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The Hencky strain energy $\|\log U\|^2$ measures the geodesic distance of the deformation gradient to $\mathrm{SO}(n)$ in the canonical left-invariant Riemannian metric on $\mathrm{GL}(n)$

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The well-known isotropic Hencky strain energy appears naturally as a distance measure of the deformation gradient to the set of rigid rotations in a canonical left-invariant Riemannian metric on the general linear group $\mathrm{GL}(n)$. Objectivity requires the Riemannian metric to be left-$\mathrm{GL}(n)$ invariant, isotropy requires the Riemannian metric to be right-$\mathrm{O}(n)$ invariant. The latter two conditions are satisfied for a three-parameter family of Riemannian metrics on the tangent space of $\mathrm{GL}(n)$. Surprisingly, the final result is basically independent of the chosen parameters.

In deriving the result, geodesics on $\mathrm{GL}(n)$ have to be parametrized and a novel minimization problem, involving the matrix logarithm for non-symmetric arguments, has to be solved.

1 Introduction

We show that the isotropic Hencky strain energy $\mu\|\mathrm{dev}\log U\|^2 + \frac{\kappa}{2}[\mathrm{tr}(\log U)]^2$ for the symmetric Biot-stretch $U = \sqrt{F^T F}$ measures the geodesic distance of the deformation gradient $F \in \mathrm{GL}^+(n)$ to $\mathrm{SO}(n)$ where $\mathrm{GL}(n)$ is viewed as a Riemannian manifold endowed with a left-invariant metric which is also right $\mathrm{O}(n)$-invariant (isotropic), and where the coefficients $\mu,\ \kappa > 0$ correspond to the shear modulus and the bulk modulus, respectively. Thus we provide yet another characterization of the polar-decomposition $F = RU$, $R \in \mathrm{SO}(n)$, $U \in \mathrm{PSym}(n)$, since the stretch U provides also the geodesic distance to $\mathrm{SO}(n)$ in the euclidean metric, i.e.,

$$\mathrm{dist}^2_{\mathrm{euclid}}(F, \mathrm{SO}(n)) := \inf_{Q\in\mathrm{SO}(n)} \mathrm{dist}^2_{\mathrm{euclid}}(F, Q) = \inf_{Q\in\mathrm{SO}(n)} \|F - Q\|^2 = \|F - R\|^2 = \|R^T F - \mathbb{1}\|^2 = \|U - \mathbb{1}\|^2$$

with $\|X\| = \sqrt{\mathrm{tr}(X^T X)}$ denoting the Frobenius matrix norm henceforth. For both the euclidean and the geodesic distance, the orthogonal factor $R = \mathrm{polar}(F)$ in the polar decomposition of F is the nearest rotation to F.

2 Left invariant Riemannian metrics on $\mathrm{GL}(n)$

Viewing $\mathrm{GL}(n)$ as a Riemannian manifold endowed with a left invariant metric

$$g_A : T_A\,\mathrm{GL}(n) \times T_A\,\mathrm{GL}(n) \to \mathbb{R} : \ \ g_A(X, Y) = \left\langle A^{-1}X,\ A^{-1}Y \right\rangle, \quad A \in \mathrm{GL}(n)\,, \tag{1}$$

for a suitable inner product $\langle\cdot,\cdot\rangle$ on the tangent space $T_{\mathbb{1}}\,\mathrm{GL}(n) = \mathfrak{gl}(n) = \mathbb{R}^{n\times n}$ at the identity $\mathbb{1}$, the distance between $F, P \in \mathrm{GL}^+(n)$ can be measured along sufficiently smooth curves. We denote by

$$\mathcal{A} = \{\gamma \in C^0([0,1]; \mathrm{GL}^+(n)) \mid \gamma \text{ piecewise differentiable, } \gamma(0) = F,\ \gamma(1) = P\} \tag{2}$$

the admissible set of curves connecting F and P, and by

$$L(\gamma) = \int_0^1 \sqrt{g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))}\ \mathrm{ds} \tag{3}$$

the length of $\gamma \in \mathcal{A}$. Then the geodesic distance

$$\mathrm{dist}_{\mathrm{geod}}(F, P) = \inf_{\gamma\in\mathcal{A}} L(\gamma) \tag{4}$$

defines a metric on $\mathrm{GL}^+(n)$. While it is generally difficult to explicitly compute this distance or to find length minimizing curves, it can be shown [1, 2] that if the Riemannian metric is defined by an inner product of the form

$$\langle X, Y\rangle = \langle X, Y\rangle_{\mu,\mu_c,\kappa} := \mu\,\langle \mathrm{dev}\,\mathrm{sym}\,X, \mathrm{dev}\,\mathrm{sym}\,Y\rangle_{n\times n} + \mu_c\,\langle \mathrm{skew}\,X, \mathrm{skew}\,Y\rangle_{n\times n} + \tfrac{\kappa}{2}\,\mathrm{tr}\,X\,\mathrm{tr}\,Y\,, \tag{5}$$

$$\|X\|^2_{\mu,\mu_c,\kappa} := \langle X, X\rangle_{\mu,\mu_c,\kappa} = \mu\|\mathrm{dev}\,\mathrm{sym}\,X\|^2 + \mu_c\|\mathrm{skew}\,X\|^2 + \tfrac{\kappa}{2}[\mathrm{tr}\,X]^2\,, \qquad \mu,\ \mu_c,\ \kappa > 0\,,$$

$$\mathrm{dev}\,X := X - \tfrac{1}{n}\,\mathrm{tr}\,X \cdot \mathbb{1}\,, \tag{6}$$

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which is the case if and only if g is right invariant under $\mathrm{O}(n)$ [3], then every geodesic γ connecting F and P is of the form

$$\gamma(t) = F\,\exp(t(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi))\;\exp(t(1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi) \tag{7}$$

for some $\xi \in \mathfrak{gl}(n)$, where $\exp : \mathfrak{gl}(n) \to \mathrm{GL}^+(n)$ denotes the matrix exponential, $\operatorname{sym}\xi = \frac{1}{2}(\xi + \xi^T)$ the symmetric part and $\operatorname{skew}\xi = \frac{1}{2}(\xi - \xi^T)$ the skew symmetric part of ξ.

Now, according to the classical Hopf-Rinow theorem, there always exists a length minimizing geodesic in $\mathcal{A}$. To obtain such a minimizer γ (and thus the distance $\operatorname{dist}_{\mathrm{geod}}(F,P) = L(\gamma)$), it therefore remains to find $\xi \in \mathfrak{gl}(n)$ with

$$P = \gamma(1) = F\,\exp(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi)\;\exp((1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi)\,. \tag{8}$$

The existence of such a ξ is clear from the above.

3 The geodesic distance to $\mathrm{SO}(n)$

Although no closed form solution to (8) is known, the equation can be used to obtain a lower bound [1]

$$\operatorname{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \inf_{Q\in\mathrm{SO}(n)} \operatorname{dist}^2_{\mathrm{geod}}(F,Q) \geq \inf_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(QF)\|^2_{\mu,\mu_c,\kappa} \tag{9}$$

for the distance of $F \in \mathrm{GL}^+(n)$ to $\mathrm{SO}(n)$, as well as an upper bound

$$\begin{aligned}\operatorname{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) &\leq \operatorname{dist}^2_{\mathrm{geod}}(F,\,\operatorname{polar}(F))\\ &\leq \|\log(\operatorname{polar}(F)^T F)\|^2_{\mu,\mu_c,\kappa} \;=\; \mu\|\operatorname{dev}\log(U)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2\,,\end{aligned} \tag{10}$$

where $F = RU$, $R = \operatorname{polar}(F) \in \mathrm{SO}(n)$, $U = \sqrt{F^TF} \in \mathrm{PSym}(n)$ denotes the polar decomposition of F. Finally, we can use an extension of a recent optimality result proved by Neff et al. [4]:

Theorem 3.1 *Let $\|\,.\,\|$ be the Frobenius matrix norm on $\mathfrak{gl}(n)$, $F \in \mathrm{GL}^+(n)$. Then the minimum*

$$\min_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(Q\cdot F)\|^2 = \|\log(\sqrt{F^TF})\|^2 \tag{11}$$

is uniquely attained at $Q = \operatorname{polar}(F)^T$.

A consequence of Theorem 3.1, combined with (9) and (10), yields our main result [5]:

Theorem 3.2 *Let g be a left invariant Riemannian metric on $\mathrm{GL}(n)$ that is also right invariant under $\mathrm{O}(n)$, and let $F \in \mathrm{GL}^+(n)$. Then:*

$$\operatorname{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \operatorname{dist}^2_{\mathrm{geod}}(F,\,\operatorname{polar}(F)) = \mu\|\operatorname{dev}\log(U)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2\,. \tag{12}$$

Furthermore, for $\mu_c = 0$ (in which case $\operatorname{dist}_{\mathrm{geod}}$ defines only a pseudometric on $\mathrm{GL}^+(n)$), Theorem 3.2 still holds.

References

[1] Alexander Mielke. Finite elastoplasticity, Lie groups and geodesics on SL(d). In Paul Newton, Philip Holmes, and Alan Weinstein, editors, *Geometry, Mechanics, and Dynamics*, pages 61–90. Springer New York, 2002.

[2] P. Neff and R. Martin. Minimal geodesics on GL(n) for left invariant Riemannian metrics which are right invariant under O(n). in preparation, 2013.

[3] R. Bryant. Personal communication, 2013. Mathematical Sciences Research Institute, Berkeley.

[4] P. Neff, Y. Nakatsukasa, and A. Fischle. The unitary polar factor $Q = U_p$ minimizes $\|\operatorname{Log}(Q^*Z)\|^2$ and $\|\operatorname{sym}_*\operatorname{Log}(Q^*Z)\|^2$ in the spectral norm in any dimension and the Frobenius matrix norm in three dimensions. *arXiv:1302.3235*, submitted, 2013.

[5] P. Neff, B. Eidel, F. Osterbrink and R. Martin. The isotropic Hencky strain energy $\|\log U\|^2$ measures the geodesic distance of the deformation gradient $F \in \mathrm{GL}^+(n)$ to $SO(n)$ in the unique left-invariant Riemannian metric on $\mathrm{GL}^+(n)$ which is also right $\mathrm{O}(n)$-invariant. in preparation, 2013.

[1] We denote by log the principal matrix logarithm, while the expression Log is used to indicate that the infimum is taken over the whole inverse image under exp, i.e. $\inf_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(QF)\|^2_{\mu,\mu_c,\kappa} = \inf\{\|X\|^2_{\mu,\mu_c,\kappa} :\, X \in \mathfrak{gl}(n),\ \exp(X) = QF\}$.