Construction of anisotropic polyconvex energies and applications to thin shells

V. Ebbing, D. Balzani, J. Schröder, P. Neff, F. Gruttmann

Institut für Mechanik, Abteilung Bauwissenschaften, Fakultät für Ingenieurwissenschaften, Universität Duisburg-Essen, Universitätsstraße 15, 45117 Essen, Germany
AG6, Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstraße 7, 64289 Darmstadt, Germany
Institut für Werkstoffe und Mechanik im Bauwesen, Technische Universität Darmstadt, Petersenstraße 12, 64287 Darmstadt, Germany

Abstract

In computer simulations where constitutive equations are considered anisotropic polyconvex energies can preferably be used because the existence of minimizers is then automatically guaranteed. In this work we investigate the capability to simulate anisotropy effects of anisotropic thin shells using polyconvex anisotropic energies. The construction of the considered polyconvex transversely isotropic energy is based on specific structural tensors. The iterative enforcement of the zero normal stress condition at the integration points allows the consideration of arbitrary three-dimensional constitutive equations. As a representative example we compare results for isotropic and anisotropic plates.

1. Introduction

Lightweight structures are widely employed in architecture, engineering and building constructions and find application e.g. in long span roofs for stadiums and exhibition structures. Typical lightweight constructions include shell structures made of a woven fiber composite. This consists of a relatively weak rubber-like matrix in which natural or synthetic fibers of high stiffness are embedded. Their optimal design necessitates the knowledge of their highly nonlinear stress–strain behavior including large strains and a strong anisotropy effect due to the fiber-reinforcement of the material. For this reason, we focus on the numerical treatment of the simulation of anisotropic hyperelastic thin shells under consideration of polyconvex energy functions. By satisfying the polyconvexity of the energy function the Legendre–Hadamard ellipticity is ensured as well as the existence of minimizers, provided that also the coercivity condition is satisfied, see Ball [1]. Since such above mentioned lightweight structures have a very small thickness compared to the dimension in other directions shell elements are used. The nonlinear shell theory is based on the Reissner–Mindlin kinematic along with inextensible director vectors. The first existence results for geometrically exact true two-dimensional shell models based on the Reissner–Mindlin kinematics have been given in [5,6] for isotropy. Here, we consider, however, a general three-dimensional anisotropic model, where the dimensional reduction appears implicitly in the numerical algorithm. For the variational framework we consider a three field variational functional taking into account independent displacements, enhanced strains and stress resultants, where the latter field is eliminated by the evaluation of some orthogonality conditions. The iterative enforcement of the zero normal stress condition at the integration points allows consideration of arbitrary three-dimensional constitutive equations. An analysis of thin shells using anisotropic polyconvex energy densities is given in Balzani et al. [2]. There the structural tensor is defined by $M = a \otimes a$, where $a$ is the preferred direction of the material. In this work polyconvex energies are considered which are constructed by the method proposed in Schröder et al. [7]. This method is based on specific structural tensors.

2. Thin shell: variational formulation

We investigate the influence of anisotropy modeled by the proposed polyconvex strain energy functions in [7] on the deformation of a thin shell. Due to the fact that low order elements like quadrilaterals based on a standard displacement interpolation are usually characterized by locking phenomena, we use a mixed finite element with an improved convergence behavior. The present finite element formulation is based on a three field variational functional as introduced by Simo and Amero [8] for large strains using enhanced displacement gradients. Here we follow the approach for nonlinear shells in Betsch et al. [3], where the
shell strains $\varepsilon$ are enhanced by $\varepsilon = \varepsilon(v) + \varepsilon_0$ denotes the enhanced part and $v$ is the vector of displacements and rotations. The shell is loaded statically by surface loads $p$ on $\Omega$ and by boundary forces $t$ on the boundary $\Gamma$. The variational framework for the enhanced assumed strain method is the following three field variational functional in a Lagrangean representation

$$
\Pi(v, \varepsilon, a) = \int_{\Omega} W(\varepsilon) - \alpha^2 \varepsilon dA - \int_{\Gamma} v^T p dA - \int_{\Gamma_2} v^T t ds.
$$

After variation with respect to all arguments some orthogonality conditions are introduced in order to eliminate the independent stresses from the set of equations. Then we approximate the field variables by using the isoparametric concept and bi-linear ansatz functions for the position and director vectors and linearize the weak form; for details we refer to Balzani et al. [2].

3. Polyconvex anisotropic energy function

In the framework of finite elasticity the mathematical treatment of boundary value problems is based on the direct methods of variations, i.e., finding a deformation $\varphi$ which minimizes the elastic free energy function $W(F = \text{Grad} \varphi)$ subject to specific boundary conditions. To guarantee the existence of minimizers of some variational principles in finite elasticity the variational functional must be sequentially weakly lower semicontinuous (s.w.l.s.) and coercive. Polyconvex functions are always s.w.l.s., quasiconvex and rank-one convex, see Ball [1]. By considering smooth energy functions, the latter condition ensures the ellipticity of the corresponding acoustic tensor, i.e., material stability is then guaranteed. In the framework of anisotropic finite elasticity the principle of objectivity and the principle of material symmetry play an important role concerning the construction of the constitutive equations. The principle of objectivity is fulfilled since we use the reduced constitutive equations in terms of the right Cauchy-Green tensor $C := F^T F$, i.e., $\psi(C) = W(F)$ defined by unit reference volume. The principle of material symmetry enforces the invariance of the constitutive equations with respect to the transformations $Q \in G \subset O(3)$ of the material symmetry group $G \subset O(3)$, where $O(3)$ is the full orthogonal group. Thus, the following conditions for the reduced constitutive equations must hold:

$$
\psi(C) = \psi(QCQ^T) \quad \forall \ Q \in G.
$$

The anisotropy of the material can be described by structural tensors $G$, see Boehler [4], which are invariant with respect to transformations of the underlying material symmetry group $G = QQ^T \forall Q \in G \subset O(3)$. Inserting the structural tensor as further tensorial argument into the free energy function (2) then

$$
\psi(C, G) = \psi(QCQ^T, QQ^T) \quad \forall \ Q \in O(3)
$$

holds. Thus, the scalar-valued isotropic tensor function (3) can be expressed by the principal invariants $I_1, I_2, I_3$ and mixed invariants $J_4, J_5$, i.e.,

$$
I_1 := \text{tr} C, \ I_2 := \text{tr} [\text{CoF} C], \ I_3 := \det C, \\
J_4 := \text{tr} [CG], \ J_5 := \text{tr} [\text{CoF} CG].
$$

For the description of a reinforced material with one fiber type we assume the material behavior to be transversely isotropic. For the construction of the corresponding energy function we consider the framework proposed in [7]. There the transversely isotropic structural tensor has the form

$$
G^a = \text{diag}(a, b, b) \quad \text{with} \quad a, b > 0,
$$

which preserves the material symmetry group $G_a := \{I, Q(0, a)|0 < a < 2\pi\}$, where $a$ denotes the fiber direction and $Q(0, a)$ are all rotations about the $a$-axis. Due to their structure fiber-reinforced materials with one fiber type are here described by an energy function decomposed into an isotropic part for the matrix material and a transversely isotropic part for the fiber family. This leads to the general representation

$$
\psi = \psi^{iso}(I_1, I_2, I_3) + \psi^a(J_4, J_5).
$$

with the coercive polyconvex isotropic Mooney–Rivlin model

$$
\psi^{iso} = \alpha_1 I_1 + \alpha_2 I_2 + \delta_1 I_3 - \delta_2 \ln(\sqrt{I_3}) \quad \forall \ \alpha_1, \alpha_2, \delta_1, \delta_2 \geq 0
$$

and the polyconvex anisotropic part $\psi^a$

$$
\psi^a = \frac{1}{\alpha_3 (\text{tr}G^a)^3} \eta_1 (J_4^2 + J_5^2), \quad \forall \ \eta_1 \geq 0, \alpha_3 \geq 1,
$$

which has been introduced in [7]. In order to fulfill the condition of a stress-free reference configuration, i.e., $S(C = I) = 0$, the restriction $\delta_2 = 2\alpha_1 + 4\alpha_2 + 2\alpha_1 + 2\eta_1$ in (7) must be taken into account.

4. Thin hexagonal plate subjected to pressure

In this numerical example, a thin hexagonal plate made of an (i) isotropic and (ii–iii) two types of transversely isotropic hyperelastic materials is subjected to a follower load $p$ (pressure at the undersurface), see Fig. 1a. The length of the plate depicted in Fig. 1a is set to $l = 600.0$ cm and the thickness to 0.2 cm. We consider the finite-element discretization with 800 four-node shell elements as shown in Fig. 1b. In the following analysis we compare the results for (i) an isotropic plate, described by (6) with $G^{ii} = I$, with the solutions (ii–iii) of the anisotropic plates, using (6) with $G^{iii} = \text{diag}(a, b, b)$ and $G^{iii} = R C^{iii} R^T$, respectively. $R$ is the transformation matrix from the local orientation system to the global coordinate system. Furthermore, the non-zero material parameters are given by

$$
\alpha_1 = 8 \text{ kN/cm}^2, \delta_1 = 10 \text{ kN/cm}^2, \alpha_3 = 50, \eta_1 = 10 \text{ kN/cm}^2, a = 1, b = 0.1.
$$

Fig. 1. Thin hexagonal plate: (a) System with boundary conditions, (b) discretization with 800 four-noded shell elements, (c) first and (d) second type of orientation of the preferred direction $a$. 
The load parameter \( p \) is increased until a maximum vertical displacement of approximately 75 cm is obtained in all cases, see Fig. 2. Fig. 2 depicts the contour plot of the out of plane displacements for the isotropic and anisotropic plate. A comparison of these plots shows the significant differences between these cases. In the isotropic case we observe a four-fold symmetric distribution of the vertical displacements. In the first anisotropic case there is a rhombic distribution in the center of the plate, whereas in the second anisotropic case a pair of identical circles occurs. For both anisotropic plates we obtain a two-fold symmetry. The anisotropy effect can be also pointed out by comparing the deformations of the considered plates, see Fig. 2. While the deformation of the isotropic plate is symmetric with view to all coordinate axes, a kink of the first anisotropic plate at its widest part in X-direction can be noticed. The last anisotropic plate deforms to a pair of identical hills.

5. Conclusion

This paper presents an analysis of thin shells using anisotropic polyconvex energy functions based on specific structural tensors. In the framework of polyconvexity the existence of minimizers for the bulk problem is guaranteed (if the coercivity condition is also fulfilled). For the incorporation of this complex constitutive laws in the considered shell formulation an interface for a general three-dimensional constitutive law is used. A representative example is discussed, where significant differences between the deformation of the isotropic and the anisotropic plates are pointed out.

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References