A first regularity result for the Armstrong–Frederick cyclic hardening plasticity model with Cosserat effects

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\begin{abstract}

The purpose of this article is to prove the Hölder continuity up to the boundary of the displacement vector and the microrotation matrix for the quasistatic, rate-independent Armstrong–Frederick cyclic hardening plasticity model with Cosserat effects. This model is of non-monotone and non-associated type. In the case of two space dimensions we use the hole-filling technique of Widman and Morrey’s Dirichlet growth theorem.

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\end{abstract}

\section{Introduction}

In the paper we investigate the important regularity question for models in elasto-plasticity. Various model systems in the finite strain and small strain case have been proposed in the articles [31–36,38]. In this contribution we focus on the infinitesimal plasticity models in the framework proposed by H.-D. Alber and his group (see [2,7,8,10,11,33]). This framework is perfectly adapted for inelastic deformation processes of metals that are characterized by a monotone flow rule (associated plasticity). In that case the finite difference method was very useful to prove regularity of stresses in the Prandtl–Reuss and Norton–Hoff models [4,5,15,18,39], because this method allows to cancel the monotone nonlinearities. Using this method H.-D. Alber and S. Nesenenko [3] have shown \( L^\infty(\mathcal{H}^{1/3-\delta})\)-regularity for stresses and plastic strain for coercive models of viscoplasticity with variable coefficients. Next, D. Knees in [28] obtains the stresses in the space \( L^\infty(\mathcal{H}^{1/2-\delta})\). A similar result was proved by P. Kamiński in [25] and [26] for coercive and self-controlling (non-coercive) viscoplastic models. Moreover, in [12] an \( H^{1}_{\text{loc}}\)-regularity result for the stresses

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and strains in Cosserat elasto-plasticity was proved, cf. [37]. See also [20] and [21] for the local regularity in the Hencky model.

The Armstrong–Frederick (AF) cyclic hardening plasticity model with Cosserat effects was formulated first in the article [14] and it is of non-monotone and not of gradient type (non-associated flow rule – see [9] and [14] for more details). Hence the finite difference method is useless to study regularity of solutions for this model. The idea of the present paper is to show the Hölder continuity up to the boundary of the displacement and the microrotation matrix in AF-model with Cosserat effects in the case of two space dimensions (the existence of the energy solutions for this model was also proved in [14]). We will use a very old method, which was exposed in [29] and [30]. Those works presented first fundamental theorems about existence and regularity of solutions of two-dimensional elliptic systems.

We derive Morrey’s condition up to the boundary of the basic domain $\Omega$. In order to get it we will use Widman’s hole filling trick proposed by K.O. Widman in the paper [41].

Morrey’s methods were used in [17] where the author showed the existence of a Hölder continuous solution for a class of two-dimensional non-linear elliptic systems. Moreover these methods were also used to prove the Hölder continuity for the displacements in isotropic and kinematic hardening with von Mises yield criterion in [19].

To our knowledge this article presents the first regularity result for models from the theory of inelastic deformations of metals, which are non-monotone, non-associated but coercive – for the definitions we refer to [2].

2. The Armstrong–Frederick model with Cosserat effects

Here we formulate the Armstrong–Frederick model with Cosserat effects in the case of two space dimensions.

Let us assume that $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded domain with boundary of class $C^2$. The structure of the model equations is the same as introduced in [14]. For the mechanical results for Cosserat plasticity we refer to [33,35] – see also [13], where the non-monotone model of poroplasticity with Cosserat effects was introduced. Hence we deal with the following system of equations

$$\begin{align*}
\text{div}_x T &= -f, \\
T &= 2\mu(\varepsilon(u) - \varepsilon^p) + 2\mu_c(\text{skew}(\nabla_x u) - A) + \lambda \text{tr}(\varepsilon(u) - \varepsilon^p) \mathbb{I}, \\
-l_c \Delta_x \text{axl}(A) &= \mu_c \text{axl}(\text{skew}(\nabla_x u) - A), \\
\varepsilon^p_i &\in \partial I_{K(b)}(T_E), \\
T_E &= 2\mu(\varepsilon(u) - \varepsilon^p) + \lambda \text{tr}(\varepsilon(u) - \varepsilon^p) \mathbb{I}, \\
b_i &= c \varepsilon^p_i - d|\varepsilon^p_i| b,
\end{align*}$$

(2.1)

where the unknowns are: the displacement vector field $u : \Omega \times [0,T] \to \mathbb{R}^3$, the microrotation matrix $A : \Omega \times [0,T] \to \mathfrak{so}(3)$ ($\mathfrak{so}(3)$ is the set of skew-symmetric $3 \times 3$ matrices) and the vector of internal variables $z = (\varepsilon^p, b) : \Omega \times [0,T] \to S^3_{\text{dev}} \times S^3_{\text{dev}}$ ($\varepsilon^p$ is the classical infinitesimal symmetric plastic strain tensor, $b$ is the symmetric backstress tensor and the space $S^3_{\text{dev}}$ denotes the set of symmetric $3 \times 3$ matrices with vanishing trace). $\varepsilon(u) = \text{sym}(\nabla_x u)$ denotes the symmetric part of the gradient of the displacement.

Eqs. (2.1) are studied for $x \in \Omega \subset \mathbb{R}^n$, $n = 2, 3$ and $t \in [0,T]$, where $t$ denotes the time.

The set of admissible elastic stresses $K(b(x,t))$ is defined in the form

$$K(b) = \{T_E \in S^3 : |\text{dev}(T_E) - b| \leq \sigma_y\},$$
where \( \text{dev}(T_E) = T_E - \frac{1}{3} \text{tr}(T_E) \cdot \mathbb{I} \), \( \sigma_y \) is a material parameter (the yield limit) and \( \mathbb{I} \) denotes the identity \( 3 \times 3 \) matrix. The function \( I_K(b) \) is the indicator function of the admissible set \( K(b) \) and \( \partial I_K(b) \) is the subgradient of the convex, proper, lower semicontinuous function \( I_K(b) \).

The function \( f : \Omega \times [0, T] \to \mathbb{R}^3 \) describes the density of the applied body forces, the parameters \( \mu, \lambda \) are positive Lamé constants, \( \mu_c > 0 \) is the Cosserat couple modulus and \( l_c > 0 \) is a material parameter with dimensions \([m^2]\), describing a length scale of the model due to the Cosserat effects. \( c, d > 0 \) are material constants.

The operator \( \text{skew}(T) = \frac{1}{2}(T - T^T) \) denotes the skew-symmetric part of a \( 3 \times 3 \) tensor. The operator \( \text{axl} : \mathfrak{so}(3) \to \mathbb{R}^3 \) establishes the identification of a skew-symmetric matrix with a vector in \( \mathbb{R}^3 \). This means that if we take \( A \in \mathfrak{so}(3) \), which is in the form \( A = ((0, \alpha, \beta), (-\alpha, 0, \gamma), (-\beta, -\gamma, 0)) \), then \( \text{axl}(A) = (\alpha, \beta, \gamma) \).

Notice that the system (2.1) is a modification of the Melan–Prager model, which is well known in the literature and it can also be seen as an approximation of the Prandtl–Reuss model. The expression \(|\varepsilon^p|b\) is a perturbation of the Melan–Prager model – if \( d = 0 \) then we obtain the classical Melan–Prager linear kinematic hardening model – details can be found in [9] and [14].

The system (2.1) is considered with the Dirichlet boundary condition for the displacement:

\[
\begin{align*}
u(x, t) &= g_D(x, t) & \text{for } x \in \partial \Omega \text{ and } t \geq 0
\end{align*}
\]

and with the Dirichlet boundary condition for the microrotation:

\[
\begin{align*}
A(x, t) &= A_D(x, t) & \text{for } x \in \partial \Omega \text{ and } t \geq 0.
\end{align*}
\]

Finally, we consider the system (2.1) with the following initial conditions

\[
\begin{align*}
\varepsilon^p(x, 0) &= \varepsilon^{p,0}(x), & b(x, 0) &= b^0(x).
\end{align*}
\]

The free energy function associated with the system (2.1) is given by the formula

\[
\begin{align*}
\rho \psi(\varepsilon, \varepsilon^p, A, b) &= \mu \|\varepsilon(u) - \varepsilon^p\|^2 + \mu_c \|\text{skew}(\nabla_x u) - A\|^2 \\
&+ \lambda \left( \text{tr}(\varepsilon(u) - \varepsilon^p) \right)^2 + 2\mu c \|\nabla_x \text{axl}(A)\|^2 + \frac{1}{2\epsilon} \|b\|^2,
\end{align*}
\]

where \( \rho \) is the mass density which we assume to be constant in time and space. The total energy is of the form:

\[
\mathcal{E}(\varepsilon, \varepsilon^p, A, b)(t) = \int_{\Omega} \rho \psi(\varepsilon(x, t), \varepsilon^p(x, t), A(x, t), b(x, t)) \, dx.
\]

Section 2 of the article [9] shows that the inelastic constitutive equation occurring in (2.1) is of pre-monotone type (for the definition see [2]). We also know that the AF-model with micropolar effects is of non-monotone type and not of gradient type (non-associated flow rule).

3. Existence theory and main result

The only one existence result for the AF-models with Cosserat effects in the case of three space dimensions was obtained in the article [14]. It was shown that the limit in the Yosida approximation process satisfies the energy inequality for special test functions. Using the same techniques as in [14] we can obtain the same existence theorem in the two dimensional cases. The goal of this section is to formulate the existence theorem for the system (2.1) and the main result of this article.
Let us assume that for all \( T > 0 \) the given data \( f, g_D \) and \( A_D \) have the regularity
\[
  f \in H^1(0,T; L^2(\Omega; \mathbb{R}^3)), \quad g_D \in H^1(0,T; H^\frac{1}{2}(\partial\Omega; \mathbb{R}^3)),
\]
\[
  A_D \in H^1(0,T; H^\frac{3}{2}(\partial\Omega; \mathfrak{so}(3))).
\]
(3.1)
Additionally let us suppose that the initial data \((\varepsilon^{p,0}, b^0) \in L^2(\Omega; S^{3}_{\text{dev}}) \times L^2(\Omega; S^{3}_{\text{dev}})\) satisfy
\[
  |b^0(x)| \leq \frac{c}{d} \quad \text{and} \quad |\text{dev}(T_E^0(x)) - b^0(x)| \leq \sigma_y \quad \text{for almost all } x \in \Omega,
\]
(3.3)
where the initial stress \( T_E^0 = 2\mu(\varepsilon(u(0)) - \varepsilon^{p,0}) + \lambda \text{tr}(\varepsilon(u(0)) - \varepsilon^{p,0})\) \( \mathbb{I} \in L^2(\Omega; S^{3}) \) is the unique solution of the following linear problem
\[
  \text{div}_x T^0(x) = -f(x,0),
\]
\[
  -l_c \Delta_x \mu \text{axl}(A(x, 0)) = \mu_c \text{axl}(\text{skew}(\nabla_x u(x, 0)) - A(x, 0)),
\]
\[
  u(x, 0)_{|\partial\Omega} = g_D(x, 0), \quad A(x, 0)_{|\partial\Omega} = A_D(x, 0),
\]
(3.4)
with
\[
  T^0(x) = 2\mu(\varepsilon(u(0)) - \varepsilon^{p,0}(x)) + 2\mu_c(\text{skew}(\nabla_x u(x, 0)) - A(x, 0)) + \lambda \text{tr}(\varepsilon(u(0)) - \varepsilon^{p,0}(x))\) \( \mathbb{I} \).

Observe that the system (3.4) is the linear elliptic system for the unknowns \( u(x, 0) \) and \( A(x, 0) \). Thus the system (3.4) possesses unique weak solution \((u(x, 0), A(x, 0)) \in H^1(\Omega; \mathbb{R}^3) \times H^2(\Omega; S^{3})\). Moreover, if the given data \( f(x, 0) \) and \( \varepsilon^{p,0}(x) \) satisfy some additional regularity conditions (more information could be found in Section 5), then we conclude that \( u(x, 0) \) is Hölder continuous with some exponent \( \beta > 0 \) – see for instance [22] and [24].

Let us consider the convex set (which will be used as set of test functions further on)
\[
  \mathcal{K}^* = \left\{ \left( \text{dev}(T_E), -\frac{1}{c} b \right) \in S^{3}_{\text{dev}} \times S^{3}_{\text{dev}} : |\text{dev}(T_E) - b| + \frac{d}{2c} |b|^2 \leq \sigma_y \right\},
\]
where the constant \( \sigma_y \) is the same as in the yield condition. The theory of elasticity implies that there exists a positive definite operator \( \mathbb{C}^{-1} : S^{3} \rightarrow S^{3} \) such that \( \mathbb{C}^{-1} T_{E,t} = \varepsilon_t - \varepsilon_t \), where \( \mathbb{C} : S^{3} \rightarrow S^{3} \) is a standard elasticity tensor \( \mathbb{C}(S) = 2\mu S + \lambda \text{tr}(S)\) \( \mathbb{I} \) and the parameters \( \mu, \lambda \) are positive Lamé constants. Now we recall from [14] a notion of the definition of the energy solution for the system (2.1) (for a motivation we refer to [14]).

**Definition 3.1 (Solution concept-energy inequality).** Fix \( T > 0 \). Suppose that the given data satisfy (3.1)–(3.4). We say that a vector \((u, T, A, \varepsilon^p, b) \in L^\infty(0,T; H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; S^{3}) \times H^2(\Omega; \mathfrak{so}(3)) \times (L^2(\Omega; S^{3}_{\text{dev}}))^2)\) solves the problem (2.1)–(2.4) if
\[
  (u_t, T, A_t, \varepsilon_t^p, b_t) \in L^2(0,T; H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; S^{3}) \times H^2(\Omega; \mathfrak{so}(3)) \times (L^2(\Omega; S^{3}_{\text{dev}}))^2),
\]
Eqs. (2.1)_1 and (2.1)_3 are satisfied pointwise almost everywhere on \( \Omega \times (0,T) \) and for all test functions \((\hat{T}_E, \hat{b}) \in L^2(0,T; L^2(\Omega; S^{3}) \times L^2(\Omega; S^{3}_{\text{dev}}))\) such that
\[
  (\text{dev}(\hat{T}_E), \hat{b}) \in \mathcal{K}^*, \quad \text{div} \hat{T}_E \in L^2(0,T; L^2(\Omega; \mathbb{R}^3)),
\]
the inequality
\[
  \int_0^T \int_{\Omega} \mu \text{dev}(T_E) \text{axl}(A) - \mu_c \text{axl}(\text{skew}(\nabla_x u) - A) - \lambda \text{tr}(\varepsilon(u) - \varepsilon^{p,0})\Delta x + f \mathbb{I} \Delta x \geq 0,
\]
holds for all \( (u, T, A, \varepsilon^p, b) \) in the set of admissible solutions. The solution is an energy solution if it is a weak solution and satisfies the energy inequality.
\[
\frac{1}{2} \int_{\Omega} \mathbb{C}^{-1} T_E(x, t) T_E(x, t) \, dx + \mu_c \int_{\Omega} |\text{skew}(\nabla_x u(x, t)) - A(x, t)|^2 \, dx + 2l_c \int_{\Omega} |\nabla \text{axl}(A(x, t))|^2 \, dx
\]
\[
+ \frac{1}{2c} \int_{\Omega} |b(x, t)|^2 \, dx \leq \frac{1}{2} \int_{\Omega} \mathbb{C}^{-1} T_E^0(x) T_E^0(x) \, dx + \mu_c \int_{\Omega} |\text{skew}(\nabla_x u(x, 0)) - A(x, 0)|^2 \, dx
\]
\[
+ \frac{1}{2c} \int_{\Omega} |b(x, 0)|^2 \, dx + 2l_c \int_{\Omega} |\nabla \text{axl}(A(x, 0))|^2 \, dx + \int_{0}^{t} \int_{\Omega} u_t(x, \tau) f(x, \tau) \, dx \, d\tau
\]
\[
+ \int_{0}^{t} \int_{\Omega} \mathbb{C}^{-1} T_E, (x, \tau) \hat{T}_E(x, \tau) \, dx \, d\tau + \frac{1}{c} \int_{0}^{t} \int_{\partial \Omega} b_t(x, \tau) \hat{b}(x, \tau) \, dx \, d\tau
\]
\[
+ 4l_c \int_{0}^{t} \int_{\partial \Omega} \nabla \text{axl}(A(x, \tau)) \cdot n \text{axl}(A_{D, t}(x, \tau)) \, dS \, d\tau
\]
is satisfied for all \( t \in (0, T) \), where \( T_E^0 \in L^2(\Omega; \mathbb{S}^3) \) and \( (u(0), A(0)) \in H^1(\Omega; \mathbb{R}^3) \times H^2(\Omega; \mathfrak{so}(3)) \) are unique solution of the problem (3.4).

**Theorem 3.2 (Existence result).** Let \( n = 2, 3 \). Moreover, let us assume that the given data and initial data satisfy the properties, which are specified in (3.1)–(3.4). Then there exists a global in time solution (in the sense of Definition 3.1) of the system (2.1) with boundary conditions (2.2), (2.3) and initial conditions (2.4).

In the case of three space dimensions the proof of Theorem 3.2 can be found in [14]. Using the same techniques and arguments as in [14] we obtain the proof of Theorem 3.2 in two dimensional cases, hence it will be omitted. The next section will only very briefly present the main steps of the proof of Theorem 3.2.

The goal of this article is to prove a higher regularity of the displacement vector \( u \) and the microrotation tensor \( A \), which are the solutions of the system (2.1) in the sense of Definition 3.1. Let us denote by \( \mathcal{C}^{0, \alpha}(\Omega; \mathbb{R}^3) \) the space of all Hölder continuous functions up to the boundary with exponent \( \alpha > 0 \). The following theorem is the main result of this article.

**Theorem 3.3 (Main result).** Let \( n = 2 \) and let us assume that for all \( T > 0 \) the given data \( f, g_D \), \( A_D \) have the regularity

\[
f \in W^{1, \infty}(0, T; L^2(\Omega; \mathbb{R}^3)), \quad g_D \in W^{1, \infty}(0, T; H^{1/2}(\partial \Omega; \mathbb{R}^3)), \quad A_D \in H^1(0, T; H^{1/2}(\partial \Omega; \mathfrak{so}(3)))
\]

and that there exists function \( w \in W^{1, \infty}(0, T; H^1(\Omega; \mathbb{R}^3)) \) such that \( w_{1|_{\partial \Omega}} = g_D, 1_{|_{\partial \Omega}} \), satisfying for all \( R > 0 \) and \( x_0 \in \Omega \) the estimate

\[
\int_{B(x_0, R) \cap \Omega} |\nabla w_t(x, t)|^2 \, dx \leq KR^\gamma \quad \text{for almost all } t \in (0, T),
\]

where \( \gamma > 0 \) is some positive number and where \( B(x_0, R) \subset \mathbb{R}^2 \) denotes the open ball with the center \( x_0 \in \mathbb{R}^2 \) and the radius \( R > 0 \) (the constant \( K > 0 \) does not depend on the radius \( R \)). Additionally let us suppose that
the initial data \((\varepsilon^{p,0}, b^0) \in L^2(\Omega; S^3_{\text{dev}}) \times L^2(\Omega; S^3_{\text{dev}})\) satisfy (3.3) and (3.4). Then there exists \(0 < \alpha < 1\) such that

\[
\begin{align*}
  u & \in C^{0,\alpha}([0, T]; C^{0,\alpha}(\bar{\Omega}; \mathbb{R}^3)) \quad \text{and} \quad A \in C^{0,\alpha}([0, T]; C^{0,\alpha}(\bar{\Omega}; \mathfrak{so}(3))),
\end{align*}
\]

where the displacement vector \(u\) and the microrotation tensor \(A\) are the solutions (in the sense of Definition 3.1) of the system (2.1) with boundary conditions (2.2), (2.3) and initial condition (2.4).

Remark. Theorem 3.3 needs some additional regularity assumptions on the initial data \(f(x,0)\) and \(\varepsilon^{p,0}(x)\). During the proof of the local Hölder continuity of the displacement vector \(u\) it could be noticed that such assumptions are hard to formulate. This situation will be commented in Section 5.

Theorem 3.3 presents the first regularity result for non-monotone models from elasto-plasticity. The proof of Theorem 3.3 is based on the method of Morrey, which was presented in [29] and [30]. It is divided into three sections. First, we use the Yosida approximation to the maximal monotone part of the inelastic constitutive equation and we very shortly show the main steps of the proof of Theorem 3.2. Next, we prove the tube-filling condition (interior case). It will be the main part of the proof of Theorem 3.3. Finally we show a Morrey’s condition for the displacement vector up to the boundary \(\partial \Omega\).

4. Existence for each Yosida approximation step

We apply the Yosida approximation for the monotone part of the flow rule from (2.1) in order to get only a Lipschitz-nonlinearity in Eq. (2.1)_4. We consider the following initial–boundary value problem

\[
\begin{align*}
  \text{div}_x T^\nu &= -f, \\
  T^\nu &= 2\mu(\varepsilon(u^\nu) - \varepsilon^{p,\nu}) + 2\mu_c(\text{skew}(\nabla_x u^\nu) - A^\nu) + \lambda \text{tr}(\varepsilon(u^\nu) - \varepsilon^{p,\nu}) I, \\
  -l_c \Delta_x axl (A^\nu) &= \mu_c axl(\text{skew}(\nabla_x u^\nu) - A^\nu), \\
  \varepsilon^\nu_t &= \frac{1}{\nu} \left\{ [\text{dev}(T^E_\nu) - b^\nu] - \sigma_y \right\} + \frac{\text{dev}(T^E_\nu) - b^\nu}{||\text{dev}(T^E_\nu) - b^\nu||}, \\
  T^E_\nu &= 2\mu(\varepsilon(u^\nu) - \varepsilon^{p,\nu}) + \lambda \text{tr}(\varepsilon(u^\nu) - \varepsilon^{p,\nu}) I, \\
  b^\nu_t &= \varepsilon^\nu_{t \nu} - d ||\varepsilon^\nu_{t \nu}|| b^\nu. 
\end{align*}
\]

The above equations are studied for \(x \in \Omega \subset \mathbb{R}^n, n = 2, 3\) and \(t \in (0, T)\). \(\nu > 0\) is the Yosida approximation parameter and \(\{\rho\}^+ = \max\{0, \rho\}\), where \(\rho\) is a scalar function.

The system (4.1) is considered with boundary conditions:

\[
\begin{align*}
  u^\nu(x, t) &= g_D(x, t) \quad \text{for} \ x \in \partial \Omega \text{ and } t \geq 0, \\
  A^\nu(x, t) &= A_D(x, t) \quad \text{for} \ x \in \partial \Omega \text{ and } t \geq 0
\end{align*}
\]

and initial conditions

\[
\begin{align*}
  \varepsilon^{p,\nu}(x, 0) &= \varepsilon^{p,0}(x), \\
  b^\nu(x, 0) &= b^0(x).
\end{align*}
\]

Denote by \(\mathcal{E}^\nu(t)\) the total energy associated with the system (4.1)

\[
\mathcal{E}^\nu(u^\nu, \varepsilon^{p,\nu}, A^\nu, b^\nu) (t) = \int_{\Omega} \rho \mathcal{E}^\nu(u^\nu(x, t), \varepsilon^{\nu}(x, t), \varepsilon^{p,\nu}(x, t), A^\nu(x, t), b^\nu(x, t)) \, dx
\]
\[ \int_{\Omega} \left( \mu \| \varepsilon(u') - \varepsilon^{p,v} \|^2 + \mu_c \| \text{skew}(\nabla_x u') - A' \|^2 \right. \\
+ \frac{\lambda}{2} (\text{tr}(\varepsilon(u') - \varepsilon^{p,v}))^2 + 2l_c \| \nabla_x \text{axl}(A') \|^2 + \frac{1}{2c} \| b' \|^2 \) \, dx. \quad (4.4) \]

**Definition 4.1.** Fix \( T > 0 \). We say that a vector \((u, A, T, \varepsilon^p, b) \in W^{1, \infty}(0, T; H^1(\Omega; \mathbb{R}^3) \times H^2(\Omega; \mathfrak{so}(3)) \times L^2(\Omega; \mathbb{R}^3)) \) is an \( L^2 \)-strong solution of the system (4.1) if

1. \(|\varepsilon|^2 b \in L^\infty(0, T; L^2(\Omega, S^3_{\text{dev}}))\),
2. \(|\text{dev}(2\mu(\varepsilon(u(x, t)) - \varepsilon^p(x, t))) - b(x, t)| \leq \sigma_x \) for almost all \((x, t) \in \Omega \times (0, T)\),
3. Eqs. (4.1) are satisfied for almost all \((x, t) \in \Omega \times (0, T)\).

The following lemma implies the \( L^\infty \)-boundedness of the backstress \( b' \).

**Lemma 4.2.** Fix \( T > 0 \). Assume that \((u', T', A', \varepsilon^{p,v}, b') \) is an \( L^2 \)-strong solution of the problem (4.1) and \(|b^0(x)| \leq \frac{c}{d} \) for almost all \(x \in \Omega\). Then for all \( \nu > 0 \)

\[ |b'(x, t)| \leq \frac{c}{d} \quad \text{for a.e.} \ (x, t) \in \Omega \times (0, T). \]

For the proof of Lemma 4.2 we refer to [9]. Now we propose the existence of solutions for each approximation step.

**Theorem 4.3.** Fix \( T > 0 \). Suppose that all hypotheses of Theorem 3.2 are satisfied. Then for all \( \nu > 0 \) there exists a unique \( L^2 \)-strong solution (in the sense of Definition 4.1)

\[ (u', T', A', \varepsilon^{p,v}, b') \in W^{1, \infty}(0, T; H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \times H^2(\Omega; \mathfrak{so}(3)) \times (L^2(\Omega; S^3_{\text{dev}}))^2) \]

satisfying the system (4.1) with boundary conditions (4.2) and initial conditions (4.3).

The proof of Theorem 4.3 in the case of two dimensional cases is the same as for three-dimensional cases. It uses the same techniques as for the related Armstrong–Frederick model without Cosserat effects: see Section 4 of [9], therefore it will be omitted. For the complete proof of Theorem 4.3 we refer to [14].

A fundamental tool in the proof of Theorem 3.2 is the following property of the energy function which results from our Cosserat modification:

**Theorem 4.4 (Coerciveness of the energy).**

(a) (The case with zero boundary data) For all \( \nu > 0 \) the energy function (4.4) is elastically coercive with respect to \( \nabla u \). This means that \( \exists C_E > 0 \), \( \forall u \in H^1_0(\Omega) \), \( \forall A \in H^1_0(\Omega) \), \( \forall \varepsilon^p \in L^2(\Omega) \), \( \forall b \in L^2(\Omega) \)

\[ \mathcal{E}'(u, \varepsilon^p, A, b) \geq C_E (\| u \|^2_{H^1(\Omega)} + \| A \|^2_{H^1(\Omega)} + \| b \|^2_{L^2(\Omega)}). \]

(b) (The case with non-zero boundary data) Moreover, \( \exists C_E > 0 \), \( \forall g_D, A_D \in H^\frac{1}{2}(\partial \Omega) \), \( \exists C_D > 0 \), \( \forall \varepsilon^p \in L^2(\Omega) \), \( \forall b \in L^2(\Omega), \forall u \in H^1(\Omega), \forall A \in H^1(\Omega) \) with boundary conditions \( u|_{\partial \Omega} = g_D \) and \( A|_{\partial \Omega} = A_D \) it holds that

\[ \mathcal{E}'(u, \varepsilon^p, A, b) + C_D \geq C_E (\| u \|^2_{H^1(\Omega)} + \| A \|^2_{H^1(\Omega)} + \| b \|^2_{L^2(\Omega)}). \]
For the proof of Theorem 4.4 we refer to Theorem 3.2 of the article [33].

To pass to the limit in the system (4.1) and obtain the solution in the sense of Definition 3.1 we need estimates for the time derivatives of the sequence \((u^\nu, A^\nu, T^\nu, \varepsilon^\nu, b^\nu)\). The article [14] yields that the following energy estimate is sufficient to pass to the limit with the Yosida approximation.

**Theorem 4.5 (Energy estimate).** Assume that the given data and initial data satisfy (3.1)-(3.4). Then for all \(t \in (0, T)\) the following estimate

\[
\int_\Omega \frac{1}{2\nu} \left\{ |\text{dev}(T^\nu_E(t)) - b^\nu(t)| - \sigma_\nu \right\}^2 + \int_0^t \int_\Omega \mathcal{C}^{-1} T^{\nu}_{E,t}(\tau) T^{\nu}_{E,t}(\tau) \, dx \, d\tau
+ 2\mu_\nu \int_0^t \int_\Omega |\nabla \hat{u}^\nu(\tau)|^2 \, dx \, d\tau + 4\mu_\nu \int_0^t \int_\Omega |\nabla \text{axl}(A^\nu_t(\tau))|^2 \, dx \, d\tau \leq C(T)
\]

holds and \(C(T)\) does not depend on \(\nu > 0\) (it depends only on the given data and the domain \(\Omega\)).

For the proof of Theorem 4.5 we refer to Section 5 of the paper [14]. Theorem 4.5 and the elastic constitutive equations (4.1) imply that for a subsequence (again denoted by \(\nu\)) we have

\[
\begin{align*}
   u^\nu & \to u \quad \text{in} \ L^\infty(0, T; H^1(\Omega; \mathbb{R}^3)), \\
   u^\nu_t & \to u_t \quad \text{in} \ L^2(0, T; H^1(\Omega; \mathbb{R}^3)), \\
   A^\nu & \to A \quad \text{in} \ L^\infty(0, T; H^2(\Omega; \mathfrak{s}_0(3))), \\
   A^\nu_t & \to A_t \quad \text{in} \ L^2(0, T; H^2(\Omega; \mathfrak{s}_0(3))), \\
   T^\nu & \to T \quad \text{in} \ L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)), \\
   T^\nu_t & \to T_t \quad \text{in} \ L^2(0, T; L^2(\Omega; \mathfrak{s}^3_{\text{dev}})), \\
   b^\nu & \to b \quad \text{in} \ L^\infty(0, T; L^2(\Omega; \mathfrak{s}^3_{\text{dev}})), \\
   b^\nu_t & \to b_t \quad \text{in} \ L^2(0, T; L^2(\Omega; \mathfrak{s}^3_{\text{dev}})).
\end{align*}
\]

The information contained in (4.5) is enough to pass to the limit in the Yosida approximation and get the solution in the sense of Definition 3.1. The details may be found in [14].

5. Hölder continuity for displacement and microrotation (interior case)

This section is the main part of the proof of Theorem 3.3.

Let us denote by \(B_R = B_R(x_0) = B(x_0, R) \subset \mathbb{R}^n\), \(n = 2, 3\), the open ball with center \(x_0 \in \mathbb{R}^n\) and radius \(R\). Moreover, let \(B_{2R} \setminus B_R = B(x_0, 2R) \setminus B(x_0, R)\) be the open annulus with center \(x_0 \in \mathbb{R}^n\). First we formulate two lemmas that will be useful later on.

**Lemma 5.1.** For each \(1 \leq p < \infty\) there exists a constant \(C\), depending only on \(n\) and \(p\), such that

\[
\int_{B_R} |v(y) - v(z)|^p \, dy \leq CR^{n+p-1} \int_{B_R} |\nabla v(y)|^p |y - z|^{1-n} \, dy
\]

for all \(B_R \subset \mathbb{R}^n\), \(v \in C^1(B_R; \mathbb{R}^3)\) and \(z \in B_R\).

For the proof of Lemma 5.1 we refer to Section 4.5.2 of [16].
Lemma 5.2. Let $B_{2R} \subset \mathbb{R}^n$, $n = 2, 3$, be an open ball and $v \in H^1(B_{2R}; \mathbb{R}^3)$. Then, there exists a constant $\tilde{C}$, not depending on $v$, such that

$$\left( \int_{B_{2R}} |v(y) - c_R|^2 \, dy \right)^{\frac{1}{2}} \leq \tilde{C} R \left( \int_{B_{2R}} |\nabla v(y)|^2 \, dy \right)^{\frac{1}{2}},$$

where

$$c_R = \frac{1}{|B_{2R} \setminus B_R|} \int_{B_{2R} \setminus B_R} v(y) \, dy.$$

Proof. Without loss of generality we may assume that $v \in C^1(B_{2R}; \mathbb{R}^3)$ and obtain

$$\int_{B_{2R}} |v(y) - c_R|^2 \, dy = \int_{B(0,2R)} \left| \frac{1}{|B_{2R} \setminus B_R|} \int_{B_{2R} \setminus B_R} (v(y) - v(z)) \, dz \right|^2 \, dy$$

$$\leq \int_{B_{2R}} \frac{1}{|B_{2R} \setminus B_R|} \int_{B_{2R} \setminus B_R} |v(y) - v(z)|^2 \, dz \, dy$$

$$\leq \int_{B_{2R}} \frac{1}{|B_{2R} \setminus B_R|} \int_{B_{2R}} |v(y) - v(z)|^2 \, dz \, dy$$

$$\leq (\text{Lemma 5.1})$$

$$\leq \int_{B_{2R}} C R \int_{B_{2R}} |\nabla v(z)|^2 |z - y|^{-1} \, dz \, dy$$

$$\leq \tilde{C} R^2 \int_{B_{2R}} |\nabla v(z)|^2 \, dz. \quad \square$$

Notice that Lemma 5.2 is the Poincaré inequality with a special constant $c_R$ (see Section 4.5.2 of [16], where the Poincaré inequality is proven with another constant). To prove the Hölder continuity of the displacement vector first we prove the following tube-filling condition. The idea is taken from the articles [17] and [19].

Theorem 5.3. Assume that the given data and initial data satisfy all hypotheses of Theorem 3.3. Then there exist constants $C, K > 0$ that do not depend on $\nu > 0$ and there exists a positive constant $\gamma > 0$ such that the tube-filling condition

$$\int_0^t \int_{B_R} |\nabla u_\nu^\tau(\tau)|^2 \, dx \, d\tau \leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u_\nu^\tau(\tau)|^2 \, dx \, d\tau + KR^\gamma$$

is satisfied for all balls $B_R = B(x_0, R) \subset B_{2R} = B(x_0, 2R) \subset \Omega \subset \mathbb{R}^n$, $n = 2, 3$ and $t \in (0,T)$. Moreover the constants $C, K > 0$ do not depend on the radius $R$.

Remark. Here we would like to underline that the tube-filling condition from Theorem 5.3 is satisfied for $n = 2$ and $n = 3$. But the Morrey condition obtained from the tube-filling step will be preserved only for $n = 2$. 
Proof of Theorem 5.3. Let us define the following function

\[
\xi_0(s) = \begin{cases} 
1 & s \in [-R, R], \\
\frac{2R-|s|}{R} & s \in [-2R, 2R] \setminus [-R, R], \\
0 & s \notin [-2R, 2R]
\end{cases}
\]

and let \( \xi \) denote the cutoff function defined by \( \xi(x) = \xi_0(|x_0 - x|) \). Compute the time derivative

\[
\frac{d}{dt} \left( \int_{B_{2R}} \xi^2 \xi_0^{2-\nu}(t) \left( |\text{dev}(T_{E,t}^\nu(t)) - b^\nu(t)| - \sigma_\gamma \right)^2 \, dx \right)
\]

\[
= \int_{B_{2R}} \xi^2 \xi_0^{2-\nu}(t) \left( |\text{dev}(T_{E,t}^\nu(t)) - b^\nu(t)| \right) \, dx = (\text{by the elastic constitutive relation})
\]

\[
= \int_{B_{2R}} \xi^2 \xi_0^{2-\nu}(t)T_{E,t}^\nu(t) \, dx - \int_{B_{2R}} \xi^2 \xi_0^{2-\nu}(t)T_{E,t}^\nu(t) \, dx - \int_{B_{2R}} \xi^2 \xi_0^{2-\nu}(t)b^\nu(t) \, dx.
\]

(5.1)

From Lemma 4.2 we conclude the following inequality

\[
\int_{B_{2R}} \xi^2 \xi_0^{2-\nu}(t)b^\nu(t) \, dx = (\text{by the equation for the backstress})
\]

\[
= c \int_{B_{2R}} \xi^2 |\xi_0^{2-\nu}(t)|^2 \, dx + d \int_{B_{2R}} \xi^2 |\xi_0^{2-\nu}(t)| \xi_0^{2-\nu}(t) \, dx
\]

\[
\geq \int_{B_{2R}} \xi^2 |\xi_0^{2-\nu}(t)|^2 \left( c - d |b^\nu(t)| \right) \, dx \geq 0.
\]

(5.2)

Notice that

\[
\int_{B_{2R}} \xi^2 \xi_0^{2-\nu}(t)T_{E,t}^\nu(t) \, dx = \int_{B_{2R}} \xi^2 \nabla u_t^\nu(t)T_{E,t}^\nu(t) \, dx - 2\mu_c \int_{B_{2R}} \xi^2 (\text{skew}(\nabla_x u_t^\nu(t)) - A_t^\nu(t)) \text{skew}(\nabla_x u_t^\nu(t)) \, dx
\]

\[
= \int_{B_{2R}} \xi^2 \nabla u_t^\nu(t)T_{E,t}^\nu(t) \, dx - 2\mu_c \int_{B_{2R}} \xi^2 |\text{skew}(\nabla_x u_t^\nu(t)) - A_t^\nu(t)|^2 \, dx
\]

\[
- 2\mu_c \int_{B_{2R}} \xi^2 (\text{skew}(\nabla_x u_t^\nu(t)) - A_t^\nu(t))A_t^\nu(t) \, dx.
\]

(5.3)

Integrating (5.3) with respect to time over 0 to \( t \) we have

\[
\int_0^t \int_{B_{2R}} \xi^2 \xi_0^{2-\nu}(\tau)T_{E,t}^\nu(\tau) \, dx \, d\tau = \int_0^t \int_{B_{2R}} \xi^2 \nabla u_t^\nu(\tau)T_{E,t}^\nu(\tau) \, dx \, d\tau - 2\mu_c \int_0^t \int_{B_{2R}} \xi^2 |\text{skew}(\nabla_x u_t^\nu(\tau)) - A_t^\nu(\tau)|^2 \, dx \, d\tau
\]

\[
- 2\mu_c \int_0^t \int_{B_{2R}} \xi^2 (\text{skew}(\nabla_x u_t^\nu(\tau)) - A_t^\nu(\tau))A_t^\nu(\tau) \, dx \, d\tau.
\]

(5.4)

Let \( \bar{u}_t^\nu \) be the average of \( u_t^\nu \) on the set \( B_{2R} \setminus B_R \). Integrating by parts the first term on the right hand side of (5.4) we obtain
\[
\int_0^t \int_{B_{2R}} \xi^2 \nabla u_t^\nu(\tau) T_t^\nu(\tau) \, dx \, d\tau = \int_0^t \int_{B_{2R}} \xi^2 (u_t^\nu(\tau) - \bar{u}_t^\nu(\tau)) f_t(\tau) \, dx \, d\tau \\
- 2 \int_0^t \int_{B_{2R} \setminus B_R} \xi T_t^\nu(\tau) \cdot (u_t^\nu(\tau) - \bar{u}_t^\nu(\tau)) \otimes \nabla \xi \, dx \, d\tau. \quad (5.5)
\]

The first term on the right hand side of (5.5) is estimated as follows

\[
\left| 2 \int_0^t \int_{B_{2R} \setminus B_R} \xi T_t^\nu(\tau) \cdot (u_t^\nu(\tau) - \bar{u}_t^\nu(\tau)) \otimes \nabla \xi \, dx \, d\tau \right| \\
\leq \alpha \int_0^t \| \xi T_t^\nu(\tau) \|_{L^2}^2 \, d\tau + \frac{\hat{C}(\alpha)}{R^2} \int_0^t \int_{B_{2R} \setminus B_R} |u_t^\nu(\tau) - \bar{u}_t^\nu(\tau)|^2 \, dx \, d\tau \\
\leq \alpha \int_0^t \| \xi T_t^\nu(\tau) \|_{L^2}^2 \, d\tau + C(\alpha) \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u_t^\nu(\tau)|^2 \, dx \, d\tau, \quad (5.7)
\]

where \( \alpha > 0 \) is any positive constant and the constants \( \hat{C}(\alpha), C(\alpha) > 0 \) do not depend on \( \nu > 0 \). The last term on the right hand side of (5.4) is estimated as follows

\[
\left| 2 \mu_c \int_0^t \int_{B_{2R}} \xi^2 (\text{skew}(\nabla_x u_t^\nu(\tau)) - A_t^\nu(\tau)) A_t^\nu(t) \, dx \, d\tau \right| \\
\leq \hat{\alpha} \int_0^t \int_{B_{2R}} \xi^2 |\text{skew}(\nabla_x u_t^\nu(\tau)) - A_t^\nu(\tau)|^2 \, dx \, dt + \bar{C}(\hat{\alpha}) \int_0^t \int_{B_{2R}} \xi^2 |A_t^\nu(\tau)|^2 \, dx \, d\tau, \quad (5.8)
\]

where \( \hat{\alpha} > 0 \) is any positive constant and the constant \( \bar{C}(\hat{\alpha}) > 0 \) does not depend on \( \nu > 0 \). Notice that the sequence \( \{A_t^\nu\}_{\nu>0} \) is bounded in \( L^2(0, T; H^2(\Omega; so(3))) \), then from the Rellich–Kondrachov theorem (cf. [1]) we obtain that the sequence \( \{A_t^\nu\}_{\nu>0} \) is bounded in \( L^2(0, T; L^p(\Omega; so(3))) \) for all \( p > 2 \), hence

\[
\int_0^t \int_{B_{2R}} \xi^2 |A_t^\nu(\tau)|^2 \, dx \, d\tau \leq CR^\gamma \quad (5.9)
\]

for some positive number \( \gamma = \gamma(n, p) > 0 \). Integrating (5.1) with respect to time and using (5.2)–(5.9) we obtain the following inequality
\[
\int_{B_{2R}} \xi^2 \frac{1}{2\nu} \left\{ |\text{dev}(T_E(t)) - b^\nu(t)| - \sigma_y \right\}_+^2 \, dx + \int_0^t \int_{B_{2R}} \xi^2 C^{-1} T_{E,t}^\nu(\tau) T_{E,t}^\nu(\tau) \, dx \, d\tau \\
+ 2\mu_c \int_0^t \int_{B_{2R}} \xi^2 \left| \text{skew}(\nabla_x u^\nu_t(\tau)) - A^\nu_t(\tau) \right|^2 \, dx \, d\tau \\
\leq \alpha \int_0^t \int_{B_{2R}} \xi^2 |T_{E,t}^\nu(\tau)|^2 \, dx \, d\tau + \alpha \int_0^t \int_{B_{2R}} \xi^2 |\text{skew}(\nabla_x u^\nu_t(\tau)) - A^\nu_t(\tau)|^2 \, dx \, d\tau \\
+ C(\alpha) \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u^\nu_t(\tau)|^2 \, dx \, d\tau + CR^\gamma + \int_{B_{2R}} \xi^2 \frac{1}{2\nu} \left\{ |\text{dev}(T_E^\nu(0)) - b^\nu(0)| - \sigma_y \right\}_+^2 \, dx,
\]

where the constants \(C(\alpha), C > 0\) do not depend on \(\nu > 0\) and the radius \(R\) (\(\gamma\) is some positive constant). Observe that \(T^\nu(0) \in L^2(\Omega; \mathcal{S}^3)\) is the unique solution of the problem

\[
\text{div}_x T^\nu(x, 0) = -f(x, 0), \\
-l_c \Delta_x \text{axl}(A^\nu(x, 0)) = \mu_c \text{axl}(\text{skew}(\nabla_x u^\nu(x, 0)) - A^\nu(x, 0)), \\
u^\nu(x, 0)_{|\partial\Omega} = g_D(x, 0), \quad A^{x,0}(x, 0)_{|\partial\Omega} = A_D(x, 0),
\]

which implies that \(\text{dev}(T_E^\nu(0)) = \text{dev}(T_E^0)\) and \(b^\nu(0) = b^0\). From the assumption (3.3) we conclude that the last term on the right hand side of (5.10) is equal to zero. Choosing in (5.10) \(\alpha > 0\) sufficiently small we arrive that

\[
\int_{B_{2R}} \xi^2 \frac{1}{2\nu} \left\{ |\text{dev}(T_E(t)) - b^\nu(t)| - \sigma_y \right\}_+^2 \, dx + \int_0^t \int_{B_{2R}} \xi^2 C^{-1} T_{E,t}^\nu(\tau) T_{E,t}^\nu(\tau) \, dx \, d\tau \\
+ 2\mu_c \int_0^t \int_{B_{2R}} \xi^2 \left| \text{skew}(\nabla_x u^\nu_t(\tau)) - A^\nu_t(\tau) \right|^2 \, dx \, d\tau \\
\leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u^\nu_t(\tau)|^2 \, dx \, d\tau + CR^\gamma.
\]

The inequality (5.12) implies the following inequality

\[
\int_0^t \int_{B_{2R}} \xi^2 C^{-1} T_{E,t}^\nu(\tau) T_{E,t}^\nu(\tau) \, dx \, d\tau + 2\mu_c \int_0^t \int_{B_{2R}} \xi^2 \left| \text{skew}(\nabla_x u^\nu_t(\tau)) - A^\nu_t(\tau) \right|^2 \, dx \, d\tau \\
\leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u^\nu_t(\tau)|^2 \, dx \, d\tau + CR^\gamma.
\]

Notice also that
Let proofs follow. From the assumptions on the elasticity tensor $C$ we know that

$$\int_0^t \int_{B_{2R}} \xi^2 C^{-1} T_{E,t}^\nu(\tau) T_{E,t}^\nu(\tau) \, dx \, d\tau \geq D \int_0^t \int_{B_{2R}} \xi^2 |T_{E,t}^\nu(\tau)|^2 \, dx \, d\tau$$

and the constant $D > 0$ does not depend on $\nu > 0$. Using the expressions (5.14)–(5.16) in (5.13) we get

$$D \int_0^t \int_{B_{2R}} \xi^2 |\text{div } u_t^\nu(\tau)|^2 \, dx \, d\tau + \mu_c \int_0^t \int_{B_{2R}} \xi^2 |\text{skew}(\nabla_x u_t^\nu(\tau))|^2 \, dx \, d\tau$$

$$\leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u_t^\nu(\tau)|^2 \, dx \, d\tau + CR^\gamma + 2\mu_c \int_0^t \int_{B_{2R}} \xi^2 |A_t^\nu(\tau)|^2 \, dx \, d\tau.$$  \hfill (5.17)

The last term on the right hand side of (5.17) is estimated in the same way as in (5.9). To complete the proof we need to estimate the expression

$$\int_0^t \int_{B_{2R}} \xi^2 |\text{div } u_t^\nu(\tau)|^2 \, dx \, d\tau + \int_0^t \int_{B_{2R}} \xi^2 |\text{curl } u_t^\nu(\tau)|^2 \, dx \, d\tau.$$ 

Let us denote by

$$c_R = \frac{1}{|B_{2R} \setminus B_R|} \int_{B_{2R} \setminus B_R} u_t^\nu(x,t) \, dx,$$

then

$$\text{curl}(\xi(u_t^\nu - c_R)) = \nabla \xi \times (u_t^\nu - c_R) + \xi \text{ curl } u_t^\nu,$$

$$\text{div}(\xi(u - c_R)) = \nabla \xi \cdot (u_t^\nu - c_R) + \xi \text{ div } u_t^\nu.$$  \hfill (5.18)
and

$$\begin{align*}
|\xi \cdot \text{curl } u'_t|^2 &= |\text{curl}(\xi(u'_t - c_R))|^2 + |\nabla \times (u'_t - c_R)|^2 - 2|\text{curl}(\xi(u'_t - c_R)) \cdot \nabla \times (u'_t - c_R)| \\
&\geq |\text{curl}(\xi(u'_t - c_R))|^2 + |\nabla \times (u'_t - c_R)|^2 - 2|\text{curl}(\xi(u'_t - c_R))||\nabla \times (u'_t - c_R)| \\
&\geq |\text{curl}(\xi(u'_t - c_R))|^2 + |\nabla \times (u'_t - c_R)|^2 - \epsilon|\text{curl}(\xi(u'_t - c_R))|^2 - \frac{C}{\epsilon}|\nabla \times (u'_t - c_R)|^2 \\
&\geq (\text{for sufficiently small epsilon}) \\
&\geq C|\text{curl}(\xi(u'_t - c_R))|^2 - \tilde{C}|\nabla \times (u'_t - c_R)|^2.
\end{align*}$$

(5.19)

In the same manner as in (5.19) we arrive at the following inequality

$$|\xi \cdot \text{div } u'_t|^2 \geq C|\text{div}(\xi(u - c_R))|^2 - \tilde{C}|\nabla \cdot (u'_t - c_R)|^2. \quad (5.20)$$

Using (5.18)–(5.20) in (5.17) we have

$$\begin{align*}
C_1 \int_0^t \int_{B_{2R}} (|\text{div}(\xi(u'_t(\tau) - c_R))|^2 + |\text{curl}(\xi(u'_t(\tau) - c_R))|^2) \, dx \, d\tau \\
&\leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u'_t(\tau)|^2 \, dx \, d\tau + CR^\gamma + C_2 \int_0^t \int_{B_{2R} \setminus B_R} |\nabla \xi|^2 |u'_t(\tau) - c_R|^2 \, dx \, d\tau,
\end{align*}$$

where the constants $C$, $\tilde{C}$, $C_1$ and $C_2$ do not depend on $\nu > 0$. Notice that the function $\xi(u'_t - c_R) \in H^1_0(\mathbb{R}^3)$ for almost all $t > 0$, hence the well-known estimate [23, p. 36]

$$\|\nabla u\|_{L^2}^2 \leq C_{\text{div}} (\|\text{div } u\|_{L^2}^2 + \|\text{curl } u\|_{L^2}^2) \quad \text{for all } u \in H^1_0$$

(the constant $C_{\text{div}}$ does not depend on $R$ and $u$) implies the following inequality

$$\begin{align*}
C_1 \int_0^t \int_{B_{2R}} |\nabla(\xi(u'_t(\tau) - c_R))|^2 \, dx \, d\tau &\leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u'_t(\tau)|^2 \, dx \, d\tau \\
&+ C_2 \int_0^t \int_{B_{2R} \setminus B_R} |\nabla \xi|^2 |u'_t(\tau) - c_R|^2 \, dx \, d\tau + CR^\gamma. \quad (5.21)
\end{align*}$$

The expression

$$|\nabla(\xi(u'_t - c_R))|^2 = |\nabla \xi \otimes (u'_t - c_R)|^2 + |\xi \cdot \nabla u'_t|^2 + 2|\nabla \xi \cdot (u'_t - c_R) \cdot \xi \nabla u'_t|^2$$

yields the inequality

$$\begin{align*}
C_1 \int_0^t \int_{B_{2R}} \xi^2 |\nabla u'_t(\tau)|^2 \, dx \, d\tau &\leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u'_t(\tau)|^2 \, dx \, d\tau + CR^\gamma + C_2 \int_0^t \int_{B_{2R} \setminus B_R} |\nabla \xi|^2 |u'_t(\tau) - c_R|^2 \, dx \, d\tau \\
&- 2 \int_0^t \int_{B_{2R}} \nabla \xi \cdot (u'_t(\tau) - c_R) \cdot \xi \nabla u'_t(\tau) \, dx \, d\tau.
\end{align*}$$
\[
\leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau + CR^\gamma + C_2 \int_0^t \int_{B_{2R} \setminus B_R} |\nabla \xi|^2 |u_\tau^\nu(\tau) - c_R|^2 \, dx \, d\tau \\
+ C(a) \int_0^t \int_{B_{2R} \setminus B_R} |\nabla \xi|^2 |u_\tau^\nu(\tau) - c_R|^2 \, dx \, d\tau + a \int_0^t \int_{B_{2R}} \xi^2 |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau
\]

(5.22)

for all \( a > 0 \). Choosing in (5.22) \( a > 0 \) sufficiently small we obtain

\[
C_1 \int_0^t \int_{B_{2R}} \xi^2 |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau \leq C_1 \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau + CR^\gamma
\]

\[
+ C_3 \int_0^t \int_{B_{2R} \setminus B_R} |\nabla \xi|^2 |u_\tau^\nu(\tau) - c_R|^2 \, dx \, d\tau.
\]

(5.23)

The last term on the right hand side of (5.23) is estimated using Poincaré’s inequality

\[
\int_0^t \int_{B_{2R} \setminus B_R} |\nabla \xi|^2 |u_\tau^\nu(\tau) - c_R|^2 \, dx \, d\tau \leq C \frac{R^2}{R^2} \int_0^t \int_{B_{2R} \setminus B_R} |u_\tau^\nu(\tau) - c_R|^2 \, dx \, d\tau
\]

\[
\leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau,
\]

(5.24)

where the constant \( C > 0 \) does not depend on \( \nu > 0 \) and the radius \( R > 0 \). Applying (5.24) in (5.23) and the fact that \( \xi \equiv 1 \) on \( B_R \) we complete the proof. \( \square \)

To prove the local Hölder continuity for the displacement vector \( u \) we use Widman’s hole filling trick from the articles [41] and [17].

**Theorem 5.4.** Suppose that all hypotheses of Theorem 3.3 hold. Then there exist \( \alpha \in (0, 1) \) and \( \gamma > 0 \) such that

\[
\int_0^t \int_{B_R} |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau \leq 2^{2a} \frac{R^{2a}}{R_0^{2a}} \left( \int_0^t \int_{B_{R_0}} |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau + K R^\gamma \right)
\]

for all balls \( B_R = B_R(x_0, R) \subset \Omega \subset \mathbb{R}^n \), \( n = 2, 3 \) and \( t \in (0, T) \), where \( x_0 \in \Omega' \Subset \Omega \) and \( 2R \leq R_0 = \frac{1}{2} \text{dist}(\Omega', \partial \Omega) \). The constants \( K \) (independent of the radius \( R > 0 \)) and \( R_0 \) do not depend on \( \nu > 0 \).

**Proof.** From Theorem 5.3 we conclude the following tube filling condition

\[
\int_0^t \int_{B_R} |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau \leq C \int_0^t \int_{B_{2R} \setminus B_R} |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau + KR^\gamma,
\]

(5.25)

where \( \gamma > 0 \) and the constant \( K \) does not depend on \( \nu \). We add to both sides of (5.25) the expression (filling the hole) \( C \int_0^t \int_{B_R} |\nabla u_\tau^\nu(\tau)|^2 \, dx \, d\tau \) and we obtain
\[
\int_0^t \int_{B_{2R}} |\nabla u_r^\nu(\tau)|^2 \, dx \, d\tau \leq \frac{C}{1+C} \int_0^t \int_{B_{2R}} |\nabla u_r^\nu(\tau)|^2 \, dx \, d\tau + KR_0^\gamma. \tag{5.26}
\]

For \( j \geq 1 \) we set \( R_j = R_0 2^{-j} \) and by iteration we deduce that

\[
\int_0^t \int_{B_{Rj}} |\nabla u_r^\nu(\tau)|^2 \, dx \, d\tau \leq \frac{C}{1+C} \int_0^t \int_{B_{2Rj}} |\nabla u_r^\nu(\tau)|^2 \, dx \, d\tau + KR_j^\gamma
\]

\[
\leq \frac{C}{1+C} \left( \frac{C}{1+C} \int_0^t \int_{B_{4Rj}} |\nabla u_r^\nu(\tau)|^2 \, dx \, d\tau + KR_{2j}^\gamma \right) + KR_j^\gamma \leq \ldots
\]

\[
\leq \left( \frac{C}{1+C} \right)^N \int_0^t \int_{B_{R0}} |\nabla u_r^\nu(\tau)|^2 \, dx \, d\tau + K \sum_{k=1}^N R_k^\gamma \left( \frac{C}{1+C} \right)^{N-k}. \tag{5.27}
\]

Let us choose \( \alpha \) such that \( \frac{C}{1+C} = 2^{-2\alpha} \), then we obtain that

\[
K \sum_{k=1}^N R_0^\gamma 2^{-\gamma k} 2^{-2\alpha(N-k)} = KR_0^\gamma 2^{-2\alpha N} \sum_{k=1}^N 2^{-\gamma k - 2\alpha k} \leq \tilde{K} R_0^\gamma (2^{-N})^{2\alpha} \tag{5.28}
\]

for \( \alpha < \frac{\gamma}{2} \) and \( \tilde{K} < \infty \). Using (5.28) in (5.27) we obtain

\[
\int_0^t \int_{B_{Rj}} |\nabla u_r^\nu(\tau)|^2 \, dx \, d\tau \leq (2^{-N})^{2\alpha} \left( \int_0^t \int_{B_{R0}} |\nabla u_r^\nu(\tau)|^2 \, dx \, d\tau + \tilde{K} R_0^\gamma \right). \tag{5.29}
\]

From the assumption \( R \leq R_0/2 \) it is possible to find \( j \geq 1 \) such that \( R_{j+1} \leq R \leq R_j \) and \( R_j \leq 2R \). Hence \( (2R)^{-1} \leq \left( \frac{R_0}{2^j} \right)^{-1} \) and

\[
R^{-2\alpha} = (2R)^{-2\alpha} 2^{2\alpha} \leq 2^{2\alpha} \left( \frac{R_0}{2^j} \right)^{-2\alpha} \cdot (2^j)^{2\alpha} = 2^{2\alpha} R^{-2\alpha} \leq 2^{2\alpha} R_0^{-2\alpha} / R^{2\alpha}.
\]

The above inequality finishes the proof. \( \square \)

**Theorem 5.5.** Let \( n = 2 \) and suppose that all hypotheses of Theorem 3.3 are satisfied. Then

\[
u \in C^{0,\alpha'}([0,T]; C^{0,\alpha}(\Omega'; \mathbb{R}^3)),
\]

where \( \Omega' \Subset \Omega \) and Hölder exponent \( \alpha \) comes from Theorem 5.4. Moreover, \( \alpha' < 1 \) is any time Hölder exponent.

**Proof.** To prove the local Hölder continuity for the displacement vector \( u \) we will show the following Morrey’s condition (see for instance [6]): in the case of two dimensions it is in the form
\[ \int_{B_R} |\nabla u'(x,t)| \, dx \leq KR^{1+\gamma} \quad \forall B_R \subset \Omega \subset \mathbb{R}^3 \text{ and } \forall t > 0, \]

where \( 0 < \gamma < 1, \ t \in (0,T) \) and the constant \( K > 0 \) does not depend on the radius \( R > 0 \). Notice that Theorem 5.4 yields that the velocity of the displacement vector satisfies Morrey’s condition only for \( n = 2 \) but we are unable to prove the Hölder continuity of the velocity. We show it for the displacement – the idea was taken from the article [19]. From the Theorem 4.5 and coerciveness of the total energy we know that \( \nabla u'' \in H^1(0,T;L^2(\Omega;\mathbb{R}^3)) \), hence

\[
\nabla u''(x,t) = \nabla u(x,0) + \int_0^t \nabla u''(x,\tau) \, d\tau \quad \text{for all } 0 \leq t \leq T
\]

and

\[
\int_{B_R} |\nabla u''(x,t)| \, dx \leq \int_{B_R} |\nabla u(x,0)| \, dx + \int_0^t \int_{B_R} |\nabla u''(x,\tau)| \, dx \, d\tau,
\]

where the ball \( B_R \subset \Omega \) is the same as in the Theorem 5.4. Observe that the function \( u(x,0) \) is the unique solution of the elliptic system (5.11) and of course of the system (3.4). The general regularity theory for linear elliptic systems implies that if the given initial data satisfy some additional regularity assumptions, then we conclude that

\[
\int_{B_R} |\nabla u(x,0)| \, dx \leq \tilde{K} R^{1+\tilde{\alpha}}
\]

for some \( 0 < \tilde{\alpha} < 1 \) and the constant \( \tilde{K} > 0 \) does not depend on the radius \( R \). Notice that in the formulation of the main theorem we could not write explicitly the assumption on the initial data, but we know that the above inequality holds for some \( \tilde{\alpha} \). Moreover if \( \alpha \neq \tilde{\alpha} \), then the displacement vector \( u \) is Hölder continuous in space direction with exponent \( \min(\alpha, \tilde{\alpha}) \).

Using Theorem 5.4 we infer that the function \( u'' \) satisfies Morrey’s condition for \( n = 2 \), hence \( u'' \in L^\infty(0,T;C^{0,\alpha}(\Omega';\mathbb{R}^3)) \). The weak convergence of the sequence \( \{\nabla u''\}_{\nu > 0} \) in \( L^\infty(L^2) \) implies that the function \( u \) satisfies Morrey’s condition and \( u \in L^\infty(0,T;C^{\alpha,\alpha}(\Omega';\mathbb{R}^3)) \). To prove the Hölder continuity with respect to time we estimate the following difference (the idea is taken again from [19])

\[
|u(x_0,t_1) - u(x_0,t_2)| \leq \left| u(x_0,t_1) - \int_{B_R} u(x,t_1) \, dx \right| + \left| u(x_0,t_2) - \int_{B_R} u(x,t_2) \, dx \right|
\]

\[
+ \left| \int_{B_R} u(x,t_1) \, dx - \int_{B_R} u(x,t_2) \, dx \right|.
\]

The Hölder continuity with respect to space of the function \( u \) implies that

\[
\left| u(x_0,t_i) - \int_{B_R} u(x,t_i) \, dx \right| \leq KR^\alpha \quad (i = 1, 2)
\]

for almost all \( t > 0 \). Let us choose \( R > 0 \) such that \( |t_2 - t_1|^\frac{1}{2} = R^{\alpha+1} \), then we obtain
\[ |u(x_0, t_1) - u(x_0, t_2)| \leq 2KR^\alpha + \frac{C}{R^2} \left| \int_{t_1}^{t_2} \int_{B_R} u_t(x, t) \, dx \, dt \right| \]
\[ \leq 2KR^\alpha + CR^{-1} \int_{t_1}^{t_2} \left( \int_{B_R} |u_t(x, t)|^2 \, dx \right)^{\frac{1}{2}} \, dt \]
\[ \leq 2KR^\alpha + CR^{-1}|t_2 - t_1|^{\frac{1}{2}} \left( \int_{t_1}^{t_2} \int_{B_R} |u_t(x, t)|^2 \, dx \, dt \right)^{\frac{1}{2}}. \]

We know that \( u_t \in L^2(0, T; H^1(\Omega; \mathbb{R}^3)) \), hence the choice of the radius \( R \) yields that
\[ |u(x_0, t_1) - u(x_0, t_2)| \leq K|t_2 - t_1|^{\frac{\alpha}{2}}. \]

The information above implies that \( u \in L^\infty(0, T; C^{0,\alpha}(\Omega'; \mathbb{R}^3)) \cap C^{0,\beta}([0, T]; L^\infty(\Omega'; \mathbb{R}^3)). \) This yields \( u \in C^{0,\gamma}([0, T] \times \Omega') \) for \( \gamma = \min(\alpha, \beta) \). \( \square \)

6. Hölder continuity of the displacement up to the boundary

To prove the Hölder continuity up to the boundary we need some assumptions about the boundary \( \partial \Omega \).

In this article we must assume that the boundary of \( \Omega \) is \( C^2 \)-class, because in the theory of existence of solution for the AF-model with Cosserat effects we need to use the regularity theory for linear elliptic system.

**Proof of Theorem 3.3.** On the beginning let us note that from the existence Theorem 3.2 we obtain that \( A \in H^1(0, T; H^2(\Omega; \mathfrak{so}(3))) \). In two dimensions we have the following embeddings (cf. [1] or [40]):

\[ H^1(0, T; H^2(\Omega; \mathfrak{so}(3))) \subset C^{0,\frac{1}{2}}([0, T]; H^2(\Omega; \mathfrak{so}(3))) \subset C^{0,\frac{1}{2}}([0, T]; C^{0,\alpha}(\bar{\Omega}; \mathfrak{so}(3))) \]

for all \( \alpha \in (0, 1) \). To prove the Hölder continuity of the displacement vector \( u \) up to the boundary we divide the proof into two steps:

**Step 1.** Assume that \( B_{2R} = B(x_0, 2R) = B_{2R}(x_0) \subset \Omega \subset \mathbb{R}^2 \). If we consider the same cutoff function as in the proof of Theorem 5.3, then the values of \( u \) on the boundary are irrelevant and the proof is the same as in the last section.

**Step 2.** Assume that \( B_{2R} \cap \partial \Omega \neq \emptyset \). Let us take a point \( x'_0 \in \partial \Omega \) such that \( x'_0 \in \partial \Omega \cap B_{2R}(x_0) \), where \( B_R(x_0) \subset B_{4R}(x'_0) \). Let us consider a standard Lipschitz continuous cutoff function \( \tau \) such that
\[ \tau = 1 \text{ on } B_{4R}(x'_0), \]
\[ \tau = 0 \text{ on } \mathbb{R}^2 \setminus B_{8R}(x'_0), \]
\[ |\nabla \tau| \leq CR^{-1} \text{ on } B_{8R}(x'_0) \setminus B_{4R}(x'_0). \]

Regularity with respect to time of the data \( g_D \) implies that there exists function \( w \in W^{1,\infty}(0, T; H^1(\Omega; \mathbb{R}^3)) \) such that \( w_{t_0,\partial} = g_{D,t} \). Differentiating with respect to time Eqs. (4.1)\(_1\) and (4.1)\(_3\), next multiplying Eq. (4.1)\(_1\) by \( \tau^2(u'_t - w_t) \), Eq. (4.1)\(_3\) by \( \tau^2 \) and integrating those two equations with respect to space we obtain...
Integrating by parts the first equation of \((6.1)\) we have

\[
- l_c \int_{B_{SR}(x_0') \cap \Omega} \tau^2 \Delta_x \text{axl}(A'_t) \text{axl}(A'_t) \, dx = \mu_c \int_{B_{SR}(x_0') \cap \Omega} \tau^2 \text{axl}(\nabla_x u'_{\nu}) - A'_t \, \text{axl}(A'_t) \, dx.
\]

Adding those two equations we get

\[
2\mu \int_{B_{SR}(x_0') \cap \Omega} \tau^2 \varepsilon(u_{\nu}' - \varepsilon_t^{p,\nu})^2 \, dx + \lambda \int_{B_{SR}(x_0') \cap \Omega} \tau^2 (\text{tr}(\varepsilon(u_{\nu}' - \varepsilon_t^{p,\nu}))^2 \, dx + 2\mu_c \int_{B_{SR}(x_0') \cap \Omega} \tau^2 |\text{skew}(\nabla_x u_t'(t)) - A'_t(t)|^2 \, dx
\]

\[
- \int_{B_{SR}(x_0') \cap \Omega} \tau^2 f_t(u_{\nu}' - w_t) \, dx - 2 \int_{T_R} \tau T_{E,t}^{-1} \cdot (u_{\nu}' - w_t) \otimes \nabla \tau \, dx + \int_{B_{SR}(x_0') \cap \Omega} \tau^2 \Delta_x \text{axl}(A'_t) \text{axl}(A'_t) \, dx.
\]

Integrating \((6.3)\) with respect to time we obtain

\[
\int_{B_{SR}(x_0') \cap \Omega} \tau^2 \frac{1}{2\nu} \left( |\text{dev}(T_{E,t}'(t)) - b''(t)| - \sigma_y \right)_+^2 \, dx + 2\mu \int_{B_{SR}(x_0') \cap \Omega} \tau^2 |\varepsilon(u_{\nu}' - \varepsilon_t^{p,\nu})|^2 \, dx \, dt
\]

\[
+ \lambda \int_0^t \int_{B_{SR}(x_0') \cap \Omega} \tau^2 (\text{tr}(\varepsilon(u_{\nu}' - \varepsilon_t^{p,\nu}))^2 \, dx \, dt + 2\mu_c \int_0^t \int_{B_{SR}(x_0') \cap \Omega} \tau^2 |\text{skew}(\nabla_x u_t'(t)) - A'_t|^2 \, dx \, dt
\]

\[
= - \int_0^t \int_{B_{SR}(x_0') \cap \Omega} \tau^2 f_t(u_{\nu}' - w_t) \, dx \, dt - 2 \int_0^t \int_{T_R} \tau T_{E,t}^{-1} \cdot (u_{\nu}' - w_t) \otimes \nabla \tau \, dx \, dt
\]

\[
+ \int_0^t \int_{B_{SR}(x_0') \cap \Omega} \tau^2 T_{E,t}^{-1} \varepsilon(w_t) \, dx \, dt - l_c \int_0^t \int_{B_{SR}(x_0') \cap \Omega} \tau^2 \Delta_x \text{axl}(A'_t) \text{axl}(A'_t) \, dx \, dt
\]

\[
+ \int_{B_{SR}(x_0') \cap \Omega} \tau^2 \frac{1}{2\nu} \left( |\text{dev}(T_{E,t}'(0)) - b''(0)| - \sigma_y \right)_+^2 \, dx.
\]

The same argument as in the proof of \textbf{Theorem 5.3} yields that the last term on the right hand side of \((6.4)\) equals zero. The first term on the right hand side of \((6.4)\) is estimated as follows.
\begin{equation}
\int_{0}^{t} \int_{B_{\delta R}(x_{0}) \cap \Omega} \tau^{2}(u_{t}^{\nu} - w_{t}) f_{t} \, dx \, d\tau \leq \int_{0}^{t} \|u_{t}^{\nu} - w_{t}\|_{L^{2}} \left\|\tau^{2} f_{t}(t)\right\|_{L^{2}} \, d\tau
\end{equation}

\leq \tilde{C}R \int_{0}^{t} \left\|\nabla u_{t}^{\nu} - \nabla w_{t}\right\|_{L^{2}} \, d\tau \leq C(T)R. \quad (6.5)

Here we apply the standard Poincaré inequality, because \(u_{t}^{\nu} - w_{t} = 0\) on the set \(B_{\delta R}(x_{0}) \cap \partial \Omega\) which has positive measure. Using Cauchy’s inequality with a small weight and applying Poincaré’s inequality to the second term on the right hand side of (6.4) we conclude

\begin{equation}
2 \int_{0}^{t} \int_{T_{R}} \tau T_{t}^{\nu} \cdot (u_{t}^{\nu} - w_{t}) \otimes \nabla \xi \, dx \, d\tau \leq a \int_{0}^{t} \left\|\tau T_{t}^{\nu}\right\|_{L^{2}}^{2} \, d\tau + \frac{\tilde{C}(a)}{R^{2}} \int_{0}^{t} \left\|u_{t}^{\nu} - w_{t}\right\|^{2} \, dx \, dt
\end{equation}

\leq a \int_{0}^{t} \left\|\tau T_{t}^{\nu}\right\|_{L^{2}}^{2} \, d\tau + C(a) \int_{0}^{t} \left\|\nabla u_{t}^{\nu} - \nabla w_{t}\right\|^{2} \, dx \, d\tau, \quad (6.6)

where \(a > 0\) is any positive constant. One before last integral on the right hand side of (6.4) is estimated in the same way as in the proof of Theorem 5.3 because \(A^{\nu}\) is the \(L^{2}\)-strong solution of (4.1). The estimates (6.5), (6.6) and the Cauchy inequality with a small weight used in the third term of (6.4) implies the following inequality

\begin{equation}
\int_{B_{\delta R}(x_{0}) \cap \Omega} \tau^{2} \frac{1}{2\nu} \left\{|\text{dev}(T_{t}^{E}(t)) - b^{\nu}(t)| - \sigma_{y}\right\}_{+}^{2} \, dx + 2\mu \int_{0}^{t} \int_{B_{\delta R}(x_{0}) \cap \Omega} \tau^{2} \left|\varepsilon(u_{t}^{\nu}) - \varepsilon_{t}^{p,\nu}\right|^{2} \, dx \, d\tau
\end{equation}

+ \lambda \int_{0}^{t} \int_{B_{\delta R}(x_{0}) \cap \Omega} \tau^{2} \left(\text{tr}(\varepsilon(u_{t}^{\nu}) - \varepsilon^{p,\nu}_{t})\right)^{2} \, dx \, d\tau + 2\mu_{c} \int_{0}^{t} \int_{B_{\delta R}(x_{0}) \cap \Omega} \tau^{2} |\text{skew}(\nabla_{x} u_{t}^{\nu}) - A_{t}^{\nu}|^{2} \, dx \, d\tau
\end{equation}

\leq C \int_{0}^{t} \int_{T_{R}} \left\|\nabla u_{t}^{\nu}(t)\right\|^{2} \, dx \, d\tau + a \int_{0}^{t} \left\|\tau T_{t}^{\nu}\right\|_{L^{2}}^{2} \, d\tau + \tilde{C} \int_{0}^{t} \int_{B_{\delta R}(x_{0}) \cap \Omega} \left\|\nabla w_{t}\right\|^{2} \, dx \, d\tau + KR^{\gamma}\quad (6.7)

where \(a, \gamma > 0\) are arbitrary positive constants and the constants \(C, \tilde{C} > 0\) do not depend on \(\nu > 0\). Choosing in (6.7) \(a > 0\) suitably small, using the additional assumption on the function \(w_{t}\) and the calculations from the proof of Theorem 5.3 we arrive at the following tube filling condition

\begin{equation}
\int_{0}^{t} \int_{B_{\delta R}(x_{0})} \left\|\nabla u_{t}^{\nu}\right\|^{2} \, dx \, d\tau \leq C \int_{0}^{t} \int_{T_{R}} \left\|\nabla u_{t}^{\nu}\right\|^{2} \, dx \, d\tau + KR^{\gamma}, \quad (6.8)
\end{equation}

where \(\gamma > 0\) is some positive constant. Let us note that

\begin{equation}
\int_{0}^{t} \int_{B_{\delta R}(x_{0}) \cap \Omega} \left\|\nabla u_{t}^{\nu}\right\|^{2} \, dx \, d\tau \leq \int_{0}^{t} \int_{B_{\delta R}(x_{0}) \cap \Omega} \left\|\nabla u_{t}^{\nu}\right\|^{2} \, dx \, d\tau
\end{equation}

and
\[
\int_0^t \int_{B_R(x_0) \cap \Omega} |\nabla u_{t\tau}'|^2 \, dx \, d\tau \leq \int_0^t \int_{B_R(x_0) \cap \Omega \setminus B_R(x_0)} |\nabla u_{t\tau}'|^2 \, dx \, d\tau.
\]

The inequalities (6.9) and (6.10) yield that

\[
\int_0^t \int_{B_R(x_0) \cap \Omega} |\nabla u_{t\tau}'|^2 \, dx \, d\tau \leq \int_0^t \int_{B_R(x_0) \cap \Omega \setminus B_R(x_0)} |\nabla u_{t\tau}'|^2 \, dx \, d\tau + KR^\gamma,
\]

therefore we can apply the tube-filling trick with \(R_j = R_016^{-j}\) (in the same manner as in the proof of Theorem 5.4) and get finally

\[
\int_0^t \int_{B_R(x_0) \cap \Omega} |\nabla u_{t\tau}'|^2 \, dx \, d\tau \leq KR^{2\alpha}
\]

for all balls \(B(x_0, R) \subset \mathbb{R}^2\), where \(K < \infty\) and \(0 < \alpha < 1\). “Morrey’s Dirichlet growth theorem” (see for example [24]) implies that \(u \in C^{0,\alpha}([0,T] \times \Omega)\). Using Lemma 1.2 of the article [27] we conclude that \(u \in C^{0,\frac{1}{2}\alpha}([0,T]; C^{0,\frac{1}{2}\alpha}(\Omega; \mathbb{R}^3))\). \(\square\)

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**References**