PROGRAM FOR THE SEMINAR ON INFINITY CATEGORIES

The aim of this term ist to learn how the notion of infinity-categories (we will use quasi-categories as an incarnation of this concept) allow to prove gluing and descent theorems for refined versions of derived categories (for which the corresponding statements may fail). Two applications of this machinery, have been very general results on formal gluing as in [2],[3] and the construction of a good six-functor formalism for algebraic stacks [9].
The ideas used in the construction come from topology and along the way we will recall some of these results. We will start with basic concepts on simplicial sets and see how categorical constructions can often be understood in this language. The notion of quasi-categories takes this point fo view and allows to replaces notions up-to-isomorphism by versions up-to-homotopy in a precise way.
Constructions of this refined structure often use methods from model categories, a notion which we will also recall. This also explains why many constructions such as limits work well for quasi-categories.

Talk 1: Introduction and Overview. Motivation: Derived categories, invert quasi-isomorphisms. This is a brutal procedure and in order to obtain a workable formalism this is done by choosing resolutions.
Still the result is brutal and there are much less results on e.g. descent once one passes to the derived category than for categories of modules ([1]).
A remedy comes from topology it is roughly: remember all (higher) homotopies involved when you want to obtain descent.
To make "higher homotopies" precise we will see quasi-categories.
To formulate sheaf-conditions we need notions of limits etc for these categories.
All of this can be motivated from topology where we have mapping spaces etc. For us spaces will simply be (special kinds of) simplicial sets.
Say something more about overall goals: Barr-Beck-Lurie theorem and maybe an application (e.g.[2],[9]).

Talk 2: From categories to simplicial sets and quasicategories. (The main reference for this talk is [15])
Recall the notion of simplicial sets - use the opportunity to introduce horns and spans:
Part I-Simplicial sets (This should be a short section, the definitions are important for the seminar. The close link between Kan complexes and topological spaces serves as a motivation only.)

(1) the category **Ord**, definition of simplicial sets, the representable simplicial sets $\Delta[n]$, the spheres $S^n := \Delta[n]/\partial\Delta[n]$, products and internal Hom, simplicial homotopy and simplicial homotopy equivalence. Mention geometric realization.
(2) Horns, Kan complexes and mention the theorem that for Kan complexes the two notions (simplicial and topological for the realization) of homotopy and homotopy groups agree. (One classical reference for the statement seems to be [5], but it is faster to ask Marc.) A simple (counter)-example: $\Delta[1]$ and Λ^1_2 are not simplicially homotopy equivalent relative their boundaries. Preliminary definition:

$$\{\textbf{Spaces}\} = \{\text{Kan complexes}\}$$

(3) If you think you have time, you can mention the adjoint pair of functors

$$|-|:\mathbf{sSets}\longleftrightarrow\mathbf{Top}:\mathrm{Sing}(-)$$

Here **Top** is the category of compactly generated topological spaces, $|-|$ is geometric realization and $\mathrm{Sing}(-)$ is the (simplicial) singular complex. Note also that $\mathrm{Sing}(T)$ is a Kan complex for every T.

Part II-Nerve of a (small) category (This is an essential part of the talk. References [4, Section 2] or [15, Section 2-4].)

(1) Definition of the nerve
(2) A few categorical notions in simplicial terms: the opposite category, functors as maps of simplicial sets, natural transformation as homotopies of maps, adjoint functors as retracts, equivalence of categories as homotopy equivalence of simplicial sets.
(3) Characterisation of the nerve: a simplicial set is the nerve of a category if and only if one has unique extension for inner horns $\Lambda_n^k \subset \Delta[n]$, $0 < k < n$.
(4) The nerve of a category C is a space (i.e. Kan complex) if and only if C is a groupoid (i.e. all morphisms in C are invertible) — this is not too surprising once you think about it.

Part III-Quasi-categories (Your aim for the talk)

(1) Define ∞-categories (a.k.a. quasicategories, see [4, 3.2,3.2] for a few words on the notions). Note that this simultaneously generalizes spaces (=Kan complexes) and nerves.
(2) Examples of categorical constructions: opposite category, product, functors, natural transformations. [15, Section 5,6], [4, Section 3,5].
(3) The homotopy category [15, Section 8]

(You can point out that Kan complexes will turn out to be our groupoids.)
As a participant I would then like to see some examples to keep in mind. [15, Section 7] has some examples (7.4,7.7,7.8 are rather easy choices) and to include one or two of these would be nice.

Talk 3: Basic categorical notions in quasicategories. Start with a brief reminder.
Part I: The example of complexes of abelian groups or modules.
It would be nice to see a serious example of a quasicategory. My favorite one would be made from complexes [11, 1.3.1.6]. If you have time, this would be very helpful (go right to the definition - for now we have to skip the alternative route Lurie sketches). When you state the formulas, please explain these !). [11, 1.3.10,1-3.11] explain that this is a quasicategory such that its homotopy category is the usual one. ([11, 1.3.2.7] is the example of usual derived categories of modules, which you should take as the prime example to think about.)
Part II: Basic categorical notions in quasicategories ([6, Section 4], or [4, Section 10,11])
The notions of final, initial objects, direct and inverse limits, fiber products, pushouts etc. have analogs in quasicategories, that have good properties (as we will hopefully see). This is explained e.g. in [10, Chapter 1.2.7-1.2.14].
For a talk it may be good to start with those notions you can motivate best. You don't need to cover all of them, but illustrate some examples, e.g. connect pushout and fiber products with cones/cones[-1] in complexes, i.e. if you take the pull-back of $0- > B$ by a morphism $A \to B$ in complexes in this setup, you get a cone - i.e. the derived pull-back.

Talk 4: Lifting properties and applications to quasicategories. As the definition of quasicategories only required the filling property for inner horns it will be useful to introduce everything that is generated from these building blocks: Define saturated classes of morphisms and our example of inner anodyne[1] morphisms (i.e., those generated by inner horns).
Lifting calculus [15, 12,13,] or [6, Section 2.3], here the main technical tool is the small object argument. Using these notions one can find a new explanation how quasicategories relate to categories [15, 16.8], (see [6, Corollary 3.7.6] for an explanation of how to think about this) and show that functors between quasicategories are again a quasicategory [6, 3.7.9] (the main result for this talk).

Talk 5: Interlude on model categories. The standard reference is [7]. But we now saw a bit of lifting calculus, so [10, Appendix A.2.1-2.5] should be enough and feasable. The main point is to explain how this structure allows to invert classes of morphisms in a category in such a way that one has some control on the result and thus can define derived functors. All along it is very helpful to keep complexes of modules as an example in mind, as well as the fundamental example: simplicial sets, with weak equivalences the maps which are isomorphisms on all "correct" homotopy groups/sets, cofibrations the degree-wise monomorphisms and fibrations the Kan fibrations.

(1) The definition of a model category (called *closed* model category in [16]) can be taken from [7, Definition 1.1.3.]. You can then jump to simplicial model categories quickly, as this is the main input for quasi-categories, see [16, Def. 2, §2, Chap II]. In [7] and in [16, Chap I], there is a discusion of "interval objects" in a model category, which allows one to have a homotopy theory without the simplicial structure. You should mention this, but then rely on the simplicial structure to give canonical objects $X \times I$ and X^I (path space) for each X in the (simplicial) model category, where I is the interval $[0, 1] = \Delta[1]$. This gives a simpler and more direct notion of homotopy. You can mention that in a simplicial model category, the two notions agree.
(2) Defining the "homotopy category" of a model category as the category with objects the fibrant and cofibrant objects in the model category and morphisms the homotopy classes of maps solves the localization problem of inverting all weak equivalences. See e.g. [7, Theorem 1.2.10.] for the statement in a (non-simplicial) model category, which you can just take over in the simplicial case. This is justified in [16, prop. 2, §2, Chap. II].
(3) Define the notion of Quillen adjunction [7, §1.3] and state how this solves the problem of defining left and right derived functors that yield an adjunction on the homotopy categories. For the precise statement of the result, see [7, Lemma 1.3.10], perhaps also mention the "naturality" of derived functors [7, theorem 1.3.7] and its analog for right derived functors.

For us this will be important in the next talk, as Joyal's model structure on simplicial sets is made such that quasicategories are the fibrant objects. This helps to define limits etc.

Talk 6: Joyal's model structure. Joyal model structure [15, Section 40] or [6, Chapter 8]. and consequences One nice topic to explain at this point is the equivalence of the model categories of simplicial sets (with the Joyal model structure) and simplicial categories. This is discussed in [10, §1.1.3, §1.1.4, §1.1.5]. Discuss the simplicial nerve construction associating a simplicial set to a simplicial category and its adjoint $\mathfrak{C}$ associating a simplicial category to a simplicial set. This shows how one can rigidify the "up to homotopy" composition in a quasi-category to a

[1]The terminology might ask for a clarification from a native speaker.

well-defined composition law in a simplicial category. Mention the theorem: there is a Quillen equivalence of model categories

$$\{\text{simplicial sets}\} \sim \{\text{simplicial categories}\}$$

You should also say a word about the weak equivalences in simplicial categories, the fact that, for a topological category, the simplicial nerve is alway a quasi-category, and the fact that the Joyal weak equivalences correspond to categorical weak equivalences [10, Definition 1.1.3.6].

Talk 7: Back to derived categories and functors. Higher algebra - Stable categories 1.1 (officially this is motivated from the notion of a spectra, but I think we can work with looking at complexes here as well and simply explain why this puts a triangulated structure on the homotopy category). 1.3.3 explains how to obtain derived functors on our new derived categories (Theorem 1.3.3.2 or 1.3.3.8), saying that right exact functors on the level of abelian categories extend canonically to t-exact functors on the corresponding quasicategories. This uses quite a bit of packaging and results form [10], so the aim is to indicate how the idea to take the functor on projective objects is carried out.

Talk 8: Adjoint functor theorems. A general machinery to construct adjoint functors is explained in [10] Chapter 5. (This is a version of Neeman's result that the construction of a $f^!$ functor follows from abstract arguments once one knows the original functor preserves coproducts [14].) The main result is Corollary 5.5.2.9. Ok, I should try to give a better guideline: The main point is not to flood us with definitions in the beginning, but to get to the result and to give some idea of the proof and use this to show how the definitions come about.

Talk 9: Interlude: The Barr-Beck theorem and descent. As one of the motivations for us is to get a gluing / descent statement for derived categories it might be good to review the categorical formulation of the argument. In categorical language this is often formulated for the dual category (i.e. descent is a statement on comonads. . .)
For me, unraveling the categorical statement to descent is almost as difficult as proving descent. For a talks I would suggest to start by recalling the argument used to prove faithfully flat descent for modules. Then explain how this is encapsulated in the comonadicity-conditions.
I would suggest the references [1] and [8]. But try to be gentle with your audience, e.g., if you define a monad to be a ring object in the tensor category of endofunctors I will probably be lost. So try to make the definitions so concrete and simple that we can follow the main example through the talk. That the descent datum can be packaged in this notion is hopefully something nice.
Section 3 of [13] might also be helpful. It would also be good if you could have a look at [11, Chapter 4.7.5] to see how Lurie explains the relation between Barr-Beck and descent (just to see how things will look in our setup).

Talk 10: Symmetric monoidal quasicategories. [11] Chapter 2 - again the main point is to explain which problems are solved by introducing the definitions. (More detailed description to come: For the Barr-Beck theorem we need to be able to talk about algebras and modules and tensor products. Also our final application is about functors preserving the tensor product. The word symmetric needs some extra care in our setup and makes this story interesting.)

Talk 11: The Barr-Beck-Lurie theorem. The theorem is [11, Theorem 4.7.3.5] and in 4.7.5 a relation to descent is given (see Corollary 4.7.5.3). Please mention that this implies that we have descent for the quasi-categroical version of the derived category of say quasi-coherent sheaves on a scheme.

Talk 12: Application: Tannaka-duality and gluing. In this talk you should explain the formal gluing statement of [2, Theorem 1.4] (or [3, Corollary 1.5] for a fancier version). (Please do not talk about spectral stacks. Algebraic spaces as in Bhatt, or if you prefer schemes).
Start by explaining the result and how this captures a formal gluing result. (Mention some of the classical examples of the formal result.)
The first key ingredient is the Tannaka duality statement Theorem 1.5. This is explained in Section 2. This of course relies on many things, but the overall strategy is explained well in Section 1 and for us it would be good to see the results we saw before in action, e.g. Lemma 2.3 uses the results from Talk 8 and 11.
The proof of the main result is then in Section 5, which in particular also gives an example showing, why the derived is essential.

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