Loss of ellipticity in additive logarithmic finite strain plasticity and related results on Hencky-type energies

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The aim of this paper is to present some results regarding the Legendre-Hadamard ellipticity and loss of ellipticity of some energies depending on the logarithmic strain tensor.

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1 Why a new Hencky-type energy?

We consider nonlinear elastic energies based on certain invariants of the logarithmic strain tensor \( \log U \), namely \( \| \text{dev}_n \log U \| \|^2 \) and \( \| \text{tr}(\log U) \|^2 \), where \( F = \nabla \varphi \) is the deformation gradient, \( U = \sqrt{F^T F} \) is the right stretch tensor, \( \text{dev}_n X = X - \frac{1}{n} \text{tr}(X) \mathbb{1}_n \) is the deviatoric part of the second order tensor \( X \in \mathbb{R}^{n \times n} \) and \( \| \cdot \| \) is the Frobenius tensor norm.

The fact that the quadratic Hencky energy \( W_H(F) = \mu \| \text{dev}_n \log U \|^2 + \kappa \| \text{tr}(\log U) \|^2 \) represents a useful tool in nonlinear elasticity is a common place [5]. Here \( \mu > 0 \) is the shear (distortional) modulus, \( \kappa = \frac{2\mu + 3\lambda}{3} > 0 \) is the bulk modulus with the first Lamé constant \( \lambda \). However, it is also well known [4] that the quadratic Hencky energy \( W_H \) is not Legendre-Hadamard elliptic, for \( n = 2 \) and for any \( \mu, \kappa > 0 \).

Motivated by this serious shortcoming, in some previous works [3,6–9] we have considered the following family of exponentiated Hencky-logarithmic strain type energies

\[
W_{\text{exH}} : \text{GL}^+(n) \to \mathbb{R}, \quad W_{\text{exH}}(F) = \mu k e^{\kappa \| \text{dev}_n \log U \|^2} + \frac{\kappa}{2} e^{\tilde{k} \text{tr}(\log U)^2},
\]

where \( k, \tilde{k} \) are dimensionless parameters. These energies approximate the classical quadratic Hencky strain energy \( W_H \) for deformation gradients \( F \) sufficiently close to the identity \( \mathbb{1}_n \).

In the planar case (\( \text{dev}_2 \log U = \log U - \frac{1}{2} \text{tr}(\log U) \mathbb{1}_2 \)), the exponentiated Hencky energies \( W_{\text{exH}} \) are polyconvex [3, 9] for \( \mu > 0, \kappa > 0, k \geq \frac{1}{4} \) and \( \tilde{k} \geq \frac{1}{8} \). Since polyconvexity of an energy implies its Legendre-Hadamard ellipticity [1], for \( n = 2 \) the exponentiated Hencky energy avoid the above mentioned shortcoming of the quadratic Hencky energy.

2 A finite strain multiplicative plasticity formulation

In the three-dimensional case, it is established [8] that for all \( k > 0 \), the function \( F \mapsto e^{k \| \text{dev}_3 \log U \|^2} \), \( F \in \text{GL}^+(3) \) is not Legendre-Hadamard elliptic. However, we discuss an interesting relation between non-ellipticity of \( W_{\text{exH}} \) in three-dimensions and finite plasticity models. Using the results from [2] we
obtain that the ellipticity domain of the isochoric energy part $W^{iso}_{el}(F) = e^k \|\text{dev}_3 \log U\|^2$ contains the domain $\{ U \in \text{Sym}^+(3) \mid \|\text{dev}_3 \log U\|^2 \leq \frac{2}{3} \}$ which is related to the von-Mises-Huber-Hencky criterion (maximum distortion to with $2\mu e^{\frac{k}{2}}$).

Thus, we turn to finite strain isotropic plasticity [6] and we couple the exponentiated Hencky energy with a finite strain multiplicative plasticity formulation. Multiplicative plasticity is stable at frozen plastic flow, i.e. if the elastic energy $F \mapsto W(F)$ is Legendre-Hadamard elliptic, it follows that the elasto-plastic formulation $F \mapsto W(F, F_p) := W(F F_p^{-1})$ remains Legendre-Hadamard elliptic w.r.t. $F$ for all given plastic distortions $F_p$. Hence, in the multiplicative setting, the elastic rank-one convexity is independent of the plastic flow. Therefore, the multiplicative approach is ideally suited as far as preservation of ellipticity properties for elastic unloading is concerned and marks a sharp contrast to the additive logarithmic modelling frameworks, as it will be seen.

3 Loss of ellipticity in additive logarithmic finite strain plasticity

Another finite plasticity model using logarithmic strains is taking the additive elastic Hencky energy in the format $\hat{W}_{el}(\log U - \log U_p) = \mu \|\text{dev}_3 [\log U - \log U_p]\|^2 + \frac{\nu}{2} \|\text{tr}([\log U])\|^2$, as a starting point, in which plastic incompressibility $\det U_p = 1$ is already included. Regarding the additive logarithmic finite strain plasticity, it is established [7] that:

**Proposition 3.1** The function $F \mapsto W(F) = e^{\|\text{dev}_2 \log U - \text{dev}_2 \log U_p\|^2}$ is not Legendre-Hadamard elliptic for some given plastic stretch $U_p \in \text{Sym}^+(2)$, while $F \mapsto W(F) = e^{\|\text{dev}_2 \log U\|^2}$ is Legendre-Hadamard elliptic.

Note that the loss of ellipticity occurs for the given plastic strain $U_p$, at elastic strains in the order of $\|\text{dev}_3 \log U_p\| \approx 2\sqrt{2}$ which means $382\%$ elastic strain. The conclusion of this section and the main conclusion of this note is that the additive logarithmic model does not preserve Legendre-Hadamard ellipticity in elastic unloading in general.

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**References**


