Seiberg-Witten theory 2

Moduli spaces of solutions to the Seiberg-Witten equations

Louis-Clément LEFÈVRE

Universität Duisburg-Essen

June 22, 2020

Recall

$\mathbb{H}$: quaternions. Unit quaternions $\simeq SU(2)$, $\mathbb{H} \simeq \mathbb{R}SU(2)$.
A pair (q_-, q_+) of unit quaternions acts on $x \in \mathbb{H}$ via

$$(q_-, q_+) \cdot x := q_+ x q_-^{-1}. \tag{1}$$

$\mathrm{Spin}(4) = SU(2) \times SU(2)$.
$\mathrm{Spin}^c(4) = \{(q_-, q_+) \in U(2) \times U(2) \mid \det(q_-) = \det(q_+)\}$.
$\pi : \mathrm{Spin}^c(4) \to SO(4)$, $\delta : \mathrm{Spin}^c(4) \to S^1$.
$\pi_\pm : \mathrm{Spin}^c(4) \to U(2)$.

$E \to M$ $SO(4)$-vector bundle with a Spin^c structure τ.
$\longrightarrow \quad \Sigma^\pm$ $U(2)$-bundles (half-spinor bundles), $\Sigma := \Sigma^- \oplus \Sigma^+$ (spinor bundle),
$L = \delta(\tau)$ S^1-bundle (determinant line bundle).

The Clifford action

Spin^c structure on $E \to$ isomorphism $\gamma^- : E \to \text{Hom}(\Sigma^-, \Sigma^+)$ defined as follows:
E: vector bundle associated to the Spin$^c(4)$-cocycles via π.
$\Sigma^{\pm}$: same via $\pi_{\pm}$
So this comes from the map $\mathbb{H} \to \text{Hom}(\mathbb{C}^2, \mathbb{C}^2)$
$(x \in \mathbb{H} \simeq \mathbb{R}SU(2)) \longmapsto (z_- \in \mathbb{C}^2 \mapsto x.z_- \in \mathbb{C}^2)$

γ^- extends to a linear map $E \to \mathfrak{su}(\Sigma) \subset \text{End}(\Sigma)$ (anti-hermitian emdomorphisms of Σ with zero trace) of the form

$$\gamma(\lambda) = \begin{pmatrix} 0 & \gamma^+(\lambda) \\ \gamma^-(\lambda) & 0 \end{pmatrix}, \quad \gamma^+(\lambda) := -\gamma^-(\lambda)^*. \tag{2}$$

This extends then to the tensor algebra of E. In particular for $\omega \in \Lambda^2(E) = \Lambda^2_+(E) \oplus \Lambda^2_-(E)$:

$$\gamma(\omega) = \begin{pmatrix} \gamma_-(\omega_-) & 0 \\ 0 & \gamma_+(\omega_+) \end{pmatrix}, \quad \gamma_{\pm} : \Lambda^2_{\pm}(E) \to \mathfrak{su}(\Sigma^{\pm}). \tag{3}$$

Recall on the Dirac operator

Now: M 4-dimensional, $E = \Lambda$ cotangent bundle.
Fix a Spin^c structure on Λ.
A: connection on $L \quad \longrightarrow \quad$ connections $\nabla_A^{\pm}, \nabla_A$ on $\Sigma^{\pm}, \Sigma$.
$\gamma : \Lambda \to \mathrm{End}(\Sigma) \quad \longleftrightarrow \quad m_\gamma : \Lambda \otimes \Sigma \to \Sigma$.

Definition (Dirac operator)

$$\not{D}_A : A^0(\Sigma) \xrightarrow{\nabla_A} A^1(\Sigma) \xrightarrow{m_\gamma} A^0(\Sigma) \tag{4}$$

Theorem (Weitzenböck formula)

$$\not{D}_A \circ \not{D}_A = \Delta_A + \frac{1}{4}\gamma(F_A) + \frac{s_g}{4}\,\mathrm{id}_\Sigma \tag{5}$$

(F_A: curvature of $A \in iA^2(M)$, s_g scalar curvature of g)

The Seiberg-Witten equations

Notations: on a complex hermitian vector space E of dimension n,
$v, w \in E$: $v \otimes \bar{w} :=$ endomorphism $u \mapsto \langle u, w \rangle v$
$u \in \mathrm{End}(E)$: $u_0 := u - \frac{1}{n}\mathrm{Tr}(u)\,\mathrm{id}_E$.
Fix: M, orientation, metric g, Spin^c structure τ.

Definition (Seiberg-Witten equations)

Equations for a pair $(A, \Psi) \in \mathscr{A}(L) \times A^0(\Sigma^+)$:

$$\not{D}_A \Psi = 0 \qquad \in A^0(\Sigma^-) \tag{6}$$

$$\gamma_+(F_A^+) = (\Psi \otimes \overline{\Psi})_0 \qquad \in A^0(\mathrm{Herm}_0(\Sigma^+)) \tag{7}$$

Perturbed equations: depend on a parameter $\beta \in A^2(M, \mathbb{R})$:

$$\not{D}_A \Psi = 0 \tag{8}$$

$$\gamma_+(F_A + 2\pi i \beta)^+ = (\Psi \otimes \overline{\Psi})_0 \tag{9}$$

Equations: SW_β^τ.
Space of solutions: $\mathscr{A}^{SW_\beta^\tau} \subset \mathscr{A} := \mathscr{A}(L) \times A^0(\Sigma^+)$ + Fréchet $\mathcal{C}^\infty$ topology.

The gauge group

Definition (Gauge group)

$$\mathscr{G} = \mathcal{C}^{\infty}(M, S^1) \tag{10}$$

$=$ group of automorphisms of the Spin^c structure preserving the $SO(4)$ structure. Acts on $\Sigma^{\pm}$ and L via

$$f \cdot \Psi := f\Psi, \quad f \cdot \lambda := f^2\lambda \tag{11}$$

and on $\mathscr{A}(L)$ via

$$A \cdot f := A + 2f^{-1}df. \tag{12}$$

Definition (Moduli space of solutions)

$$\mathscr{M}_{\beta}^{\tau} := \mathscr{A}^{SW_{\beta}^{\tau}} / \mathscr{G} \tag{13}$$

$+$ induced topology.

Fixed points

Theorem

The stabilizer of a solution $(A, \Psi) \in \mathscr{A}(L) \times A^0(\Sigma^+)$ for the $\mathscr{G}$ action is:

- *Trivial if $\Psi \neq 0$ ((A, Ψ) is called irreducible),*
- *S^1 if $\Psi = 0$ ((A, Ψ) is called reducible).*

Proof.

For a fixed point under $f \in \mathscr{G}$: $A \cdot f = A$ so $f^{-1}df = 0$ so $df = 0$.
And $f\Psi = \Psi$. □

Remark: a reducible solution is just a connection A on L with $F_A^+ = -2\pi i \beta^+$.

Definition

$[\mathscr{A}^*]^{SW_\beta^\tau} \subset \mathscr{A}^{SW_\beta^\tau}$ space of irreducible solutions.
$[\mathscr{M}_\beta^\tau]^* := [\mathscr{A}^*]^{SW_\beta^\tau} / \mathscr{G}$ moduli space of irreducible solutions.

The moduli space of solutions

Theorem

$\mathscr{M}_\beta^\tau$ is compact Hausdorff.

Tools of proof.

- Weitzenböck formula + maximum principle for Δ: bound on $|\Psi|$ (depends on β, s_g but not on A).
- "gauge fixing": choose A in some bounded subset up to gauge.
- "Sobolev bootstrapping".

□

The deformation complex

Goal: for a pair $p = (A, \Psi) \in \mathscr{A}$ (β fixed), define a complex

$$C_p := 0 \longrightarrow C^0 \xrightarrow{D_p^0} C^1 \xrightarrow{D_p^1} C^2 \longrightarrow 0 \tag{14}$$

$$C^0 := iA^0(M) \tag{15}$$

$=$ tangent space to $\mathscr{G}$ at the identity.

$$C^1 := iA^1(M) \oplus A^0(\Sigma^+) \tag{16}$$

$=$ tangent space to $\mathscr{A} = \mathscr{A}(L) \times A^0(\Sigma^+)$ (affine space)

$$C^2 := A^0(\mathrm{Herm}_0(\Sigma^+)) \oplus A^0(\Sigma^-) \simeq iA^2_+(M) \oplus A^0(\Sigma^-) \tag{17}$$

$=$ vector space, $\not{D}_A \Psi \in A^0(\Sigma^-)$, $\gamma_+(F_A + 2\pi i \beta)^+ - (\Psi \otimes \overline{\Psi})_0 \in A^0(\mathrm{Herm}_0(\Sigma^+))$.

Differentials depend on p:

$$D_p^0(\varphi) := \begin{pmatrix} 2d\varphi \\ -\varphi\Psi \end{pmatrix} \tag{18}$$

$=$ differential at the identity of the orbit map $f \in \mathscr{G} \mapsto p \cdot f$.

$$D_p^1(\alpha, \varphi) := \begin{pmatrix} \gamma(d^+\alpha) - (\Psi\bar{\psi})_0 - (\psi\overline{\Psi})_0 \\ \not{D}_A\varphi + \frac{1}{2}\gamma(\alpha)\Psi \end{pmatrix} \tag{19}$$

$=$ differential at p of

$$sw_\beta^\tau : p \mapsto \left(\gamma_+(F_A + 2\pi i\beta)^+ - (\Psi \otimes \overline{\Psi})_0,\ \not{D}_A\right). \tag{20}$$

Lemma

If p is a solution to SW_β^τ then $D_p^1 \circ D_p^0 = 0$.

Idea: solutions to SW_β^τ are the vanishing locus of sw_β^τ.

Theorem

- *A solution p is irreducible iff* $H^0(C_p) = 0$.
- *For an irreducible solution p, the Zariski tangent space to* $[\mathscr{M}_\beta^\tau]^*$ *at* $[p]$ *is* $H^1(C_p)$.
- *For an irreducible solution p,* $[p]$ *is a smooth point of* $[\mathscr{M}_\beta^\tau]^*$ *iff* $H^2(C_p) = 0$.

The Kuranishi method

Theorem

Let p be an irreducible solution. There exist neighborhoods U_p of 0 in $H^1(C_p)$, V_p of $[p]$ in $[\mathscr{M}_\beta^\tau]^*$, an analytic map $\chi_p : U_p \to H^2(C_p)$ with $\chi_p(0) = 0$, $d\chi_p(0) = 0$, and a homeomorphism $U_p \supset Z(\chi_p) \xrightarrow{h_p} V_p$ with $h_p(0) = [p]$.

Corollary

$[\mathscr{M}_\beta^\tau]^*$ is a (finite-dimensional) real analytic space.

For reducible solutions: U_p is S^1-invariant, χ_p is S^1-invariant, and h_p induces a homeomorphism $U_p/S^1 \supset Z(\chi_p)/S^1 \xrightarrow{h_p} V_p$.
Expected dimension: index theory

$$w_c := \frac{1}{4}\left(c^2 - 2e(M) - 3\sigma(M)\right) \tag{21}$$

where $c = c_1(L)$, e: Euler Characteristic, σ: signature.

Genericness

Theorem

For "generic" β, $[\mathscr{M}_\beta^\tau]^*$ is a smooth manifold.
There is a condition on (g, β) under which $\mathscr{M}_\beta^\tau$ is a smooth manifold. This condition depends on the anti-self-duality of certain harmonic representatives (with respect to g) of cohomology classes. It is a "generic" condition if $b_+(M) > 0$ and "generic in families" if $b_+(M) > 1$.

Furthermore $[\mathscr{M}_\beta^\tau]^*$ is oriented.
Seiberg-Witten invariants: numbers obtained by pairing homology 1-cycles on $\mathscr{M}_\beta^\tau$ (for β such that this is smooth). For $b_+(M) > 1$ this depends only on the discrete data (τ, orientation of $\mathscr{M}_\beta^\tau$) but non on the continuous data (g, β).
For $b_+(M) = 1$: wall-crossing phemonema.

Donaldson theorem on the intersection form

Theorem

If the intersection form q_M on M is definite negative, then it is standard (diagonalizable over $\mathbb{Z}$).

Proof.

Case where M is simply connected and q_M is even (i.e. $q_M(x,x)$ is even).
M admits a Spin structure (Spin^c with L trivial): by Wu formula on characteristic classes $x \cdot x = w_2(M) \cdot x \; \forall x \in H^2(M, \mathbb{Z}_2)$ so here $w_2(M) = 0$ mod 2 so it is the reduction mod2 of the first Chern class of the trivial bundle.
With our hypothesis $b_+ = 0$, the moduli space is not smooth. There is a reducible connection $p_\beta = (A_\beta, 0)$.
For β generic, and in a neighborhood of p_β: find a contradiction...
q_M not even: use a result of number theory of Elkies to characterize non-standard forms. q_M is standard if $c_1(L)^2 + b_2(M) \leq 0$ for all Spin^c structure with determinant L. □