THE STEINBERG CURVE

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ABSTRACT. Let E and E' be elliptic curves over $\mathbb{C}$. We construct non-torsion 0-cycles in the kernel of the Albanese mapping $\mathrm{CH}_0(E \times E')_{\deg 0} \to E \times E'$, which are not detectable by a certain class of cohomology theories, including the cohomology of the analytic motivic complex involving the dilogarithm function defined by S. Bloch in [3]. This is in contrast to the étale version of Bloch's complex defined by S. Lichtenbaum [9], which contains the Chow group.

0. INTRODUCTION

Let X be a smooth projective algebraic variety over $\mathbb{C}$, and let X_{an} be the associated compact complex manifold. The equivalence of the category of algebraic coherent sheaves on X with the category of analytic sheaves on X_{an}, proved in [16], yields the isomorphism

$$\mathrm{CH}^1(X) \cong H^1(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}}),$$

where $\mathrm{CH}^1(X)$ is the group of divisors modulo linear equivalence, and $\mathcal{O}_{X_{\mathrm{an}}}$ is the sheaf of analytic functions. This isomorphism, together with the exponential sequence

$$0 \to (2\pi i)\mathbb{Z} \to \mathcal{O}_{X_{\mathrm{an}}} \to \mathcal{O}^*_{X_{\mathrm{an}}} \to 1,$$

yields a direct connection of $\mathrm{CH}^1(X)$ with the Hodge theory of X_{an}.

It is natural to ask if the group of codimension p cycles on X modulo rational equivalence, $\mathrm{CH}^p(X)$, admits a similar description for $p > 1$. Presumably with this in mind, S. Bloch has introduced a complex for the analytic topology, denoted $\mathcal{B}(2)$. There is a natural map

$$\mathbb{H}^*(X_{\mathrm{an}}, \mathcal{B}(2)) \to H^*_{\mathcal{D}}(X, \mathbb{Z}(2)),$$

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where $H^*_{\mathcal{D}}(X,\mathbb{Z}(2))$ is the weight two Deligne cohomology, as well as a cycle class

$$\mathrm{CH}^2(X)\to\mathbb{H}^4(X_{\mathrm{an}},\mathcal{B}(2)),$$

factorizing the cycle class to Deligne cohomology $\mathbb{H}^4(X_{\mathrm{an}},\mathcal{B}(2))$ [5]. Our purpose in this article is to construct non-torsion cycles in $\mathrm{CH}^2(X)$ (X a product of two elliptic curves) which vanish in $\mathbb{H}^4(X_{\mathrm{an}},\mathcal{B}(2))$. In fact, we construct non-torsion cycles which vanish in $\mathbb{H}^4(X_{\mathrm{an}},\Gamma(2)_{\mathrm{an}})$, where $\Gamma(2)_{\mathrm{an}}$ is any complex of sheaves on X_{an} satisfying a modest list of axioms.

The cycles we construct come from the *Steinberg curve* on the product $E\times E'$ of elliptic curves. In the degenerate case of the product of nodal cubic curves $E_0\times E_0$, the Steinberg curve is just the line $x+y=1$ on $\mathbb{C}^*\times\mathbb{C}^*$, where $\mathbb{C}^*$ is the non-singular locus of E_0. In general, we have the Tate parametrizations $\mathbb{C}^*\times\mathbb{C}^*\to E\times E'$, and the Steinberg curve is the image of $\{x+y=1\}\subset\mathbb{C}^*\times\mathbb{C}^*$ in $E\times E'$. It turns out that, using the definition of $\mathrm{CH}_0(E_0\times E_0)$ given by [7], $K_2(\mathbb{C})$ is a summand of $\mathrm{CH}_0(E_0\times E_0)_0$, with the Steinberg curve parametrizing the classical Steinberg relation. However, the fact that the Steinberg curve is *non-algebraic* unless $E=E'=E_0$ implies that the analog of the Steinberg relation is *not* satisfied in $\mathrm{CH}_0(E\times E')$, unless $E=E'=E_0$. Since one can use the cover $\mathbb{C}^*\times\mathbb{C}^*\to E\times E'$ to compute cohomology in the analytic topology, the analog of the Steinberg relation *is* satisfied in $\mathbb{H}^4(E_{\mathrm{an}}\times E'_{\mathrm{an}},\Gamma(2)_{\mathrm{an}})$, where $\Gamma(2)_{\mathrm{an}}$ is as above.

Our results are in contrast with various results on cohomology theories defined via the étale topology, or using coefficients mod n. For instance, Lichtenbaum [9] has defined an étale version of weight two motivic cohomology, which receives the codimension two Chow groups injectively. Also, Raskind and Spieß [14] have shown that, for smooth elliptic curves E, E' defined over a p-adic field k, there is a surjective map of $K_2(k)/n$ onto the mod n Albanese kernel in $\mathrm{CH}_0(E\times E')/n$ for n prime to p.

We recall some well-known facts on the Tate parametrization of elliptic curves in §1. In §2 we introduce the Steinberg curve, and show that it parametrizes the Steinberg relation in $\mathrm{CH}_0(E_0\times E_0)$. We then consider $\mathrm{CH}_0(E\times E')$, with at least one of E,E' smooth, and show that the 0-cycle in the Albanese kernel corresponding to a point u of the Steinberg curve is non-torsion in $\mathrm{CH}_0(E\times E')$ (outside of countably many points u). In §3, we list our axioms for the complex $\Gamma(2)_{\mathrm{an}}$, and show that the "Steinberg relation" holds in $\mathbb{H}^4(E_{\mathrm{an}}\times E'_{\mathrm{an}},\Gamma(2)_{\mathrm{an}})$. In §4, we consider the problem of constructing a non-torsion cycle which vanishes in both $\mathbb{H}^4(X_{\mathrm{an}},\mathcal{B}(2))$ and in the absolute Hodge cohomology

$H^2(X, \Omega^2_{X/\mathbb{Q}})$. We give such an example for $X = E \times E_0$, $E \neq E_0$, but we are not able to handle the smooth case.

1. Tate curves and line bundles

For a scheme X over $\mathbb{C}$, we let X_{an} denote the set of $\mathbb{C}$-points with the classical topology. We let $\mathcal{O}_{X_{\mathrm{an}}}$ denote the sheaf of holomorphic functions on X_{an}.

We begin by describing a construction of the universal analytic Tate curve over $\mathbb{C}$. We first form the analytic manifold $\hat{\mathcal{C}}^*$ as the quotient of the disjoint union $\sqcup_{i=-\infty}^{\infty} U_i$, with each $U_i = \mathbb{C}^2$, by the equivalence relation

$$(x, y) \in U_i \setminus \{Y = 0\} \sim (\frac{1}{y}, xy^2) \in U_{i+1} \setminus \{X = 0\}.$$

The function $\tilde{\pi}(x, y) = xy$ is globally defined on $\hat{\mathcal{C}}^*$. Letting $D \subset \mathbb{C}$ be the disk $\{|z| < 1\}$, we define $\mathcal{C}^* = \tilde{\pi}^{-1}(D)$, so $\tilde{\pi}$ restricts to the analytic map $\pi : \mathcal{C}^* \to D$. We let $\tilde{0} : D \to \mathcal{C}^*$ be the section $z \mapsto (z, 1) \in U_0$.

Let $D^* \subset D$ be the punctured disk $z \neq 0$. Since the map $(x, y) \mapsto (\frac{1}{y}, xy^2)$ is an automorphism of $(\mathbb{C}^*)^2$, the open submanifold $\pi^{-1}(D^*)$ of $\mathcal{C}^*$ is isomorphic to $(\mathbb{C}^*)^2$, and the restriction of the map π is just the map $(x, y) \mapsto xy$. Thus, the projection $p_2 : (\mathbb{C}^*)^2 \to \mathbb{C}^*$ gives an isomorphism of the fiber $\mathcal{C}^*_t := \pi^{-1}(t)$ with $\mathbb{C}^*$, for $t \in D^*$.

The fiber $\pi^{-1}(0)$, on the other hand, is an infinite union of projective lines. Indeed, define the map $f_i : \mathbb{CP}^1 \to \mathcal{C}^*_0$ by sending $(a : 1) \in \mathbb{CP}^1 \setminus \infty$ to $(0, a) \in U_i$, and $\infty = (1 : 0)$ to $(0, 0) \in U_{i+1}$, and let $C_i = f_i(\mathbb{CP}^1)$. Then $\pi^{-1}(0) = \cup_{i=-\infty}^{\infty} C_i$, with $\infty \in C_i$ joined with $0 \in C_{i+1}$. Note in particular that the value $\tilde{0}(0)$ of the zero section avoids the singularities of $\pi^{-1}(0)$.

Define the automorphism ϕ of $\mathcal{C}^*$ over D by sending $(x, y) \in U_i$ to $(x, y) \in U_{i-1}$. This gives the action of $\mathbb{Z}$ on $\mathcal{C}^*$, with n acting by ϕ^n. It is easy to see that this action is free and proper, so the quotient space $\mathcal{E} := \mathcal{C}^*/\mathbb{Z}$ exists as a bundle $\pi : \mathcal{E} \to D$. The section $\tilde{0} : D \to \mathcal{C}^*$ induces the section $0 : D \to \mathcal{E}$.

Take $t \in D^*$. Identifying $\mathcal{C}^*_t$ with $\mathbb{C}^*$ as above, we see that ϕ restricts to the automorphism $z \mapsto tz$. Thus, the fiber $\mathcal{E}_t := \pi^{-1}(t)$ for $t \in D^*$ is the *Tate elliptic curve* $\mathbb{C}^*/t^{\mathbb{Z}}$, with identity $0(t)$. On $\mathcal{C}^*_0$, however, ϕ is the union of the "identity" isomorphisms $C_i \to C_{i-1}$. Thus $\phi(\infty \in C_i) = 0 \in C_i$, so the restriction of $\mathcal{C}^*_0 \to \mathcal{E}_0$ to C_0 identifies $\mathcal{E}_0$ with the nodal curve $\mathbb{CP}^1/0 \sim \infty$. We let $* \in \mathcal{E}_0$ denote the singular point. Then $\tilde{0}(0) \in \mathcal{E}_0 \setminus *$.

The map $(t, w) \in D \times \mathbb{C}^* \mapsto (\frac{t}{w}, w) \in U_0$ gives an isomorphism $\psi : D \times \mathbb{C}^* \to U_0 \setminus \{Y = 0\}$ over D. The composition

$$D \times \mathbb{C}^* \to U_0 \setminus \{Y = 0\} \subset \mathcal{C}^* \xrightarrow{q} \mathcal{E}$$

defines the map $p : D \times \mathbb{C}^* \to \mathcal{E}$ over D, with image $\mathcal{E} \setminus \{*\}$.

Take $u \in \mathbb{C}^*$. We have the local system on $\mathcal{E}$

$$\mathcal{L}_u := \mathcal{C}^* \times \mathbb{C}/(z, \lambda) \sim (\phi(z), u\lambda) \to \mathcal{E},$$

and the associated holomorphic line bundle $\mathcal{L}_u^{\mathrm{an}}$ on $\mathcal{E}$.

Let E_t be the algebraic elliptic curve associated to the analytic variety $\mathcal{E}_t$, let $L_u(t)$ and $L_u^{\mathrm{an}}(t)$ denote the restriction of $\mathcal{L}_u$ and $\mathcal{L}_u^{\mathrm{an}}$ to $\mathcal{E}_t$, and let $L_u^{\mathrm{alg}}(t)$ be the algebraic line bundle on E_t corresponding to $L_u^{\mathrm{an}}(t)$ via [16]. The restriction of p to $t \times \mathbb{C}^*$ defines the map $p_t : \mathbb{C}^* \to E_{t\mathrm{an}}$. For $t \neq 0$, p_t is a covering space of $E_{t\mathrm{an}}$. The map $p_0 : \mathbb{C}^* \to E_{0\mathrm{an}}$ is the analytic map associated to the algebraic open immersion

$$\mathbb{P}^1 \setminus \{0, \infty\} \xrightarrow{j} \mathbb{P}^1 \to \mathbb{P}^1/0 \sim \infty = E_0.$$

If E is an elliptic curve over $\mathbb{C}$, then $E_{\mathrm{an}} \cong \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice spanned by 1 and some τ in the upper half plane. Taking $t = e^{2\pi i\tau}$ gives the isomorphism $E_{\mathrm{an}} \cong \mathcal{E}_t$, so each elliptic curve over $\mathbb{C}$ occurs as an E_t for some (in fact for infinitely many) $t \in D^*$.

Sending $u \in \mathbb{C}^*$ to the isomorphism class of $L_u^{\mathrm{alg}}(t)$ defines a homomorphism $\tilde{p}_t : \mathbb{C}^* \to \mathrm{Pic}(E_t)$. We denote the identity $0(t) \in E_t$ simply by 0 if t is given.

Lemma 1.1. *For all $t \in D$, $c_1(L_u^{\mathrm{alg}}(t)) = (p_t(u)) - (0)$.*

Proof. We first handle the case $t \neq 0$. Let $q : \mathbb{C} \to E := E_t$ be the map $q(z) = p_t(e^{2\pi iz})$, let $\tau \in \mathbb{C}$ be an element with $e^{2\pi i\tau} = t$, and let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and τ. The map q identifies E with $\mathbb{C}/\Lambda$, and $L_u(t)$ with the local system defined by the homomorphism $\rho : \Lambda \to \mathbb{C}^*$, $\rho(a + b\tau) = u^b$.

There is a unique cocycle θ in $Z^1(\Lambda, H^0(\mathbb{C}, \mathcal{O}^*_{\mathbb{C}_{\mathrm{an}}}))$ with $\theta(1) = 1$, $\theta(\tau) = e^{-2\pi iz}$; let L be the corresponding holomorphic line bundle on E. Computing $c_1^{\mathrm{top}}(L) \in H^2(E, \mathbb{Z})$ by using the exponential sequence, we find that $\deg(L) = 1$. By Riemann-Roch, we have $H^0(E, L) = \mathbb{C}$; let $\Theta(z)$ be the corresponding global holomorphic function on $\mathbb{C}$, i.e.,

$$\Theta(z + 1) = \Theta(z),\ \Theta(z + \tau) = e^{-2\pi iz}\Theta(z),$$

and the divisor of Θ on E is (x), with $L \cong \mathcal{O}_E(x)$.

Take $v, w \in \mathbb{C}$ with $u = e^{2\pi i v}$ and $q(w) = x$. Let $f(z) = \frac{\Theta(z+w-v)}{\Theta(z+w)}$. Then

$$f(z+1) = f(z), \ f(z+\tau) = uf(z),$$

and $\mathrm{Div}(f) = (p(u)) - (0)$. Thus, multiplication by f defines an isomorphism

$$\times f : \mathcal{O}_{E_{\mathrm{an}}}((p(u)) - (0)) \to L_u^{\mathrm{an}}.$$

The proof for $E_0 = \mathbb{P}^1/0 \sim \infty$ is essentially the same, where we replace $\frac{\Theta(z+w-v)}{\Theta(z+w)}$ with the rational function $\frac{X-u}{X-1}$. □

Thus, the image of $\tilde{p}_t$ in $\mathrm{Pic}(E_t)$ is $\mathrm{Pic}^0(E)$. After identifying the smooth locus of E_t^0 of E_t with $\mathrm{Pic}^0(E_t)$ by sending $x \in E_t^0$ to the class of the invertible sheaf $\mathcal{O}_{E_t}((x) - (0))$, we have $\tilde{p}_t = p_t$.

2. The Albanese kernel and the Steinberg relation

Let X be a smooth projective variety. We let $\mathrm{CH}_0(X)$ denote the group of zero cycles on X, modulo rational equivalence, $F^1\mathrm{CH}_0(X)$ the subgroup of cycles of degree zero, and $F^2\mathrm{CH}_0(X)$ the kernel of the Albanese map $\alpha_X : F^1\mathrm{CH}_0(X) \to \mathrm{Alb}(X)$. The choice of a point $0 \in X$ gives a splitting to the inclusion $F^1\mathrm{CH}_0(X) \to \mathrm{CH}_0(X)$.

Let E, E' be smooth elliptic curves. As $\mathrm{Alb}(E \times E') = E \times E'$, the inclusion $F^2\mathrm{CH}_0(E \times E') \to F^1\mathrm{CH}_0(E \times E')$ is split by sending $(x, y) - (0, 0)$ to $(x, y) - (x, 0) - (0, y) + (0, 0)$. Thus $F^2\mathrm{CH}_0(E \times E')$ is generated by zero-cycles of the form $(x, y) - (x, 0) - (0, y) + (0, 0)$. Choosing an isomorphism $E \cong E_t$, $E' \cong E_{t'}$, we have the covering spaces $p : \mathbb{C}^* \to E_{\mathrm{an}}$, $p' : \mathbb{C}^* \to E'_{\mathrm{an}}$, and the map

$$\begin{aligned} p * p' : \mathbb{C}^* \otimes \mathbb{C}^* &\to F^2\mathrm{CH}_0(E \times E') \\ u \otimes v &\mapsto p(u) * p'(v) := \\ (p(u), p'(v)) - (p(u), 0) &- (0, p'(v)) + (0, 0). \end{aligned} \tag{2.1}$$

By the theorem of the cube [11], the map $p * p'$ is a well-defined group homomorphism, and thus is surjective.

In case one or both of E, E' is the singular curve E_0, we will need to use the the theory of zero-cycles mod rational equivalence defined in [7]. If X is a reduced, quasi-projective variety over a field k with singular locus X_{sing}, the group $\mathrm{CH}_0(X)$ (denoted $\mathrm{CH}_0(X, X_{\mathrm{sing}})$ in [7]) is defined as the quotient of the free abelian group on the regular closed points of X, modulo the subgroup generated by zero-cycles of the form $\mathrm{Div} f$, where f is a rational function on a dimension one closed subscheme D of X such that

1. No irreducible component of D is contained in $X_{\rm sing}$.
2. In a neighborhood of each point of $D\cap X_{\rm sing}$, the subscheme D is a complete intersection.
3. f is in the subgroup $\mathcal{O}^*_{D,D\cap X_{\rm sing}}$ of $k(D)^*$.

It follows in particular from these conditions that Div$f$ is a sum of regular points of X.

For X a reduced curve, sending a regular closed point $x\in X$ to the invertible sheaf $\mathcal{O}_X(x)$ extends to give an isomorphism $\mathrm{CH}_0(X)\cong \mathrm{Pic}(X)$.

We extend the definition of $F^i\mathrm{CH}_0$ to $E\times E'$ with either $E=E_0$, $E'=E_0$ or $E=E'=E_0$, by defining $F^1\mathrm{CH}_0(E\times E')$ as the subgroup of $\mathrm{CH}_0(E\times E')$ generated by the differences $[x]-[y]$, and $F^2\mathrm{CH}_0(E\times E')$ the subgroup generated by expressions $[(x,y)]-[(x,0)]-[(0,y)]+[(0,0)]$, where x is a smooth point of E and y a smooth point of E'. The surjection $p*p':\mathbb{C}^*\otimes\mathbb{C}^*\to F^2\mathrm{CH}_0(E\times E')$ is then defined by the same formula as (2.1).

Proposition 2.1 (The Steinberg relation)**.** *Take $E=E'=E_0$. Then*

1. *There is an isomorphism $\mathrm{CH}_0(E_0\times E_0)\cong\mathbb{Z}\oplus(\mathbb{C}^*\times\mathbb{C}^*)\oplus K_2(\mathbb{C})$, sending $F^2\mathrm{CH}_0(E_0\times E_0)$ onto the summand $K_2(\mathbb{C})$.*
2. *$p(u)*p(1-u)=0$ in $\mathrm{CH}_0(E_0\times E_0)$ for all $u\in\mathbb{C}\setminus\{0,1\}$.*

Proof. Let X be a quasi-projective surface over a field k. By [8], there is an isomorphism $\phi: H^2(X,\mathcal{K}_2)\to\mathrm{CH}_0(X)$. The product $\mathcal{O}_X^*\otimes\mathcal{O}_X^*\to\mathcal{K}_2$ gives the cup product

$$H^1(X,\mathcal{O}_X^*)\otimes H^1(X,\mathcal{O}_X^*)\xrightarrow{\cup}H^2(X,\mathcal{K}_2).$$

In addition, let D, D' be Cartier divisors which intersect properly on X, and suppose that $\mathrm{supp}\,D\cap\mathrm{supp}\,D'\cap X_{\rm sing}=\emptyset$. Then

$$\phi(\mathcal{O}_X(D)\cup\mathcal{O}_X(D'))=[D\cdot D'], \tag{2.2}$$

where $\cdot$ is the intersection product and $[-]$ denotes the class in CH_0.

Since $L_u^{\rm alg}=\mathcal{O}_{E_0}(p(u)-0)$, (2.2) implies

$$p(u)*p(1-u)=\rho(p_1^*L_u^{\rm alg}\cup p_2^*L_{1-u}^{\rm alg}),$$

so to prove (2), it suffices to show that $p_1^*L_u^{\rm alg}\cup p_2^*L_{1-u}^{\rm alg}=0$ in $H^2(E_0\times E_0,\mathcal{K}_2)$.

Write X for $E_0\times E_0$. Let $\bar{\mathcal{K}}_2$ be the image of $\mathcal{K}_2$ in the constant sheaf $K_2(\mathbb{C}(X))$. By Gersten's conjecture [13, §7, Theorem 5.11], the surjection $\pi:\mathcal{K}_2\to\bar{\mathcal{K}}_2$ is an isomorphism at each regular point of X, hence π induces an isomorphism on H^2.

Let $q:\mathbb{P}^1\to E_0$ be the normalization, giving the normalization $q\times q:\mathbb{P}^1\times\mathbb{P}^1\to X$. Let $i:*\to E_0$ be the inclusion of the singular point. We have the exact sequence of sheaves on E_0

$$q_*\mathcal{K}_1 \xrightarrow{\beta} i_*K_1(\mathbb{C}) \to 0 \tag{2.3}$$

and the exact sequence of sheaves on X:

$$(q\times q)_*\mathcal{K}_2 \xrightarrow{\alpha} (i\times q)_*\mathcal{K}_2 \oplus (q\times i)_*\mathcal{K}_2 \to (i\times i)_*K_2(\mathbb{C}) \to 0, \tag{2.4}$$

with augmentations $\epsilon_1 : \mathcal{K}_1 \to (2.3)$, $\epsilon_2 : \bar{\mathcal{K}}_2 \to (2.4)$. The various cup products in K-theory give the map of complexes

$$p_1^*(2.3) \otimes p_2^*(2.3) \to (2.4), \tag{2.5}$$

compatible with the cup product

$$p_1^*\mathcal{K}_1 \otimes p_2^*\mathcal{K}_1 \to \bar{\mathcal{K}}_2. \tag{2.6}$$

The augmentation $\epsilon_1 : \mathcal{K}_1 \to \ker\beta$ is an isomorphism. The augmentation $\epsilon_2 : \bar{\mathcal{K}}_2 \to \ker\alpha$ is an injection, and the cokernel is supported on $*\times *$. Indeed, by [6, lemma 1.15 and corollary 1.16], there is an isomorphism of sheaves on $X \setminus \{*\times *\}$,

$$\mathcal{I}/\mathcal{I}^2 \otimes (q\times q)_*\Omega^1_{(q\times q)^{-1}(X_{\rm sing})/X_{\rm sing}} \cong \ker\alpha/\bar{\mathcal{K}}_2,$$

where $\mathcal{I}$ is the ideal sheaf of $X_{\rm sing}$. Since $(q\times q)^{-1}(X_{\rm sing}) \to X_{\rm sing}$ is étale away from $*\times *$, the relative differentials vanish, verifying our claim. Thus, $\epsilon_2 : \bar{\mathcal{K}}_2 \to \ker\alpha$ induces an isomorphism on H^2, and the complexes (2.3) and (2.4) give rise to maps

$$\delta_2 : K_2(\mathbb{C}) \to H^2(X, \ker\alpha) = H^2(X, \bar{\mathcal{K}}_2) = H^2(X, \mathcal{K}_2)$$
$$\delta_1 : \mathbb{C}^* = K_1(\mathbb{C}) \to H^1(E_0, \mathcal{K}_1).$$

The compatibility of (2.5) with (2.6) yields the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{C}^*\otimes\mathbb{C}^* & \xrightarrow{\ \cup\ } & K_2(\mathbb{C}) \\ \downarrow{\scriptstyle \delta_1\otimes\delta_1} & & \downarrow{\scriptstyle \delta_2} \\ H^1(E_0,\mathcal{K}_1)\otimes H^1(E_0,\mathcal{K}_1) & \xrightarrow[p_1^*\cup p_2^*]{} & H^2(X,\mathcal{K}_2). \end{array}$$

Since $L_v^{\rm alg} = \delta_1(v)$ for each $v \in \mathbb{C}^*$, we have

$$p_1^*L_u^{\rm alg} \cup p_2^*L_{1-u}^{\rm alg} = \delta_2(\{u, 1-u\}) = 0.$$

completing the proof of (2). Similarly, since

$$p_1^*L_u^{\rm alg} \cup p_2^*L_v^{\rm alg} = \delta_2(\{u, v\}),$$

we see that $F^2{\rm CH}_0(X)$ is the image of $K_2(\mathbb{C})$ in $H^2(X, \mathcal{K}_2)$.

For (1), we have the isomorphisms

$$H^2(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{K}_2) \cong \mathbb{Z},\ \ H^1(\mathbb{P}^1, \mathcal{K}_2) \cong \mathbb{C}^*,$$

the generator of $H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2)$ being the class of a point in $\mathrm{CH}_0(\mathbb{P}^1\times\mathbb{P}^1)$, and the map $\mathbb{C}^*\to H^1(\mathbb{P}^1,\mathcal{K}_2)$ being induced by the Gysin map $\mathbb{C}^* = H^0(\mathrm{Spec}\,\mathbb{C},\mathcal{K}_1)\to H^1(\mathbb{P}^1,\mathcal{K}_2)$ for the inclusion of a point (this follows from the projective bundle formula for $\mathcal{K}$-cohomology). Using Gersten's conjecture *loc. cit.* and the Gersten resolution of $\mathcal{K}_2$ [13, §7, Proposition 5.8], we have

$$R^j(q\times q)_*\mathcal{K}_2 = 0 = R^j(i\times q)_*\mathcal{K}_2;\quad j>0$$

and

$$H^j(\mathbb{P}^1,\mathcal{K}_2)=0;\quad j>1.$$

Fix a smooth point y of E_0. The inclusion $y\times y\to X$ induces the map $\mathbb{Z}\to\mathrm{CH}_0(X)$. The inclusions $E_0\times y\to X$, $y\times E_0\to X$ induce the map

$$\mathbb{C}^*\times\mathbb{C}^* = F^1\mathrm{CH}_0(E_0)\times F^1\mathrm{CH}_0(E_0)\to\mathrm{CH}_0(X);$$

one checks that these maps correspond to the terms $H^2(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2)$ and $H^1(\mathbb{P}^1,\mathcal{K}_2)\times H^1(\mathbb{P}^1,\mathcal{K}_2)$ in the spectral sequence arising from the resolution (2.4) of $\ker\alpha$. Thus, this spectral sequence gives the exact sequence

$$H^1(X,\ker\alpha)\xrightarrow{\gamma} H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2)\to\mathbb{Z}\oplus\mathbb{C}^*\times\mathbb{C}^*\oplus K_2(\mathbb{C})$$
$$\to H^2(X,\mathcal{K}_2)\to 0.$$

To complete the proof of (1), we need only show that γ is surjective.

Let $x=q^{-1}(y)$, giving the inclusions $i_1:\mathbb{P}^1\times x\to\mathbb{P}^1\times\mathbb{P}^1$, $i_2: x\times\mathbb{P}^1\to\mathbb{P}^1\times\mathbb{P}^1$. By the projective bundle formula, $H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2)$ is isomorphic to $\mathbb{C}^*\oplus\mathbb{C}^*$, with each $\mathbb{C}^*$ given as the image under Gysin of the maps

$$i_{j*}:\mathbb{C}^* = H^0(\mathbb{P}^1,\mathcal{K}_1)\to H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2);\quad j=1,2.$$

We can factor say i_{1*} as the composition

$$\mathbb{C}^* = H^0(\mathrm{Spec}\,\mathbb{C},\mathcal{K}_1)\xrightarrow{i_{x*}} H^1(\mathbb{P}^1,\mathcal{K}_2)\xrightarrow{p_2^*} H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2).$$

Since y is a smooth point of E_0, we have

$$H^1_y(E_0,\mathcal{K}_2)\cong H^1_x(\mathbb{P}^1,\mathcal{K}_2)\cong H^0(x,\mathcal{K}_1),$$

so we have the Gysin map $H^0(\mathrm{Spec}\,\mathbb{C},\mathcal{K}_1)\xrightarrow{i_{y*}} H^1(E_0,\mathcal{K}_2)$ with $q^*\circ i_{y*}=i_{x*}$. Also, the map

$$(q\times q)^*\circ p_2^*:\mathcal{K}_{2E_0}\to\mathcal{K}_{2\mathbb{P}^1\times\mathbb{P}^1}$$

factors through $\ker\alpha$. Thus, we see that the factor $i_{1*}(\mathbb{C}^*)$ of $H^1(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{K}_2)$ is in the image of γ. The factor $i_{2*}(\mathbb{C}^*)$ is handled similarly. □

In contrast to Proposition 2.1, the Steinberg relation is *not* satisfied in $\mathrm{CH}_0(E \times E')$ if at least one of E, E' is smooth. To show this, we first require the following lemma:

Lemma 2.2. *Let $s : \mathbb{C} \setminus \{0,1\} \to E \times E'$ be the analytic map $s(u) = (p(u), p'(1-u))$. Then $s(\mathbb{C} \setminus \{0,1\})$ is not contained in any algebraic curve on $E \times E'$, except in case $E = E' = E_0$.*

Proof. We first consider the case in which both E and E' are smooth elliptic curves, $E = E_t$, $E' = E_{t'}$, where t and t' are in $\mathbb{C}^*$ and $|t| < 1$, $|t'| < 1$. We have the maps

$$p : \mathbb{C}^* \to E,\ p' : \mathbb{C}^* \to E',$$

which are group homomorphisms with $\ker p = t^{\mathbb{Z}}$, $\ker p' = t'^{\mathbb{Z}}$.

Suppose that $s(\mathbb{C}^*)$ is contained in an algebraic curve $D \subset E \times E'$. For each $x \in E$, $(x \times E') \cap D$ is a finite set, hence, for each $u \in \mathbb{C} \setminus \{0,1\}$, the set of points of $\mathbb{C}^* \times \mathbb{C}^*$ of the form $(t^n u, 1 - t^n u)$ has finite image in $E \times E'$. Thus, for each u, there are integers n, m and p, depending on u, such that $n \neq m$ and

$$1 - t^m u = t'^p (1 - t^n u). \tag{2.7}$$

Since there are uncountably many u, there is a single choice of n, m and p for which (2.7) holds for uncountably many u. But then

$$(t'^p t^n - t^m) u = t'^p - 1. \tag{2.8}$$

If $t'^p t^n - t^m = 0$, then $|t'| = 1$, contradicting the condition $|t'| < 1$. If $t'^p t^n - t^m \neq 0$, then we can solve (2.8) for u, so (2.7) only holds for this single u, a contradiction.

If say $E' = E_0$, then $p' : \mathbb{C}^* \to E'$ is injective, and we have the infinite set of points $p'(1 - t^n u)$ in the image of s, all lying over the single point $p(u)$. □

Theorem 2.3. *Let $E = E_t$, $E' = E_{t'}$, with at least one of E, E' non-singular. Then, for all u outside a countable subset of $\mathbb{C} \setminus \{0,1\}$, $p(u) * p'(1-u)$ is not a torsion element in $F^2\mathrm{CH}_0(E \times E')$.*

Proof. We first give the proof in case E and E' are both non-singular. For a quasi-projective $\mathbb{C}$-scheme X, we let $S^n X$ denote the nth symmetric power of X. For X smooth, we have the map

$$\begin{aligned} \rho_n : S^n X(\mathbb{C}) \times S^n X(\mathbb{C}) &\to \mathrm{CH}_0(X) \\ (\sum_{i=1}^n x_i, \sum_{j=1}^n y_j) &\mapsto [\sum_{i=1}^n x_i - \sum_{j=1}^n y_j]. \end{aligned}$$

For each integer $n \geq 1$, we have the morphism

$$\phi_n : E \times E' \to S^{2n}(E \times E') \times S^{2n}(E \times E')$$
$$(x, y) \mapsto (n(x, y) + n(0, 0), n(x, 0) + n(0, y)),$$

By [15, Theorem 1], $(\rho_{2n} \circ \phi_n)^{-1}(0)$ is a countable union of Zariski closed subsets of $E \times E'$.

On the other hand, since $p_g(E \times E') = 1$, the Albanese kernel $F^2\mathrm{CH}_0(E\times E')$ is "infinite dimensional" [10]; in particular, $F^2\mathrm{CH}_0(E\times E')_{\mathbb{Q}} \neq 0$. Since $F^2\mathrm{CH}_0(E \times E')$ is generated by cycles of the form $p(u) * p(v)$, it follows that $(\rho_{2n} \circ \phi_n)^{-1}(0)$ is a countable union of *proper* closed subsets of $E \times E'$. If D is a proper Zariski closed subset of $E \times E'$, then, by Lemma 2.2, $s^{-1}(D)$ is a proper closed analytic subset of $\mathbb{C} \setminus \{0, 1\}$, hence $s^{-1}(D)$ is countable. Thus, the set of $u \in \mathbb{C} \setminus \{0, 1\}$ such that $p(u) * p'(1 - u)$ is torsion is countable, which completes the proof in case both E and E' are non-singular.

If say $E' = E_0$, we use essentially the same proof. We let X be the open subscheme $E \times (E_0 \setminus \{*\})$ of $E \times E_0$. We have the map $\rho_n : S^nX(\mathbb{C}) \times S^nX(\mathbb{C}) \to \mathrm{CH}_0(E \times E_0)$ defined as above. By [7, Theorem 4.3], $(\rho_{2n} \circ \phi_n)^{-1}(0)$ is a countable union of closed subsets D_i of X. By [17], we have the similar infinite dimensionality result for $\mathrm{CH}_0(E \times E_0)$ as in the smooth case, from which it follows that each D_i is a proper closed subset of X. Thus, the closure of each D_i in $E \times E_0$ is a proper algebraic subset of $E \times E_0$. The same argument as in the smooth case finishes the proof. □

3. Indetectability

The zero-cycle $p(u) * p(1-u)$ is indetectable by cohomology theories based on the sheaf $\mathcal{O}^*_{E_{\mathrm{an}}\times E'_{\mathrm{an}}}$. We first consider the following abstract situation.

The exponential sequence

$$0 \to \mathbb{Z}(1) \xrightarrow{\iota} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 1$$

defines a projective resolution of the group $\mathbb{C}^*$, so the complex

$$\mathbb{C}^*(1) \xrightarrow{\iota\otimes\mathrm{id}} \mathbb{C} \otimes \mathbb{C}^*$$

represents the derived tensor product $\mathbb{C}^* \otimes^L \mathbb{C}^*$. Let $\Gamma_0(2)$ be the complex:

$$\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}] \oplus \mathbb{C}^*(1) \to \mathbb{C} \otimes \mathbb{C}^*$$
$$(u, 2\pi in \otimes z) \mapsto \log(1 - u) \otimes u + 2\pi in \otimes z,$$

with $\mathbb{C} \otimes \mathbb{C}^*$ in degree two (we make some choice of $\log(1 - u)$ for each $u \in \mathbb{C} \setminus \{0, 1\}$).

Let $X = E \times E'$, and let $\Gamma(2)_{\mathrm{an}}$ be a complex of sheaves on X_{an} with the following properties:

(3.1)

1. There is a group homomorphism $\mathrm{cl} : \mathrm{CH}_0(X) \to \mathbb{H}^4(X_{\mathrm{an}}, \Gamma(2)_{\mathrm{an}})$.
2. There is a map $\rho : \mathcal{O}^*_{X_{\mathrm{an}}} \otimes^L \mathcal{O}^*_{X_{\mathrm{an}}}[-2] \to \Gamma(2)_{\mathrm{an}}$ in the derived category of sheaves $D(\mathrm{Sh}_{X_{\mathrm{an}}})$.
3. The composition
$$\mathbb{C}^* \otimes^L \mathbb{C}^*[-2] \to \mathcal{O}^*_{X_{\mathrm{an}}} \otimes^L \mathcal{O}^*_{X_{\mathrm{an}}}[-2] \to \Gamma(2)_{\mathrm{an}}$$
extends to a map $\Gamma_0(2) \to \Gamma(2)_{\mathrm{an}}$ in $D(\mathrm{Sh}_{X_{\mathrm{an}}})$.
4. The composition
$$\mathrm{Pic}(X) \otimes \mathrm{Pic}(X) \cong H^1(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}}) \otimes H^1(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}}) \xrightarrow{\cup} \mathbb{H}^2(X_{\mathrm{an}}, \mathcal{O}^*_{X_{\mathrm{an}}} \otimes^L \mathcal{O}^*_{X_{\mathrm{an}}}) \xrightarrow{\rho} \mathbb{H}^4(X_{\mathrm{an}}, \Gamma(2)_{\mathrm{an}})$$
agrees with the composition
$$\mathrm{Pic}(X) \otimes \mathrm{Pic}(X) \xrightarrow{\cup} \mathrm{CH}_0(X) \xrightarrow{\mathrm{cl}} \mathbb{H}^4(X_{\mathrm{an}}, \Gamma(2)_{\mathrm{an}}).$$

Remark 3.1. To justify the axioms above, at least in case X is smooth, we note the following: Let Y be a scheme smooth and of finite type over a field k. There is a complex of sheaves $\Gamma_Y(q)$ on Y_{Zar} whose hypercohomolgy computes the *motivic cohomology* of Y,
$$H^p(Y, \mathbb{Z}(q)) = \mathbb{H}^p(Y_{\mathrm{Zar}}, \Gamma_X(q)).$$
The complexes $\Gamma_Y(q)$ have products $\Gamma_Y(q) \otimes^L \Gamma_Y(q') \to \Gamma_Y(q+q')$ in the derived category, and the assignment $Y \mapsto \Gamma_Y(q)$ extends to a functor from smooth $\mathbb{C}$-schemes of finite type to the derived category of sheaves on the big Zariski site of smooth schemes of finite type over $\mathbb{C}$. $\Gamma_Y(0) = \mathbb{Z}_{Y_{\mathrm{Zar}}}$ and $\Gamma_Y(1) = \mathcal{O}^*_Y[-1]$ in the derived category. For details, see e.g. [19].

Suppose we have complexes of sheaves on Y_{an}, $\Gamma_Y(q)_{\mathrm{an}}$, for $q = 1, 2$, functorial in Y, with natural products $\Gamma_Y(1)_{\mathrm{an}} \otimes^L \Gamma_Y(1)_{\mathrm{an}} \to \Gamma_Y(2)_{\mathrm{an}}$, and with $\Gamma_Y(1) = \mathcal{O}^*_{Y_{\mathrm{an}}}$. Suppose in addition we have maps in the derived category of sheaves on the big Zariski site of smooth quasi-projective $\mathbb{C}$-schemes
$$\theta_q : \Gamma_{(-)}(q) \to R\epsilon_* \Gamma(-)(q)_{\mathrm{an}}; q = 1, 2,$$
where ϵ is the change of topology morphism, such that the θ are compatible with the products. Finally, suppose that θ_1 is the canonical map adjoint to the inclusions $\epsilon^* \mathcal{O}^*_Y \to \mathcal{O}^*_{Y_{\mathrm{an}}}$. Then $\Gamma(2)_{\mathrm{an}} := \Gamma_X(2)_{\mathrm{an}}$

satisfies the axioms above. Indeed, since $K_2(\mathbb{C}) = H^2(\operatorname{Spec}\mathbb{C}, \mathbb{Z}(2)) = H^2(\Gamma_{\operatorname{Spec}\mathbb{C}}(2))$ (see [12], [18]) the product map

$$\mathbb{C}^* \otimes^L \mathbb{C}^*[-2] \to \Gamma_{\operatorname{Spec}\mathbb{C}}(2)$$

extends to a map $\Gamma_0(2) \to \Gamma_{\operatorname{Spec}\mathbb{C}}(2)$. Composing this with the map $p_X^* : \Gamma_{\operatorname{Spec}\mathbb{C}}(2) \to \Gamma_X(2)$ given by the structure morphism, and then with $\theta_2(X)$, verifies axiom (3). The remaining axioms follow from the isomorphisms

$$H^2(X, \mathbb{Z}(1)) \cong \mathrm{CH}^1(X) \cong \mathrm{Pic}(X), H^4(X, \mathbb{Z}(2)) \cong \mathrm{CH}^2(X),$$

compatible with the various products.

Theorem 3.2. *Let $E = E_t$ and $E' = E_{t'}$, and let $\Gamma(2)_{\mathrm{an}}$ be a complex of sheaves on $E_{\mathrm{an}} \times E'_{\mathrm{an}}$ satisfying the conditions* (3.1). *Then* $\mathrm{cl}(p(u) * p(1-u)) = 0$ *for all* $u \in \mathbb{C} \setminus \{0, 1\}$.

Proof. We give the proof in case both E and E' are non-singular; the singular case is similar, but easier, and is left to the reader.

Since

$$p(u) * p(1-u) = [p_1^* c_1(L_u^{\mathrm{alg}})] \cap [p_2^* c_1(L_{1-u}^{\mathrm{alg}})],$$

it follows from (3.1)(4) that we need to show that $\rho([L_u^{\mathrm{an}}] \cup [L_{1-u}^{\mathrm{an}}]) = 0$. The class $[L_u^{\mathrm{an}}] \in H^1(E_{\mathrm{an}}, \mathcal{O}_{E_{\mathrm{an}}}^*)$ is the image of $[L_u] \in H^1(E_{\mathrm{an}}, \mathbb{C}^*)$ under the map of sheaves $\mathbb{C}^* \to \mathcal{O}_{E_{\mathrm{an}}}^*$, and similarly for L_{1-u} and L_{1-u}^{an}. Thus, by (3.1)(3), it suffices to see that $p_1^*[L_u] \cup p_2^*[L_{1-u}] \in \mathbb{H}^2(E_{\mathrm{an}} \times E'_{\mathrm{an}}, \mathbb{C}^* \otimes^L \mathbb{C}^*)$ vanishes in $\mathbb{H}^4(E_{\mathrm{an}} \times E'_{\mathrm{an}}, \Gamma_0(2))$.

The $\mathbb{Z}$-covers $p : \mathbb{C}^* \to E = E_t$, $p' : \mathbb{C}^* \to E' = E_{t'}$ give natural maps

$$\alpha : H^*(\mathbb{Z}, H^0(\mathbb{C}^*, \mathbb{C}^*)) \to H^*(E_{\mathrm{an}}, \mathbb{C}^*),$$
$$\beta : H^*(\mathbb{Z}, H^0(\mathbb{C}^*, \mathbb{C}^*)) \to H^*(E'_{\mathrm{an}}, \mathbb{C}^*).$$

Similarly, the $\mathbb{Z}^2$-cover $p \times p' : \mathbb{C}^* \times \mathbb{C}^* \to E \times E'$ gives the natural map

$$\gamma : \mathbb{H}^*(\mathbb{Z}^2, H^0(\mathbb{C}^* \times \mathbb{C}^*, \Gamma_0(2))) \to \mathbb{H}^*(E_{\mathrm{an}} \times E'_{\mathrm{an}}, \Gamma_0(2)).$$

Letting $\iota : \mathbb{C}^* \otimes^L \mathbb{C}^*[-2] \to \Gamma_0(2)$ denote the natural map, the maps above are compatible with the respective cup products:

$$\iota \circ (\alpha(a) \cup \beta(b)) = \gamma \circ \iota(a \cup b).$$

Each $v \in \mathbb{C}^*$ gives the corresponding homomorphism $v : \mathbb{Z} \to \mathbb{C}^*$, $v(n) = v^n$. Since $[L_u] \in H^1(E_{\mathrm{an}}, \mathbb{C}^*)$ is $\alpha(u : \mathbb{Z} \to \mathbb{C}^*)$ and $[L_{1-u}] \in H^1(E'_{\mathrm{an}}, \mathbb{C}^*)$ is $\beta(1-u : \mathbb{Z} \to \mathbb{C}^*)$, it suffices to show that $\iota(p_1^* u \cup p_2^*(1-u)) = 0$ in $\mathbb{H}^4(\mathbb{Z}^2, \Gamma_0(2))$, where $p_1^* u, p_2^*(1-u) : \mathbb{Z}^2 \to \mathbb{C}^*$ are the respective homomorphisms $(a, b) \mapsto u^a$, and $(a, b) \mapsto (1-u)^b$.

We have the spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}^2, H^q(\Gamma_0(2))) \Longrightarrow \mathbb{H}^{p+q}(\mathbb{Z}^2, \Gamma_0(2)).$$

Since $\mathbb{Z}^2$ has cohomological dimension two, and since $H^q(\Gamma_0(2)) = 0$ for $q \neq 1, 2$, it follows that the natural map $\mathbb{H}^4(\mathbb{Z}^2, \Gamma_0(2)) \to H^2(\mathbb{Z}^2, H^2(\Gamma_0(2)))$ is an isomorphism. Since $H^2(\Gamma_0(2)) = K_2(\mathbb{C})$, we need to show that the image of $p_1^* u \cup p_2^*(1-u)$ in $H^2(\mathbb{Z}^2, K_2(\mathbb{C}))$ is zero.

By definition of the cup product in group cohomology, we have

$$\begin{aligned}[p_1^* u \cup p_2^*(1-u)]((a,b),(c,d)) &= p_1^* u(a,b) \otimes p_2^*(1-u)(c-a, d-b) \\ &= u^a \otimes (1-u)^{d-b},\end{aligned}$$

which clearly vanishes in $K_2(\mathbb{C})$. ☐

As an immediate consequence of Theorem 3.2, we have

Corollary 3.3. *Let E, E' and $\Gamma(2)_{\mathrm{an}}$ be as in Theorem 3.2. Then the composition*

$$\mathbb{C}^* \otimes \mathbb{C}^* \xrightarrow{p*p'} \mathrm{CH}_0(E \times E') \xrightarrow{\mathrm{cl}} \mathbb{H}^4(E_{\mathrm{an}} \times E'_{\mathrm{an}}, \Gamma(2)_{\mathrm{an}})$$

factors through the surjection $\mathbb{C}^* \otimes \mathbb{C}^* \to K_2(\mathbb{C})$.

Example 3.4. In [3], S. Bloch defines a quotient complex $\mathcal{B}(2)_X$ of the analytic complex $\mathcal{O}^*_{X_{\mathrm{an}}}(1) \xrightarrow{\iota \otimes \mathrm{id}} \mathcal{O}_{X_{\mathrm{an}}} \otimes \mathcal{O}^*_{X_{\mathrm{an}}}$ fulfilling $\mathcal{H}^i(\mathcal{B}(2)) = 0$ for $i \neq 1, 2$,

$$\mathcal{H}^1(\mathcal{B}(2)) = \mathrm{Im}\Big(r : K_{3,\mathrm{ind}}(\mathbb{C}) \to \mathbb{C}/\mathbb{Z}(2)\Big) =: \Delta^*(1),$$

where r is the regulator map, and $\mathcal{H}^2(\mathcal{B}(2)) = \mathcal{K}_{2,\mathrm{an}}$. He shows in the same article that $r(K_{3,\mathrm{ind}}(\mathbb{C})) = r(K_{3,\mathrm{ind}}(\bar{\mathbb{Q}}))$, thus $\Delta^*(1)$ is a countable subgroup of $\mathbb{C}/\mathbb{Z}(2)$, and also that $\mathcal{B}(2)$ maps to the complex $\mathbb{Z}(2) \to \mathcal{O}_{X_{\mathrm{an}}} \to \Omega^1_{X_{\mathrm{an}}}$ which computes the Deligne cohomology $H^*_{\mathcal{D}}(X, 2)$ when X is projective smooth over $\mathbb{C}$. In fact, the cycle map $\mathrm{CH}^2(X) \to H^4_{\mathcal{D}}(X, 2)$ is shown to factor through $\mathbb{H}^4(X_{\mathrm{an}}, \mathcal{B}(2))$ [5]. Bloch asked in [4] whether the cycle map $\mathrm{CH}^2(X) \to \mathbb{H}^4(X_{\mathrm{an}}, \mathcal{B}(2))$ could possibly be injective. The computations of this article show that it is not. Indeed, the complex $\mathcal{B}(2)_X$ is defined as

$$\mathcal{B}(2)_X := \mathcal{O}^*_{X_{\mathrm{an}}}(1) \xrightarrow{\iota \otimes \mathrm{id}} \mathcal{O}_{X_{\mathrm{an}}} \otimes \mathcal{O}^*_{X_{\mathrm{an}}} / \epsilon(\mathbb{Z}[\mathbb{C} \setminus \{0,1\}]),$$

where $\epsilon : \mathbb{Z}[\mathbb{C} \setminus \{0,1\}] \to \mathbb{C} \otimes \mathbb{C}^*$ is the map defined on generators $a \in \mathbb{C} \setminus \{0,1\}$ by

$$\epsilon(a) = \log(1-a) \otimes a - [2\pi i \otimes \exp\Big(\frac{-1}{2\pi i} \int_0^a \log(1-t) \frac{dt}{t}\Big)].$$

Let us take $\Gamma(2)_{\rm an} = \mathcal{B}(2)$. We now verify the conditions 3.1. The complex $\mathcal{O}^*_{X_{\rm an}}(1) \xrightarrow{\iota\otimes \rm id} \mathcal{O}_{X_{\rm an}} \otimes \mathcal{O}^*_{X_{\rm an}}$ represents $\mathcal{O}^*_{X_{\rm an}} \otimes^L \mathcal{O}^*_{X_{\rm an}}$, so the evident surjection of complexes gives us a map $\mathcal{O}^*_{X_{\rm an}} \otimes^L \mathcal{O}^*_{X_{\rm an}} \to \mathcal{B}(2)_X$. Also, the complexes $\mathcal{B}(2)_X$ are clearly contravariantly functorial in X, so to verify (3), it suffices to extend the map $\mathbb{C}^* \otimes^L \mathbb{C}^* \to \mathcal{B}(2)_{\rm Spec\,\mathbb{C}}$ to a map $\Gamma_0(2) \to \mathcal{B}(2)_{\rm Spec\,\mathbb{C}}$. We have the evident surjection

$$(\mathbb{C}^*(1) \to \mathbb{C} \otimes \mathbb{C}^*) \to \mathcal{B}(2)_{\rm Spec\,\mathbb{C}},$$

which we extend to the map $\Gamma_0(2) \to \mathcal{B}(2)_{\rm Spec\,\mathbb{C}}$ by using the map $\tilde\epsilon : \mathbb{Z}[\mathbb{C} \setminus \{0,1\}] \to \mathbb{C}^*(1)$ defined on generators by

$$\tilde\epsilon(a) = 2\pi i \otimes \exp\Big(\frac{-1}{2\pi i}\int_0^a \log(1-t)\frac{dt}{t}\Big).$$

The condition (1) is given by [5]. Indeed, one computes the Leray spectral sequence associated to $\alpha : X_{\rm an} \to X_{\rm Zar}$ and the first term entering $\mathbb{H}^4(\mathcal{B}(2))$ is

$$E^{2,2} = H^2_{\rm Zar}(\mathbb{R}^2\alpha_*\mathcal{B}(2)) = H^2(\mathcal{K}_{2,\mathbb{Z}}),$$

where $\mathcal{K}_{2,\mathbb{Z}} := {\rm Ker}\Big(\alpha_*\mathcal{K}_{2,\rm an} \xrightarrow{d\log\wedge d\log} H^2(\mathbb{C}/\mathbb{Z}(2))\Big)$. Then the cycle map cl is induced by $\mathcal{K}_2 \to \mathcal{K}_{2,\mathbb{Z}}$ on $X_{\rm Zar}$, which is obviously compatible with the product in Pic. Thus we have (4).

Hence we can apply Theorem 2.3 to yield a 0-cycle $p(u)*p(1-u)$ on $E \times E'$, where both E and E' are smooth elliptic curves, which is non-torsion in the Chow group ${\rm CH}_0(E \times E')$, but which dies in $\mathbb{H}^4(\mathcal{B}(2))$ by Theorem 3.2.

In [9], S. Lichtenbaum constructs an étale version $\Gamma(2)$ of S. Bloch's analytic complex $\mathcal{B}(2)$, the cohomology of which contains ${\rm CH}^2(X)$. This contrasts with the examples discussed above.

Over a p-adic field, W. Raskind and M. Spieß ([14]) show that the Albanese kernel modulo n of a product of two Tate elliptic curves is dominated by $K_2(k)/n$. This result is not immediately comparable to ours, but is obviously related.

Remark 3.5. Since, $K_2(\bar{\mathbb{Q}}) = 0$, it follows from Corollary 3.3 that ${\rm cl}(p(u)*p(v)) = 0$ in $\mathbb{H}^4(E_{\rm an} \times E'_{\rm an}, \Gamma(2)_{\rm an})$ for all $u, v \in \bar{\mathbb{Q}}^*$, and all $\Gamma(2)_{\rm an}$ satisfying the conditions (3.1), in particular for $\Gamma(2)_{\rm an} = \mathcal{B}(2)$. It would be interesting to know if $p(u)*p(v) \in F^2{\rm CH}_0(E \times E')$ is non-torsion for some $u, v \in \bar{\mathbb{Q}}^*$.

4. The Relative Situation

In this section, we study the cycles constructed in section 2 on $X = E \times E_0$, where as there, E is smooth, and E_0 is a nodal curve. We extend the definition of Bloch's complex $\mathcal{B}(2)$ to this case by using a *relative* complex $\bar{\mathcal{B}}(2)$, and use Theorem 3.2 to show that the cycles $p(u) * p(1-u)$ die in $\mathbb{H}^4(X_{\text{an}}, \bar{\mathcal{B}}(2))$. Using some results from transcendence theory, we are able to construct examples of non-torsion cycles on X which not only die in $\mathbb{H}^4(X_{\text{an}}, \bar{\mathcal{B}}(2))$, but vanish as well in the absolute Hodge cohomology $H^2(X, \Omega^2_{X/\mathbb{Q}})$.

Let $\nu = 1 \times q : E \times \mathbb{P}^1 \to X$ be the normalization. We define

$$\bar{\mathcal{K}_2} = \text{Ker}\Big(\nu_* \mathcal{K}_2 \xrightarrow{|_{E\times 0} - |_{E\times\infty}} \mathcal{K}_2|_E\Big) \tag{4.1}$$

Lemma 4.1. *One has*

$$\text{CH}_0(X) = H^2(X, \bar{\mathcal{K}_2}),$$

and the Chow group $\text{CH}_0(X)$ *fits into an exact sequence*

$$0 \to H^1(E, \mathcal{K}_2) \xrightarrow{\gamma} \text{CH}_0(X) \xrightarrow{\nu^*} \text{CH}_0(E \times \mathbb{P}^1) = \text{Pic}(E) \otimes \text{Pic}(\mathbb{P}^1) \to 0.$$

Moreover, the map γ *is defined by*

$$\gamma(\sum_{x \in E^{(1)}} x \otimes \lambda_x) = \sum_{x \in E^{(1)}} (x, p_0(\lambda_x)) - (x, 0).$$

Proof. As in the proof of Proposition 2.1, the map $\nu^* : \mathcal{K}_2 \to \bar{\mathcal{K}_2}$ is surjective, and the kernel is supported in codimension 1. Thus ν^* induces an isomorphism on H^2.

On the other hand,

$$H^1(E \times \mathbb{P}^1, \mathcal{K}_2) = H^1(E, \mathcal{K}_2) \oplus H^0(E, \mathcal{K}_1) \cup c_1(\mathcal{O}(1)).$$

The term $H^1(E, \mathcal{K}_2)$ maps to $0 \in H^1(E, \mathcal{K}_2)$ via the difference of the restrictions to $E \times 0$ and $E \times \infty$, while $c_1(\mathcal{O}(1))$ restricts to 0 to either $E \times 0$ or $E \times \infty$. This shows the long exact sequence associated to the short one defining $\bar{\mathcal{K}_2}$.

Finally, the value $\gamma(x \otimes \lambda_x)$ of the map is given by the boundary morphism $\mathbb{C}^* \to H^1(X, \mathcal{O}_X^*)$ induced by the normalization sequence

$$0 \to \mathcal{O}_X^* \to q_* \mathcal{O}_{\mathbb{P}^1}^* \xrightarrow{|_0 - |_\infty} \mathbb{C}^* \to 0$$

on the right argument λ_x. The formula for γ thus follows from Lemma 1.1. □

Let $\text{Nm} : H^1(E, \mathcal{K}_2) \to \mathbb{C}^*$ be the norm map, defined by

$$\text{Nm}\Big(\sum_{x \in E^{(1)}} x \otimes \lambda_x\Big) := \prod_{x \in E^{(1)}} \lambda_x. \tag{4.2}$$

We set

$$V(E) = \text{Ker}\text{Nm}. \tag{4.3}$$

One has

Lemma 4.2. $F^2\mathrm{CH}_0(X) = \gamma\Big(V(E)\Big)$.

Proof. By the definition given in §2, $F^2\mathrm{CH}_0(X)$ is generated by the expressions $[(x,y)] - [(x,0)] - [(0,y)] + [(0,0)]$, with $x \in E(\mathbb{C})$ and $y \in E_0(\mathbb{C}) \setminus \{*\}$. By the formula for γ given in Lemma 4.1, this expression is $\gamma(x \otimes y - 0 \otimes y)$, after identifying $y \in \mathbb{C}^*$ with $p_0(y) \in E_0(\mathbb{C})$. Clearly $V(E)$ is generated by the elements of $H^1(E, \mathcal{K}_2)$ of the form $x \otimes y - 0 \otimes y$, whence the lemma. □

Next, we want to map $\mathrm{CH}_0(X)$ to a relative version of S. Bloch's analytic motivic cohomology. So we define

$$\bar{\mathcal{B}}(2) := \mathrm{Ker}\Big(\nu_*\mathcal{B}(2) \xrightarrow{|_{E\times 0} - |_{E\times\infty}} \mathcal{B}(2)|_E\Big). \tag{4.4}$$

In particular, $\bar{\mathcal{B}}(2)$ is an extension of

$$\bar{\mathcal{K}}_{2,\mathrm{an}} = \mathrm{Ker}\Big(\nu_*\mathcal{K}_{2,\mathrm{an}} \xrightarrow{|_{E\times 0} - |_{E\times\infty}} \mathcal{K}_{2,\mathrm{an}}|_E\Big),$$

placed in degree 2, by $\Delta^*(1)$, placed in degree 1. In other words, $\mathcal{B}(2)$ is the pull-back of $\bar{\mathcal{B}}(2)$ via the map $\nu^* : \mathcal{K}_{2,\mathrm{an}} \to \bar{\mathcal{K}}_{2,\mathrm{an}}$, and in particular, it receives the complex $\Gamma_0(2)$ as explained in the example 3.4.

Considering again the Leray spectral sequence attached to the identity $\alpha : X_{\mathrm{an}} \to X_{\mathrm{zar}}$, we see that

$$\bar{\mathcal{K}}_{2,\mathbb{Z}} := \mathrm{Ker}\Big(\alpha_*\bar{\mathcal{K}}_{2,\mathrm{an}} \to \mathcal{H}^2(\mathbb{C}/\mathbb{Z}(2))\Big) \tag{4.5}$$

receives $\bar{\mathcal{K}}_2$ and that the first map of the spectral sequence is then

$$H^2(X, \bar{\mathcal{K}}_{2,\mathbb{Z}}) \to \mathbb{H}^4(X_{\mathrm{an}}, \bar{\mathcal{B}}(2)). \tag{4.6}$$

In conclusion, we have shown

Lemma 4.3. *One has a cycle map*

$$\psi_X : \mathrm{CH}_0(X) \to \mathbb{H}^4(X_{\mathrm{an}}, \bar{\mathcal{B}}(2))$$

compatible with the cycle map

$$\psi_{E\times\mathbb{P}^1} : \mathrm{CH}_0(E \times \mathbb{P}^1) \to \mathbb{H}^4((E \times \mathbb{P}^1)_{\mathrm{an}}, \mathcal{B}(2))$$

on the normalization. Moreover, ψ_X fulfills the conditions described in 3.1.

Proof. We just have to verify the condition 4 of 3.1. From the normalization sequence

$$0 \to \mathcal{O}_X^* \to \nu_*\mathcal{O}_{E\times\mathbb{P}^1}^* \xrightarrow{|_{E\times 0} - |_{E\times\infty}} \mathcal{O}_E^* \to 0,$$

one has a natural map

$$\mathcal{O}^*_{X_{\mathrm{an}}} \otimes \mathcal{O}^*_{X_{\mathrm{an}}} \to \bar{\mathcal{K}}_{2,\mathrm{an}}$$

which obviously fulfills 3.1(4). □

Now we can apply Theorem 3.2 to conclude

Theorem 4.4. *The 0-cycles defined by the Steinberg curve on $E \times E_0$ die in the analytic motivic cohomology $\mathbb{H}^4(X_{\mathrm{an}}, \bar{\mathcal{B}}(2))$.*

Let K be a subfield of $\mathbb{C}$. We next consider for any algebraic variety Z defined over K, the cycle map with values in the absolute Hodge cohomology

$$H^m(Z, \mathcal{K}_2) \xrightarrow{d\log \wedge d\log} H^m(Z, \Omega^2_{Z/\mathbb{Q}}) \tag{4.7}$$

induced by the absolute $d\log$ map

$$\mathcal{O}^*_Z \xrightarrow{d\log} \Omega^1_{Z/\mathbb{Q}}. \tag{4.8}$$

This cycle map is obviously compatible with the map γ, and with extension of scalars.

Let $E \to \operatorname{Spec} K$ be an elliptic curve over a subfield K of $\mathbb{C}$. We have the exact sheaf sequence

$$0 \to \mathcal{O}_E \otimes \Omega^1_{K/\mathbb{Q}} \to \Omega^1_{E/\mathbb{Q}} \to \Omega^1_{E/K} \to 0,$$

which induces a two-term filtration $F^*\Omega^2_{E/\mathbb{Q}}$ of $\Omega^2_{E/\mathbb{Q}}$ with $F^2\Omega^2_{E/\mathbb{Q}} = \mathcal{O}_E \otimes \Omega^2_{K/\mathbb{Q}}$. This gives us the natural maps

$$\gamma_1 : H^*(E, \mathcal{O}_E) \otimes \Omega^1_{K/\mathbb{Q}} \to H^*(E, \Omega^1_{E/\mathbb{Q}})$$
$$\gamma_2 : H^*(E, \mathcal{O}_E) \otimes \Omega^2_{K/\mathbb{Q}} \to H^*(E, \Omega^2_{E/\mathbb{Q}}).$$

We have the norm map $\mathrm{Nm} : H^1(E, \mathcal{K}_2) \to H^0(K, \mathcal{K}_1) = K^*$ as in 4.2, but over K; we let $V(E) \subset H^1(E, \mathcal{K}_2)$ be the kernel of Nm (see (4.3)).

Lemma 4.5. *Let K be an algebraically closed subfield of $\mathbb{C}$, $E \to \operatorname{Spec} K$ an elliptic curve over K. Then the cycle map with values in absolute Hodge cohomology maps $V(X)$ to the subgroup $\gamma_2[H^1(E, \mathcal{O}_E) \otimes \Omega^2_{E/\mathbb{Q}}]$ of $H^1(E, \Omega^2_{E/\mathbb{Q}})$.*

Proof. The kernel of the composition

$$\mathrm{Pic}(E) = H^1(E, \mathcal{K}_1) \xrightarrow{d\log} H^1(E, \Omega^1_{E/\mathbb{Q}}) \to H^1(E, \Omega^1_{E/K}) \cong K$$

is the composition

$$\mathrm{Pic}(E) \xrightarrow{\deg} \mathbb{Z} \subset K,$$

hence the $d\log$ map sends $\mathrm{Pic}^0(E)$ to the subgroup $\gamma_1[H^1(E,\mathcal{O}_E)\otimes \Omega^1_{K/\mathbb{Q}}]$ of $H^1(E,\Omega^1_{E/\mathbb{Q}})$.

Take $\tau\in\mathrm{Pic}^0(E)$, $u\in H^0(E,\mathcal{K}_1)=K^*$, and let $\xi=\tau\cup u\in H^1(E,\mathcal{K}_2)$. Then

$$d\log(\xi)=d\log(\tau)\cup d\log(u).$$

Since $d\log: K^*\to\Omega^1_{K/\mathbb{Q}}$ is just the absolute $d\log$ map, we see that $d\log(\xi)$ lands in the image of the cup product map

$$[H^1(E,\mathcal{O}_E)\otimes\Omega^1_{K/\mathbb{Q}}]\otimes\Omega^1_{K/\mathbb{Q}}\to H^1(E,\Omega^2_{E/\mathbb{Q}}),$$

which is $\gamma_2(H^1(E,\mathcal{O}_E)\otimes\Omega^2_{K/\mathbb{Q}})$.

Since K is algebraically closed, the cup product $\mathrm{Pic}(E)\otimes K^*\to H^1(E,\mathcal{K}_2)$ is surjective, from which one sees that the cup product maps $\mathrm{Pic}^0(E)\otimes K^*$ onto $V(E)$. Combining this with the computation above completes the proof. □

From the surjectivity of the cup product $\mathrm{Pic}^0(E)\otimes K^*\to V(E)$ for K algebraically closed, we see that the injection $H^1(E,\mathcal{K}_2)\to\mathrm{CH}_0(X)$ sends $V(E)$ isomorphically onto $F^2\mathrm{CH}_0(X)$.

Let K be a subfield of $\mathbb{C}$. We say that an element ξ of $\mathrm{CH}_0(X)$ is *defined over* K if there is an K-scheme X^0, an element ξ^0 of $\mathrm{CH}_0(X^0)$ and an isomorphism $\alpha: X^0_{\mathbb{C}}\to X$ such that $\xi=\alpha_*(\xi^0_{\mathbb{C}})$. From Lemma 4.5 and the compatibility of $d\log$ with extension of scalars, we have

Lemma 4.6. *Take $K=\mathbb{C}$, and let ξ be an element of $F^2\mathrm{CH}_0(X)=V(E)$. If ξ is defined over a field of transcendence degree one over $\mathbb{Q}$, then ξ vanishes under the cycle map to absolute Hodge cohomology.*

Corollary 4.7. *If E is an elliptic curve with complex multiplication, then there are non-torsion cycles $\xi\in F^2\mathrm{CH}_0(X)$ dying in the analytic motivic cohomology as well as in absolute Hodge cohomology.*

Proof. By the remark above, we may replace $F^2\mathrm{CH}_0(X)$ with $V(E)$. Let $\bar{E}$ be a model for E, with equation $y^2=4x^3-ax-b$ defined over a number field $K\subset\mathbb{C}$. Let $\omega=\frac{dx}{y}$ be the standard global one-form on $\bar{E}$.

Choosing an isomorphism $\bar{E}_{\mathbb{C}}\cong E_{\mathbb{C}}$ defines the period lattice $L_\omega\subset\mathbb{C}$ for ω. Choose a basis for L_ω of the form $\{\Omega,\tau\Omega\}$, and let $t=e^{2\pi i\tau}$. Let

$$\mathcal{P}:\mathbb{C}\to\mathbb{CP}^1$$

be the Weierstraß P-function for the lattice L_ω.

The map $\times\Omega^{-1} : \mathbb{C} \to \mathbb{C}$ gives rise to the isomorphism of Riemann surfaces $\alpha_{\mathrm{an}} : \bar{E}^{\mathrm{an}}_{\mathbb{C}} \to E^{\mathrm{an}}_t$ making the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\times\Omega^{-1}} & \mathbb{C} \\ \Big\downarrow {\scriptstyle (\mathcal{P},\mathcal{P}')} & & \Big\downarrow {\scriptstyle \exp} \\ & & \mathbb{C}^* \\ & & \Big\downarrow {\scriptstyle p} \\ \bar{E}^{\mathrm{an}}_{\mathbb{C}} & \xrightarrow[\alpha_{\mathrm{an}}]{} & E^{\mathrm{an}}_t \end{array}$$

commute, i.e.,

$$p(u) = \alpha_{\mathrm{an}}(\mathcal{P}(\frac{\Omega}{2\pi i}\mathrm{log}u), \mathcal{P}'(\frac{\Omega}{2\pi i}\mathrm{log}u)).$$

We let

$$\alpha : \bar{E}_{\mathbb{C}} \to E_t$$

be the corresponding isomorphism of algebraic elliptic curves over $\mathbb{C}$.

By [1, théorème 1], $\mathcal{P}(\frac{\Omega}{2\pi i}\mathrm{log}u)$ has transcendence degree 1 over $\bar{\mathbb{Q}}$ for all $u \in \mathbb{N}$, $u \geq 2$. (We thank Y. André for giving us this reference). Fix a $u \geq 2$, let K be the algebraic closure of the field $\mathbb{Q}(\mathcal{P}(\frac{\Omega}{2\pi i}\mathrm{log}u))$, and let $x \in \bar{E}(K)$ be the point $(\mathcal{P}(\frac{\Omega}{2\pi i}\mathrm{log}u), \mathcal{P}'(\frac{\Omega}{2\pi i}\mathrm{log}u))$. Then x is a generic point of $\bar{E}$ over $\bar{\mathbb{Q}}$.

We take

$$\xi := p(u) * p(1-u).$$

By construction, $\xi = \alpha(\xi_K \times_K \mathbb{C})$, where $\xi_K \in H^1(\bar{E}, \mathcal{K}_2)$ is the element $[(x) - (0)] \cup [1-u]$. Here $[(x) - (0)]$ denotes the class in $\mathrm{Pic}(E) = H^1(\bar{E}, \mathcal{K}_1)$, and $[1-u]$ denotes the class in $H^0(\bar{E}, \mathcal{K}_1) = K^*$. Since K has transcendence degree one over $\bar{\mathbb{Q}}$, the class of ξ in the absolute Hodge cohomology of E vanishes, by Lemma 4.6. By Theorem 4.4, ξ dies in the analytic motivic cohomology of E as well. It remains to show that ξ is a non-torsion element of $H^1(E_K, \mathcal{K}_2)$.

We give an analytic proof of this using the regulator map with values in Deligne-Beilinson cohomology.

Let Y be a smooth projective surface over $\mathbb{C}$, and let $\mathrm{NS}(Y)$ denote the Néron-Severi group of divisors modulo homological equivalence. Then Hodge theory implies that

$$\mathrm{NS}(Y) = \{(z, \varphi) \in (H^2(Y_{\mathrm{an}}, \mathbb{Z}(1)) \times F^1H^2(Y_{\mathrm{an}}, \mathbb{C})), z \otimes \mathbb{C} = \varphi\},$$

and that

$$\mathrm{NS}(Y) \cap F^2H^2_{DR}(Y) = \emptyset.$$

We note that the map $\mathrm{Pic}(Y)\otimes\mathbb{C}^*\to H^3_{\mathcal{D}}(Y,\mathbb{Z}(2))$ induced by the cup product in Deligne cohomology factors through $\mathrm{NS}(Y)\otimes\mathbb{C}^*$, and that the induced map $\iota:\mathrm{NS}(Y)\otimes\mathbb{C}^*\to H^3_{\mathcal{D}}(Y,\mathbb{Z}(2))$ is injective. Indeed,

$$H^3_{\mathcal{D}}(Y,\mathbb{Z}(2))=H^2(Y_{\mathrm{an}},\mathbb{C}/\mathbb{Z}(2))/F^2.$$

Now take $Y=E\times E$, and let $U\subset E$ be the complement of a non-empty finite set Σ of points of E. Let $[E\times 0]$ be the class of $E\times 0$ in $\mathrm{NS}(Y)$, and let $\gamma:\mathbb{C}^*\to\mathrm{NS}(Y)\otimes\mathbb{C}^*$ be the map $\gamma(v)=[E\times 0]\otimes v$. Let

$$\iota_U:\mathrm{NS}(Y)\otimes\mathbb{C}^*\to H^3_{\mathcal{D}}(E\times U,\mathbb{Z}(2))$$

be the composition of ι with the restriction map $H^3_{\mathcal{D}}(Y,\mathbb{Z}(2))\to H^3_{\mathcal{D}}(E\times U,\mathbb{Z}(2))$. We claim that the sequence

$$\mathbb{C}^*\xrightarrow{\gamma}\mathrm{NS}(Y)\otimes\mathbb{C}^*\xrightarrow{\iota_U}H^3_{\mathcal{D}}(E\times U,\mathbb{Z}(2))$$

is exact. Indeed, we have the localization sequence

$$\oplus_{s\in\Sigma}H^1_{\mathcal{D}}(E\times s,\mathbb{Z}(1))\xrightarrow{\oplus_s\iota_s}H^3_{\mathcal{D}}(Y,\mathbb{Z}(2))\to H^3_{\mathcal{D}}(E\times U,\mathbb{Z}(2))\to,$$

the isomorphism $H^1_{\mathcal{D}}(E\times s,\mathbb{Z}(1))\cong\mathbb{C}^*$ and the identity

$$\iota_s(v)=\gamma(v),\ v\in\mathbb{C}^*,$$

which proves our claim.

In particular, let $[\Xi]=[\Delta-\{0\}\times E]\otimes v$, where Δ is the diagonal, v is an element of $\mathbb{C}^*$ which is not a root of unity, and $[\Delta-\{0\}\times E]$ is the class in $\mathrm{NS}(Y)$. Since $[\Delta-\{0\}\times E]$ is not torsion in $\mathrm{NS}(Y)/[E\times\{0\}]$, we see that $[\Xi]$ has non-torsion image $[\Xi_{\mathbb{C}(E)}]$ in

$$H^3_{\mathcal{D}}(E\times_{\mathbb{C}}\mathbb{C}(E),\mathbb{Z}(2)):=\varinjlim_{\emptyset\neq U\subset E}H^3_{\mathcal{D}}(E\times U,\mathbb{Z}(2)),$$

where the limit is over non-empty Zariski open subsets U of E.

Let Ξ be the image of $(\Delta-0\times E)\otimes v$ in $H^1(Y,\mathcal{K}_2)$. Then $[\Xi]$ is the image of Ξ under the regulator map $H^1(Y,\mathcal{K}_2)\to H^3_{\mathcal{D}}(Y,\mathbb{Z}(2))$. Similarly, letting $\Xi_{\mathbb{C}(E)}$ be the pull-back of Ξ to $E\times_{\mathbb{C}}\mathbb{C}(E)$, $[\Xi_{\mathbb{C}(E)}]$ is the image of $\Xi_{\mathbb{C}(E)}$ under the regulator map $H^1(E\times_{\mathbb{C}}\mathbb{C}(E),\mathcal{K}_2)\to H^3_{\mathcal{D}}(E\times_{\mathbb{C}}\mathbb{C}(E),\mathbb{Z}(2))$. Thus, $\Xi_{\mathbb{C}(E)}$ is a non-torsion element of $H^1(E\times_{\mathbb{C}}\mathbb{C}(E),\mathcal{K}_2)$ for each non-torsion element $v\in\mathbb{C}^*$.

Let $\bar\Delta$ be the diagonal in $\bar E\times\bar E$, let $\bar\xi$ be the image of $(\bar\Delta-0\times\bar E)\otimes(1-u)$ in $H^1(E,\mathcal{K}_2)$, and let $\bar\xi_{\bar{\mathbb{Q}}(E)}$ be the image of $\bar\xi$ in $H^1(\bar E\times_{\bar{\mathbb{Q}}}\bar{\mathbb{Q}}(\bar E),\mathcal{K}_2)$. Clearly, after choosing a complex embedding $\bar{\mathbb{Q}}\subset\mathbb{C}$, $\Xi_{\mathbb{C}(E)}$ (for $v=1-u$) is the image of $\bar\xi_{\bar{\mathbb{Q}}(E)}$ under the extension of scalars $\bar{\mathbb{Q}}(\bar E)\to\mathbb{C}(\bar E)\cong\mathbb{C}(E)$, hence $\bar\xi_{\bar{\mathbb{Q}}(E)}$ is a non-torsion element of $H^1(\bar E\times_{\bar{\mathbb{Q}}}\bar{\mathbb{Q}}(\bar E),\mathcal{K}_2)$.

Since x is a geometric generic point of $\bar{E}$ over $\bar{\mathbb{Q}}$, there is an embedding $\sigma : \bar{\mathbb{Q}}(E) \to \mathbb{C}$ such that $x : \operatorname{Spec}\mathbb{C} \to \bar{E}$ is the composition $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\bar{\mathbb{Q}}(E) \to \bar{E}$. Thus, ξ is the image of $\bar{\xi}$ under $(\mathrm{id}\times x)^* : H^1(\bar{E}\times_{\bar{\mathbb{Q}}}\bar{E}, \mathcal{K}_2) \to H^1(E, \mathcal{K}_2)$, and hence ξ is the image of $\bar{\xi}_{\bar{\mathbb{Q}}(E)}$ under the map $\mathrm{id}\times\sigma_* : H^1(\bar{E}\times_{\bar{\mathbb{Q}}}\bar{\mathbb{Q}}(\bar{E}), \mathcal{K}_2) \to H^1(E, \mathcal{K}_2)$ induced by the extension of scalars σ.

Since the kernel of $\mathrm{id}\times\sigma_*$ is torsion, it follows that ξ is a non-torsion element of $H^1(E, \mathcal{K}_2)$, as desired. □

Remark 4.8. Going back to $X = E\times E'$, where both elliptic curves are smooth, we are lacking the transcendence theorem which would force the existence of a cycle $0 \neq \xi = p(u) * p(1-u) \in F^2\mathrm{CH}_0(X)$ dying both in $\mathbb{H}^4(X, \mathcal{B}(2))$ and in absolute Hodge cohomology.

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