On isotropy conditions in second gradient materials

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In gradient elasticity, isotropy and frame-indifference requirements are sensitive to the homogeneity of the applied rotation field \( Q \in \text{SO}(3) \). This is in contrast to standard elasticity, where only first gradients of the deformation are under consideration. We use a diffeomorphism to show the effect of inhomogeneous coordinate transformation to the form-invariance requirement of elastic energy. From a classical geometric rigidity result follows that the appearance of a right-local \( \text{SO}(3) \)-invariance condition is not the general condition for isotropy. The correct statement for isotropy in second gradient elasticity should be a right-global \( \text{SO}(3) \)-invariance condition.

1 Global versus local rotational invariance for isotropy

In hyperelasticity, the difference between form-invariance under compatible transformations of the reference configuration with rigid rotations \( \bar{Q} \) (isotropy) and right-invariance under inhomogeneous rotation fields \( Q = Q(x) \in \text{SO}(3) \) becomes visible only in higher gradient elasticity. To see this, consider coordinates \( x \in \mathbb{R}^3 \) transformed to \( \xi \in \mathbb{R}^3 \) via the diffeomorphism \( \zeta : B \subset \mathbb{R}^3 \mapsto \mathbb{R}^3 \)

\[
\begin{align*}
x &= \zeta(\xi), & \xi &= \zeta^{-1}(x), & x &= \zeta(\zeta^{-1}(x)),
\end{align*}
\]

(1)

see also [1,2]. Connected to the coordinate transformation (1) we consider the deformation expressed in these new coordinates via setting

\[
\varphi^\xi(\xi) := \varphi(\zeta(\xi)), & \quad \varphi^\xi(\zeta^{-1}(x)) = \varphi(x).
\]

(2)

Let the elastic energy of the body \( B \subset \mathbb{R}^3 \) depend on first and second gradients of the deformation \( \varphi(x) \). We say that the elastic energy is form-invariant with respect to the (referential) coordinate transformation \( \zeta \) if and only if

\[
\int_{x \in B} W(\nabla_x[\varphi(x)], \nabla_{\nabla_x}[\nabla_x[\varphi(x)]]) \, dx = \int_{\xi \in \zeta^{-1}(B)} \left( \nabla_{\xi}[\nabla^\xi(\xi)], \nabla_{\zeta}[\nabla^\xi(\xi)] \right) \, d\xi.
\]

(3)

For the first and second derivative with respect to \( x \) we obtain from eq.(1)

\[
I = \nabla_x[\zeta(\zeta^{-1}(x))], & \quad \nabla_x[\zeta^{-1}(x)] = (\nabla_x[\zeta(\xi)])^{-1},
\]

(4)

and

\[
\nabla_x[\zeta(\zeta^{-1}(x))] \nabla_x[\zeta^{-1}(x)] = -\nabla_{\zeta}[\nabla_x[\zeta(\xi)] \nabla_x[\zeta^{-1}(x)] \nabla_x[\zeta^{-1}(x)]],
\]

(5)

yielding

\[
\nabla_x[\zeta(\zeta^{-1}(x))] \nabla_x[\zeta^{-1}(x)] = -(\nabla_{\zeta}[\zeta(\xi)])^{-1} \nabla_{\zeta}[\nabla_x[\zeta(\xi)] \nabla_x[\zeta^{-1}(x)] \nabla_x[\zeta^{-1}(x)]] \nabla_{\zeta}[\zeta(\xi)]^{-1} (\nabla_{\zeta}[\zeta(\xi)])^{-1},
\]

(6)

Thus, (3) is form-invariant with respect to the (referential) coordinate transformation \( \zeta \) if and only if

\[
\int_{\xi \in \zeta^{-1}(B)} W(\nabla_{\xi}[\nabla^\xi(\xi)] (\nabla_{\zeta}[\zeta(\xi)])^{-1}, \nabla_{\zeta}[\nabla_x[\zeta(\xi)] \nabla_x[\zeta^{-1}(x)] \nabla_x[\zeta^{-1}(x)]] \nabla_{\zeta}[\zeta(\xi)]^{-1} (\nabla_{\zeta}[\zeta(\xi)])^{-1}) \]

\[
\det(\nabla_{\zeta}[\zeta(\xi)]) \, d\xi
\]

(7)

\[
\int_{\xi \in \zeta^{-1}(B)} W(\nabla_{\xi}[\nabla^\xi(\xi)], \nabla_{\zeta}[\nabla_x[\zeta(\xi)] \nabla_x[\zeta^{-1}(x)] \nabla_x[\zeta^{-1}(x)]]) \, d\xi.
\]

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Equality (7) can be specified to
\[
\det(\text{Grad}_\xi[\zeta(\xi)]) = 1, \quad \text{Grad}_\xi[\zeta(\xi)] \in \mathcal{G} \subseteq \text{SO}(3) \quad \forall \xi \in \zeta^{-1}(B),
\]
where \( \mathcal{G} \) is the symmetry group of the material. We set \((\text{Grad}_\xi[\zeta(\xi)])^{-1} = Q(\xi)\) and obtain as first concise form-invariance statement for material symmetry
\[
\begin{align*}
\int & W\left(\text{Grad}_\xi[\varphi^b(\xi)] Q(\xi), \text{GRAD}_\xi[\text{Grad}_\xi[\varphi^b(\xi)]] Q(\xi) Q(\xi)\right) \\
& \xi \in \zeta^{-1}(B) \\
& \quad - \text{Grad}_\xi[\varphi^b(\xi)] Q(\xi) \text{GRAD}_\xi[Q^T(\xi)] Q(\xi) Q(\xi) \quad 1 \ d\xi \\
& = \int W\left(\text{Grad}_\xi[\varphi^b(\xi)], \text{GRAD}_\xi[\text{Grad}_\xi[\varphi^b(\xi)]]\right) \ d\xi \quad \forall Q(\xi) \in \mathcal{G},
\end{align*}
\]
(9)
which we will call **right-local SO(3)-invariance** since the rotations in eq.(9) are allowed to be inhomogeneous. However, requiring that
\[
(\text{Grad}_\xi[\zeta(\xi)])^{-1} = Q^T(\xi) \in \text{SO}(3) \iff \text{Grad}_\xi[\zeta(\xi)] = Q(\xi) \in \text{SO}(3)
\]
(10)
means, by a classical geometric rigidity result, see e.g. [3], that
\[
\text{Grad}_\xi[\zeta(\xi)] = Q(\xi) \in \text{SO}(3) \iff Q(\xi) = \overline{Q} = \text{const. and } \zeta(\xi) = \overline{Q} \xi + \overline{b},
\]
(11)
and therefore \(\text{GRAD}_\xi[\text{Grad}_\xi[\zeta(\xi)]] = 0\). Assuming furthermore that \(B\) is a ball of homogeneous material, we have \(\zeta^{-1}(B) = B\), and the correct statement for isotropy, in our view, is then
\[
\begin{align*}
\int & W\left(\text{Grad}_\xi[\varphi^b(\xi)] \overline{Q}^T, \text{GRAD}_\xi[\text{Grad}_\xi[\varphi^b(\xi)]] \overline{Q}^T \overline{Q}^T\right) \ d\xi \\
& \xi \in B \\
& = \int W\left(\text{Grad}_\xi[\varphi^b(\xi)], \text{GRAD}_\xi[\text{Grad}_\xi[\varphi^b(\xi)]]\right) \ d\xi \quad \forall \overline{Q} \in \text{SO}(3).
\end{align*}
\]
(12)
We denominate the latter condition as **right-global SO(3)-invariance**, which, for us, is **isotropy**. We appreciate that the right-local SO(3)-invariance condition (9) is much too restrictive in that arbitrary, inhomogeneous rotation fields are allowed instead of only constant rotations \(\overline{Q}\). The reader should carefully note that we started by using a coordinate transformation \(x = \zeta(\xi)\) and therefore we require in the end that \(\zeta(\xi) = \overline{Q} \xi + \overline{b}\). There is no other coordinate transformation \(\zeta\) such that \(\text{Grad}_\xi[\zeta(\xi)] = Q(\xi) \in \text{SO}(3)\) everywhere, provided a minimum level of smoothness is assumed.

In the local theory the above discussion cannot distinguish between constant or non-constant rotations, since the gradient of the rotation \(Q(\xi)\) is not involved. The latter might explain why one may be inclined to allow non-constant rotation fields in (9), which is forbidden for higher gradient materials [4].

**References**


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