New Thoughts in Nonlinear Elasticity Theory via Hencky's Logarithmic Strain Tensor

Patrizio Neff, Robert J. Martin and Bernhard Eidel

Abstract We consider the two logarithmic strain measures $\omega_{\mathrm{iso}} = \|\mathrm{dev}_n \log U\|$ and $\omega_{\mathrm{vol}} = |\mathrm{tr}(\log U)|$, which are isotropic invariants of the Hencky strain tensor $\log U = \log(F^T F)$, and show that they can be uniquely characterized by purely geometric methods based on the geodesic distance on the general linear group $\mathrm{GL}(n)$. Here, F is the deformation gradient, $U = \sqrt{F^T F}$ is the right Biot-stretch tensor, log denotes the principal matrix logarithm, $\|\,.\,\|$ is the Frobenius matrix norm, tr is the trace operator and $\mathrm{dev}_n X = X - \frac{1}{n}\,\mathrm{tr}(X)\cdot \mathbb{1}$ is the n-dimensional deviator of $X \in \mathbb{R}^{n\times n}$. This characterization identifies the Hencky (or true) strain tensor as the natural nonlinear extension of the linear (infinitesimal) strain tensor $\varepsilon = \mathrm{sym}\,\nabla u$, which is the symmetric part of the displacement gradient ∇u, and reveals a close geometric relation between the classical quadratic isotropic energy potential in linear elasticity and the geometrically nonlinear quadratic isotropic Hencky energy. Our deduction involves a new fundamental logarithmic minimization property of the orthogonal polar factor R, where $F = RU$ is the polar decomposition of F.

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1 Strain and strain measures in nonlinear elasticity

The concept of *strain* is of fundamental importance in elasticity theory. In linearized elasticity, one assumes that the Cauchy stress tensor σ is a linear function of the symmetric infinitesimal strain tensor

$$\varepsilon = \operatorname{sym} \nabla u = \operatorname{sym}(\nabla\varphi - \mathbb{1}) = \operatorname{sym}(F - \mathbb{1})\,,$$

where $\varphi\colon \Omega \to \mathbb{R}^n$ is the deformation of an elastic body with a given reference configuration $\Omega \subset \mathbb{R}^n$, $\varphi(x) = x + u(x)$ with the displacement u, $F = \nabla\varphi$ is the deformation gradient, $\operatorname{sym} \nabla u = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetric part of the displacement gradient ∇u and $\mathbb{1} \in \mathrm{GL}^+(n)$ is the identity tensor in the group of invertible tensors with positive determinant. In geometrically nonlinear elasticity models, it is no longer necessary to postulate a linear connection between some stress and some strain. However, nonlinear strain tensors are often used in order to simplify the stress response function, and many constitutive laws are expressed in terms of linear relations between certain strains and stresses [2, 3, 6].

There are different definitions of what exactly the term "strain" encompasses: while Truesdell and Toupin [42, p. 268] consider "*any uniquely invertible isotropic second order tensor function of* [*the right Cauchy-Green deformation tensor* $C = F^T F$]" to be a strain tensor, it is commonly assumed [20, p. 230] (cf. [21, 22, 5, 36]) that a (material or Lagrangian[1]) strain takes the form of a *primary matrix function* of the right Biot-stretch tensor $U = \sqrt{F^T F}$ of the deformation gradient $F \in \mathrm{GL}^+(n)$, i.e. an isotropic tensor function $E\colon \mathrm{Sym}^+(n) \to \mathrm{Sym}(n)$ from the set of positive definite tensors to the set of symmetric tensors of the form

$$E(U) = \sum_{i=1}^{n} \mathrm{e}(\lambda_i) \cdot e_i \otimes e_i \qquad \text{for} \qquad U = \sum_{i=1}^{n} \lambda_i \cdot e_i \otimes e_i \tag{1}$$

with a *scale function* $\mathrm{e}\colon (0,\infty) \to \mathbb{R}$, where $\otimes$ denotes the tensor product, λ_i are the eigenvalues and e_i are the corresponding eigenvectors of U.

The general idea underlying these definitions is clear: strain is a measure of deformation (i.e. the change in form and size) of a body with respect to a chosen (arbitrary) reference configuration. Furthermore, the strain of the deformation gradient $F \in \mathrm{GL}^+(n)$ should correspond only to the *non-rotational* part of F. In particular, the strain must vanish if and only if F is a pure rotation, i.e. if and only if $F \in \mathrm{SO}(n)$, where $\mathrm{SO}(n) = \{Q \in \mathrm{GL}(n) \,|\, Q^T Q = \mathbb{1}, \det Q = 1\}$ denotes the special orthogonal group. This ensures that the only strain-free deformations are rigid body movements [33].

In contrast to *strain* or *strain tensor*, we use the term ***strain measure*** to refer to a nonnegative real-valued function $\omega\colon \mathrm{GL}^+(n) \to [0,\infty)$ depending on the deformation gradient which vanishes if and only if F is a pure rotation, i.e. $\omega(F) = 0$ if and only if $F \in \mathrm{SO}(n)$.

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[1] Similarly, a *spatial* or *Eulerian* strain tensor $\widehat{E}(V)$ depends on the left Biot-stretch tensor $V = \sqrt{F F^T}$ (cf. [14]).

In the following we consider the question of what strain measures are appropriate for the theory of nonlinear isotropic elasticity. Since, by our definition, a strain measure attains zero if and only if $F \in \mathrm{SO}(n)$, a simple geometric approach is to consider a *distance function* on the group $\mathrm{GL}^+(n)$ of admissible deformation gradients, i.e. a function $\mathrm{dist}\colon \mathrm{GL}^+(n) \times \mathrm{GL}^+(n) \to [0,\infty)$ with $\mathrm{dist}(A,B) = \mathrm{dist}(B,A)$ which satisfies the triangle inequality and vanishes if and only if its arguments are identical. Such a distance function induces a "natural" strain measure on $\mathrm{GL}^+(n)$ by means of the distance to the special orthogonal group $\mathrm{SO}(n)$:

$$\omega(F) := \mathrm{dist}(F, \mathrm{SO}(n)) := \inf_{Q \in \mathrm{SO}(n)} \mathrm{dist}(F,Q)\,. \tag{2}$$

In this way, the search for an appropriate strain measure reduces to the task of finding a *natural, intrinsic distance function* on $\mathrm{GL}^+(n)$.

2 Euclidean strain measures

2.1 The Euclidean strain measure in linear isotropic elasticity

An approach similar to the definition of strain measures via distance functions on $\mathrm{GL}^+(n)$, as stated in equation (2), can be employed in linearized elasticity theory: let $\varphi(x) = x + u(x)$ with the displacement u. Then the *infinitesimal strain measure* may be obtained by taking the distance of the displacement gradient $\nabla u \in \mathbb{R}^{n\times n}$ to the set of *linearized rotations* $\mathfrak{so}(n) = \{A \in \mathbb{R}^{n\times n} : A^T = -A\}$, which is the vector space of skew symmetric matrices. An obvious choice for a distance measure on the linear space $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ of $n\times n$-matrices is the *Euclidean distance* induced by the canonical Frobenius norm

$$\|X\| = \sqrt{\mathrm{tr}(X^T X)} = \sqrt{\sum_{i,j=1}^{n} X_{ij}^2}\,.$$

We use the more general weighted norm defined by

$$\|X\|^2_{\mu,\mu_c,\kappa} = \mu\,\|\mathrm{dev}_n \,\mathrm{sym} X\|^2 + \mu_c\,\|\mathrm{skew} X\|^2 + \frac{\kappa}{2}\,[\mathrm{tr}(X)]^2\,, \quad \mu,\mu_c,\kappa > 0\,, \tag{3}$$

which separately weights the *deviatoric* (or *trace free*) *symmetric part* $\mathrm{dev}_n \,\mathrm{sym} X = \mathrm{sym} X - \frac{1}{n}\,\mathrm{tr}(\mathrm{sym} X)\cdot \mathbb{1}$, the *spherical part* $\frac{1}{n}\,\mathrm{tr}(X)\cdot\mathbb{1}$, and the *skew symmetric part* $\mathrm{skew} X = \frac{1}{2}(X - X^T)$ of X; note that $\|X\|_{\mu,\mu_c,\kappa} = \|X\|$ for $\mu = \mu_c = 1, \kappa = \frac{2}{n}$, and that $\|\,.\,\|_{\mu,\mu_c,\kappa}$ is induced by the inner product

$$\langle X,Y\rangle_{\mu,\mu_c,\kappa} = \mu\,\langle \mathrm{dev}_n \,\mathrm{sym} X, \mathrm{dev}_n \,\mathrm{sym} Y\rangle + \mu_c\,\langle \mathrm{skew} X, \mathrm{skew} Y\rangle + \tfrac{\kappa}{2}\,\mathrm{tr}(X)\,\mathrm{tr}(Y) \tag{4}$$

on $\mathbb{R}^{n\times n}$, where $\langle X,Y\rangle = \operatorname{tr}(X^T Y)$ denotes the canonical inner product. In fact, every isotropic inner product on $\mathbb{R}^{n\times n}$, i.e. every inner product $\langle\cdot,\cdot\rangle_{\text{iso}}$ with

$$\langle Q^T X\, Q,\; Q^T Y\, Q\rangle_{\text{iso}} = \langle X,Y\rangle_{\text{iso}}$$

for all $X,Y\in\mathbb{R}^{n\times n}$ and all $Q\in \mathrm{O}(n)$, is of the form (4), cf. [11]. The suggestive choice of variables μ and κ, which represent the *shear modulus* and the *bulk modulus*, respectively, will prove to be justified later on. The remaining parameter μ_c will be called the *spin modulus*.

Of course, the element of best approximation in $\mathfrak{so}(n)$ to ∇u with respect to the weighted Euclidean distance $\operatorname{dist}_{\text{Euclid},\mu,\mu_c,\kappa}(X,Y) = \|X-Y\|_{\mu,\mu_c,\kappa}$ is given by the associated orthogonal projection of ∇u to $\mathfrak{so}(n)$.Since $\mathfrak{so}(n)$ and the space $\operatorname{Sym}(n)$ of symmetric matrices are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mu,\mu_c,\kappa}$, this projection is given by the *continuum rotation*, i.e. the skew symmetric part $\operatorname{skew}\nabla u = \frac{1}{2}(\nabla u - (\nabla u)^T)$ of ∇u, the axial vector of which is $\operatorname{curl} u$. Thus the distance is

$$\begin{aligned}\operatorname{dist}_{\text{Euclid},\mu,\mu_c,\kappa}(\nabla u,\mathfrak{so}(n)) :&= \inf_{A\in\mathfrak{so}(n)} \|\nabla u - A\|_{\mu,\mu_c,\kappa}\\ &= \|\nabla u - \operatorname{skew}\nabla u\|_{\mu,\mu_c,\kappa} = \|\operatorname{sym}\nabla u\|_{\mu,\mu_c,\kappa}\,. \qquad (5)\end{aligned}$$

We therefore find

$$\begin{aligned}\operatorname{dist}^2_{\text{Euclid},\mu,\mu_c,\kappa}(\nabla u,\mathfrak{so}(n)) &= \|\operatorname{sym}\nabla u\|^2_{\mu,\mu_c,\kappa}\\ &= \mu\,\|\operatorname{dev}_n \operatorname{sym}\nabla u\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}(\operatorname{sym}\nabla u)]^2\\ &= \mu\,\|\operatorname{dev}_n \varepsilon\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}(\varepsilon)]^2 = W_{\text{lin}}(\nabla u)\end{aligned}$$

for the linear strain tensor $\varepsilon = \operatorname{sym}\nabla u$, which is the quadratic isotropic elastic energy.

2.2 *The Euclidean strain measure in nonlinear isotropic elasticity*

In order to obtain a strain measure in the geometrically nonlinear case, we must compute the distance

$$\operatorname{dist}(\nabla\varphi,\mathrm{SO}(n)) = \operatorname{dist}(F,\mathrm{SO}(n)) = \inf_{Q\in\mathrm{SO}(n)} \operatorname{dist}(F,Q)$$

of the deformation gradient $F=\nabla\varphi\in\mathrm{GL}^+(n)$ to the actual set of pure rotations $\mathrm{SO}(n)\subset\mathrm{GL}^+(n)$. It is therefore necessary to choose a distance function on $\mathrm{GL}^+(n)$; an obvious choice is the restriction of the Euclidean distance on $\mathbb{R}^{n\times n}$ to $\mathrm{GL}^+(n)$. For the canonical Frobenius norm $\|\,.\,\|$, the Euclidean distance between $F,P\in\mathrm{GL}^+(n)$ is

$$\operatorname{dist}_{\text{Euclid}}(F,P) = \|F-P\| = \sqrt{\operatorname{tr}[(F-P)^T(F-P)]}\,.$$

Now let $Q \in \mathrm{SO}(n)$. Since $\|\,.\,\|$ is orthogonally invariant, i.e. $\|\widehat{Q}X\| = \|X\widehat{Q}\| = \|X\|$ for all $X \in \mathbb{R}^{n\times n}$, $\widehat{Q} \in \mathrm{O}(n)$, we find

$$\mathrm{dist}_{\mathrm{Euclid}}(F,Q) = \|F-Q\| = \|Q^T(F-Q)\| = \|Q^T F - \mathbb{1}\|\,. \tag{6}$$

Thus the computation of the strain measure induced by the Euclidean distance on $\mathrm{GL}^+(n)$ reduces to the *matrix nearness problem* [19]

$$\mathrm{dist}_{\mathrm{Euclid}}(F,\mathrm{SO}(n)) = \inf_{Q\in\mathrm{SO}(n)} \|F-Q\| = \min_{Q\in\mathrm{SO}(n)} \|Q^T F - \mathbb{1}\|\,.$$

By a well-known optimality result discovered by Giuseppe Grioli [15] (cf. [32, 16, 27, 9]), also called "Grioli's Theorem" by Truesdell and Toupin [42, p. 290], this minimum is attained for the orthogonal polar factor R.

Theorem 1 (Grioli's Theorem [15, 32, 42]). *Let* $F \in \mathrm{GL}^+(n)$. *Then*

$$\min_{Q\in\mathrm{SO}(n)} \|Q^T F - \mathbb{1}\| = \|R^T F - \mathbb{1}\| = \|\sqrt{F^T F} - \mathbb{1}\| = \|U - \mathbb{1}\|\,,$$

where $F = RU$ *is the polar decomposition of* F *with* $R = \mathrm{polar}(F) \in \mathrm{SO}(n)$ *and* $U = \sqrt{F^T F} \in \mathrm{Sym}^+(n)$. *The minimum is uniquely attained at the orthogonal polar factor* R.

Thus for nonlinear elasticity, the restriction of the Euclidean distance to $\mathrm{GL}^+(n)$ yields the strain measure

$$\mathrm{dist}_{\mathrm{Euclid}}(F,\mathrm{SO}(n)) = \|U - \mathbb{1}\|\,.$$

In analogy to the linear case, we obtain

$$\mathrm{dist}^2_{\mathrm{Euclid}}(F,\mathrm{SO}(n)) = \|U-\mathbb{1}\|^2 = \|E_{1/2}\|^2\,, \tag{7}$$

where $E_{1/2} = U - \mathbb{1}$ is the Biot strain tensor. Note the similarity between this expression and the *Saint-Venant-Kirchhoff* energy [24]

$$\|E_1\|^2_{\mu,\mu_c,\kappa} = \mu\,\|\mathrm{dev}_3\,E_1\|^2 + \frac{\kappa}{2}\,[\mathrm{tr}(E_1)]^2\,, \tag{8}$$

where $E_1 = \frac{1}{2}(C-\mathbb{1}) = \frac{1}{2}(U^2-\mathbb{1})$ is the Green-Lagrangian strain.

However, the resulting strain measure $\omega(U) = \mathrm{dist}_{\mathrm{Euclid}}(F,\mathrm{SO}(n)) = \|U-\mathbb{1}\|$ does not truly seem appropriate for finite elasticity theory: for $U \to 0$ we find $\|U - \mathbb{1}\| \to \|\mathbb{1}\| = \sqrt{n} < \infty$, thus singular deformations do not necessarily correspond to an infinite measure ω. Furthermore, the above computations are not compatible with the weighted norm introduced in Section 2.1: in general [31, 12, 13],

$$\min_{Q\in\mathrm{SO}(n)} \|F-Q\|^2_{\mu,\mu_c,\kappa} \neq \min_{Q\in\mathrm{SO}(n)} \|Q^T F - \mathbb{1}\|^2_{\mu,\mu_c,\kappa} \neq \|\sqrt{F^T F} - \mathbb{1}\|^2_{\mu,\mu_c,\kappa}\,, \tag{9}$$

thus the Euclidean distance of F to $\mathrm{SO}(n)$ with respect to $\|\,.\,\|_{\mu,\mu_c,\kappa}$ does not equal $\|\sqrt{F^TF}-\mathbb{1}\|_{\mu,\mu_c,\kappa}$ in general. In these cases, the element of best approximation is not the orthogonal polar factor $R=\mathrm{polar}(F)$.

We also observe that the Euclidean distance is not an *intrinsic* distance measure on $\mathrm{GL}^+(n)$: in general, $A-B\notin\mathrm{GL}^+(n)$ for $A,B\in\mathrm{GL}^+(n)$, hence the term $\|A-B\|$ depends on the underlying linear structure of $\mathbb{R}^{n\times n}$.

Most importantly, because $\mathrm{GL}^+(n)$ is not convex, the straight line $\{A+t\,(B-A)\,|\,t\in[0,1]\}$ connecting A and B is not necessarily contained in $\mathrm{GL}^+(n)$, which shows that the characterization of the Euclidean distance as the length of a shortest connecting curve is also not possible in a way intrinsic to $\mathrm{GL}^+(n)$, as the intuitive sketches in Figures 1 and 2 indicate.

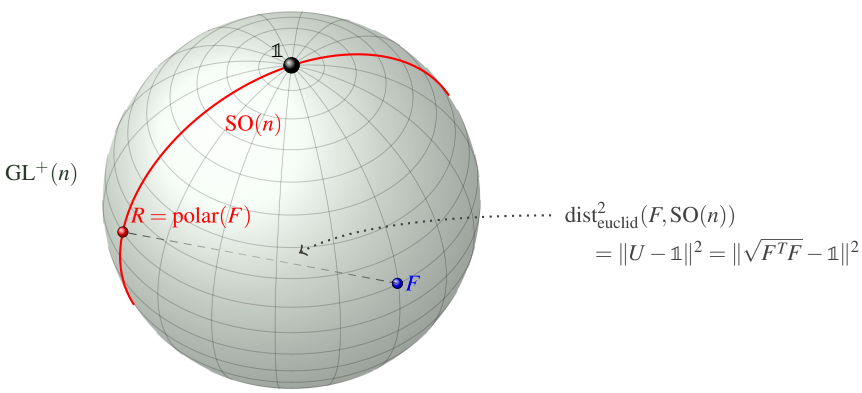

Fig. 1: The Euclidean distance as an extrinsic measure on $\mathrm{GL}^+(n)$; note that the representation of the manifold $\mathrm{GL}^+(n)$ as a sphere only serves to demonstrate the necessity of an intrinsic distance measure and does not reflect the actual topological properties of $\mathrm{GL}^+(n)$.

These issues amply demonstrate that the Euclidean distance can only be regarded as an *extrinsic* distance measure on the general linear group. We therefore need to expand our view to allow for a more appropriate, truly *intrinsic* distance measure on $\mathrm{GL}^+(n)$.

3 The Riemannian strain measure in nonlinear isotropic elasticity

3.1 $\mathrm{GL}^+(n)$ *as a Riemannian manifold*

In order to find an intrinsic distance function on $\mathrm{GL}^+(n)$ that alleviates the drawbacks of the Euclidean distance, we endow $\mathrm{GL}(n)$ with a *Riemannian metric* g, which is defined by an inner product $g_A\colon T_A\,\mathrm{GL}(n)\times T_A\,\mathrm{GL}(n)\to\mathbb{R}$ on each tangent space $T_A\,\mathrm{GL}(n)$, $A\in\mathrm{GL}(n)$. Then the length of a sufficiently smooth curve $\gamma\colon [0,1]\to\mathrm{GL}(n)$ is given by $L(\gamma)=\int_0^1\sqrt{g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}\,\mathrm{d}t$, where $\dot\gamma(t)=\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)$, and the *geodesic distance* (cf. Figure 2) between $A,B\in\mathrm{GL}^+(n)$ is defined as the infimum over the lengths of all (twice continuously differentiable) curves connecting A to B:

$$\mathrm{dist}_{\mathrm{geod}}(A,B)=\inf\{L(\gamma)\,|\,\gamma\in C^2([0,1];\mathrm{GL}^+(n)),\ \gamma(0)=A,\ \gamma(1)=B\}\,.$$

Our search for an appropriate strain measure is thereby reduced to the task of finding

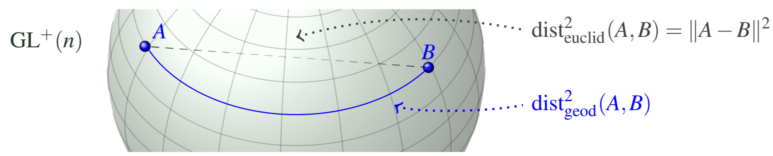

Fig. 2: The geodesic (intrinsic) distance compared to the Euclidean (extrinsic) distance.

an appropriate Riemannian metric on $\mathrm{GL}(n)$. Although it might appear as an obvious choice, the metric $\breve g$ with

$$\breve g_A(X,Y):=\langle X,Y\rangle \qquad \text{for all } A\in\mathrm{GL}^+(n),\ X,Y\in\mathbb{R}^{n\times n} \tag{10}$$

provides no improvement over the already discussed Euclidean distance on $\mathrm{GL}^+(n)$: since the length of a curve γ with respect to $\breve g$ is its classical (Euclidean) length, the shortest connecting curves with respect to $\breve g$ are straight lines of the form $t\mapsto A+t(B-A)$ with $A,B\in\mathrm{GL}^+(n)$. Locally, the geodesic distance induced by $\breve g$ is therefore equal to the Euclidean distance, and thus many of the shortcomings of the Euclidean distance apply to the geodesic distance induced by $\breve g$ as well.

In order to find a more viable Riemannian metric g on $\mathrm{GL}(n)$, we consider the mechanical interpretation of the induced geodesic distance $\mathrm{dist}_{\mathrm{geod}}$: while our focus lies on the strain measure induced by g, that is the geodesic distance of the deformation gradient F to the special orthogonal group $\mathrm{SO}(n)$, the distance $\mathrm{dist}_{\mathrm{geod}}(F_1,F_2)$ between two deformation gradients F_1,F_2 can also be motivated directly as a *measure of difference* between two linear (or *homogeneous*) deformations F_1,F_2 of the same body Ω. More generally, we can define a difference measure between two inhomogeneous deformations $\varphi_1,\varphi_2:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$ via

$$\mathrm{dist}(\varphi_1,\varphi_2):=\int_\Omega \mathrm{dist}_{\mathrm{geod}}(\nabla\varphi_1(x),\nabla\varphi_2(x))\,\mathrm{d}x \tag{11}$$

under suitable regularity conditions for φ_1,φ_2 (e.g. if φ_1,φ_2 are sufficiently smooth with $\det\nabla\varphi_i>0$ up to the boundary).

In order to find an appropriate Riemannian metric g on $\mathrm{GL}(n)$, we must discuss the required properties of this "difference measure". First, the requirements of objectivity (left-invariance) and isotropy (right-invariance) suggest that the metric g should be *bi*-$\mathrm{O}(n)$-*invariant*, i.e. satisfy

$$\underbrace{g_{QA}(QX,QY) = \overbrace{g_A(X,Y)}_{\text{objectivity}} = g_{AQ}(XQ,YQ)}^{\text{isotropy}} \tag{12}$$

for all $Q \in \mathrm{O}(n)$, $A \in \mathrm{GL}(n)$ and $X,Y \in T_A\,\mathrm{GL}(n)$, to ensure that $\mathrm{dist}_{\mathrm{geod}}(A,B) = \mathrm{dist}_{\mathrm{geod}}(QA,QB) = \mathrm{dist}_{\mathrm{geod}}(AQ,BQ)$.

However, these requirements do not sufficiently determine a specific Riemannian metric. For example, (12) is satisfied by the metric $\breve{g}$ defined in (10) as well as by the metric $\check{\breve{g}}$ with $\check{\breve{g}}_A(X,Y) = \langle A^T X, A^T Y\rangle$. In order to rule out unsuitable metrics, we need to impose further restrictions on g. If we consider the distance measure $\mathrm{dist}(\varphi_1,\varphi_2)$ between two deformations φ_1,φ_2 introduced in (11), a number of further invariances can be motivated: if we require that the distance is not changed by the superposition of a homogeneous deformation, i.e. that

$$\mathrm{dist}(B\cdot\varphi_1, B\cdot\varphi_2) = \mathrm{dist}(\varphi_1,\varphi_2)$$

for all constant $B \in \mathrm{GL}(n)$, then g must be *left*-$\mathrm{GL}(n)$-*invariant*, i.e.

$$g_{BA}(BX,BY) = g_A(X,Y) \tag{13}$$

for all $A,B \in \mathrm{GL}(n)$ and $X,Y \in T_A\,\mathrm{GL}(n)$.

It can easily be shown [26] that a Riemannian metric g is left-$\mathrm{GL}(n)$-invariant as well as right-$\mathrm{O}(n)$-invariant if and only if g is of the form

$$g_A(X,Y) = \langle A^{-1}X, A^{-1}Y\rangle_{\mu,\mu_c,\kappa}\,, \tag{14}$$

where $\langle\cdot,\cdot\rangle_{\mu,\mu_c,\kappa}$ is the fixed inner product on the tangent space $\mathfrak{gl}(n) = T_{\mathbb{1}}\,\mathrm{GL}(n) = \mathbb{R}^{n\times n}$ at the identity with

$$\langle X,Y\rangle_{\mu,\mu_c,\kappa} = \mu\,\langle \mathrm{dev}_n \operatorname{sym} X, \mathrm{dev}_n \operatorname{sym} Y\rangle + \mu_c\langle \operatorname{skew} X, \operatorname{skew} Y\rangle + \tfrac{\kappa}{2}\operatorname{tr}(X)\operatorname{tr}(Y)$$

for constant positive parameters $\mu,\mu_c,\kappa > 0$, and where $\langle X,Y\rangle = \operatorname{tr}(X^TY)$ denotes the canonical inner product on $\mathfrak{gl}(n) = \mathbb{R}^{n\times n}$. In the following, we will always assume that $\mathrm{GL}(n)$ is endowed with a Riemannian metric of the form (14) unless indicated otherwise.

In order to find the geodesic distance

$$\mathrm{dist}_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \inf_{Q\in\mathrm{SO}(n)} \mathrm{dist}_{\mathrm{geod}}(F,Q)$$

of $F \in \mathrm{GL}^+(n)$ to $\mathrm{SO}(n)$, we need to consider the *geodesic curves* on $\mathrm{GL}^+(n)$. It has been shown [26, 28, 17, 1] that every geodesic on $\mathrm{GL}^+(n)$ with respect to the left-$\mathrm{GL}(n)$-invariant Riemannian metric (14) is of the form

$$\gamma_F^\xi(t) = F\,\exp(t(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi))\,\exp(t(1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi) \tag{15}$$

with $F \in \mathrm{GL}^+(n)$ and some $\xi \in \mathfrak{gl}(n)$, where exp denotes the matrix exponential. Since the geodesic curves are defined globally, $\mathrm{GL}^+(n)$ is *geodesically complete* with respect to the metric g. We can therefore apply the Hopf-Rinow theorem [23, 26] to find that for all $F,P \in \mathrm{GL}^+(n)$ there exists a *length minimizing geodesic* γ_F^ξ connecting F and P. Without loss of generality, we can assume that γ_F^ξ is defined on the interval $[0,1]$. Then the end points of γ_F^ξ are

$$\gamma_F^\xi(0) = F \quad \text{and} \quad P = \gamma_F^\xi(1) = F\,\exp(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi)\,\exp((1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi)\,,$$

and the length of the geodesic γ_F^ξ starting in F with initial tangent $F\,\xi \in T_F\,\mathrm{GL}^+(n)$ (cf. (15) and Figure 3) is given by [26]

$$L(\gamma_F^\xi) = \|\xi\|_{\mu,\mu_c,\kappa}\,.$$

The geodesic distance between F and P can therefore be characterized as

$$\mathrm{dist}_{\mathrm{geod}}(F,P) = \min\{\|\xi\|_{\mu,\mu_c,\kappa} \mid \xi \in \mathfrak{gl}(n) : \gamma_F^\xi(1) = P\}\,,$$

that is the minimum of $\|\xi\|_{\mu,\mu_c,\kappa}$ over all $\xi \in \mathfrak{gl}(n)$ which connect F and P, i.e. satisfy

$$\exp(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi)\,\exp((1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi) = F^{-1}P\,. \tag{16}$$

Although some numerical computations have been employed [43] to approximate the geodesic distance in the special case of the canonical left-$\mathrm{GL}(n)$-invariant metric, i.e. for $\mu = \mu_c = 1$, $\kappa = \frac{2}{n}$, there is no known closed-form solution to the highly nonlinear system (16) in terms of ξ for given $F,P \in \mathrm{GL}^+(n)$ and thus no known method of directly computing $\mathrm{dist}_{\mathrm{geod}}(F,P)$ in the general case exists. However, this parametrization of the geodesic curves will still allow us to obtain a lower bound on the distance of F to $\mathrm{SO}(n)$.

3.2 The geodesic distance to $\mathrm{SO}(n)$

Having defined the geodesic distance on $\mathrm{GL}^+(n)$, we can now consider the geodesic strain measure, i.e. the geodesic distance of the deformation gradient F to $\mathrm{SO}(n)$:

$$\mathrm{dist}_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \inf_{Q\in\mathrm{SO}(n)} \mathrm{dist}_{\mathrm{geod}}(F,Q)\,. \tag{17}$$

Now, let $F = RU$ denote the polar decomposition of F with $U \in \mathrm{Sym}^+(n)$ and $R \in \mathrm{SO}(n)$. In order to establish a simple upper bound on the geodesic distance $\mathrm{dist}_{\mathrm{geod}}(F,\mathrm{SO}(n))$, we construct a particular curve γ_R connecting F to its orthogonal factor $R \in \mathrm{SO}(n)$ and compute its length $L(\gamma_R)$. For

$$\gamma_R(t) := R\,\exp((1-t)\,\log U)\,,$$

where $\log U \in \mathrm{Sym}(n)$ is the principal matrix logarithm of U, we find

$$\gamma_R(0) = R\,\exp(\log U) = RU = F \qquad \text{and} \qquad \gamma_R(1) = R\,\exp(0) = R \in \mathrm{SO}(n)\,.$$

It is easy to confirm that γ_R is in fact a geodesic as given in (15) with $\xi = \log U \in \mathrm{Sym}(n)$, thus the length of γ_R is given by $L(\gamma_R) = \|\log U\|_{\mu,\mu_c,\kappa}$. We can thereby establish the *upper bound*

$$\mathrm{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \inf_{Q\in\mathrm{SO}(n)} \mathrm{dist}^2_{\mathrm{geod}}(F,Q) \le \mathrm{dist}^2_{\mathrm{geod}}(F,R) \tag{18}$$

$$\le L^2(\gamma_R) = \|\log U\|^2_{\mu,\mu_c,\kappa} = \mu\,\|\mathrm{dev}_n \log U\|^2 + \frac{\kappa}{2}\,[\mathrm{tr}(\log U)]^2 \tag{19}$$

for the geodesic distance of F to $\mathrm{SO}(n)$.

Our task in the remainder of this section is to show that the right hand side of inequality (19) is *also a lower bound* for the (squared) geodesic strain measure, i.e. that, altogether,

$$\mathrm{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \mu\,\|\mathrm{dev}_n \log U\|^2 + \frac{\kappa}{2}\,[\mathrm{tr}(\log U)]^2\,.$$

However, while the orthogonal polar factor R is the element of best approximation in the Euclidean case (for $\mu = \mu_c = 1$, $\kappa = \frac{2}{n}$) due to Grioli's Theorem, it is not clear whether R is indeed the element in $\mathrm{SO}(n)$ with the shortest geodesic distance to F (and thus whether equality holds in (18)). Furthermore, it is not even immediately obvious that the geodesic distance between F and R is actually given by the right hand side of (19), since a shorter connecting geodesic might exist (and hence inequality might hold in (19)).

Nonetheless, the following fundamental logarithmic minimization property of the orthogonal polar factor, combined with the computations in Section 3.1, allows us to show that (19) is indeed also a lower bound for $\mathrm{dist}_{\mathrm{geod}}(F,\mathrm{SO}(n))$.

Proposition 2. *Let $F = R\sqrt{F^TF}$ be the polar decomposition of $F \in \mathrm{GL}^+(n)$ with $R \in \mathrm{SO}(n)$ and let $\|\,.\,\|$ denote the Frobenius norm on $\mathbb{R}^{n\times n}$. Then*

$$\inf_{Q\in\mathrm{SO}(n)} \|\mathrm{sym}\,\mathrm{Log}(Q^TF)\| = \|\mathrm{sym}\log(R^TF)\| = \|\log\sqrt{F^TF}\|\,,$$

where

$$\inf_{Q\in\mathrm{SO}(n)} \|\mathrm{sym}\,\mathrm{Log}(Q^TF)\| := \inf_{Q\in\mathrm{SO}(n)} \inf\{\|\mathrm{sym}\,X\| \mid X \in \mathbb{R}^{n\times n}\,,\ \exp(X) = Q^TF\}$$

is defined as the infimum of $\|\mathrm{sym}\,.\,\|$ over "all real matrix logarithms" of Q^TF.

Proposition 2, which can be seen as the natural logarithmic analogue of Grioli's Theorem (cf. Section 2.2), was first shown for dimensions $n = 2,3$ by Neff et al. [35] using the so-called sum-of-squared-logarithms inequality [7, 37, 10, 8]. A generalization to all unitarily invariant norms and complex logarithms for arbitrary dimension was given by Lankeit, Neff and Nakatsukasa [25]. We also require the following corollary involving the weighted Frobenius norm, which is not orthogonally invariant.

Corollary 3. *Let*

$$\|X\|^2_{\mu,\mu_c,\kappa} = \mu\,\|\mathrm{dev}_n\,\mathrm{sym}X\|^2 + \mu_c\,\|\mathrm{skew}X\|^2 + \frac{\kappa}{2}\,[\mathrm{tr}(X)]^2\,, \qquad \mu,\mu_c,\kappa > 0\,,$$

for all $X \in \mathbb{R}^{n\times n}$, where $\|\,.\,\|$ is the Frobenius matrix norm. Then

$$\inf_{Q\in\mathrm{SO}(n)} \|\mathrm{sym}\,\mathrm{Log}(Q^T F)\|_{\mu,\mu_c,\kappa} = \|\log\sqrt{F^T F}\|_{\mu,\mu_c,\kappa}\,.$$

We are now ready to prove our main result.

Theorem 4. *Let g be the left-$\mathrm{GL}(n)$-invariant, right-$\mathrm{O}(n)$-invariant Riemannian metric on $\mathrm{GL}(n)$ defined by*

$$g_A(X,Y) = \langle A^{-1}X, A^{-1}Y\rangle_{\mu,\mu_c,\kappa}\,, \qquad \mu,\mu_c,\kappa > 0\,,$$

for $A \in \mathrm{GL}(n)$ and $X,Y \in \mathbb{R}^{n\times n}$, where

$$\langle X,Y\rangle_{\mu,\mu_c,\kappa} = \mu\,\langle \mathrm{dev}_n\,\mathrm{sym}X, \mathrm{dev}_n\,\mathrm{sym}Y\rangle + \mu_c\langle \mathrm{skew}X, \mathrm{skew}Y\rangle + \tfrac{\kappa}{2}\,\mathrm{tr}(X)\,\mathrm{tr}(Y)\,. \tag{20}$$

Then for all $F \in \mathrm{GL}^+(n)$, the geodesic distance of F to the special orthogonal group $\mathrm{SO}(n)$ induced by g is given by

$$\mathrm{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \mu\,\|\mathrm{dev}_n \log U\|^2 + \frac{\kappa}{2}\,[\mathrm{tr}(\log U)]^2\,, \tag{21}$$

where log *is the principal matrix logarithm,* $\mathrm{tr}(X) = \sum_{i=1}^n X_{i,i}$ *denotes the trace and* $\mathrm{dev}_n X = X - \frac{1}{n}\,\mathrm{tr}(X)\cdot\mathbb{1}$ *is the n-dimensional deviatoric part of $X \in \mathbb{R}^{n\times n}$. In particular, the geodesic distance does not depend on the* spin modulus μ_c.

Remark 5. It can also be shown [30] that the orthogonal factor $R \in \mathrm{SO}(n)$ of the polar decomposition $F = RU$ is the unique element of best approximation in $\mathrm{SO}(n)$, i.e. that for $Q \in \mathrm{SO}(n)$, $\mathrm{dist}_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \mathrm{dist}_{\mathrm{geod}}(F,Q)$ if and only if $Q = R$.

Proof (of Theorem 4). Let $F \in \mathrm{GL}^+(n)$ and $\widehat{Q} \in \mathrm{SO}(n)$. Then according to our previous considerations (cf. Section 3.1) there exists $\xi \in \mathfrak{gl}(n)$ with

$$\exp(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi)\,\exp((1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi) = F^{-1}\widehat{Q} \tag{22}$$

and

$$\|\xi\|_{\mu,\mu_c,\kappa} = \operatorname{dist}_{\text{geod}}(F,\widehat{Q})\,. \tag{23}$$

In order to find a lower estimate on $\|\xi\|_{\mu,\mu_c,\kappa}$ (and thus on $\operatorname{dist}_{\text{geod}}(F,\widehat{Q})$), we compute

$$\begin{aligned} & \exp(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi)\,\exp((1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi) = F^{-1}\widehat{Q} \\ \implies\quad & \exp((1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi)^{-1}\,\exp(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi)^{-1} = \widehat{Q}^T F \\ \implies\quad & \exp(-\operatorname{sym}\xi + \tfrac{\mu_c}{\mu}\operatorname{skew}\xi) = \exp(\underbrace{(1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi}_{\in\mathfrak{so}(n)})\,\widehat{Q}^T F\,. \end{aligned}$$

Since $\exp(W)\in\mathrm{SO}(n)$ for all skew symmetric $W\in\mathfrak{so}(n)$, we find

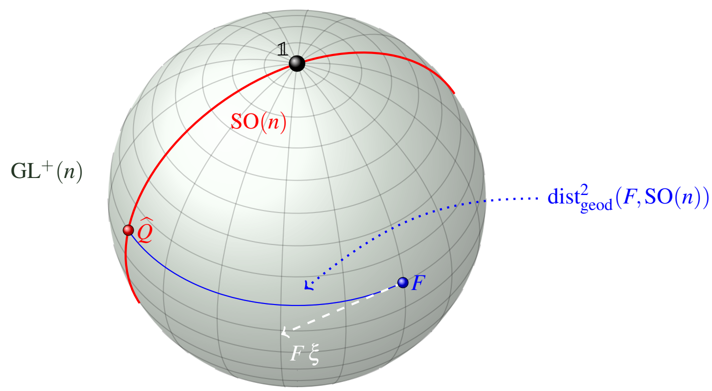

Fig. 3: The geodesic (intrinsic) distance to $\mathrm{SO}(n)$; neither the element $\widehat{Q}$ of best approximation nor the initial tangent $F\,\xi\in T_F\,\mathrm{GL}^+(n)$ of the connecting geodesic is known beforehand.

$$\exp(\underbrace{-\operatorname{sym}\xi + \tfrac{\mu_c}{\mu}\operatorname{skew}\xi}_{=:Y}) = Q_\xi^T F \tag{24}$$

with $Q_\xi = \widehat{Q}\exp(-(1+\frac{\mu_c}{\mu})\operatorname{skew}\xi)\in\mathrm{SO}(n)$; note that $\operatorname{sym}Y = -\operatorname{sym}\xi$. According to (24), $Y = -\operatorname{sym}\xi + \frac{\mu_c}{\mu}\operatorname{skew}\xi$ is "a logarithm"[2] of $Q_\xi^T F$. The weighted Frobenius norm of the symmetric part of $Y = -\operatorname{sym}\xi + \frac{\mu_c}{\mu}\operatorname{skew}\xi$ is therefore bounded below by the infimum of $\|\operatorname{sym}X\|_{\mu,\mu_c,\kappa}$ over "all logarithms" X of $Q_\xi^T F$:

[2] Loosely speaking, we use the term "a logarithm of $A\in\mathrm{GL}^+(n)$" to denote any (real) solution X of the matrix equation $\exp X = A$.

$$
\begin{aligned}
\|\operatorname{sym}\xi\|_{\mu,\mu_c,\kappa} &= \|\operatorname{sym}Y\|_{\mu,\mu_c,\kappa} \\
&\overset{(24)}{\geq} \quad \inf\{\|\operatorname{sym}X\|_{\mu,\mu_c,\kappa} \,|\, X\in\mathbb{R}^{n\times n},\ \exp(X)=Q_\xi^T F\} \\
&\geq \inf_{Q\in\mathrm{SO}(n)} \inf\{\|\operatorname{sym}X\|_{\mu,\mu_c,\kappa} \,|\, X\in\mathbb{R}^{n\times n},\ \exp(X)=Q^T F\} \\
&= \inf_{Q\in\mathrm{SO}(n)} \|\operatorname{sym}\operatorname{Log}(Q^T F)\|_{\mu,\mu_c,\kappa}\,. \qquad (25)
\end{aligned}
$$

We can now apply Corollary 3 to find

$$
\begin{aligned}
\operatorname{dist}^2_{\mathrm{geod}}(F,\widehat{Q}) = \|\xi\|^2_{\mu,\mu_c,\kappa} &= \mu\,\|\operatorname{dev}_n\operatorname{sym}\xi\|^2 + \mu_c\,\|\operatorname{skew}\xi\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}(\operatorname{sym}\xi)]^2 \\
&\geq \mu\,\|\operatorname{dev}_n\operatorname{sym}\xi\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}(\operatorname{sym}\xi)]^2 \qquad (26)\\
&= \|\operatorname{sym}\xi\|^2_{\mu,\mu_c,\kappa} \\
&\overset{(25)}{\geq} \inf_{Q\in\mathrm{SO}(n)} \|\operatorname{sym}\operatorname{Log}(Q^T F)\|^2_{\mu,\mu_c,\kappa} \\
&\overset{\text{Corollary 3}}{=} \quad \mu\,\|\log\sqrt{F^T F}\|^2_{\mu,\mu_c,\kappa} \\
&= \mu\,\|\operatorname{dev}_n\log U\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}(\log U)]^2
\end{aligned}
$$

for $U=\sqrt{F^T F}$. Since this inequality is independent of $\widehat{Q}$ and holds for all $\widehat{Q}\in\mathrm{SO}(n)$, we obtain the desired lower bound

$$
\operatorname{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \inf_{\widehat{Q}\in\mathrm{SO}(n)} \operatorname{dist}^2_{\mathrm{geod}}(F,\widehat{Q}) \geq \mu\,\|\operatorname{dev}_n\log U\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}(\log U)]^2
$$

on the geodesic distance of F to $\mathrm{SO}(n)$. Together with the upper bound already established in (19), we finally find

$$
\operatorname{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \operatorname{dist}^2_{\mathrm{geod}}(F,R) = \mu\,\|\operatorname{dev}_n\log U\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}(\log U)]^2\,. \qquad \square
$$

According to Theorem 4, the squared geodesic distance between F and $\mathrm{SO}(n)$ with respect to any left-$\mathrm{GL}(n)$-invariant, right-$\mathrm{O}(n)$-invariant Riemannian metric on $\mathrm{GL}(n)$ is the *isotropic quadratic Hencky energy*

$$
W_{\mathrm{H}}(F) = \mu\,\|\operatorname{dev}_n\log U\|^2 + \frac{\kappa}{2}\,[\operatorname{tr}(\log U)]^2\,,
$$

where the parameters $\mu,\kappa>0$ represent the shear modulus and the bulk modulus, respectively. The Hencky energy function was introduced in 1929 by H. Hencky [18], who derived it from geometrical considerations as well: his deduction was based on a set of axioms including a law of superposition for the stress response function [29], an approach previously employed by G. F. Becker [4, 34] in 1893 and later followed in a more general context by H. Richter [39], cf. [40, 38, 41].

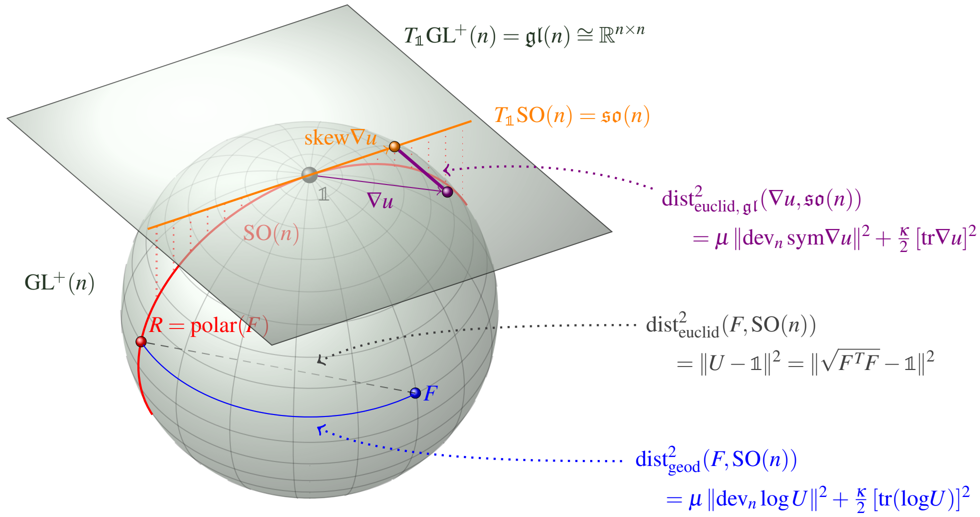

Fig. 4: The isotropic Hencky energy of F measures the geodesic distance between F and $\mathrm{SO}(n)$. The linear Euclidean strain measure is obtained via linearization of the tangent space $\mathfrak{gl}(n)$ at $\mathbb{1}$.

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