j'} T_j$, $j, j' \in J$. The open subscheme $T_i \neq 0$ of $\mathbb{A}^n_{R,J}$ is isomorphic over $\mathbb{A}^n_R$ to $\mu_{J,i} : \mathbb{A}^n_R \to \mathbb{A}^n_R$, with coordinates Y_j,

$$Y_j = \begin{cases} T_j/T_i, & j \in J \setminus \{i\} \\ X_j, & j \notin J \setminus \{i\}. \end{cases}$$

Let $\phi_{J,i} : \mathbb{R}^n \to \mathbb{R}^n$ be the map

$$\phi_{J,i}(r_1, \ldots, r_n) = (r'_1, \ldots, r'_n),$$

with

$$r'_j = \begin{cases} r_j & \text{for } j \notin J \setminus \{i\} \\ r_j + r_i & \text{for } j \in J \setminus \{i\}. \end{cases}$$

Lemma 5.2. *Take $\omega \in \mathbb{R}^n_+$, and let Y be an irreducible, reduced codimension two closed subscheme of $\mathbb{A}^n_{\mathcal{O}}$, generically in good position. Let $Y' = \mu_{J,i}^{-1}(Y)$, $\omega' = \phi_{J,i}(\omega)$. Then*

$$\mu_{J,i}^*(|\mathrm{ld}_{\omega'}(Y)|) = |\mathrm{ld}_\omega(Y')|.$$

Proof. By Theorem 2.14, we may assume that ω is in $\mathbb{Z}^n_+$. From §2.19 have the $\mathbb{G}_m$-actions

$$\rho_\omega : \mathbb{G}_m \times \mathbb{A}^n \to \mathbb{A}^n$$
$$\rho_{\omega'} : \mathbb{G}_m \times \mathbb{A}^n \to \mathbb{A}^n$$

giving us the commutative diagram

(5.1)
$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{A}^n & \xrightarrow{\rho_\omega} & \mathbb{A}^n \\ {\scriptstyle \mathrm{id}\times\mu_{J,i}}\downarrow & & \downarrow{\scriptstyle \mu_{J,i}} \\ \mathbb{G}_m \times \mathbb{A}^n & \xrightarrow[\rho_{\omega'}]{} & \mathbb{A}^n \end{array}$$

and the subschemes $\overline{\rho_\omega^{-1}(Y')}$, $\overline{\rho_{\omega'}^{-1}(Y)}$ of $\mathbb{A}^1 \times \mathbb{A}^n$.

Let $F = F(J)$, $E = \mu_{J,i}^{-1}(F)$. From the commutativity of the diagram (5.1), we have

$$\rho_\omega^{-1}(Y') = (\mathrm{id}_{\mathbb{G}_m} \times \mu_{J,i})^{-1}(\rho_{\omega'}^{-1}(Y)),$$

and hence

$$\overline{\rho_\omega^{-1}(Y')} \subset (\mathrm{id}_{\mathbb{A}^1} \times \mu_{J,i})^{-1}(\overline{\rho_{\omega'}^{-1}(Y)}).$$

Since Y is irreducible and generically in good position, $Y' \cap \mathbb{A}^n_K$ is irreducible and generically reduced, hence $\overline{\rho_\omega^{-1}(Y')} \cap \mathbb{A}^1 \times \mathbb{A}^n_K$ is also irreducible and generically reduced. Similarly, $(\mathrm{id}_{\mathbb{A}^1} \times \mu_{J,i})^{-1}(\overline{\rho_{\omega'}^{-1}(Y)}) \cap \mathbb{A}^1 \times \mathbb{A}^n_K$ is irreducible and generically reduced. Since $\mu_{J,i} : \mathbb{A}^n_{\mathcal{O}} \setminus E \to \mathbb{A}^n_{\mathcal{O}} \setminus F$ is an isomorphism, we have

$$|\overline{\rho_\omega^{-1}(Y')}| \cap (\mathbb{A}^1 \times \mathbb{A}^n_{\mathcal{O}} \setminus 0 \times E_k) = (\mathrm{id}_{\mathbb{A}^1} \times \mu_{J,i})^*(|\overline{\rho_{\omega'}^{-1}(Y)}|) \cap (\mathbb{A}^1 \times \mathbb{A}^n_{\mathcal{O}} \setminus 0 \times E_k).$$

Since $0 \times E_k$ has codimension three in $\mathbb{A}^1 \times \mathbb{A}^n_{\mathcal{O}}$, we have the identity of codimension two cycles

$$|\overline{\rho_\omega^{-1}(Y')}| = (\mathrm{id}_{\mathbb{A}^1} \times \mu_{J,i})^*(|\overline{\rho_{\omega'}^{-1}(Y)}|).$$

Pulling back by the inclusion $\iota_0 : \mathbb{A}^n_{\mathcal{O}} \to \mathbb{A}^1 \times \mathbb{A}^n_{\mathcal{O}}$ and applying Lemma 2.20 gives

$$|\mathrm{ld}_\omega(Y')| = \mu_{J,i}^*(|\mathrm{ld}_{\omega'}(Y)|).$$

□

Proposition 5.3. *Take $\omega \in \mathbb{R}_+^n$, and let $Z \geq 0$ be a codimension two cycle in $\mathbb{A}_{\mathcal{O}}^n$, generically in good position. Then*

$$\mathrm{mult}_\omega^{(2)}(\mu_{J,i}^*(Z)) = \mathrm{mult}_{\phi_{J,i}(\omega)}^{(2)}(Z).$$

Proof. Let $Z' = \mu_{J,i}^*(Z)$, $\omega' = \phi_{J,i}(\omega)$. It follows from the definition of the map $\mu_{J,i}$ that $\mu_{J,i}(F_k(\omega))$ is contained in $F_k(\omega')$; after localization, we may assume that $F_k(\omega') = 0_k$. By Lemma 3.9 and Lemma 3.10, the cycle $\mathrm{ld}_{\omega'}(Z)$ is a codimension one cycle in $\mathbb{A}_k^n$; let h be a defining equation.

By Lemma 5.2, we have $\mathrm{ld}_\omega(Z') = \mu_{J,i}^*(\mathrm{ld}_\omega(Z))$, hence $\mathrm{ld}_\omega(Z')$ is a codimension one cycle in $\mathbb{A}_k^n$ with defining equation $\mu_{J,i}^*(h)$.

Let $\phi_{J,i}^t$ be the transpose of $\phi_{J,i}$ with respect to the standard inner product on $\mathbb{R}^n$, i.e.,

$$\phi_{J,i}^t(r_1, \dots, r_n) = (r_1'', \dots, r_n'')$$

with

$$r_j'' = \begin{cases} r_j & \text{for } j \neq i \\ \sum_{j \in J} r_j & \text{for } j = i. \end{cases}$$

Since $\mu_{J,i}^*(X^I) = X^{\phi_{J,i}^t(I)}$, we have $\deg_{\phi_{J,i}(\omega)}(X^I) = \deg_\omega(\mu_{J,i}^*(X^I))$, whence

$$\deg_\omega(\mathrm{ld}_\omega(\mu_{J,i}^*(h))) = \deg_{\omega'}(\mathrm{ld}_{\omega'}(h)).$$

The proposition follows from this and the identities

$$\mathrm{mult}_\omega^{(2)}(Z') = \deg_\omega(\mathrm{ld}_\omega(\mu_{J,i}^*(h)))$$
$$\mathrm{mult}_{\omega'}^{(2)}(Z) = \deg_{\omega'}(\mathrm{ld}_{\omega'}(h)).$$

□

For a subset S of $\mathbb{R}_+^n$, we let S^+ denote the positive convex hull of S, i.e., the convex hull of the union of the sets $s + \mathbb{R}_+^n$, for $s \in S$. If S is convex, then the union of the sets $s + \mathbb{R}_+^n$ is already convex.

Theorem 5.4. *Let $Z \geq 0$ be a codimension two cycle in $\mathbb{A}_{\mathcal{O}}^n$, generically in good position, let J be a subset of $\{1, \dots, n\}$ and $i \in J$. Then*

$$\mathrm{Np}(\mu_{J,i}^*(Z)) = \phi_{J,i}^t(\mathrm{Np}(Z))^+.$$

Proof. Let $\mathrm{Np}(Z)^* \subset (\phi_{J,i}^t)^{-1}(\mathbb{R}_+^n)$ be the intersection of the half-spaces

$$\mathrm{Np}(Z)^* := \cap_{\omega \in \phi_{J,i}(\mathbb{R}_+^n)} \{x \in (\phi_{J,i}^t)^{-1}(\mathbb{R}_+^n) \mid l_\omega(x) \geq \mathrm{mult}_\omega^{(2)}(Z)\}.$$

Since $\mathrm{mult}_\omega^{(2)}(Z)$ is the characteristic function of $\mathrm{Np}(Z)$, it follows that

$$\phi_{J,i}^t(\mathrm{Np}(Z))^+ = \phi_{J,i}^t(\mathrm{Np}(Z)^*).$$

On the other hand, by Proposition 5.3, $\mathrm{mult}_\omega^{(2)}(\mu_{J,i}^*(Z)) = \mathrm{mult}_{\phi_{J,i}(\omega)}^{(2)}(Z)$, hence

$$\begin{aligned} p \text{ is in } \mathrm{Np}(Z)^* &\iff \phi_{J,i}(\omega) \cdot p \geq \mathrm{mult}_{\phi_{J,i}(\omega)}^{(2)}(Z) \text{ for all } \omega \in \mathbb{R}_+^n \\ &\iff \omega \cdot \phi_{J,i}^t(p) \geq \mathrm{mult}_\omega^{(2)}(\mu_{J,i}^*(Z)) \text{ for all } \omega \in \mathbb{R}_+^n \\ &\iff \phi_{J,i}^t(p) \text{ is in } \mathrm{Np}(\mu_{J,i}^*(Z)). \end{aligned}$$

□

For a subset J of $\{1,\dots,n\}$, and an element $i\in J$, we let $\Phi_{J,i}:\mathbb{R}^n\to\mathbb{R}^n$ be the transformation $\Phi_{J,i}(r_1,\dots,r_n)=(r'_1,\dots,r'_n)$ with

$$r'_j=\begin{cases} r_j & \text{for } j\neq i\\ (\sum_{j\in J} r_j)-1 & \text{for } j=i.\end{cases}$$

Corollary 5.5. *Let $Z\geq 0$ be a codimension two cycle in $\mathbb{A}^n_{\mathcal{O}}$, generically in good position, let J be a subset of $\{1,\dots,n\}$ and $i\in J$. If $|J|>1$, let $E_k\subset\mathbb{A}^n_k$ be the exceptional divisor of $\mu_{J,i}:\mathbb{A}^n_k\to\mathbb{A}^n_k$; if $J=\{i\}$, we let $E_k\subset A^n_k$ be the divisor with defining equation X_i. Then*

1. *$\mu^*_{J,i}(Z)-E_k\geq 0$ if and only if $\mathrm{Np}(Z)$ is contained in the half-space*
$$\{(r_1,\dots,r_n)\in\mathbb{R}^n_+\mid \sum_{j\in J} r_j\geq 1\}.$$
2. *If the condition in (1) is satisfied, then $\mathrm{Np}(\mu^*_{J,i}(Z)-E)$ is the positive convex hull of $\Phi_{J,i}(\mathrm{Np}(Z))$.*

Proof. Assume at first that $|J|>1$. By the positivity part of Serre's intersection multiplicity theorem (for local rings smooth over a DVR, see [7]) it follows that $\mu^*_{J,i}(Z)-E\geq 0$ if and only if $\mathrm{supp}(Z)$ contains the center $F(J)$ of the blow-up $\mu_{J,i}$. Part (1) is thus a consequence of Corollary 4.9, noting that $F(J)=F(\omega_J)$.

For (2), the exceptional divisor E_k on $\mathbb{A}^n_k$ has defining equation X_i. It follows from (3.2)(3) that

$$\mathrm{mult}^{(2)}_\omega(E_k)=\omega\cdot e_i,$$

where e_i is the ith standard basis vector in $\mathbb{R}^n$. By the additivity (3.2)(1) of $\mathrm{mult}^{(2)}_\omega(-)$, it follows that

$$\mathrm{mult}^{(2)}_\omega(\mu^*_{J,i}(Z)-E_k)=\mathrm{mult}^{(2)}_\omega(\mu^*_{J,i}(Z))-\omega\cdot e_i,$$

from which it easily follows that $\mathrm{Np}(\mu^*_{J,i}(Z)-E_k)$ is the translate of $\mathrm{Np}(\mu^*_{J,i}(Z))$ by $-e_i$. The result is then a direct consequence of the description of $\mathrm{Np}(\mu^*_{J,i}(Z))$ given in Theorem 5.4.

In case $J=\{i\}$, the map $\mu_{J,i}$ is the identity, and the map $\Phi_{\{i\},i}$ is the translation by $-e_i$. The assertion (1) follows immediately from Corollary 4.9, and (2) follows as above from the identity

$$\mathrm{mult}^{(2)}_\omega(Z-E_k)=\mathrm{mult}^{(2)}_\omega(Z)-\omega\cdot e_i.$$

□

5.6. **Hironaka's game.** In [5], Spivakovsky considers the following game: Let P be a positively convex polyhedron in $\mathbb{R}^n_+$ with all vertices in $\mathbb{Q}^n$. There are two players, A and B. A moves by choosing a non-empty subset J of $\{1,\dots,n\}$ such that P is contained in the half-space $\sum_{j\in J} r_j\geq 1$ (such a J is called *allowable*). B moves by choosing an element $i\in J$, forming the positively convex polyhedron $P':=\Phi_{J,i}(P)^+$. A wins at this stage if P' contains a point $(r_1,\dots,r_n)$ with $\sum_i r_i\leq 1$; if not they keep playing with P' replacing P. The main result of [5] is that for each starting polyhedron P, A has a winning strategy, i.e., after finitely many moves, A wins.

In [3], Bloch considers a modification of this game, where the moves are the same, but where A wins if the new polyhedron P' contains the origin. Let us explain his construction of a winning strategy for player A, assuming Spivakovsky's strategy; we assume that the starting polyhedron has *integral* vertices.

Using Spivakovsky's strategy, we may assume that P contains a point with $\sum_i r_i = 1$, i.e., P contains one of the basis vectors e_i. We may also assume that each point of P satisfies $\sum_i r_i \geq 1$. By reordering the coordinates, we may assume that P contains the basis vectors $e_1, \ldots, e_s$. Suppose $s = n$. In this case, A takes $J = \{1, \ldots, n\}$, which is clearly allowable. Since e_i is in P for all i, the origin is in $\Phi_{J,i}(P)$, regardless of which i B chooses, so A wins.

In general, we proceed by descending induction on s. We let $P_{|>s}$ be the intersection of P with the $\mathbb{R}_+^{n-s}$ defined by $r_1 = \ldots = r_s = 0$. If $P_{|>s}$ is empty, then every element of P satisfies $\sum_{j=1}^{s} r_j \geq 1$. A takes $J = \{1, \ldots, s\}$. As above, J is allowable, and A wins.

If $P_{|>s}$ is non-empty, then $P_{|>s}$ is a positively convex polyhedron in $\mathbb{R}_+^{n-s}$ with integral vertices, so we may apply Spivakovsky's winning strategy to $P_{|>s}$, where A now chooses an allowable subset $J_{|>s}$ of $\{s+1, \ldots, n\}$. If $J_{|>s}$ is allowable for $P_{|>s}$, then $J := \{1, \ldots, s\} \cup J_{|>s}$ is allowable for P; indeed, if $\sum_{j\in J} r_j < 1$ for a point $p = (r_1, \ldots, r_n) \in P$, then $\sum_{j\in J} r_j = 0$, hence $r_1 = \ldots = r_s = 0$, p is in $P_{|>s}$, and $\sum_{j\in J_{|>s}} r_j = 0$. In addition, we have

$$(\Phi_{J,i}(P)^+)_{|>s} = \Phi_{J_{|>s},i}(P_{|>s})^+$$

for each $i \in J_{|>s}$. Finally, e_j is in $\Phi_{J,i}(P)^+$ for each $j = 1, \ldots, s$. Thus, we can play the two games side by side, until A wins Hironaka's game for $P_{|>s}$. If the origin is in $P_{|>s}$, then the origin is also in P, so A wins Bloch's game immediately; if e_j is in $P_{|>s}$ for some $j > s$, then e_j is in P, and A has increased s. Thus, Spivakovsky's winning strategy for Hironaka's game gives a winning strategy for Bloch's game as well.

5.7. **Proof of the main theorem.** We can now give the proof of Theorem 0.2. Let $Z > 0$ be a codimension two cycle in $\mathbb{A}_{\mathcal{O}}^n$, generically in good position. By Theorem 4.8, the Newton polygon $\mathrm{Np}(Z)$ is a positively convex polygon in $\mathbb{R}_+^n$ with integral vertices. By Corollary 4.9, $\mathrm{supp}(Z)$ contains $F_k(J)$ if and only if $\mathrm{Np}(Z)$ is contained in the half-space $\sum_{j\in J} r_j \geq 1$, i.e., if and only if J is allowable for $\mathrm{Np}(Z)$; in particular, 0_k is in $\mathrm{supp}(Z)$ if and only if $\mathrm{Np}(Z)$ does not contain the origin. By Corollary 5.5, J is allowable for $\mathrm{Np}(Z)$ if and only if $\mu_{J,i}^*(Z) > E_k$, and in this case, we have

$$\mathrm{Np}(\mu_{J,i}^*(Z) - E_k) = \Phi_{J,i}(\mathrm{Np}(Z))^+.$$

We have thus translated the blow-up game for Z described in §0 into the Bloch game for the polyhedron $\mathrm{Np}(Z)$. The strategy for winning the Bloch game for $\mathrm{Np}(Z)$ described in §5.6 thus gives a strategy for winning the blow-up game for Z, completing the proof of Theorem 0.2.

6. Moving maps by blowing up

We conclude this paper with our main application, the proof of Theorem 0.3. The proof is taken, with minor changes, from [3], where the theorem is proved for $\mathcal{O}$ a field (see [3, Theorem 2.1.2]); we will give here a proof for both a field and a DVR for the reader's convenience. We retain the notation from §0.

6.1. **Blowing up faces.** We first prove some elementary facts on the category of blow-ups. Fix a Dedekind domain A, let $B = \operatorname{Spec} A$, and let $S \to B$ be a smooth B-scheme with a reduced strict relative normal crossing divisor ∂S. Following [3], let $\mathfrak{B}_S$ be the full sub-category of the category of S-schemes, where the objects are morphisms $p : S' \to S$ which are iterated blow-ups of faces. We write $\partial S'$ for the reduced strict normal crossing subscheme $p^{-1}(\partial S)$ of S'. Since each structure morphism $S' \to S$ in $\mathfrak{B}_S$ is birational, and the objects of $\mathfrak{B}_S$ are all regular, there is at most one morphism $g : S_1 \to S_2$ between objects $S_i \to S$ of $\mathfrak{B}_S$.

Lemma 6.2. *Let $S \to B$, $V \to B$ be smooth B-schemes with respective reduced strict relative normal crossing divisors $\partial S = \sum_i \partial S_i$ and $\partial V = \sum_j \partial V_j$, with the ∂S_i and ∂V_j irreducible. Let $f : V \to S$ be a B-morphism (not necessarily in $\mathfrak{B}_S$) such that, for each i, $f^*(\partial S_i) = \sum_j n_{ij}\partial V_j$ for suitable integers $n_{ij} \geq 0$, and let $p : S' \to S$ be a morphism in $\mathfrak{B}_S$. Then there is a morphism $p' : V' \to V$ in $\mathfrak{B}_V$ and a commutative diagram*

$$\begin{array}{ccc} V' & \xrightarrow{p'} & V \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[p]{} & S \end{array}$$

such that, for each i, $f'^*(\partial S'_i) = \sum_j m_{ij}\partial V'_j$ for suitable integers $m_{ij} \geq 0$. Furthermore, if the morphism p is an isomorphism over $S \setminus \cup_{j\in J}\partial S_j$ for some J, then we may find p' which is an isomorphism over $V \setminus f^{-1}(\cup_{j\in J}\partial S_j)$.

Proof. Each structure morphism in $\mathfrak{B}_S$ is an iterated blow-up of faces; by induction on the number of blow-ups used to construct p, we reduce to the case of the blow-up of a face ∂S_I of S. We first proceed by induction on $|I|$.

We may assume that $I = \{1 < \ldots < r\}$. If the map $S' \to S$ is an isomorphism over $S \setminus \cup_{j\in J}\partial S_j$, then $I \cap J \neq \emptyset$; we may assume that r is in J. By the universal property of the blow-up, the map f' exists if and only if $p'^{-1}(f^{-1}(\partial S_I))$ is locally principle. Let $q : T \to S$ be the blow-up of the face $\partial S_{r-1,r}$, and let $E \subset T$ be the exceptional divisor. Then

$$q^{-1}(\partial S_I) = q^{-1}(\partial S_{1,\ldots,r-2}) \cap E = \partial T_{1,\ldots,r-2} \cap E.$$

By induction on r, we see that the pull-back of ∂S_I by a suitable composition of blow-ups of codimension two faces is locally principal, i.e., the blow-up of S along ∂S_I is dominated by a composition of blow-ups of codimension two faces, all of which lie over S_r. We may therefore assume that $I = \{1, 2\}$ and 2 is in J.

We must therefore show that, by an iterated blow-up of faces of V lying over S_2, we can make the intersection

$$(\sum_j n_{1j}\partial V_j) \cap (\sum_j n_{2j}\partial V_j)$$

locally principal. We proceed step by step, blowing up one face at a time. After each step, we may remove the largest common Cartier divisor from the two sums; after doing this, we may also remove a term from the first sum which has empty intersection with all terms in the second sum. In particular, we may assume that the first sum is over $j = 1, \ldots, s$, and the second sum is over $j = s + 1, \ldots, m$.

We proceed by induction: first on the maximum of the indices n_{ij}, and then on the number of occurences of the maximal n_{ij}. It suffices to show that we can always lower the number of occurences of the maximal n_{ij} by blowing up faces contained in the above intersection. We may suppose that $n_{2,s+1}$ is maximal. If we blow up the intersection $V_1 \cap V_{s+1}$, giving the exceptional divisor E, we may remove the common Cartier divisor $n_{11}E$ from the intersection, so the pull-back becomes

$$(\sum_{j=1}^{s} n_{1j}\partial V'_j) \cap \big(\sum_{j=s+1}^{m} n_{2j}\partial V'_j + (n_{2,s+1} - n_{11})\partial V'_{m+1}\big)$$

where $\partial V'_j$ is the proper transform of ∂V_j, and $\partial V'_{m+1}$ is the exceptional divisor E. We note that $\partial V'_1 \cap \partial V'_{s+1} = \emptyset$. Blowing up the intersection $\partial V'_2 \cap \partial V'_{s+1}$, and repeating this proceedure for all the divisors occuring in the first sum, we end up with an intersection of the form

$$(\sum_{j=1}^{s} n_{1j}\partial V''_j) \cap \big(\sum_{j=s+1}^{m} n_{2j}\partial V''_j + \sum_{j=1}^{s}(n_{2,s+1} - n_{1j})\partial V''_{m+j}\big)$$

with the $\partial V''_j$ all distinct and irreducible, forming a reduced strict relative normal crossing divisor. Since now $\partial V''_{s+1}$ has empty intersection with each $\partial V''_j$, $j = 1, \ldots, s$, we may remove $\partial V''_{s+1}$ from the second sum, thereby lowering the number of components with multiplicity equal to $n_{2,s+1}$, and completing the proof. □

Proposition 6.3. *Let $f_i : S_i \to S$ be in $\mathfrak{B}_S$, $i = 1, \ldots, r$. Then there is an object $f : S' \to S$ of $\mathfrak{B}_S$, and S-morphisms $g_i : S' \to S_i$, $i = 1, \ldots, r$, i.e., the category $\mathfrak{B}_S$ is left filtering. Moreover, if there is a set J such that each f_i is an isomorphism over $S \setminus \cup_{j\in J}\partial S_j$, then there is an f as above which is an isomorphism over $S \setminus \cup_{j\in J}\partial S_j$.*

Proof. An elementary induction reduces us to the case $r = 2$; the result in this case is a special case of Lemma 6.2 □

Let $S' \to S$ be in $\mathfrak{B}_S$. A *vertex* of S' is a face of S' of dimension zero over B.

Lemma 6.4. *Let $f : S_1 \to S_2$ be a morphism in $\mathfrak{B}_S$. Then f maps vertices to vertices.*

Proof. Let v be a vertex. By definition, v is smooth over B of relative dimension zero. Since a base-change from B to v extends to a functor $\mathfrak{B}_S \to \mathfrak{B}_{S\times_B v}$, we may assume that the map $v \to B$ is an isomorphism. Since all maps in $\mathfrak{B}_S$ are B-morphisms, we may assume that $B = \operatorname{Spec} F$ for some field F. Since all the maps in $\mathfrak{B}_S$ are S-morphisms, we may replace S with the spectrum of the completion of the local ring of S at v, and we may assume that the divisors in ∂S which pass through v are given by the vanishing of analytic coordinates x_i, $i = 1, \ldots, r = \dim_F S$.

We have the action of $\mathbb{G}_m^r$ on S, given by

$$(t_1, \ldots, t_r) \cdot (x_1, \ldots, x_r) = (t_1x_1, \ldots, t_rx_r),$$

and the faces through v are exactly the orbit closures of this action. It follows by induction on the number of blow-ups that there is a unique extension of this action to a $\mathbb{G}_m^r$-action on each $f : S' \to S$ in $\mathfrak{B}_S$, functorial in S', so that the faces of S' are exactly the orbit closures of this action. The vertices of S' are thus the closed orbits, hence are functorial over $\mathfrak{B}_S$. □

6.5. **Some reductions.** The proof of Theorem 0.3 is by a series of reductions, finally reducing to Theorem 0.2. We first reduce to the case of a projective morphism $f : Z \to \mathbb{A}^n_{\mathcal{O}}$, where $\mathcal{O}$ is a DVR, and $\partial\mathbb{A}^n_{\mathcal{O}}$ is the union of coordinate hyperplanes.

Fix a morphism $f : Z \to S$ as in the statement of Theorem 0.3. Suppose first of all that f already intersects all faces of S properly, and let $p : S' \to S$ be an iterated blow-up of faces. Then the projection of the fiber product $Z \times_S S' \to S'$ intersects all faces of S' properly, hence the proper transforms $p^{-1}[f] : p^{-1}[Z] \to S'$ does as well. Thus, in general, once we find an iterated blow-up of faces $p : S' \to S$ for which $p^{-1}[f]$ intersects all faces of S' properly, then the same is true for all further blow-ups of faces of S'.

Next, suppose that we have a finite open cover $Z = \cup_{i=1}^m U_i$ of Z, and that we can find iterated blow-ups of faces $p_i : S_i \to S$ such that $p_i^{-1}[f_i] : p_i^{-1}[U_i] \to S_i$ intersects all faces properly, where $f_i : U_i \to S$ is the restriction of f to U_i. By Proposition 6.3, we may find an $p : S' \to S$ in $\mathfrak{B}_S$ dominating all the S_i, hence, by the remarks above, we may replace all the S_i with S'. It is then clear that $p^{-1}[f] : p^{-1}[Z] \to S'$ intersects all faces of S' properly.

Similarly, suppose we have an étale open cover $\{j_i : V_i \to S\}$ of S, and suppose we have $p_i : T_i \to V_i$ in $\mathfrak{B}_{V_i}$ such that $p_i^{-1}[f_i] : p^{-1}[Z_i] \to T_i$ intersects all faces of T_i properly, $i = 1, \ldots, m$, where $Z_i = Z \times_S V_i$, $f_i : Z_i \to V_i$ is the projection, and $\partial(V_i) = j_i^{-1}(\partial S)$. Let F be a face of V_i. Then F is étale over a face F' of S, and $j_i^{-1}(F')$ is a disjoint union of faces of V_i. Thus, the blow-up $V_i' \to V$ of V_i along F is dominated by a blow-up $V_i'' \to V$ which is étale over the blow-up of S along F. By induction on the number of blow-ups, the same remains true for each iterated blow-up of faces of V_i. Thus, we may assume that each T_i is étale over some blow-up $S_i \to S$ of S.

We may dominate each S_i by a single $p : S' \to S$ in $\mathfrak{B}_S$, and the pull-backs of the T_i form an étale cover $\{U_i \to S'\}$ of S'. It is clear by the remarks at the beginning of this section that the pull-back of the map $p^{-1}[f] : p^{-1}[Z] \to S'$ to each U_i intersects all faces of U_i properly; since the U_i cover S', $p^{-1}[f]$ intersects all faces of S' properly.

Thus, the problem we need to solve is Zariski local on Z and étale local on S; we may in particular assume that Z is affine over S. If $\bar{Z} \supset Z$ is an S-projective closure of Z, and we can solve our problem for $\bar{Z}$, then we have solved it for Z as well, so we may assume that Z is projective over S, if we like. Similarly, if $q : S \to S'$ is étale, with $\partial S = q^{-1}(\partial S')$, we need only solve our problem for $q \circ f : Z \to S'$.

Each smooth $S \to B$ with reduced strict relative normal crossing divisor ∂S is étale locally isomorphic to $(\mathbb{A}^n_{\mathcal{O}}, \partial\mathbb{A}^n_{\mathcal{O}})$, with $\mathcal{O}$ a DVR or a field, and $\partial\mathbb{A}^n_{\mathcal{O}}$ a union of some coordinate hyperplanes. Thus, we have the following reduction:

Lemma 6.6. *To prove Theorem 0.3, it suffices to consider the case $f : Z \to \mathbb{A}^n_{\mathcal{O}}$, where $\mathcal{O}$ is a DVR or a field, f is projective, and $\partial\mathbb{A}^n_{\mathcal{O}}$ is the union of all coordinate hyperplanes of $\mathbb{A}^n_{\mathcal{O}}$.*

Proof. The discussion above reduces us to the case of $f : Z \to \mathbb{A}^n_{\mathcal{O}}$, where $\mathcal{O}$ is a DVR or a field, f is projective, and $\partial\mathbb{A}^n_{\mathcal{O}}$ is a union of *some* coordinate hyperplanes of $\mathbb{A}^n_{\mathcal{O}}$, say the hyperplanes $X_i = 0$, $i = 1, \ldots, r$. Let $\pi : \mathbb{A}^n_{\mathcal{O}} \to \mathbb{A}^r_{\mathcal{O}}$ be the projection on the first r factors,and let $\partial\mathbb{A}^r_{\mathcal{O}}$ be the union of all coordinate hyperplanes. Since π is smooth with irreducible fibers, and $\pi^{-1}(\partial\mathbb{A}^r_{\mathcal{O}}) = \partial\mathbb{A}^n_{\mathcal{O}}$, it is easy to see that $Y \mapsto Y \times_{\mathbb{A}^r_{\mathcal{O}}} \mathbb{A}^n_{\mathcal{O}}$ defines an isomorphism of categories $\mathfrak{B}_{\mathbb{A}^r_{\mathcal{O}}} \to \mathfrak{B}_{\mathbb{A}^n_{\mathcal{O}}}$. Clearly, if we solve our problem for $\pi \circ f : Z \to \mathbb{A}^r_{\mathcal{O}}$, the corresponding blow-up of $\mathbb{A}^n_{\mathcal{O}}$ solves our problem for f. We may then cover Z by affine schemes, and take $\mathbb{A}^r_{\mathcal{O}}$-projective closures, which gives the desired reduction. □

6.7. **Reduction step two.** We now reduce to the case of a closed embedding. We consider projective morphisms $f : Z \to S$, $S := \mathbb{A}^n_{\mathcal{O}}$, and we proceed by induction on the maximum of the Krull dimension d of a component of Z. We may suppose that Z is irreducible, and that Z is a closed subscheme of $\mathbb{P}^N_S$.

Suppose that $d \leq 1$. It is easy to see in this case that $f : Z \to S$ intersects all faces properly if and only if $f(Z) \subset S$ meets all faces properly, so we may assume that $d \geq 2$.

We first suppose that $\mathcal{O}$ is a DVR with residue field k and quotient field K. Let $\mathcal{O}(t)$ be the local ring of $\operatorname{Spec} k[t_0, \ldots, t_N]$ in $\operatorname{Spec} \mathcal{O}[t_0, \ldots, t_N]$, where the t_i are variables. Then $\mathcal{O} \to \mathcal{O}(t)$ is a local extension of DVR's; the residue field $k(t)$ and quotient field $K(t)$ of $\mathcal{O}(t)$ are the pure transcendental extensions $k(t_0, \ldots, t_N)$ and $K(t_0, \ldots, t_N)$ of k and K, respectively. For an $\mathcal{O}$-scheme Y, we let $Y(t)$ denote the base-extension $Y \times_{\mathcal{O}} \mathcal{O}(t)$.

Let $H \subset \mathbb{P}^N_{\mathcal{O}(t)}$ be the hyperplane with equation $\sum_{i=0}^N t_i X_i = 0$, where the X_i are the homogeneous coordinates on $\mathbb{P}^N$. The base-extension $Z(t)$ of Z is naturally a closed subscheme of $\mathbb{P}^N_{S(t)} = \mathbb{P}^N_{\mathcal{O}(t)} \times_{\mathcal{O}(t)} S(t)$. Let $Y = Z(t) \cap H \times S(t)$, and let $Z' = f(Z)$. Let $g : Y \to S(t)$ be the restriction of $f(t)$ to Y. Y has Krull dimension $d - 1$.

As base-extension from $\mathfrak{B}_S$ to $\mathfrak{B}_{S(t)}$ is an equivalence of categories, we may ignore this base-extension when we talk of iterated blow-ups of faces of S or $S(t)$. Let $T \to S$ be an iterated blow-up of faces such that $p^{-1}[Z'] \subset T$ and $p^{-1}[g] : p^{-1}[Y] \to T(t)$ intersect all faces properly, which exists by our induction hypothesis and our reduction hypothesis. We claim that $p^{-1}[f] : p^{-1}[Z] \to T$ intersects all faces properly. To see this, we first need

Lemma 6.8. *Let x be an Ω-rational point of H for some $\mathcal{O}(t)$-field $\mathcal{O}(t) \to \Omega$, and let $\pi : H \to \mathbb{P}^N_{\mathcal{O}}$ be the projection. Then $\pi(x)$ is neither a $\bar{K}$-rational point of $\mathbb{P}^N_{\mathcal{O}}$, nor a $\bar{k}$-rational point of $\mathbb{P}^N_k$.*

Proof. If $\pi(x)$ were $\bar{K}$ rational, then there are element $x_0, \ldots, x_N \in \bar{K}$, not all zero, with $\sum_i t_i x_i = 0$ in $\bar{K}[t_0, \ldots, t_N]$, which is clearly impossible. The same proof works for $\bar{k}$. □

Lemma 6.9. $p^{-1}[Z(t)] \cap H \times_{\mathcal{O}} T = p^{-1}[Y]$.

Proof. Clearly $p^{-1}[Z(t)] \cap H \times_{\mathcal{O}} T \supset p^{-1}[Y]$, so it suffices to show that the left-hand side is irreducible (as an $\mathcal{O}(t)$-scheme). Over $T \setminus \partial T$, the map p is an isomorphism, hence $p^{-1}[Z(t)] \cap H \times_{\mathcal{O}} T$ and $p^{-1}[Y]$ are equal over $T \setminus \partial T$. Also, $p^{-1}[Z(t)] \cap H \times_{\mathcal{O}} T$ is a divisor on $p^{-1}[Z(t)]$. Suppose there is an irreducible component R of $p^{-1}[Z(t)] \cap H \times_{\mathcal{O}} T$ not contained in $p^{-1}[Y]$. Then R is an irreducible component of $p^{-1}[Z(t)] \cap \mathbb{P}^N_{\mathcal{O}(t)} \times \partial T$. But this latter scheme is the extension to $\mathcal{O}(t)$ of the pure codimension one closed subscheme $p^{-1}[Z] \cap \mathbb{P}^N_{\mathcal{O}} \times_{\mathcal{O}} \partial T$ of $p^{-1}[Z]$. Since K is algebraically closed in $K(t)$ and k is algebraically closed in $k(t)$, R is the base-extension to $\mathcal{O}(t)$ of an irreducible component R_0 of $p^{-1}[Z] \cap \mathbb{P}^N_{\mathcal{O}} \times_{\mathcal{O}} \partial T$. In particular, since R_0 has points over $\bar{K}$ or $\bar{k}$, there must be an Ω-point x of R, for some field Ω, $\mathcal{O}(t) \to \Omega$, with projection to $\mathbb{P}^N_{\mathcal{O}}$ a $\bar{K}$-rational or $\bar{k}$-rational point of $\mathbb{P}^N_{\mathcal{O}}$. By Lemma 6.8, this is impossible. □

We now show that $\hat{f} := p^{-1}[f] : p^{-1}[Z] \to T$ intersects all faces properly. Let F be a face of T. Suppose that $W \to F$ is generically finite, for each irreducible component W of $\hat{f}^{-1}(F)$. Since $p^{-1}[Z']$ is the image of $p^{-1}[Z]$, it follows that

$$\dim(W) \leq \dim_{\mathcal{O}}(F) + d - n.$$

Now suppose there is an irreducible component W of $\hat{f}^{-1}(F)$ such that the generic fiber dimension of W over $\hat{f}(W)$ is at least one. It follows from Lemma 6.9 that

$$W(t) \cap (H \times_{\mathcal{O}(t)} T) \subset (p^{-1}[Y] \cap \mathbb{P}^N_{\mathcal{O}(t)} \times_{\mathcal{O}} F),$$

so

$$\dim(W(t) \cap (H \times_{\mathcal{O}(t)} T)) \leq \dim_{\mathcal{O}}(F) + d - 1 - n.$$

Since the generic fiber dimension of $W \subset \mathbb{P}^N_{\mathcal{O}} \times_{\mathcal{O}} T$ over $\hat{f}(W) = p_2(W)$ is at least one, it follows that each irreducible component of $W(t) \cap (H \times_{\mathcal{O}(t)} T)$ has Krull dimension at most $\dim(W) - 1$. Thus

$$\dim(W) \leq \dim_{\mathcal{O}}(F) + d - n,$$

as before. Thus, $p^{-1}[Z]$ intersects all faces of T properly, as claimed.

The proof in case $\mathcal{O}$ is a field is the same.

6.10. **Reduction step three.** Let d_0 be the Krull dimension of $\mathcal{O}$, i.e., $d_0 = 0$ if $\mathcal{O}$ is a field, $d_0 = 1$ if $\mathcal{O}$ is a DVR. We reduce to the case of a closed embedding $Z \subset \mathbb{A}^n_{\mathcal{O}}$ of codimension $\leq d_0 + 1$. We proceed by descending induction on the codimension, the case of codimension $n + 1$ being obvious.

Let $Z \subset \mathbb{A}^n_{\mathcal{O}}$ be an irreducible closed subscheme of codimension d. If Z has codimension $> d_0+1$, we can find an irreducible closed codimension d_0+1 subscheme W with $Z \subset W \subset \mathbb{A}^n_{\mathcal{O}}$. Assuming the result for W, we find an iterated blow-up of faces $p : S' \to \mathbb{A}^n_{\mathcal{O}}$ such that $p^{-1}[W]$ intersects all faces of S' properly; in particular, $p^{-1}[W]$ avoids all vertices of S'. We may cover S' by open subsets, S'_v, v a vertex, such that $(S'_v, \partial S'_v)$ is isomorphic to $(\mathbb{A}^n_{\mathcal{O}}, \partial\mathbb{A}^n_{\mathcal{O}})$.

As in §6.5, it suffices to prove the result for $p^{-1}[Z] \cap S'_v$ for each v. Thus, we may assume that Z is an irreducible codimension d closed subscheme of $\mathbb{A}^n_{\mathcal{O}}$, with $Z \cap 0_{\mathcal{O}} = \emptyset$. We may then blow-up the vertex $0_{\mathcal{O}}$, and project the resulting scheme to $\mathbb{P}^{n-1}_{\mathcal{O}}$,

$$\begin{array}{ccc} & S' & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \pi} \\ \mathbb{A}^n_{\mathcal{O}} & & \mathbb{P}^{n-1}_{\mathcal{O}}. \end{array}$$

It suffices to prove the result for $Z' := p^{-1}(Z)$ in $(S', \partial S')$, where we set $\partial S' = \cup_{i=1}^n p^{-1}[(X_i = 0)]$; the result for Z' follows from the result for $\overline{\pi(Z')} \subset \mathbb{P}^{n-1}_{\mathcal{O}}$, with $\partial\mathbb{P}^{n-1}_{\mathcal{O}}$ the union of the coordinate hyperplanes, since S' is smooth over $\mathbb{P}^{n-1}_{\mathcal{O}}$ with irreducible fibers, and $\pi^{-1}(\partial\mathbb{P}^{n-1}_{\mathcal{O}}) = \partial S'$. We may then cover $\mathbb{P}^{n-1}_{\mathcal{O}}$ by the standard affine open cover; the reduction of §6.5 thus reduces to the case of $Z'' \subset \mathbb{A}^{n-1}_{\mathcal{O}}$ of codimension $d - 1$.

6.11. **Completion of the proof.** By Lemma 6.4, we have the functor

$$\mathcal{V}er : \mathfrak{B}_S \to \mathbf{Sets}$$

with $\mathcal{V}er(S' \to S)$ the set of vertices of S'. By Proposition 6.3, the category $\mathfrak{B}_S$ is left-filtering. The category $\mathfrak{B}_S$ has only countably many objects; indeed, each $S' \to S$ in $\mathfrak{B}_S$ has only finitely many faces, from which it follows that, for each N, there are only finitely many $p : S' \to S$ in $\mathfrak{B}_S$ such that p is a composition of at most N blow-ups. Thus, the inverse limit $\widehat{\mathcal{V}er} := \lim\limits_{\substack{\leftarrow \\ \mathfrak{B}_S}} \mathcal{V}er$ is a pro-finite set.

Lemma 6.12. *Let Z be a codimension $d_0 + 1$ closed subscheme of $\mathbb{A}^n_{\mathcal{O}}$. Suppose that, for each element $\tilde{v} \in \widehat{\mathcal{V}er}$, there is an iterated blow-up of faces $p : S' \to S$ (depending perhaps on $\tilde{v}$), such that $p^{-1}[Z] \cap \tilde{v}(S') = \emptyset$. Then there is an iterated blow-up of faces $q : S'' \to S$, such that $q^{-1}[Z] \cap v = \emptyset$ for all vertices v of S''.*

Proof. For each $p : S' \to S$ in $\mathfrak{B}_S$, the set of $\tilde{v}' \in \widehat{\mathcal{V}er}$ such that $\tilde{v}'(S') = \tilde{v}(S')$ is an open neighborhood $U_p(\tilde{v})$ of $\tilde{v}$ in $\widehat{\mathcal{V}er}$. Let $\mathcal{U}$ of be the collection of open neighborhoods $U_p(\tilde{v})$ such that $p^{-1}[Z]$ $p^{-1}[Z] \cap \tilde{v}(S') = \emptyset$; by hypothesis, $\mathcal{U}$ is an open cover of $\widehat{\mathcal{V}er}$. Since $\widehat{\mathcal{V}er}$ is pro-finite, it is compact, hence there is a finite subcover of $\mathcal{U}$, say $\{U_{p_i}(\tilde{v}_i) \mid i = 1, \dots, r\}$. A choice of $q : S'' \to S$ which dominates all the $p_i : S_i \to S$ is the desired blow-up. □

We can now complete the verification of Theorem 0.3 for subschemes of $\mathbb{A}^n_{\mathcal{O}}$ of codimension $\le d_0 + 1$. Suppose first that $\mathcal{O}$ is a field, so we need only consider a codimension one subscheme Z of $S := \mathbb{A}^n_{\mathcal{O}}$. We may consider a point $\tilde{v} \in \widehat{\mathcal{V}er}$ as the Player B in the Hironaka/Bloch game, as explained in §0 and §5.6. Since Player A has a winning strategy (Theorem 0.1), there is a $p : S' \to S$ in $\mathfrak{B}_S$ such that $p^{-1}[Z]$ avoids the vertex $\tilde{v}(S')$. By Lemma 6.12, there is a $q : S'' \to S$ in $\mathfrak{B}_S$ such that $q^{-1}[Z]$ avoids all vertices of S''. Since $q^{-1}[Z]$ has codimension one, this is equivalent to $q^{-1}[Z]$ intersecting all faces of S'' properly. This completes the proof of Theorem 0.3 in case $\mathcal{O}$ is a field.

Now suppose that $\mathcal{O}$ is a DVR, with quotient field K. Let $Z \subset \mathbb{A}^n_{\mathcal{O}}$ have codimension ≤ 2. By Theorem 0.3 in the case of a field, there is an $p : S' \to S$ in $\mathfrak{B}_S$ such that $p^{-1}[Z]_K$ intersects all faces of S'_K properly. In case $\mathrm{codim} Z = 1$, this automatically implies that $p^{-1}[Z]$ intersect all faces of S' properly.

Now suppose Z has codimension two. We may cover S' by open subsets S'_v, v a vertex of S', such that $(S'_v, \partial S'_v) \cong (\mathbb{A}^n_{\mathcal{O}}, \partial \mathbb{A}^n_{\mathcal{O}})$. Using the localization arguments of §6.5, we are reduced to the case of a codimension two subscheme Z of $\mathbb{A}^n_{\mathcal{O}}$ which is generically in good position. Using the translation of a point $\tilde{v}$ of $\widehat{\mathcal{V}er}$ into the codimension two version of the two player game described in §0, we may apply Theorem 0.2 to conclude that there is, for each $\tilde{v} \in \widehat{\mathcal{V}er}$, a $p : S' \to S$ in $\mathfrak{B}_S$ such that $p^{-1}[Z] \cap \tilde{v}(S') = \emptyset$. By Lemma 6.12, there is a $q : S'' \to S$ in $\mathfrak{B}_S$ such that $q^{-1}[Z] \cap v = \emptyset$ for all vertices v of S''. Since Z is generically in good position, this implies that $q^{-1}[Z]$ intersects all faces of S'' properly. This completes the verification of the case of codimension two, and the proof of Theorem 0.3.

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