Seiberg-Witten theory

Gauge theory and topology of 4-manifolds

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Introduction

Goal

Study the topology and the differentiable structures of 4-manifolds by studying moduli spaces of solutions to certain systems of partial differential equations over the differentiable manifold.

Reference

Introduction à la théorie de jauge, Andrei TELEMAN. Cours spécialisés, Société Mathématique de France, 2012.

4-manifolds

M: compact, connected, oriented, topological manifold, of dimension n.

Intersection pairing

Bilinear pairing

$$H^k(M,\mathbb{Z}) \times H^{n-k}(M,\mathbb{Z}) \longrightarrow H^n(M,\mathbb{Z}) \simeq \mathbb{Z}. \tag{1}$$

Poincaré duality: this is non-degenerate, perfect pairing.

$n = 4$

$$q_M : H^2(M,\mathbb{Z})/\operatorname{Torsion} \times H^2(M,\mathbb{Z})/\operatorname{Torsion} \longrightarrow \mathbb{Z} \tag{2}$$

is a bilinear pairing, symmetric, unimodular.
Its signature is called the *signature* of M.
Changing the orientation reverses the signature.

Simply connected 4-manifolds

If M is simply connected:

- $H^1(M,\mathbb{Z}) = H^3(M,\mathbb{Z}) = 0$,
- $H^2(M,\mathbb{Z})$ has no torsion.

So: $(H^2(M,\mathbb{Z}), q_M)$ is the remaining topological invariant, with rank $b_2(M)$.

Theorem (Freedman)

Every symmetric bilinear unimodular form over $\mathbb{Z}$ is the intersection form of some simply-connected 4-manifold.

Theorem (Donaldson)

If M is a differentiable 4-manifold with negative definite intersection form, then its intersection form is standard (diagonalizable?): there is a basis (h_i) of $H^2(M,\mathbb{Z})/\text{Torsion}$ such that $q_M(h_i, h_j) = -\delta_{i,j}$.

Corollary

There exist topological 4-manifolds without differentiable structure, e. g. the manifold M with $q_M \simeq E_8$.

Hodge-star operator

M: compact, connected, oriented, differentiable manifold, of dimension n.
Λ_M: cotangent bundle of M.
$g = \langle\ ,\ \rangle$: Riemannian metric on M.

Definition (Hodge-star operator)

$$* : \Lambda^k_{x,M} \longrightarrow \Lambda^{n-k}_{x,M} \tag{3}$$

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \mathrm{vol}_g \tag{4}$$

Definition (Laplace operator)

$$d : A^k(M) \longrightarrow A^{k+1}(M) \tag{5}$$

$$d^* : A^{k+1}(M) \longrightarrow A^k(M) \tag{6}$$

$$d^* := -(-1)^{nk} * d * \tag{7}$$

$$\Delta := dd^* + d^*d \tag{8}$$

(9)

Harmonic forms

Definition

A k-form α is *harmonic* iff $\Delta(\alpha) = 0$, iff (M compact) $d(\alpha) = d^*(\alpha) = 0$.
Space of harmonic forms: $\mathbb{H}^k(M)$.

Theorem

The space of harmonic forms is finite-dimensional and each cohomology class is represented by a unique harmonic form:

$$H^k(M, \mathbb{R}) \simeq \mathbb{H}^k(M). \tag{10}$$

Corollary

For $n = 4$, $k = 2$*:* $*^2 = \mathrm{id}$ *on* Λ^2_M*. So*

$$\Lambda^2_M := \Lambda^2_+ \oplus \Lambda^2_- \tag{11}$$

$$\mathbb{H}^2(M) := \mathbb{H}^2_+ \oplus \mathbb{H}^2_- \tag{12}$$

$$b_\pm(M) := \dim(\mathbb{H}^2_\pm) \tag{13}$$

Harmonic forms and signature

For $\alpha \in \mathbb{H}^2_\pm$:

$$q_M^{\mathbb{R}}(\alpha, \alpha) = \int_M \alpha \wedge \alpha = \pm \int_M \alpha \wedge *\alpha = \pm \|\alpha\|^2. \tag{14}$$

Theorem

$(b_+(M), b_-(M))$ is the signature of $q_M^{\mathbb{R}}$.

Connections on vector bundles

Let E be a vector bundle over M (of real or complex vector spaces).

Definition

A *connection* on E is a linear morphism

$$\nabla : A^0(E) \longrightarrow A^1(E) \tag{15}$$

such that, for a local function ($\mathcal{C}^\infty$) f on M and a section s of E

$$\nabla(fs) = df \otimes s + f\nabla(s). \tag{16}$$

The space of connections $\mathscr{A}(E)$ is an affine space over $A^1(\mathrm{End}(E))$: locally $\nabla = d + \omega$, $\omega \in A^1(\mathrm{End}(E))$.

Definition

A connection ∇ on E extends to $d^\nabla : A^k(E) \to A^{k+1}(E)$ via

$$d^\nabla(\alpha \otimes s) := d(\alpha) \otimes s + (-1)^k \alpha \wedge \nabla(s) \tag{17}$$

(locally, α: k-form, s: section of E).

Definition

A connection ∇ on E induces a connection on $\mathrm{End}(E)$ via

$$\nabla^{\mathrm{End}}(u)(s) := \nabla(u(s)) - u(\nabla(s)) \tag{18}$$

(u: local section of $\mathrm{End}(E)$, s: local section of E).

Definition

The *gauge group* of E is the set of global ($\mathcal{C}^\infty$) sections of $\mathrm{Aut}(E)$. It acts on the right on $\mathscr{A}(E)$ via

$$\nabla \cdot f := f^{-1} \circ \nabla \circ f = \nabla + f^{-1} \nabla^{\mathrm{End}}(f). \tag{19}$$

Definition

The *curvature* of the connection ∇ is

$$F_{\nabla} = \nabla \circ \nabla : A^0(E) \longrightarrow A^2(E). \tag{20}$$

It is in fact in $A^2(\mathrm{End}(E))$.

Lemma

For an element f of the gauge group of E,

$$F_{\nabla . f} := f^{-1} \circ F_{\nabla} \circ f. \tag{21}$$

Hermitian vector bundles

Now assume that E has an hermitian metric h.

Definition

A connection ∇ is *compatible with the metric* if for sections s, t of E

$$dh(s,t) = h(\nabla s, t) + h(s, \nabla t) \tag{22}$$

Lemma

These form an affine space over $A^1(\mathfrak{u}(E))$ *(*$\mathfrak{u}(E) \subset \mathrm{End}(E)$*: anti-hermitian endomorphisms). The curvature is in* $A^2(\mathfrak{u}(E))$

On differential forms with values in E there is an operation

$$\wedge : A^k(E) \times A^\ell(E) \longrightarrow A^{k+\ell} \tag{23}$$

combining the wedge product of forms and the metric on E. So there is also a Hodge-$*$ operator

$$* : A^k(E) \longrightarrow A^{n-k}(E) \tag{24}$$

such that $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \operatorname{vol}_g$.
Given a metric connection ∇ there is a Laplace operator Δ_∇

$$\Delta_\nabla := d^\nabla \circ (d^\nabla)^* + (d^\nabla)^* \circ d^\nabla : A^k(E) \longrightarrow A^k(E). \tag{25}$$

Corollary

The curvature of a metric connections ∇ on M of dimension 4 for decomposes into self-dual and anti-self-dual parts:

$$F_\nabla = F_\nabla^+ + F_\nabla^-. \tag{26}$$

Spin structure

E: real vector bundle of rank r over M, defined by cocycles (g_{ij}) ($\mathcal{C}^\infty$ functions on U_{ij} with values in $GL(r, \mathbb{R})$).
A metric h on E gives a *restriction of the structure group*, i.e. equivalent cocycles with values in $O(r) \subset GL(r, \mathbb{R})$.
An orientation of E gives a restriction of the structure group to $SO(r)$.

Definition

Recall that $SO(r)$ is connected ($r > 1$) and $\pi_1(SO(r)) = \mathbb{Z}$ for $r = 2$ and $\mathbb{Z}_2$ for $r > 2$.
The *Spin group* $\mathrm{Spin}(r)$ is the double cover of $SO(r)$ (hence the universal cover for $r > 2$).
A *Spin structure* on E is a choice of lifting of the cocycles g_{ij} from $SO(r)$ to $\mathrm{Spin}(r)$.

To E are associated Stiefel-Whitney classes $w_i(E) \in H^i(M, \mathbb{Z}_2)$.
Recall: E is orientable iff $w_1(E) = 0$.

Theorem

E admits a Spin structure iff $w_2(E) = 0$.
If E admits Spin structures, then the set of equivalence classes of Spin structures is a $\mathbb{Z}_2$-affine space over $H^1(M, \mathbb{Z}_2)$.

Proof.

Cech cohomology. □

Spin^c structure

Definition

The Spin^c *group* is the group $\mathrm{Spin}^c(r) := (\mathrm{Spin}(r) \times S^1)/\mathbb{Z}_2$. It carries a projection π to $SO(r)$ and δ (called the *determinant*) to S^1.
A Spin^c structure on E is a choice of lifting of the cocycles to $\mathrm{Spin}^c(r)$. This defines via δ a hermitian line bundle called the *determinant line bundle*.

Lemma

The determinant line bundle of a Spin^c structure is trivial iff the Spin^c *structure comes from a* Spin *structure.*

Theorem

E admits a Spin^c *structure with determinant L iff $w_2(E)$ equals the image of $c_1(L) \in H^2(M,\mathbb{Z})$ in $H^2(M,\mathbb{Z}_2)$.*
If E admits a Spin^c *structure, the set of isomorphism classes of* Spin^c *structures is a torsor under the group $H^2(M,\mathbb{Z})$.*

Definition

A Spin (resp. Spin^c) structure on an oriented Riemannian manifold M is a Spin (resp. Spin^c) structure on its cotangent bundle Λ_M (via the metric $\Lambda_M \simeq T_M$).

Theorem

Any compact oriented Riemannian manifols of dimension 4 *admits a* Spin^c *structure.*

Spinor bundles

For $r = 2$: $SO(2) = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ so $\mathrm{Spin}(2) = S^1$ and
$\pi : \mathrm{Spin}(2) \to SO(2)$ is $z \mapsto z^2$.
To generalize this: quaternions $\mathbb{H}$.
$q = a + bi + cj + dk$ with $i^2 = j^2 = k^2 = ijk = -1$ and norm
$|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$.
$\{q \in \mathbb{H} \mid |q| = 1\} = S^3 \simeq SU(2)$ via

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{27}$$

$q \in \mathbb{H}$ of norm 1 acts on $x \in \mathbb{H}$ by isometries via

$$q \cdot x := qxq^{-1}. \tag{28}$$

This leaves invariant $1^{\perp} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. This defines a $2:1$ map to $SO(3)$. So
$\mathrm{Spin}(3) = S^3 = SU(2)$.

Now for $r = 4$: a pair (q_-, q_+) of unit quaternions acts on $x \in \mathbb{H}$ via

$$(q_-, q_+) \cdot x := q_+ x q_-^{-1}. \tag{29}$$

This action defines a $2:1$ map to $SO(4)$.

Theorem

$\mathrm{Spin}(4) = SU(2) \times SU(2)$.
$\mathrm{Spin}^c(4) = \{(q_-, q_+) \in U(2) \times U(2) \mid \det(q_-) = \det(q_+)\}$.

Corollary

If E is a vector bundle of rank 4 over M with a Spin^c structure, one can introduce two hermitian bundles of rank 2 (i.e. $U(2)$-bundles) Σ^+, Σ^- with the same determinant $L =$ determinant line bundle. These are the half-spinor bundles. Then $\Sigma := \Sigma^- \oplus \Sigma^+$ is the spinor bundle.

The Dirac operator

Lemma

If E admits a Spin^c *structure, the data of a connection on E and a connection on L (determinant) induces canonically a connection on the half-spinor bundles.*

Proof.

Easier to see with principal bundles. □

Now recall that on T_M there is a particular canonical connection, the Levi-Civita connection, such that

$$\nabla_Y X - \nabla_X Y = [X, Y]. \tag{30}$$

Corollary

Any connection A on L induces canonically connections ∇_A^-, ∇_A^+ on the half-spinor bundles associated to a Spin^c *structure on Λ_M with determinant L.*

Now we work only with M of dimension 4 and with Spin^c structures on M, with determinant L. Let A be a connection on L.

Definition

The connection A induces, via ∇_A^-, ∇_A^+, an operator

$$\not{D}_A : A^0(\Sigma) \longrightarrow A^0(\Sigma). \tag{31}$$

It is the *Dirac operator*. It interchanges positive and negative spinors. It is an elliptic operator of order one.

It behaves like a square root of the Laplace operator: there exists a Weitzenböck formula, expressing the difference between $\not{D}_A \circ \not{D}_A$ and Δ_A, with one term with F_A ($= 0$ for a Spin structure) and one term with the scalar curvature of g ($= 0$ in flat $\mathbb{R}^4$).

The Seiberg-Witten equations

Fix M, with an orientation, a metric g, a Spin^c structure with determinant L. These are equations for a pair (A, Ψ) where $A \in \mathscr{A}(L)$ (connection on L) and $\Psi \in A^0(\Sigma^+)$ (positive spinor).
The moduli space of solutions is quotient of the set of solutions by the action of the gauge group of the Spin^c structure: $\mathscr{G} = \mathcal{C}^\infty(M, S^1)$ acting on both A and Ψ.
First equation: $\not{D}_A(\Psi) = 0$.
Second equation: uses the curvature of A.
Bonus: for a metric of signature $(3, 1)$ instead of $(4, 0)$, A is an electromagnetic potential and Ψ is the wave function of a Dirac electron interacting in the electromagnetic field.