The $\Gamma$-limit of a finite-strain Cosserat model for asymptotically thin domains and a consequence for the Cosserat couple modulus $\mu_c$

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We study the behaviour of a geometrically exact 3D Cosserat continuum model for an asymptotically flat domain. Despite the inherent nonlinearity, the $\Gamma$-limit of a corresponding canonically rescaled problem on a domain with constant thickness can be explicitly calculated. This "membrane" limit exhibits no bending contributions scaling with $H^2$ (similar to classical approaches) but features a transverse shear resistance scaling with $h$ for strictly positive Cosserat couple modulus $\mu_c > 0$. This result is physically unacceptable for a zero-thickness "membrane" limit model. Therefore it is suggested that the physically consistent value of the Cosserat couple modulus $\mu_c$ is zero. In this case, however, the $\Gamma$-limit looses coercivity for the midsurface deformation in $H^{1,2}(\omega, \mathbb{R}^3)$. For numerical purposes then, a transverse shear resistance can be reintroduced, establishing coercivity.

1 The finite-strain 3D-Cosserat model in variational form

We consider a fully frame-indifferent finite-strain Cosserat [2] formulation on an asymptotically thin domain $\Omega_h = \omega \times [-\frac{h}{2}, \frac{h}{2}]$, where $h > 0$ is the characteristic thickness and $\omega \subset \mathbb{R}^2$ is the referential mid surface. The two-field Cosserat problem will be introduced in a variational setting. The task is to find a pair $(\varphi, R) : \Omega_h \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3 \times SO(3, \mathbb{R})$ of deformation $\varphi$ and independent microrotation $R$ minimizing the energy functional $I$,

$$I(\varphi, R) = \int_{\Omega_h} W(\nabla(\varphi)) + \mu L_c ||D_x R||^2 \, dV \rightarrow \min \text{ w.r.t. } (\varphi, R), \quad \varphi_{\Gamma_h} = g_{\Gamma_h}, \quad R_{\Gamma_h} \text{ free},$$

$$W(\nabla(\varphi)) = \mu ||\text{sym}(\nabla(\varphi - 1))||^2 + \nu \text{tr}[\text{sym}(\nabla(\varphi - 1))|^2 + \mu_c||\text{skew}(\nabla(\varphi - 1))||^2,$$

$$\nabla = R^T \nabla \varphi,$$

$$D_x R := (\nabla_{\Gamma_h}(R_{e1})) (\nabla_{\Gamma_h}(R_{e2})) (\nabla_{\Gamma_h}(R_{e3})), \quad \Gamma_h = \gamma_0 \times [-\frac{h}{2}, \frac{h}{2}],$$

with Dirichlet boundary condition of place for the deformation $\varphi$ on a part of the lateral boundary $\Gamma_h$ with $\gamma_0 : \mathbb{R} \rightarrow \partial \omega \subset \mathbb{R}^2$ and everywhere Neumann conditions on the Cosserat rotations $R$. The parameters $\mu, \nu > 0$ are the classical Lamé constants of isotropic elasticity, the additional parameter $\mu_c \geq 0$ is called the Cosserat couple modulus, whose value is controversial. The parameter $L_c > 0$ (with dimension length) introduces an internal length which is characteristic for the material, e.g. related to the grain size in a polycrystal. The internal length $L_c > 0$ is responsible for size effects in the sense that smaller samples are relatively stiffer than larger samples.

In this setting, the variational problem (1.1) admits minimizers for any given thickness $h > 0$ and for all $\infty \geq \mu_c \geq 0$ ($\mu_c = \infty$ formally implies a symmetry constraint). For more information and mathematical existence results concerning this Cosserat bulk model we refer to [7, 6, 4, 9]. In the following, we are interested in characterizing the behaviour of minimizers to (1.1) as $h \rightarrow 0$.

2 The rescaled Cosserat model

In order to do so, it is customary to consider a corresponding rescaled problem, i.e. transforming the problem (1.1) on a domain with constant thickness. This is achieved by letting $\Omega_t = \omega \times [-\frac{1}{2}, \frac{1}{2}]$ and defining the rescaled deformations and rotations by $\varphi^t(x, y, z) := \varphi(x, y, h z), \quad R^t(x, y, z) := R(x, y, h z)$. The rescaled variational problem reads then

$$P^t(\varphi^t, R^t) = h \int_{\Omega_t} W(\nabla^t(\varphi^t)) + \mu L_c^t ||D_x R^t||^2 \, dV \rightarrow \min \text{ w.r.t. } (\varphi^t, R^t), \quad \varphi^t_{\Gamma_t} = g^t_{\Gamma_t}, \quad R^t_{\Gamma_t} \text{ free},$$

$$\nabla^t = R^t \nabla \varphi^t,$$

$$\nabla \varphi^t := (\partial_x \varphi^t \partial_y \varphi^t \partial_z \varphi^t)^T, \quad \nabla \varphi^t := (\partial_x \varphi^t \partial_y \varphi^t \partial_z \varphi^t)^T \left(= \nabla \varphi\right),$$

$$D_x R^t := (\nabla^t_{\Gamma_t}(R^t_{e1})) (\nabla^t_{\Gamma_t}(R^t_{e2})) (\nabla^t_{\Gamma_t}(R^t_{e3})), \quad \Gamma_t = \gamma_0 \times [-\frac{1}{2}, \frac{1}{2}],$$

for strictly positive Cosserat couple modulus $\mu_c$. For numerical purposes then, a transverse shear resistance can be reintroduced, establishing coercivity.

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and we consider the \textit{sequence of variational problems} $I^p_h(\varphi^p, \overline{R}^p) := \frac{1}{h} I^p(\varphi^p, \overline{R}^p)$.

3 \hspace{2pt} The $\Gamma$-limit Cosserat "membrane" model

We define the metric space $X = L^r(\Omega_1, \mathbb{R}^3) \times L^p(\Omega_1, SO(3, \mathbb{R}))$, $r = p' = \frac{2p}{p + 2}$, $p > 3$ and note the compact embeddings $H^{1,2}(\Omega_1, \mathbb{R}^3) \subset L^r(\Omega_1, \mathbb{R}^3)$, $W^{1,p}(\Omega_1, SO(3, \mathbb{R})) \subset L^p(\Omega_1, SO(3, \mathbb{R}))$. The following result has been obtained in [8]. The $\Gamma$-limit [3, 1] to the sequence $I^p_h(\varphi^p, \overline{R}^p)$ : $X \to \mathbb{R}^+$ is given by the variational problem (after de-scaling) for the \textit{midsurface deformation} $m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ and \textit{the independent microrotation} of the plate $\overline{R} : \omega \subset \mathbb{R}^2 \to SO(3, \mathbb{R})$:

\begin{equation}
I_0(m, \overline{R}) = \int_\omega h W^\text{hom}((\nabla m, \overline{R}) + h \mu L_c \text{c}^\nabla |\overline{N}|^2 \text{d} \omega \to \min \text{ w.r.t. } (m, \overline{R}),
\end{equation}

\begin{align}
W^\text{hom}((\nabla m, \overline{R}) &= \mu \| \text{sym}((\overline{R}_1[\overline{R}_2]^T \nabla m - \text{II}_2))\|^2 + \mu_c \| \text{skew}((\overline{R}_1[\overline{R}_2]^T \nabla m - \text{II}_2))\|^2 \\
+ 2 \frac{\mu_c}{\mu + \mu_c} \left( \sqrt{\text{sym}((\overline{R}_3 m_z)^2)} + \sqrt{\text{sym}((\overline{R}_3 m_y)^2)} \right) + \frac{\mu \lambda}{2 \mu + \lambda} \text{tr} \left[ \text{sym}((\overline{R}_1[\overline{R}_2]^T \nabla m - \text{II}_2))\right]^2,
\end{align}

where we set $\overline{R}_1 = \overline{R}_e_1$. Note that $\frac{\mu c}{\mu + \mu_c} = 1/2 \mathcal{H}(\mu, \lambda/2)$, where $\mathcal{H}$ denotes the \textit{harmonic mean}.

This variational limit formulation looses coercivity for the midsurface deformation $m \in H^{1,2}(\omega, \mathbb{R}^3)$ if $\mu_c = 0$. However, this loss of coercivity is not related to the missing drill-energy contribution but only due to the missing transverse shear term in that case. The proof of this $\Gamma$-limit result is first obtained for $\mu_c > 0$ (in which case equicoercivity for the sequence $I^p_h$ over $X$ greatly facilitates the task) and thereafter it is shown, that the result remains true also for $\mu c = 0$ where, however, one is faced with an unusual loss of equicoercivity of this sequence. For dimensionally reduced Cosserat models based on a formal ansatz we refer to [5] and references therein.

4 \hspace{2pt} A surprising consequence for the Cosserat couple modulus $\mu_c$

The $\Gamma$-limit describes rigourously the limit of zero-thickness, hence a two-dimensional object. Such a "membrane"-model should neither have bending-resistance (scaling with $h^3$) nor transverse shear resistance, since both effects can only be explained by some remaining small (but finite) thickness. The $\Gamma$-limit does not have a bending resistance. The resistance $\tau$ against transverse shearing is, however, proportional to $\tau \sim 2 \mu \frac{\mu c}{\mu + \mu c} \left( \sqrt{\text{sym}((\overline{R}_3 m_z)^2)} + \sqrt{\text{sym}((\overline{R}_3 m_y)^2)} \right)$. This strongly suggests that $\mu_c \equiv 0$ is the physically consistent value, thus providing us with an answer to the controversy about the value of $\mu_c$. From a practical point of view, for the computation of thin structures with a remaining finite thickness $h > 0$, one should use the Cosserat $\Gamma$-limit model (3.1) with $\mu_c = 0$ but augment the stretch energy expression $W^\text{hom}$ exclusively with some transverse shear contribution. This will restore coercivity for $m \in H^{1,2}(\omega, \mathbb{R}^3)$ and lead to stable computations.

References