COHOMOLOGY OF THE MODULI STACK OF COHERENT SHEAVES ON A CURVE

Abstract. We compute the cohomology of the moduli stack of coherent sheaves on a curve and find that it is a free graded algebra on infinitely many generators.

1. Introduction

The moduli stack of vector bundles on a curve and its cohomological invariants have been much studied. In particular Atiyah–Bott showed in [1] that its cohomology is freely generated by the Künneth components of the Chern classes of the universal vector bundle. In Laumon’s work on the geometric Langlands correspondence [5] the larger stack of coherent sheaves plays an important role.

In this note we compute the cohomology of the stack of coherent sheaves on a smooth projective curve \( C \), defined over some field \( k \), and thereby answer a question of Olivier Schiffmann, related to the aforementioned work.

We find that the cohomology of the moduli stack of coherent sheaves of fixed degree and positive generic rank is again freely generated by the Künneth components of the Chern classes of the universal family of coherent sheaves. However, since for a family of coherent sheaves on a curve any higher Chern class can be non-trivial, we find that – unlike in the case of vector bundles – the result does not depend on the generic rank of the coherent sheaf.

To state the result, let us fix some notation. We will denote étale cohomology groups with \( \mathbb{Q} \) coefficients by \( H^* \), i.e., \( H^* \) denotes étale cohomology with \( \mathbb{Q} \) coefficients. If the ground field \( k \) is chosen to be \( \mathbb{C} \), the same result will hold for singular cohomology with rational coefficients.

Theorem 1. Let \( \text{Coh}_d^d \) denote the stack of coherent sheaves of degree \( d \) and generic rank \( n \) on \( C \).

1. For \( n = 0 \) we have
   \[ H^*(\text{Coh}_0^d) = H^*(C) \otimes \mathbb{Q} \llbracket C \rrbracket \]

   and
   \[ H^*(\text{Coh}_n^d) = \text{Sym}^d(H^*(\text{Coh}_0^d)). \]

2. For \( n > 0 \) and \( d \in \mathbb{Z} \) we have
   \[ H^*(\text{Coh}_n^d) = \mathbb{Q}[a_1, a_2, \ldots] \otimes \bigwedge [b_{ij}]_{i \in N, j = 1, \ldots, 2g} \otimes \mathbb{Q}[f_1, f_2, \ldots]. \]

Here \( a_i \in H^{2i}(\text{Coh}_0^d) \), \( b_{ij} \in H^{2i-1}(\text{Coh}_0^d) \) and \( f_i \in H^{2i-2}(\text{Coh}_0^d) \) and \( \bigwedge \) denotes the exterior algebra generated by the given elements.

If \( k = \mathbb{F}_q \) and the eigenvalues of Frobenius on \( H^1(C) \) are \( \alpha_j \) then the \( a_i, b_{ij}, f_i \) can be chosen to be eigenvectors for the action of Frobenius with eigenvalues \( q^i, q^{i-1} \alpha_j, q^{i-1} \) respectively.

Remark 1. The generators \( a_i, b_{ij}, f_i \) occurring as generators in the theorem, are the Atiyah–Bott classes, defined as follows: For any \( n \geq 0 \), let us denote by \( \text{F}_n^{univ} \) the universal coherent sheaf on \( \text{Coh}_n^d \times C \). Since \( \text{Coh}_n^d \) is a smooth stack, locally of
finite type over $k$ (see e.g. [6, Théorème 4.6.2.1]) we can consider the Chern classes (see section 2):

$$c_i(F^d_{n,univ}) \in H^{2i}(\text{Coh}_n \times C) = \bigoplus_{k=0}^{2} H^{2i-k}(\text{Coh}_n^d) \otimes H^k(C).$$

Let $1 \in H^0(C), \gamma_1, \ldots, \gamma_2g \in H^1(C), [pt] \in H^2(C)$ be a basis of $H^*(C)$, then we can define the classes $c_i, a_i^j, b_j$ by the formula:

$$c_i(F^d_{n,univ}) = a_i \otimes 1 + \sum_{k=1}^{2g} b_i^j \otimes \gamma_j + f_i \otimes [pt].$$

As we will see below these classes generate $H^*(\text{Coh}_n^d)$ for all $n \geq 0$.

2. Preliminaries

Before giving the proof of our result, let us recall some well known results on the cohomology of stacks and Chern classes for bundles on stacks that we will use. First, since the stack $\text{Coh}_n^d$ is only locally of finite type, it will be useful to reduce our problem to substacks of finite type:

**Lemma 2.** Let $\mathcal{M}$ be a smooth algebraic stack locally of finite type over a field $k$ and $\mathcal{Z} \subset \mathcal{M}$ a closed substack of codimension $\geq c$. Then the restriction $H^i(\mathcal{M}) \rightarrow H^i(\mathcal{M} - \mathcal{Z})$ is an isomorphism for $i < 2c - 1$ and it is injective for $i = 2c - 1$.

**Proof.** Denote by $j: \mathcal{M} - \mathcal{Z} \rightarrow \mathcal{M}$ the inclusion. It is sufficient to show that $R^kj_*\mathbb{Q} = 0$ for $0 < k < 2c - 1$. By the base change theorem for smooth maps, this can be checked locally on an atlas $X \rightarrow \mathcal{M}$, so the result follows from the corresponding result for schemes. \qed

By definition, a vector bundle $\mathcal{E}$ on an algebraic stack $\mathcal{M}$ defines a map $f_\mathcal{E}: \mathcal{M} \rightarrow B\text{GL}_n$. Since $H^*(B\text{GL}_n) = \mathbb{Q}[c_1, \ldots, c_n]$ (see [2, Theorem 2.3.2]), cohomological Chern classes of $\mathcal{E}$ can be defined as $c_i(\mathcal{E}) := f_\mathcal{E}(c_i)$ and the Chern-series of $\mathcal{E}$ is defined as $c_1(\mathcal{E}) := 1 + \sum_{i=1}^{\infty} c_i(\mathcal{E}) t^i$. The series $c_i(\mathcal{E})$ is multiplicative on short exact sequences and therefore induces a notion of Chern classes for coherent sheaves on stacks that admit a resolution by vector bundles. By the preceding lemma this notion extends to coherent sheaves such that the restriction of the sheaf to any substack of finite type admits a resolution by vector bundles. For example this property is satisfied for the universal coherent sheaf $F^d_{n,univ}$ on $\text{Coh}_n^d \times C$.

Finally we will need a stack-theoretic analog of [1, Proposition 13.4]. In order to formulate it, let us recall that if $\mathcal{M} \rightarrow \mathcal{M}$ is a $\mathbb{G}_m$-gerbe and $\mathcal{E}$ is a vector bundle on $\mathcal{M}$, then $\mathbb{G}_m$ acts naturally on all fibers of $\mathcal{E}$. If this action is given as multiplication by the $n$-th power of scalars, the bundle is said to be of weight $n$.

**Lemma 3.** Let $\mathcal{M} \rightarrow \mathcal{M}$ be a morphism of algebraic stacks and suppose that $\mathcal{M}$ is a $\mathbb{G}_m$-gerbe over $\mathcal{M}$. Let $\mathcal{E}$ a vector bundle on $\mathcal{M}$ that is of weight $n \neq 0$ with respect to the $\mathbb{G}_m$-gerbe structure. Then we have:

1. $H^*(\mathcal{M}) = H^*(\mathcal{M})[c_1(\mathcal{E})]$.
2. The Chern classes $c_i(\mathcal{E})$ for $i = 1, \ldots, \text{rank}(\mathcal{E})$ are not zero divisors in $H^*(\mathcal{M})$.

**Proof.** Recall that $H^*(B\mathbb{G}_m) = \mathbb{Q}[c_1]$, where $c_1$ is the first Chern class of the universal line bundle on $B\mathbb{G}_m$. Since the universal line bundle is of weight 1, the first claim is true for $\mathcal{M} = B\mathbb{G}_m \rightarrow \mathcal{M} = \text{Spec}(k)$.

The general case follows from this and the argument of Leray-Hirsch, since the first Chern class of $\mathcal{E}$ is an element of the cohomology of $\mathcal{M}$ that restricts to a generator on all fibers (see e.g. [3]).
The second part follows from this, because the restriction to a fiber of \( \pi \) defines the evaluation map \( ev: H^*(\mathcal{M}) = H^*(\overline{\mathcal{M}})_{[c_1(\mathcal{E})]} \to \overline{\mathcal{Q}}_{\ell}[c_1] \). Since the restriction of \( \mathcal{E} \) to a fiber is isomorphic to \( (\mathcal{L}_{univ})^n \), the image of \( c_i(\mathcal{E}) \) under this map is not a zero divisor, so that \( c_i(\mathcal{E}) = a[c_1]^k + \beta \) with \( \beta \in \ker(ev) \). Hence \( c_i(\mathcal{E}) \) is not a zero divisor.

3. PROOF OF THE THEOREM

The first part of Theorem 1 follows from [5, Section 3]. Let us briefly recall the argument. For \( d = 1 \) we have a canonical isomorphism \( C \times BG_m \cong \text{Coh}_0^d \), mapping a point of \( C \) to the skyscraper sheaf defined by the point. Since \( H^*(BG_m) \cong \overline{\mathcal{Q}}_{\ell}[c_1] \) (e.g. [2]) the claim for \( d = 1 \) follows.

For \( d > 1 \) we follow [5] and consider the stack \( \overline{\text{Coh}}_0^d = (0 \subseteq T_1 \subseteq \cdots \subseteq T_d | T_i \in \text{Coh}_0^1) \).

The map \( \text{gr} : \overline{\text{Coh}}_0^d \to \coprod_{i=1}^d \text{Coh}_0^1 \), given by \( \text{gr}(\mathcal{T}_i) = (\mathcal{T}_i/\mathcal{T}_{i-1})_{i=1,\ldots,d} \) is a smooth fibration with contractible fibers, so that it induces an isomorphism in cohomology. Moreover by [5, Theorem 3.3.1], the forgetful map \( p: \text{Coh}_0^d \to \text{Coh}_0^d \) is small and generically an étale \( S_d \)-covering, so \( p_*(\mathcal{Q}_\ell) \) carries an action of the symmetric group \( S_d \) and we have \( p_*(\mathcal{Q}_\ell)^{S_d} \cong \mathcal{Q}_\ell \). Therefore \( H^*(\text{Coh}_0^d) \cong H^*(\text{Coh}_0^d)^{S_d} \), proving the first claim.

To show (2) note that the classes \( a_i, b_i, f_i \) define a morphism of graded algebras:

\[
\begin{align*}
AB: \overline{\mathbb{Q}}_{\ell}[a_1,a_2,\ldots] \otimes \bigwedge [b_i^e]_{i=\ell,j=1,\ldots,2g} \otimes \mathbb{Q}_{\ell}[f_1,f_2,\ldots] & \to H^*(\text{Coh}_0^d).
\end{align*}
\]

Before showing that the morphism is injective, let us first check that the graded components of the two rings have the same dimension. To this end, let us stratify \( \text{Coh}_0^d \) according to the length of the torsion

\[
\begin{align*}
\text{Coh}_n^{d,e} := \langle \mathcal{F} | \text{length}(\text{Tors}(\mathcal{F})) \leq e \rangle.
\end{align*}
\]

We have natural maps:

\[
\begin{align*}
\text{gr}: \text{Coh}_n^{d,e} & \to \text{Bun}_n^{d-e} \times \text{Coh}_0^d
\end{align*}
\]

given by mapping a sheaf \( \mathcal{F} \) to \( (\mathcal{F}/\text{Tors}(\mathcal{F}), \text{Tors}(\mathcal{F})) \), and the direct sum of sheaves induces a map

\[
\oplus: \text{Bun}_n^{d-e} \times \text{Coh}_0^d \to \text{Coh}_n^{d,e}.
\]

The map \( \oplus \) is a vector bundle whose fiber over a sheaf \( \mathcal{F} \in \text{Coh}_n^{d,e} \) is given by \( \text{Hom}(\mathcal{F}/\text{Tors}(\mathcal{F}), \text{Tors}(\mathcal{F})) \), because any coherent sheaf on \( C \) is non-canonically isomorphic to the direct sum of its torsion subsheaf and its torsion free quotient. Since \( \text{Bun}_n^{d-e} \) and \( \text{Coh}_e^d \) are smooth, this implies that \( \text{Coh}_n^{d,e} \) is a smooth, locally closed substack of \( \text{Coh}_n^d \) of codimension \( ne \) and that

\[
H^*(\text{Coh}_n^{d,e}) = H^*(\text{Bun}_n^{d-e}) \otimes H^*(\text{Coh}_0^d).
\]

To deduce the additive structure of \( H^*(\text{Coh}_n^d) \) we use an analog of an argument of Atiyah–Bott [1]: First note that since \( \text{Coh}_n^{d,\leq e} \) is an open dense substack of \( \text{Coh}_n^d \), such that the complement is of codimension \( n(e + 1) \), we have by Lemma 2

\[
H^i(\text{Coh}_n^d) \cong H^i(\text{Coh}_n^{d,e}) \text{ for } i < 2n(e + 1).
\]

Consider the Gysin sequence [2, Corollary 2.1.3] for the pair \( (\text{Coh}_n^{d,e}, \text{Coh}_n^{d,e}) \):

\[
\cdots \to H^{e-2ne}(\text{Coh}_n^{d,e}) \to H^*(\text{Coh}_n^{d,\leq e}) \to H^*(\text{Coh}_n^{d,\leq e-1}) \to \cdots.
\]
We claim that this long exact sequence splits into short exact sequences. This holds because the composition

\[ H^{*-2ae}(\text{Coh}_n^{d,=e}) \to H^*(\text{Coh}_n^{d,\leq e}) \to H^*(\text{Coh}_n^{d,=e}) \]

is given by the cup product with the top Chern class of the normal bundle of \( \text{Coh}_n^{d,=e} \) \( \subset \text{Coh}_n^{d,\leq e} \). The normal bundle is given by

\[ \text{Ext}^1(\text{Tors}(\mathcal{F}_{\text{univ}}), \mathcal{F}_{\text{univ}}/\text{Tors}(\mathcal{F}_{\text{univ}})). \]

If we restrict this bundle to \( \text{Bun}_n^{d-e} \times \text{Coh}_n^e \) we see that the central \( \mathbb{G}_m \)-automorphisms given by multiplication by scalars on one of the factors act non-trivially on the fibers of this bundle. Lemma 3 thus implies that the top Chern class of the bundle is not a zero-divisor in \( H^*(\text{Bun}_n^{d-e}) \otimes H^*(\text{Coh}_n^0) \), so that additively we have

\[ H^*(\text{Coh}_n^d) \cong \bigoplus_{e \geq 0} H^{*-2ae}((\text{Bun}_n^d) \times \text{Coh}_n^0). \]

By the result of Atiyah–Bott [1, Theorem 2.15] (see e.g. [4] for an algebraic argument) we know that the Poincaré series of \( \text{Bun}_n^d \) is given by

\[ P_{\text{Bun}_n^d}(t) = \sum_{i=0}^{\infty} \dim (H^i(\text{Bun}_n^d)) t^i = \frac{\prod_{i=1}^{n}(1 + t^{2i-1}2g)}{\prod_{i=1}^{n}(1 - t^{2i}) \prod_{i=2}^{n}(1 - t^{2i-2})}. \]

And since \( H^*(\text{Coh}_n^0) = \text{Sym}^e(H^*(\text{C}}) \cdot \mathbb{C} \) \( ]] \) we have

\[ Z(\text{Coh}_n^1, z) := \sum_{e=0}^{\infty} P_{\text{Coh}_n^1}(t) z^e = \frac{\prod_{i=1}^{\infty}(1 + z t^{2i-1}2g)}{\prod_{i=1}^{\infty}(1 - z t^{2i}) \prod_{i=1}^{\infty}(1 - z t^{2i-2})}. \]

Thus the Poincaré series of \( \text{Coh}_n^d \) is given by:

\[ P_{\text{Coh}_n^d}(t) = P_{\text{Bun}_n^d}(t) \left( \sum_{e=0}^{\infty} t^{2ae} P_{\text{Coh}_n^1}(t) \right) = P_{\text{Bun}_n^d}(t) Z(\text{Coh}_n^1, t^e) \]

\[ = \frac{\prod_{i=1}^{\infty}(1 + t^{2i-1}2g)}{\prod_{i=1}^{\infty}(1 - t^{2i}) \prod_{i=2}^{\infty}(1 - t^{2i-2})}. \]

This coincides with the Poincaré series of the graded ring freely generated by the Atiyah–Bott classes.

To complete the proof of the theorem, we are left to prove that the morphism \( AB \) is injective. To this end, pick a line bundle \( \mathcal{L} \) such that deg(\( \mathcal{L} \)) = \(- l < 0 \). For any \( k > 0 \) this defines a map \( i_k : \text{Coh}_0^{d+knl} \to \text{Coh}_n^d \), given by \( T \to T \otimes (\mathcal{L}^k)^{n} \). We claim that the composition of \( AB \) with the induced map

\[ i_k^* : H^*(\text{Coh}_n^d) \to H^*(\text{Coh}_0^{d+knl}) \cong \text{Sym}^{d+knl}(H^*(\text{C}}) \cdot \mathbb{C} \]

is injective for \( * \leq d + knl \).

We have \( (i_k \times Id_C)^* \left( c_t(\mathcal{F}_{\text{univ}_n}) \right) = c_t(\mathcal{F}_{\text{univ}_n}^{d+knl} \oplus (\mathcal{L}^k)^{n} ) \) and

\begin{equation}
(3.1) \quad c_t(\mathcal{F}_{\text{univ}_n}^{d} \oplus (\mathcal{L}^k)^{n} ) = c_t(\mathcal{F}_{\text{univ}_n}^{d} \cdot (1 + kc_1(\mathcal{L}))^n ).
\end{equation}

Thus we need to compute \( c_t(\mathcal{F}_{\text{univ}_n}^{d} \oplus (\mathcal{L}^k)^{n} ) \). The universal torsion sheaf \( \mathcal{F}_n^{1} \) on \( \text{Coh}_0^1 \times C \cong B\mathbb{G}_m \times C \times C \) can be described as follows. Let \( \mathcal{L} := p_1^* \mathcal{L}_{\text{univ}} \) denote the pull back of the universal line bundle on \( B\mathbb{G}_m \), denote by \( \Delta \) the diagonal in \( C \times C \) and \( \mathcal{L}(-\Delta) := \mathcal{L}_{\text{univ}} \otimes O(-\Delta) \). Then we have an exact sequence

\[ 0 \to \mathcal{L}(-\Delta) \to \mathcal{L} \to \mathcal{F}_{\text{univ}_n}^{1} \to 0. \]
Thus we find $c_i(F_{0,\text{univ}}^1) = 1 + \sum_{i=1}^{\infty} c_i(F_{0,\text{univ}}) t^i = \frac{c_i(L)}{c_i(L^C(-\Delta))}$. By definition we know that $c_1(L_{\text{univ}}) = c_1 \in H^*(BG_m) = \mathbb{T}_L[c_1]$. Therefore

$$c_i(F_{0,\text{univ}}^1) = (1 + c_1 t)/(1 - ((\Delta) - c_1) t) = 1 + \sum_{i=1}^{\infty} ((\Delta) - c_1)^{i-1} [\Delta] t^i$$

Let us denote by $\tau_j \in H^1(C)$ the basis that is Poincaré dual to the basis $\gamma_j$. Then $[\Delta] = 1 \otimes [pt] + \sum_{j=1}^g \tau_j \otimes \gamma_j + [pt] \otimes 1 \in H^*(C) \otimes H^*(C)$, where $\tau_j \cup \gamma_j = [pt]$ and $[\Delta]^2 = (2 - 2g) [pt] \otimes [pt]$. Putting $c := -c_1$ we find:

$$c_i(F_{0,\text{univ}}^1) = (c^{i-1} + (i-1)(2-2g)c^{i-2}[pt]) \otimes [pt] + \sum_{j=1}^g (c^{i-1} \tau_j) \otimes \gamma_{2g-j} + (c^{i-1}[pt]) \otimes 1.$$

Thus we see that the Künneth components of $c_i(F_{0,\text{univ}}^1)$ form a basis for $H^*(Coh_0^1) = H^*(C)[c]$. Also, the Chern series of a direct sum is the product of the Chern series of the summands, so that the image of $c_i(F_{0,\text{univ}}^d)$ under the inclusion

$$i: H^*(Coh_0^d) \hookrightarrow H^*(Coh_0^1)^{\otimes d}$$

is given by the product of the Chern series of the universal bundles on the factors. Let us write $c_i(F_{0,\text{univ}}^d) = A_i \otimes 1 + B_i \otimes \gamma_j + F_i \otimes [pt]$ and denote by $A_{i,k} := pr_k^*(A_i) \in H^*(\prod_{i=1}^d Coh_0^1)$ and similarly $B_{i,k} := pr_k^*(B_i), F_{i,k} := pr_k^*(F_i)$. Then we have:

$$i(c_i(F_{0,\text{univ}}^d)) = \sum_{i_1 + \cdots + i_d = i} A_{i_1,1} \cdots A_{i_d,d} \otimes 1$$

$$+ \sum_{i_1 + \cdots + i_d = i} \sum_{k=1}^d A_{i_1,1} \cdots B_{i_k,k} \cdots A_{i_d,d} \otimes \gamma_j$$

$$+ \sum_{i_1 + \cdots + i_d = i} \sum_{k=1}^d A_{i_1,1} \cdots F_{i_k,k} \cdots A_{i_d,d} \otimes [pt]$$

$$+ \sum_{i_1 + \cdots + i_d = i} \sum_{k \neq l} \pm A_{i_1,1} \cdots B_{i_k,k} B_{i_l,l}^{2g-j} A_{i_d,d} \otimes [pt]$$

We claim that this implies that the Künneth components of $c_i(F_{0,\text{univ}}^d)$ generate the subring of $S_d$-invariant elements in $H^*(Coh_0^1)^{\otimes d}$. We argue by induction. Since $A_i = c_i^{-1}[pt], B_i = c_i^{-1}\tau_i$, we see that $i(a_1) = \sum_{k=1}^d pr_k^*[\tau_i], i(b_i) = \sum_{k=1}^d pr_k^*(\tau_i)$ are the first elementary symmetric functions of a basis of $H^*(C)$. Moreover, the products $A_i A_k, A_i B_k, B_i B_k$ vanish. Therefore $i(a_i) = \sum_{k=1}^d A_i A_k + r_i$ where $r_i$ is a product of symmetric functions of lower degree and similarly $i(b_i) = \sum_{k=1}^d pr_k^*(B_i) + r_i'$. Finally $i(F_i) = \sum_{k=1}^d pr_k^*(c_i^{-1}) + r''_i$, where $r''_i$ is a symmetric function that is contained in the ideal generated by $\sum pr_k^*[pt]$ and the classes $\sum_{k=1}^d pr_k^*(\tau_i)$. This shows that the Künneth components of $c_i(F_{0,\text{univ}}^d)$ generate $H^*(Coh_0^d)$ and that relations between the classes are contained in cohomological degrees $> d$. By (3.1) this calculation also implies that for any $k > 0$ the composition of $AB$ with the induced map

$$H^*(Coh_0^d) \to H^*(Coh_0^{d+knl}) \cong \text{Sym}_d^{d+knl} (H^*(C)[c_1])$$

is injective for $* \leq d + knl$. This proves the theorem.
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References