



Construction of Anisotropic Polyconvex Energies on the Basis of Crystallographic Motivated Structural Tensors

Motivation

- For the description of isotropic as well as for transversely isotropic and orthotropic material behavior energy functions that are polyconvex already exist.
- The main goal is to provide a new method for the construction of anisotropic polyconvex hyperelastic models for arbitrary anisotropy classes.
- Basic literature: BALL [1], SCHRÖDER & NEFF [3,4], SCHRÖDER, NEFF & EBBING [5].
- A priori stress-free reference configuration: ITSKOV & AKSEL [2].

Polyconvex Energy Functions for Arbitrary Anisotropy –Anisotropy: The principle of material symmetry requires

$$\psi(\mathbf{C}) = \psi(\mathbf{Q}\mathbf{C}\mathbf{Q}^T) \quad \forall \quad \mathbf{Q} \in \mathcal{G} \subset \text{O}(3).$$

Introduction of an anisotropic metric tensor

$$\mathbf{G} := \mathbf{H}\mathbf{H}^T,$$

which is per definition symmetric and positive definite. \mathbf{G} reflects the symmetry properties of the underlying crystal class, i.e.

$$\mathbf{G} = \mathbf{Q}\mathbf{G}\mathbf{Q}^T \quad \forall \quad \mathbf{Q} \in \mathcal{G} \subset \text{O}(3)$$

must hold. Thus, energy functions in terms of the scalar products

$$\mathbf{C} \cdot \mathbf{G} = \mathbf{Q}\mathbf{C}\mathbf{Q}^T \cdot \mathbf{G} = \mathbf{C} \cdot \mathbf{Q}^T\mathbf{G}\mathbf{Q} \quad \forall \quad \mathbf{Q} \in \mathcal{G}$$

automatically satisfy the principle of material symmetry.

Representation of metric tensors, e.g.:

Monoclinic system, $\mathbf{G}^m \in \mathcal{G}^m$: Triclinic system, $\mathbf{G}^a \in \mathcal{G}^a$:

$$\mathbf{G}^m = \begin{bmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad \mathbf{G}^a = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}.$$

The parameters a, b, c and d are additional material parameters.

–Polyconvexity: Anisotropic invariants

$$J_4 = \mathbf{C} \cdot \mathbf{G} \quad \text{and} \quad J_5 = \text{cof} \mathbf{C} \cdot \mathbf{G}$$

are polyconvex.

Proof: $(\mathbf{C} \cdot \mathbf{G})^k = \langle \mathbf{F}\mathbf{H}, \mathbf{F}\mathbf{H} \rangle^k, \forall k \geq 1$:

$$\begin{aligned} D_{\mathbf{F}}^2(\langle \mathbf{F}\mathbf{H}, \mathbf{F}\mathbf{H} \rangle^k) \cdot (\boldsymbol{\xi}, \boldsymbol{\xi}) &= 2k \|\mathbf{F}\mathbf{H}\|^{2k-2} \|\boldsymbol{\xi}\mathbf{H}\|^2 \\ &+ 4k(k-1) \|\mathbf{F}\mathbf{H}\|^{2k-4} \\ &\langle \mathbf{F}\mathbf{H}, \boldsymbol{\xi}\mathbf{H} \rangle^2 \geq 0. \end{aligned}$$

Generic anisotropic polyconvex coercive free energy function

$$\begin{aligned} \psi^{aniso} = & \sum_{r=1}^n \sum_{j=1}^m \xi_{rj} \left[\frac{1}{\alpha_{rj} + 1} \frac{1}{(g_j)^{\alpha_{rj}}} (J_{4j})^{\alpha_{rj}+1} \right. \\ & \left. + \frac{1}{\beta_{rj} + 1} \frac{1}{(g_j)^{\beta_{rj}}} (J_{5j})^{\beta_{rj}+1} + \frac{g_j}{\gamma_{rj}} (I_3)^{-\gamma_{rj}} \right], \end{aligned}$$

with $g_j = \text{tr} \mathbf{G}_j$, $J_{4j} = \mathbf{C} \cdot \mathbf{G}_j$, $J_{5j} = \text{cof} \mathbf{C} \cdot \mathbf{G}_j$ and $\alpha_{rj}, \beta_{rj}, \xi_{rj} \geq 0, \gamma_{rj} \geq -\frac{1}{2}$. Proof of coercivity: see [5].

Second Piola-Kirchhoff stresses

$$\begin{aligned} \mathbf{S} = & \sum_{r=1}^n \sum_{j=1}^m 2\xi_{rj} \left[(-g_j I_3^{-\gamma_{rj}} + \frac{1}{(g_j)^{\beta_{rj}}} J_{5j}^{\beta_{rj}+1}) \mathbf{C}^{-1} \right. \\ & \left. + \frac{1}{(g_j)^{\alpha_{rj}}} J_{4j}^{\alpha_{rj}} \mathbf{G}_j - \frac{1}{(g_j)^{\beta_{rj}}} J_{5j}^{\beta_{rj}} I_3 \mathbf{C}^{-1} \mathbf{G}_j \mathbf{C}^{-1} \right], \end{aligned}$$

with $\mathbf{S}(\mathbf{C} = \mathbf{1}) = \mathbf{0}$.

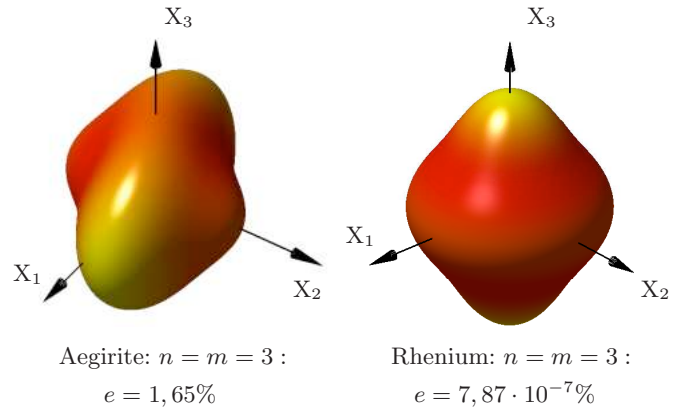
Anisotropic Moduli-Fitting to Referential Data

Minimizing of error function

$$e = \frac{\|\mathbf{C}^{(V)comp} - \mathbf{C}^{(V)exp}\|}{\|\mathbf{C}^{(V)exp}\|},$$

with $\mathbf{C}^{(V)comp} = 4\partial_{\mathbf{C}\mathbf{C}}\psi^{aniso}$ in Voigt notation.

Characteristic Surfaces of Young's Moduli:



References

- [1] BALL, J.M. [1977], "Convexity Conditions and Existence Theorems in Non-Linear Elasticity", *Arch. Rat. Mech. Anal.*, 63, 337–403
- [2] ITSKOV, M. & AKSEL, N. [2004], "A Class of Orthotropic and Transversely Isotropic Hyperelastic Constitutive Models Based on a Polyconvex Strain Energy Function", *Int. J. Solid. Struct.*, 41, 3833–3848
- [3] SCHRÖDER, J. & NEFF, P. [2001], "On the Construction of Polyconvex Anisotropic Free Energy Functions", *Proceedings of the IUTAM Symposium on Computational Mechanics of Solid Materials at Large Strains*, 171–180
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- [5] SCHRÖDER, J., NEFF, P. & EBBING, V. [2008], "Anisotropic Polyconvex Energies on the Basis of Crystallographic Motivated Structural Tensors", submitted to *J. Mech. Phys. Solids*