Relations of constants for isotropic linear Cosserat elasticity

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1 The linear elastic Cosserat model in variational form

In the following, $(\nabla u)_{ij} = \partial_{x_j} u_i$ denotes the differential of $u : \mathbb{R}^3 \mapsto \mathbb{R}^3$. For the **displacement** $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the **skew-symmetric infinitesimal microrotation** $\overline{A} : \Omega \subset \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ consider the **two-field** minimization problem

$$I(u, \overline{A}) = \int_\Omega W_{\mathrm{mp}}(\overline{\varepsilon}) + W_{\mathrm{curv}}(\nabla \operatorname{axl} \overline{A}) - \langle f, u\rangle - \langle \overline{M}, \overline{A}\rangle \, \mathrm{dx} \tag{1.1}$$
$$- \int_{\Gamma_S} \langle f_S, u\rangle - \langle \overline{M}_S, \overline{A}\rangle \, \mathrm{dS} \mapsto \quad \min . \text{ w.r.t. } (u, \overline{A}),$$

under the following constitutive requirements and boundary conditions[1]

$$\begin{aligned}
\overline{\varepsilon} &= \nabla u - \overline{A}, \quad u_{|\Gamma} = u_{\mathrm{d}}\,, \\
W_{\mathrm{mp}}(\overline{\varepsilon}) &= \mu \,\|\operatorname{sym} \overline{\varepsilon}\|^2 + \mu_c \,\|\operatorname{skew} \overline{\varepsilon}\|^2 + \frac{\lambda}{2} \operatorname{tr}[\operatorname{sym} \overline{\varepsilon}]^2 \qquad \textbf{strain energy} \\
&= \mu \,\|\operatorname{sym} \nabla u\|^2 + \mu_c \,\|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2} \operatorname{tr}[\operatorname{sym} \nabla u]^2 \qquad (1.3)\\
&= \mu \,\|\operatorname{dev}\operatorname{sym} \nabla u\|^2 + \mu_c \,\|\operatorname{skew}(\nabla u - \overline{A})\|^2 + \frac{2\mu + 3\lambda}{6} \operatorname{tr}[\operatorname{sym} \nabla u]^2 \\
&= \mu \,\|\operatorname{sym} \nabla u\|^2 + \frac{\mu_c}{2} \,\|\operatorname{curl} u - 2 \operatorname{axl} \overline{A}\|^2_{\mathbb{R}^3} + \frac{\lambda}{2} \,(\operatorname{Div} u)^2\,, \\
\phi &:= \operatorname{axl} \overline{A} \in \mathbb{R}^3, \quad \overline{\mathfrak{k}} = \nabla \phi\,, \quad \|\operatorname{curl} \phi\|^2_{\mathbb{R}^3} = 4 \|\operatorname{axl} \operatorname{skew} \nabla \phi\|^2_{\mathbb{R}^3} = 2 \|\operatorname{skew} \nabla \phi\|^2_{\mathbb{M}^{3\times 3}}\,, \\
W_{\mathrm{curv}}(\nabla \phi) &= \frac{\gamma + \beta}{2} \|\operatorname{sym} \nabla \phi\|^2 + \frac{\gamma - \beta}{2} \|\operatorname{skew} \nabla \phi\|^2 + \frac{\alpha}{2} \operatorname{tr}[\nabla \phi]^2 \qquad \textbf{curvature energy} \\
&= \frac{\gamma + \beta}{2} \|\operatorname{dev}\operatorname{sym} \nabla \phi\|^2 + \frac{\gamma - \beta}{2} \|\operatorname{skew} \nabla \phi\|^2 + \frac{3\alpha + (\beta + \gamma)}{6} \operatorname{tr}[\nabla \phi]^2 \\
&= \frac{\gamma}{2} \|\nabla \phi\|^2 + \frac{\beta}{2} \langle \nabla \phi, \nabla \phi^T\rangle + \frac{\alpha}{2} \operatorname{tr}[\nabla \phi]^2 \\
&= \frac{\gamma + \beta}{2} \|\operatorname{sym} \nabla \phi\|^2 + \frac{\gamma - \beta}{4} \|\operatorname{curl} \phi\|^2_{\mathbb{R}^3} + \frac{\alpha}{2} \,(\operatorname{Div} \phi)^2\,.
\end{aligned}$$

Here, $f, \overline{M}$ are volume force and volume couples, respectively; $f_s, \overline{M}_S$ are surface tractions and surface couples at $\Gamma_S \subset \partial\Omega$, respectively, while u_{d} are Dirichlet boundary conditions for displacement at $\Gamma \subset \partial\Omega$.[2] The strain energy W_{mp} and the curvature energy W_{curv} are

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[1] More detailed than strictly necessary in order to accommodate the different representations in the literature. Note that $\operatorname{axl} \overline{A} \times \xi = \overline{A}.\xi$ for all $\xi \in \mathbb{R}^3$, such that

$$\operatorname{axl} \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} := \begin{pmatrix} -\gamma \\ \beta \\ -\alpha \end{pmatrix}, \quad \overline{A}_{ij} = \sum_{k=1}^{3} -\varepsilon_{ijk} \cdot (\operatorname{axl} \overline{A})_k\,, \tag{1.2}$$

where ε_{ijk} is the totally antisymmetric permutation tensor. Here, $\overline{A}.\xi$ denotes the application of the matrix $\overline{A}$ to the vector ξ and $a \times b$ is the usual cross-product. Note that it is always possible to prescribe essential boundary values for $\overline{A}$ but we abstain from such a prescription throughout.

[2] For simplicity only we assume that $\Gamma \cap \Gamma_S = \emptyset$ and that surface tractions and surface couples are prescribed at the same portion of the boundary. Much more general combinations could be considered.

the most general isotropic, centro-symmetric quadratic forms in the **non-symmetric strain tensor** $\overline{\varepsilon} = \nabla u - \overline{A}$ and the **micropolar curvature tensor** $\overline{\mathfrak{k}} = \nabla \operatorname{axl} \overline{A}$ (curvature-twist tensor). The parameters μ, λ[MPa] are the classical Lamé moduli and α, β, γ are additional micropolar moduli with dimension $[\mathrm{Pa} \cdot \mathrm{m}^2] = [\mathrm{N}]$ of a force. It is usually clearer to write $\alpha, \beta, \gamma \sim \mu L_c^2 \alpha', \mu L_c^2 \beta', \mu L_c^2 \gamma'$ with corresponding non-dimensional parameters α', β', γ' and a material length scale $L_c > 0\,[\mathrm{m}]$.

The additional parameter $\mu_c \geq 0$[MPa] in the strain energy is the **Cosserat couple modulus**. For $\mu_c = 0$ the two fields of displacement and microrotations decouple and one is left formally with classical linear elasticity for the displacement u.

1.1 The linear elastic Cosserat balance equations: hyperelasticity

Taking free variations of the energy in (1.1) w.r.t. both displacement $u \in \mathbb{R}^3$ and infinitesimal microrotation $\overline{A} \in \mathfrak{so}(3)$, one arrives at the equilibrium system (the Euler-Lagrange equations of (1.1))

$$\begin{aligned}
&\operatorname{Div} \sigma = f\,, \qquad -\operatorname{Div} m = 4\,\mu_c \cdot \operatorname{axl} \operatorname{skew} \overline{\varepsilon} + \operatorname{axl} \operatorname{skew}(\overline{M})\,, \quad \overline{\varepsilon} = \nabla u - \overline{A},\\
&\quad \sigma = 2\mu \cdot \operatorname{sym} \overline{\varepsilon} + 2\mu_c \cdot \operatorname{skew} \overline{\varepsilon} + \lambda \cdot \operatorname{tr}[\overline{\varepsilon}] \cdot 1\!\!1 = (\mu + \mu_c) \cdot \overline{\varepsilon} + (\mu - \mu_c) \cdot \overline{\varepsilon}^T + \lambda \cdot \operatorname{tr}[\overline{\varepsilon}] \cdot 1\!\!1\,,\\
&\quad m = \gamma\, \nabla \phi + \beta\, \nabla \phi^T + \alpha \operatorname{tr}[\nabla \phi] \cdot 1\!\!1, \qquad \phi = \operatorname{axl} \overline{A}\,, \qquad (1.4)\\
&\quad u_{|\Gamma} = u_{\mathrm{d}}\,, \quad \sigma.\vec{n}_{|\Gamma_S} = f_S\,, \quad m.\vec{n}_{|\Gamma_S} = \frac{1}{2} \operatorname{axl}(\operatorname{skew}(\overline{M}_S))\,,\\
&\quad \sigma.\vec{n}_{|\partial\Omega \setminus (\Gamma_S \cup \Gamma)} = 0\,, \quad m.\vec{n}_{|\partial\Omega \setminus (\Gamma_S \cup \Gamma)} = 0\,.
\end{aligned}$$

Here, m is the **couple stress tensor**. For comparison, in [4, p.111] or [1, 9, 5] the elastic moduli in our notation are defined to be $\mu = \mu^* + \frac{\kappa}{2}$, $\mu_c = \frac{\kappa}{2}$.[3] But in this last definition (see [2]), μ^* cannot be regarded as one of the classical Lamé constants.[4] [5] We note that under the usual positivity requirements on the curvature energy, the couple stress/ curvature relation can be pointwise inverted. In [4] the role of β and γ is reversed.

2 Constitutive restrictions for Cosserat hyperelasticity

2.1 Pointwise positivity of the micropolar energy

For a mathematical treatment in the hyperelastic case it is often assumed that for arbitrary nonzero strain and curvature $\overline{\varepsilon}, \overline{\mathfrak{k}} \in \mathbb{M}^{3\times 3}$ one has the **local positivity condition**

$$\forall\, \overline{\varepsilon}, \overline{\mathfrak{k}} \neq 0: \qquad W_{\mathrm{mp}}(\overline{\varepsilon}) > 0\,, \qquad W_{\mathrm{curv}}(\overline{\mathfrak{k}}) > 0\,. \tag{2.1}$$

This condition is most often invoked as the basis of uniqueness proofs in static micropolar elasticity, see e.g. [7, 6, 4, 3]. By splitting $\overline{\varepsilon}$ in its deviatoric and volumetric part, i.e. writing

$$\overline{\varepsilon} = \operatorname{dev} \operatorname{sym} \overline{\varepsilon} + \operatorname{skew} \overline{\varepsilon} + \frac{1}{3} \operatorname{tr}[\overline{\varepsilon}] \cdot 1\!\!1 \tag{2.2}$$

and inserting this into the energy W_{mp} one gets

$$W_{\mathrm{mp}}(\overline{\varepsilon}) = \mu\, \| \operatorname{dev} \operatorname{sym} \overline{\varepsilon} \|^2 + \mu_c\, \| \operatorname{skew} \overline{\varepsilon} \|^2 + \frac{2\mu + 3\lambda}{6} \operatorname{tr}[\overline{\varepsilon}]^2\,. \tag{2.3}$$

Since all three contributions in (2.2) can be chosen independent of each other, one obtains from (2.1) the **pointwise positive-definiteness condition**

$$\begin{aligned}
&\mu > 0\,, \quad 2\mu + 3\lambda > 0\,, \quad \mu_c > 0\,,\\
&\gamma + \beta > 0\,, \quad (\gamma + \beta) + 3\alpha > 0\,, \quad \gamma - \beta > 0\,, \quad (\Rightarrow\ \gamma > 0)\,,
\end{aligned} \tag{2.4}$$

[3] In [8, 4] the Cauchy stress tensor σ is defined as $\sigma = (\mu^* + \kappa)\,\overline{\varepsilon} + \mu^*\, \overline{\varepsilon}^T + \lambda \operatorname{tr}[\overline{\varepsilon}] \cdot 1\!\!1$ with given constants μ^*, κ, λ and one must identify $\mu^* + \kappa = \mu + \mu_c$, $\quad \mu^* = \mu - \mu_c$.

[4] A simple definition of the Lamé constants in micropolar elasticity is that they should coincide with the classical Lamé constants for symmetric situations. Equivalently, they are obtained by the classical formula $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, where E and ν are uniquely determined from uniform traction where Cosserat effects are absent.

[5] Unfortunately, while authors are consistent in their usage of material parameters, one should be careful when identifying the actually used parameters with his own usage. The different representations in (1.3) might be useful for this purpose.

where the argument pertaining to the curvature energy W_{curv} is exactly similar, cf. [8, (2.9)]. In effect, one ensures **uniform convexity** of the integrand w.r.t $\overline{\varepsilon}, \overline{\mathfrak{k}}$. In this case, then, the stress/strain and couple-stress/curvature relation can be inverted, simplifying the mathematical treatment considerably.

By a **thermodynamical stability argument** [4] one may similarly infer the **non-negativity** of the energy (material stability), leading only to

$$\begin{aligned} &\mu \geq 0\,, \quad 2\mu + 3\lambda \geq 0\,, \quad \mu_c \geq 0\,, \\ &\gamma + \beta \geq 0\,, \quad (\gamma+\beta) + 3\alpha \geq 0\,, \quad \gamma - \beta \geq 0\,, \quad (\Rightarrow\ \gamma \geq 0)\,, \end{aligned} \tag{2.5}$$

which allows for classical linear elasticity but which condition alone is not strong enough to guarantee existence and uniqueness of the corresponding boundary value problem. Nevertheless, all constitutive restrictions on a linear Cosserat solid must at least be consistent with (2.5) from a purely physical point of view.

2.2 Legendre-Hadamard ellipticity conditions for the Cosserat model

For a dynamic problem, another condition, implying **real wave speeds** in wave propagation problems, is useful. This is the Legendre-Hadamard ellipticity condition. Let us investigate the restrictions which it imposes on the constitutive parameters of the Cosserat model. In the following we treat the generic case of a quadratic form which can then be applied to the balance of linear and angular momentum system. The generic quadratic form is

$$W(\nabla\phi) := a_1\,\|\operatorname{sym}\nabla\phi\|^2 + a_2\,\|\operatorname{skew}\nabla\phi\|^2 + a_3\operatorname{tr}\left[\nabla\phi\right]^2. \tag{2.6}$$

Replacing $\nabla\phi$ by the rank one dyadic product $\xi\otimes\eta$ we obtain

$$\begin{aligned} &a_1\,\|\operatorname{sym}\xi\otimes\eta\|^2 + a_2\,\|\operatorname{skew}\xi\otimes\eta\|^2 + a_3\operatorname{tr}\left[\xi\otimes\eta\right]^2 \\ &= \frac{a_1}{4}\,\|\xi\otimes\eta+\eta\otimes\xi\|^2 + \frac{a_2}{4}\,\|\xi\otimes\eta-\eta\otimes\xi\|^2 + a_3\,\langle\xi,\eta\rangle^2 \qquad (2.7) \\ &= \frac{a_1}{4}\,\left(2\,\|\xi\otimes\eta\|^2 + 2\,\langle\xi\otimes\eta,\eta\otimes\xi\rangle\right) + \frac{a_2}{4}\,\left(2\,\|\xi\otimes\eta\|^2 - 2\,\langle\xi\otimes\eta,\eta\otimes\xi\rangle\right) + a_3\,\langle\xi,\eta\rangle^2 \\ &= \frac{a_1}{4}\,\left(2\,\|\xi\|^2\|\eta\|^2 + 2\,\langle\xi,\eta\rangle^2\right) + \frac{a_2}{4}\,\left(2\,\|\xi\|^2\|\eta\|^2 - 2\,\langle\xi,\eta\rangle^2\right) + a_3\,\langle\xi,\eta\rangle^2 \\ &= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2 + \frac{a_1-a_2+2\,a_3}{2}\langle\xi,\eta\rangle^2 \\ &= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2 + \frac{a_1-a_2+2\,a_3}{2}\|\xi\|^2\|\eta\|^2\cos^2\vartheta \\ &= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2\,(\sin^2\vartheta+\cos^2\vartheta) + \frac{a_1-a_2+2\,a_3}{2}\|\xi\|^2\|\eta\|^2\cos^2\vartheta \\ &= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2\,\sin^2\vartheta + (\frac{a_1+a_2}{2} + \frac{a_1-a_2+2\,a_3}{2})\|\xi\|^2\|\eta\|^2\cos^2\vartheta \\ &= \frac{a_1+a_2}{2}\,\|\xi\|^2\|\eta\|^2\,\sin^2\vartheta + \frac{a_1+2\,a_3}{2}\|\xi\|^2\|\eta\|^2\cos^2\vartheta\,. \end{aligned}$$

Thus

$$D^2W(\nabla\phi).(\xi\otimes\eta,\xi\otimes\eta) = (a_1+a_2)\,\|\xi\|^2\|\eta\|^2\,\sin^2\vartheta + (a_1+2\,a_3)\,\|\xi\|^2\|\eta\|^2\cos^2\vartheta\,, \tag{2.8}$$

and Legendre-Hadamard ellipticity demands that the acoustic tensor $Q(\xi):\mathbb{R}^3\mapsto\mathbb{R}^3$, defined through $D^2W(\nabla\phi).(\xi\otimes\eta,\xi\otimes\eta) = \langle\eta, Q(\xi).\eta\rangle_{\mathbb{R}^3}$ is strictly positive definite for any nonzero wave direction $\xi\in\mathbb{R}^3$. We infer the necessary and sufficient conditions for strict Legendre-Hadamard ellipticity of the quadratic form (2.6)

$$a_1+a_2>0\,, \quad a_1+2\,a_3>0\,. \tag{2.9}$$

Applying this result to both the strain energy and curvature energy in (1.3) we obtain the **Legendre-Hadamard ellipticity condition for linear, isotropic Cosserat solids**

$$\begin{aligned} \mu+\mu_c>0\,, &\quad \mu+\lambda>0\,, \\ \gamma>0\,, &\quad \gamma+\beta+\alpha>0\,. \end{aligned} \tag{2.10}$$

In the case of $\mu_c=0$ we recover the well known ellipticity condition for linear elasticity. It is clear that (2.4) is sufficient for (2.10). But (2.10) alone is not sufficient for well-posedness of the Cosserat boundary value problem.

2.3 Further relations for micropolar constants

In the literature on Cosserat or micropolar solids the following abbreviations and definitions are frequently encountered. As a convenience for the reader, we collect these technical constants here.

$$\begin{aligned}
\Psi &:= \frac{\beta+\gamma}{\alpha+\beta+\gamma}\,, \qquad \text{non-dimensional polar ratio}\,, \quad 0 \le \Psi \le \frac{3}{2}\,,\\
\ell_t^2 &:= \left(\frac{\beta+\gamma}{2\mu^*+\kappa}\right) = \frac{\beta+\gamma}{2\,\mu}\,, \qquad \text{"characteristic length for torsion"}\,,\\
\ell_b^2 &= \frac{\gamma}{2\,(2\mu^*+\kappa)} = \frac{\gamma}{4\mu}\,, \qquad \text{"characteristic length for bending"}\,,\\
p^2 &:= \frac{2\,\kappa}{\alpha+\beta+\gamma} = \frac{4\,\mu_c}{\alpha+\beta+\gamma}\,, \qquad \kappa := 2\,\mu_c\,,\\
N^2 &:= \frac{\mu_c}{\mu+\mu_c} = \frac{\kappa}{2\,(\mu^*+\kappa)}\,, \quad \text{Cosserat coupling number,} \quad 0 \le N \le 1\,.\\
\nu &= \frac{\lambda}{2\mu^*+2\lambda+\kappa} = \frac{\lambda}{2\,(\mu+\lambda)}\,, \qquad \text{classical Poisson ratio.}
\end{aligned} \tag{2.11}$$

For every physical material, it is essential that small samples still have bounded rigidity. This may or may not be true for Cosserat models, depending on the values of Cosserat parameters. Based on analytic solution formulas for simple three-dimensional Cosserat boundary value problems it has been shown in [10] that for bounded stiffness for arbitrary slender specimens we must have

1. torsion of a cylinder: either $\beta+\gamma=0$ or $\Psi = \frac{\beta+\gamma}{\alpha+\beta+\gamma} = \frac{3}{2}$.
2. bending of a cylinder: $(\beta+\gamma)\,(\gamma-\beta)=0$.

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