Symmetric Cauchy stresses *do not imply* symmetric Biot strains in weak formulations of isotropic hyperelasticity with rotational degrees of freedom.

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Overview

Section

Notation

Deformation of a Body

- Reference configuration: $\Omega \subset \mathbb{R}^3$
- Deformation $\varphi : \Omega \to \Omega_{\mathrm{def}} \subset \mathbb{R}^3$
- Deformation gradient: $F := \nabla\varphi \in \mathrm{GL}^+(3,\mathbb{R})$

Theorem (Right Polar Decomposition of F)

$\forall F \in \mathrm{GL}^+(3,\mathbb{R}).\ \exists!\ (R,U) \in \mathrm{SO}(3) \times \mathrm{PSym}(3,\mathbb{R})$ *with* $F = RU$.

$\Longrightarrow \quad R^T F = U = \sqrt{F^T F}$ *and* $R = FU^{-1}$.

Notation

- Polar rotation: R, $\mathrm{polar}(F) \in \mathrm{SO}(3)$ (True rotation)
- Right Biot-stretch: $U = R^T F = \sqrt{F^T F}$ ($U \in \mathrm{PSym}(3,\mathbb{R})$!)

Hyperelasticity, Objectivity and Material Isotropy

Hyperelasticity - Minimizing total stored energy

$$\text{Find minimizers } \varphi_{\min} \text{ of:} \quad \mathcal{I}(\varphi) = \int_\Omega W(\nabla\varphi)\,\mathrm{dV}$$

- Choice of $W : \mathbb{M}(3,\mathbb{R}) \to \mathbb{R}$ is a constitutive assumption
- Can apply various boundary conditions ...
- Consider only zero body forces

Isotropy and Objectivity in terms of $F := \nabla\varphi$

- $W(F)$ is **objective** if $\quad \forall Q \in \mathrm{SO}(3): \; W(QF) = W(F)$
- $W(F)$ is **isotropic** if $\quad \forall Q \in \mathrm{SO}(3): \; W(FQ) = W(F)$

Objectivity of $W(F)$

$\Rightarrow \exists W^\sharp : \mathrm{PSym}(3,\mathbb{R}) \to \mathbb{R}$ s. th. $W(F) = W^\sharp(U(F))$

Overview

Section

The isotropic Biot-model - Standard version

Basic constitutive assumptions

- Objectivity: $W(F) = W^\sharp(U)$, where $U := R^T F = \sqrt{F^T F} \in \mathrm{PSym}$
- Isotropy: $\forall Q \in \mathrm{SO}(3) : W(FQ) = W(F) \iff W^\sharp(Q^T U Q) = W^\sharp(U)$

Example: Most general isotropic quadratic energy in U

$$W^\sharp(U) = \mu \, \|U - \mathbb{1}\|^2 + \frac{\lambda}{2} \operatorname{tr}[U - \mathbb{1}]^2 \,, \quad U := \sqrt{F^T F}$$

- Linearization equivalent to classical isotropic linear elasticity
- Zero stresses in the reference configuration

Biot approach is intrinsically based on a formulation in $U := \sqrt{F^T F}$

- Have to take derivatives of U, i.e., of a matrix square root
- Can be rephrased into an equivalent theory depending on $\overline{U} := \overline{R}^T F$ via relaxation ...

Euler-Lagrange equations for the Biot-model

Free variation w.r.t. to φ

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \mathcal{I}(\varphi + t\, v) = \int_\Omega \langle D_F W(\nabla\varphi), \nabla v\rangle \,\mathrm{dV} = \int_\Omega \langle D_F[W^\sharp(U(F))], \nabla v\rangle \,\mathrm{dV} = \int_\Omega \langle D_U W^\sharp(U), D_F U(F).\nabla v\rangle \,\mathrm{dV}$$

$$= \int_\Omega \langle D_U W^\sharp(U), D_F[R(F)^T\, F].\nabla v\rangle \,\mathrm{dV} = \ldots = \int_\Omega \langle R(F)\, D_U W^\sharp(U), \nabla v\rangle$$

$$= \int_\Omega \langle \mathrm{Div}[R(F)\, D_U W^\sharp(U)], v\rangle \,\mathrm{dV}\,, \quad \forall\ v \in C_0^\infty(\Omega, \mathbb{R}^3)\,.$$

Strong form of the equilibrium equation

$$0 = \mathrm{Div}[R(F)\, D_U W^\sharp(U)]\,, \qquad R(F) = \mathrm{polar}(F)$$

A weaker form of the equilibrium equation

$$0 = \mathrm{Div}[\overline{R}(F)\, D_U W^\sharp(\overline{R}(F)^T F)]\,, \qquad \overline{R}(F)^T F \in \mathrm{Sym}$$

Euler-Lagrange equations for the Biot-model

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- Weaker: $\overline{R}^T F \in \mathrm{Sym} \quad \not\Rightarrow \quad \overline{U} = \overline{R}^T F \in \mathrm{PSym}$

Relaxation

Idea - Generalize a classical model, but preserve its solutions

- $U := R^T F = \text{polar}(F)^T F = \sqrt{F^T F}$ **constrains** rotation

Relaxation of the constraint

- Independent rotations $\overline{R} : \Omega \to \mathrm{SO}(3)$ (no physics involved!)
- Yields relaxed Biot-stretch $\overline{U} := \overline{R}^T F$ (usually not symmetric)
- Fixing $\overline{R} = \text{polar}(F)$ the non-relaxed Biot-case can be recovered

Consequences

- Can define relaxed energies $W^\sharp(\overline{U}) = W(F, \overline{R})$
- Relaxation gets rid of U but introduces $\overline{R} \in \mathrm{SO}(3)$
- One obtains a 2-field model!

The relaxed isotropic Biot-model is a Cosserat-model

Hyperelastic Cosserat-model (Neff, 2006)

$$\mathcal{I}(\varphi,\overline{R}) := \int_\Omega W_{\mathsf{mp}}(F(x),\overline{R}(x)) + W_{\mathsf{curv}}(\overline{R}(x), D_{\mathrm{x}}\overline{R}(x))\,\mathrm{dV} \mapsto \text{ min. w.r.t. } (\varphi,\overline{R})$$

Definition of the energetic components: Isotropic case

$$\begin{aligned}
W_{\mathrm{mp}}(\overline{U}) &:= W_{\mathrm{shear}}(\overline{U}) + W_{\mathrm{vol}}(\det[\overline{U}]) \\
W_{\mathrm{shear}}(\overline{U}) &:= \mu\|\,\mathrm{sym}\,(\overline{U}-\mathbb{1})\|^2 + \mu_c\|\,\mathrm{skew}\,(\overline{U}-\mathbb{1})\|^2 \\
W_{\mathrm{vol}}(\det[\overline{U}]) &:= \frac{\lambda}{4}\left((\det[\overline{U}]-1)^2 + (\det[\overline{U}]^{-1}-1)^2\right) \\
W_{\mathsf{curv}}(\overline{R}, D_{\mathrm{x}}\overline{R}) &:= 2\,\frac{\mu}{q}(1 + L_c^2\|\overline{R}^T D_{\mathrm{x}}\overline{R}^2\|)^{\frac{q}{2}}
\end{aligned}$$

Notation

$$\overline{U} := \overline{R}^T F, \quad F := \nabla\varphi, \quad D_{\mathrm{x}}\overline{R} := \left(\nabla(\overline{R}.e_1), \nabla(\overline{R}.e_2), \nabla(\overline{R}.e_3)\right)$$

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Hyperelastic Cosserat-model without curvature energy $\hat{=}\ L_c = 0$

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Hyperelastic Cosserat-model without curvature energy $\hat{=}$ $L_c = 0$

$$\mathcal{I}(\varphi, \overline{R}) := \int_\Omega W_{\text{shear}}(F, \overline{R}) + W_{\text{vol}}(F, \overline{R}) \, \mathrm{dV} \mapsto \text{ min. w.r.t. } (\varphi, \overline{R})$$

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W_{\text{curv}}(\overline{R}, D_{\text{x}} \overline{R}) &:= 2 \, \frac{\mu}{q} (1 + L_c^2 \| \overline{R}^T D_{\text{x}} \overline{R}^2 \|)^{\frac{q}{2}} \qquad \leftarrow \text{ constant if } L_c = 0
\end{aligned}$$

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$$\overline{U} := \overline{R}^T F, \quad F := \nabla \varphi, \quad D_{\text{x}} \overline{R} := \Big(\nabla(\overline{R}.e_1), \nabla(\overline{R}.e_2), \nabla(\overline{R}.e_3) \Big)$$

The relaxed isotropic Biot-model is a Cosserat-model

Hyperelastic Cosserat-model with $L_c = 0$ and $\cancel{W_{\mathrm{vol}}}$

$$\mathcal{I}(\varphi, \overline{R}) := \int_\Omega \mu \| \operatorname{sym}(\overline{R}^T F - \mathbb{1}) \|^2 + \mu_c \| \operatorname{skew}(\overline{R}^T F - \mathbb{1}) \|^2 \, \mathrm{dV} \mapsto \text{ min. w.r.t. } (\varphi, \overline{R})$$

Definition of the energetic components: Isotropic case

$$\begin{aligned}
W_{\mathrm{mp}}(\overline{U}) &:= W_{\mathrm{shear}}(\overline{U}) + W_{\mathrm{vol}}(\det[\overline{U}]) \\
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W_{\mathrm{curv}}(\overline{R}, D_{\mathrm{x}} \overline{R}) &:= 2 \, \frac{\mu}{q} (1 + L_c^2 \| \overline{R}^T D_{\mathrm{x}} \overline{R}^2 \|)^{\frac{q}{2}} \qquad \leftarrow \text{ constant if } L_c = 0
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Notation

$$\overline{U} := \overline{R}^T F, \quad F := \nabla \varphi, \quad D_{\mathrm{x}} \overline{R} := \left(\nabla(\overline{R}.e_1), \nabla(\overline{R}.e_2), \nabla(\overline{R}.e_3) \right)$$

The linear setting - $\sigma \in \mathrm{Sym} \iff \mathrm{skew}(\nabla u) = 0$

Linearizations of the fields

- $F \approx \mathbb{1} + \nabla u$, $\nabla u \ll 1$
- $\overline{R} \approx \mathbb{1} + \overline{A}$, where $\overline{A} \in \mathfrak{so}(3)$
- $\overline{U} - \mathbb{1} \approx \nabla u - \overline{A}$
- $\mathrm{polar}(F) \approx \mathbb{1} + \mathrm{skew}(\nabla u)$

Linearized "constraint"

- Finite: $\overline{R} = \mathrm{polar}(F)$
- Linear: $\overline{A} = \mathrm{skew}(\nabla u)$

Definition of $W^\sharp$ - Cosserat without curvature

$$W^\sharp(\overline{U}) := \mu \,\|\,\mathrm{sym}(\overline{U} - \mathbb{1})\|^2 + \mu_c \,\|\,\mathrm{skew}(\overline{U} - \mathbb{1})\|^2 + \frac{\lambda}{4}\left((\det[\overline{U}] - 1)^2 + (\frac{1}{\det[\overline{U}]} - 1)^2\right)$$

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- $F \approx 1\!\!1 + \nabla u$, $\nabla u \ll 1$
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- Finite: $\overline{R} = \mathrm{polar}(F)$
- Linear: $\overline{A} = \mathrm{skew}(\nabla u)$

$W^\sharp_{\mathrm{quad}}$ - Quadratic approximation of $W^\sharp$

$$W^\sharp_{\mathrm{quad}}(\nabla u - \overline{A}) := \mu\, \|\,\mathrm{sym}(\nabla u - \overline{A})\|^2 + \mu_c\, \|\,\mathrm{skew}(\nabla u - \overline{A})\|^2 + \frac{\lambda}{2}\mathrm{tr}\left[\nabla u - \overline{A}\right]^2$$

Linear case $\mu_c > 0$: $\sigma \in \mathrm{Sym} \iff \mathrm{skew}(\nabla u) = 0$

$$\sigma = D_{\nabla u} W^\sharp_{\mathrm{quad}} = 2\mu\ \mathrm{sym}(\nabla u) + 2\mu_c \underbrace{\mathrm{skew}(\nabla u - \overline{A})}_{\text{vanishes} \iff \sigma \in \mathrm{Sym}} + \lambda\, \mathrm{tr}\left[\nabla u\right] 1\!\!1$$

The linear setting - $\sigma \in \mathrm{Sym} \iff \mathrm{skew}(\nabla u) = 0$

Linearizations of the fields

- $F \approx \mathbb{1} + \nabla u, \; \nabla u \ll 1$
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Linear case $\mu_c = 0$: $\sigma \in \mathrm{Sym}$ is **independent** of $\overline{A}$ (no coupling)

$$\sigma = D_{\nabla u} W^\sharp_{\mathrm{quad}} = 2\mu \, \mathrm{sym}(\nabla u) + \lambda \, \mathrm{tr}\,[\nabla u] \cdot \mathbb{1}$$

Cauchy-stress and balance of angular momentum in continuum theories with rotational degrees of freedom

Generalized stress tensors in theories with relaxed rotations $\overline{R}$

- 1st First Piola-Kirchhoff stress tensor:

$$S_1(F,\overline{R}) := D_F W(F,\overline{R}) = \overline{R}\, D_{\overline{U}} W(\overline{U})$$

- Cauchy-stresses: $\sigma := \frac{1}{\det[F]}\, S_1(F,\overline{R})\, F^T = \frac{1}{\det[F]}\,\overline{R} \ldots =$ FIXME!

Theorem (Objectivity of $W^\sharp$ implies $\sigma \in \mathrm{Sym}$)

Let $W^\sharp = W(F,\overline{R})$ be differentiable and objective, then the associated Cauchy stress tensor σ is symmetric.

A Note on Balance of Angular Momentum

Exact volume terms are irrelevant

- $\det[\overline{U}] = \det[\overline{R}^T F]$ is independent of $\overline{R}$
- Volume term of $W^\sharp$ is irrelevant for balance of angular momentum

Consider

$W^\sharp(\overline{U}) := W_{\mu,\mu_c}(F,\overline{R}) := \mu\,\|\operatorname{sym}(\overline{U}-\mathbb{1})\|^2 + \mu_c\,\|\operatorname{skew}(\overline{U}-\mathbb{1})\|^2$

$$
\begin{aligned}
D_{\overline{U}} W^\sharp(\overline{U})\,\overline{U}^T \in \mathrm{Sym} \quad &\iff \quad D_{\overline{U}} W_{\mu,\mu_c}(\overline{U})\,\overline{U}^T \in \mathrm{Sym} \\
&\iff \quad \left[2\mu\,(\operatorname{sym}(\overline{U}-\mathbb{1})) + 2\mu_c\,\operatorname{skew}\overline{U}\right]\,\overline{U}^T \in \mathrm{Sym} \\
&\iff \quad \left[\mu\,(\overline{U}+\overline{U}^T - 2\,\mathbb{1}) + \mu_c(\overline{U}-\overline{U}^T)\right]\,\overline{U}^T \in \mathrm{Sym} \\
&\iff \quad (\mu-\mu_c)\,\overline{U}\,\overline{U} - 2\mu\,\overline{U} \in \mathrm{Sym}
\end{aligned}
$$

Equilibrium equation

$$
(\mu-\mu_c)\,[\overline{U}^2 - \overline{U}^{T,2}] - 2\mu\,[\overline{U}-\overline{U}^T] = 0
$$

Overview

Section

Example I - Assumptions and Definitions

Assumptions on the structure of F and $\overline{R}$

$$F := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_1, \lambda_2 > 0$$

$$\overline{R} := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Choose further

- $\rho := \frac{2\mu}{\mu - \mu_c}, \quad 0 \leq \mu_c < \mu$
- $\lambda_1 + \lambda_2 > \rho$
- $\alpha := \arccos\left(\frac{\rho}{\lambda_1 + \lambda_2}\right) \in \left[0, \frac{\pi}{2}\right]$

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$\Longrightarrow$ Particular forms for $\overline{R}$ and $\overline{U}$

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$\Longrightarrow$ Particular forms for $\overline{R}$ and $\overline{U}$

$\overline{R}$ is given by

$$\begin{pmatrix} \frac{\rho}{\lambda_1+\lambda_2} & -\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ \sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & \frac{\rho}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\overline{U} := \overline{R}^T F \qquad$ (relaxed Biot-stretch)

$$\begin{pmatrix} \frac{\rho\;\lambda_1}{\lambda_1+\lambda_2} & \lambda_2\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ -\lambda_1\sqrt{1 - \frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & \frac{\rho\;\lambda_2}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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- $\rho := \frac{2\,\mu}{\mu-\mu_c}, \quad 0 \leq \mu_c < \mu$
- $\lambda_1 + \lambda_2 > \rho$
- $\alpha := \arccos\left(\frac{\rho}{\lambda_1+\lambda_2}\right) \in \left[0, \frac{\pi}{2}\right]$

Note that $\overline{U}$ need not be symmetric!

$\overline{U} := \overline{R}^T F \qquad$ (relaxed Biot-stretch)

$$\begin{pmatrix} \frac{\rho\ \lambda_1}{\lambda_1+\lambda_2} & \lambda_2\sqrt{1-\frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ -\lambda_1\sqrt{1-\frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & \frac{\rho\ \lambda_2}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example II - Identities

An easy computation

$$\overline{U} - \overline{U}^T = \begin{pmatrix} 0 & (\lambda_1+\lambda_2)\sqrt{1-\frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ -(\lambda_1+\lambda_2)\sqrt{1-\frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overline{U}^2 - \overline{U}^{T,2} = \begin{pmatrix} 0 & \rho\,(\lambda_1+\lambda_2)\sqrt{1-\frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ -\rho\,(\lambda_1+\lambda_2)\sqrt{1-\frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$\overline{U}^2 - \overline{U}^{T,2} = \begin{pmatrix} 0 & \rho\,(\lambda_1+\lambda_2)\sqrt{1-\frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 \\ -\rho\,(\lambda_1+\lambda_2)\sqrt{1-\frac{\rho^2}{(\lambda_1+\lambda_2)^2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

allows us to infer

$$\overline{U}^2 - \overline{U}^{T,2} = \rho\,(\overline{U} - \overline{U}^T) = \frac{2\mu}{\mu-\mu_c}\,(\overline{U} - \overline{U}^T)$$

$$\iff (\mu-\mu_c)\,(\overline{U}^2 - \overline{U}^{T,2}) - 2\mu\,(\overline{U} - \overline{U}^T) = 0$$

$$\iff D_{\overline{U}} W^\sharp(\overline{U})\,\overline{U}^T \in \mathrm{Sym}$$

$$\iff \sigma \in \mathrm{Sym}$$

Overview

Section

Conclusions and Future Work

The finite Cosserat model and relaxed Biot model

- Absence of Cosserat curvature $\Rightarrow$ "Cosserat = relaxed Biot"
- Isotropy is insufficient to guarantee classical, symmetric Biot-stretches

Influence of μ_c as a penalization parameter

- $\mu_c < \mu$: Biot-stretches may become asymmetric
- $\mu_c \geq \mu$: Exactly enforces symmetry of the Biot stretch tensor.

Further Research

Characterize a sufficiently large class of isotropic free energies such that moment equilibrium in the relaxed Biot formulation automatically implies the symmetry of the relaxed Biot stretch $\overline{U}$.

Selected References

P. Neff, A. Fischle and I. Münch.
Symmetric Cauchy stresses do not imply symmetric Biot strains in weak formulations of isotropic hyperelasticity with rotational degrees of freedom.
Acta Mechanica, 2007, published online

H. Bufler.
On drilling degrees of freedom in nonlinear elasticity and a hyperelastic material description in terms of the stretch tensor. Part I: Theory.
Acta Mech. 113: 21-35, 1995.

P. Neff.
The Cosserat couple modulus for continuous solids is zero viz the linearized Cauchy-stress tensor is symmetric.
Preprint 2409 (available online), Zeitschrift f. Angewandte Mathematik Mechanik (ZAMM), 86(DOI 10.1002/zamm.200510281):892–912, 2006.

P. Neff.
A finite-strain elastic-plastic Cosserat theory for polycrystals with grain rotations.
Int. J. Eng. Sci., DOI 10.1016/j.ijengsci.2006.04.002, 44:574–594, 2006.

Thank you

- The End -

Thank you for your attention!