Z. Angew. Math. Phys.

DOI 10.1007/s00033-015-0495-0

Zeitschrift für angewandte Mathematik und Physik ZAMP

The exponentiated Hencky-logarithmic strain energy. Part II: Coercivity, planar polyconvexity and existence of minimizers

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Abstract. We consider a family of isotropic volumetric–isochoric decoupled strain energies

$$F \mapsto W_{\rm eH}(F) := \widehat{W}_{\rm eH}(U) := \begin{cases} \frac{\mu}{k}\, e^{k\, \|\, {\rm dev}_n \log U\|^2} + \frac{\kappa}{2\hat{k}}\, e^{\hat{k}\, [{\rm tr}(\log U)]^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \le 0, \end{cases}$$

based on the Hencky-logarithmic (true, natural) strain tensor $\log U$, where $\mu > 0$ is the infinitesimal shear modulus, $\kappa = \frac{2\mu+3\lambda}{3} > 0$ is the infinitesimal bulk modulus with λ the first Lamé constant, $k, \hat{k}$ are dimensionless parameters, $F = \nabla\varphi$ is the gradient of deformation, $U = \sqrt{F^T F}$ is the right stretch tensor and ${\rm dev}_n \log U = \log U - \frac{1}{n}{\rm tr}(\log U)\cdot 1\!\!1$ is the deviatoric part (the projection onto the traceless tensors) of the strain tensor $\log U$. For small elastic strains, the energies reduce to first order to the classical quadratic Hencky energy

$$F \mapsto W_{\rm H}(F) := \widehat{W}_{\rm H}(U) := \mu\, \|{\rm dev}_n \log U\|^2 + \frac{\kappa}{2}\, [{\rm tr}(\log U)]^2,$$

which is known to be not rank-one convex. The main result in this paper is that in plane elastostatics the energies of the family $W_{\rm eH}$ are polyconvex for $k \ge \frac{1}{3}$, $\widehat{k} \ge \frac{1}{8}$, extending a previous finding on its rank-one convexity. Our method uses a judicious application of Steigmann's polyconvexity criteria based on the representation of the energy in terms of the principal invariants of the stretch tensor U. These energies also satisfy suitable growth and coercivity conditions. We formulate the equilibrium equations, and we prove the existence of minimizers by the direct methods of the calculus of variations.

Mathematics Subject Classification. 74B20 · 74G64 · 35Q74 · 35E10.

Keywords. Finite isotropic elasticity · Logarithmic strain · Polyconvexity · Existence of minimizers · Plane elastostatics · Coercivity.

Contents

Published online: 04 February 2015

Birkhäuser

1. Introduction

1.1. Motivation

In the first part of a series of papers [32], we have introduced a nonlinear elastic energy based on certain invariants of the Hencky tensor $\log U$, namely $\|\,\mathrm{dev}_n \log U\|^2$ and $(\mathrm{tr}(\log U))^2$, where $F = \nabla\varphi$ is the gradient of deformation, $U = \sqrt{F^T F}$ is the right stretch tensor, $\log U$ is the referential (Lagrangian) logarithmic strain tensor, $\mathrm{dev}_n X = X - \frac{1}{n}\,\mathrm{tr}(X)\cdot \mathbb{1}$ is the deviatoric part (the projection onto the traceless tensors) of the second-order tensor $X \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ is the Frobenius tensor norm (see Sect. 1.4 for other notations). We have shown that this exponentiated energy expression improves several features of the formulation with respect to mathematical issues regarding well posedness. In this paper, we will discuss the polyconvexity for this family. In order to set the stage, let us briefly recapitulate some useful details. The considered exponentiated Hencky-logarithmic strain type energies are

$$
\begin{aligned}
W_{\mathrm{eH}}(F) := \widehat{W}_{\mathrm{eH}}(U) :&= \begin{cases} \underbrace{\dfrac{\mu}{k}\, e^{k\,\|\,\mathrm{dev}_n \log\, U\|^2} + \dfrac{\kappa}{2\,\widehat{k}}\, e^{\widehat{k}\,(\mathrm{tr}(\log U))^2}}_{\text{volumetric-isochoric split}} & \text{if} \quad \det\, F > 0, \\ +\infty & \text{if} \quad \det F \leq 0, \end{cases} \\
&= \begin{cases} \dfrac{\mu}{k}\, e^{k\,\|\,\log \frac{U}{\det U^{1/n}}\|^2} + \dfrac{\kappa}{2\,\widehat{k}}\, e^{\widehat{k}\,(\log \det U)^2} & \text{if} \quad \det\, F > 0, \\ +\infty & \text{if} \quad \det F \leq 0, \end{cases}
\end{aligned}
\tag{1.1}
$$

where $\mu > 0$ is the shear (distortional) modulus, $\kappa = \frac{2\mu+3\lambda}{3} > 0$ is the bulk modulus with λ the first Lamé constant and $k, \widehat{k}$ are dimensionless parameters. The immediate importance of the family (1.1) of free-energy functions is seen by looking at small (but not infinitesimally small) strains. Then, the exponentiated Hencky energy $W_{\mathrm{eH}}(\cdot)$ reduces to first order to the classical quadratic Hencky energy $\widehat{W}_{\mathrm{H}}(U)$ based on the logarithmic strain tensor $\log U$:

$$
W_{\mathrm{H}}(F) := \widehat{W}_{\mathrm{H}}(U) := \mu\,\|\mathrm{dev}_n \log U\|^2 + \frac{\kappa}{2}\,[\mathrm{tr}(\log U)]^2. \tag{1.2}
$$

Our renewed interest in the Hencky energy is motivated by a recent finding that the Hencky energy (not the logarithmic strain itself) exhibits a fundamental property. By purely differential geometric reasoning, in forthcoming papers [29,30,33] (see also [5,23]), it will be shown that

$$
\begin{aligned}
\operatorname{dist}^2_{\mathrm{geod}}\left((\det F)^{1/n}\cdot \mathbb{1}, \mathrm{SO}(n)\right) &= \operatorname{dist}^2_{\mathrm{geod},\mathbb{R}_+\cdot\mathbb{1}}\left((\det F)^{1/n}\cdot \mathbb{1}, \mathbb{1}\right) = |\log\det F|^2,\\
\operatorname{dist}^2_{\mathrm{geod}}\left(\frac{F}{(\det F)^{1/n}}, \mathrm{SO}(n)\right) &= \operatorname{dist}^2_{\mathrm{geod},\mathrm{SL(n)}}\left(\frac{F}{(\det F)^{1/n}}, \mathrm{SO}(n)\right) = \|\operatorname{dev}_n\log U\|^2, \qquad (1.3)
\end{aligned}
$$

where $\operatorname{dist}_{\mathrm{geod}}$ is the canonical left invariant geodesic distance on the Lie group $\mathrm{GL}^+(n)$ and $\operatorname{dist}_{\mathrm{geod},\mathrm{SL(n)}}$ and $\operatorname{dist}_{\mathrm{geod},\mathbb{R}_+\cdot\mathbb{1}}$ denote the corresponding geodesic distances on the Lie groups $\mathrm{SL}(n)$ and $\mathbb{R}_+\cdot\mathbb{1}$, respectively (see [30,33]).

In the first part [32], we have summarized the well-known unique features of the quadratic Hencky-strain energy W_{H} based exclusively on the natural strain tensor $\log U$. The Hencky model is definitely one of the most widely used strain energies in the small elastic strain regime [7,8,18–22]. In [32], however, we also pointed out that the quadratic Hencky energy has some serious shortcomings. For example, the quadratic Hencky energy is neither rank-one convex nor does it satisfy any suitable coercivity condition. These points being more or less well known, it is clear that there cannot exist a general mathematical well-posedness result for the quadratic Hencky model W_{H}. Of course, in the vicinity of the identity, an existence proof for small loads based on the implicit function theorem will always be possible. All in all, the status of Hencky's quadratic energy is put into doubt. This state of affairs, on the one hand the preferred use of the quadratic Hencky energy and its fundamental property (1.3) and on the other hand its mathematical shortcomings, motivated our search for a modification of Hencky's energy. Our best candidate for now is W_{eH} defined by (1.1). Up to moderate strains, for principal stretches $\lambda_i \in (0.7, 1.4)$, our new exponentiated Hencky formulation (1.1) is de facto as good as the quadratic Hencky model W_{H}, and in the large strain region, it improves several important features from a mathematical point of view. Moreover, some other properties (see [32]) such as uniqueness in the hydrostatic loading problem [10,34] confirm the status of the exponentiated Hencky formulation as a useful energy in plane elastostatics and give a new perspective in three dimensions. The main features that have been shown in [32] are that the exponentiated Hencky energy (1.1) satisfies the Legendre–Hadamard condition (rank-one convexity) in planar elastostatics, i.e., for $n = 2$. In this paper, we aim to complete this investigation by showing that the planar elastostatic formulation is, in fact, polyconvex and satisfies a coercivity estimate which allows us to show the existence of minimizers. Unfortunately, some aspects of the three-dimensional description remain open, since the formulation is not globally rank-one convex.

1.2. Polyconvexity

A very useful constitutive requirement is Ball's fundamental polyconvexity condition [1,2]. A free-energy function $W(F)$ is called polyconvex if and only if it is expressible in the form $W(F) = P(F, \operatorname{Cof} F, \det F)$, $P : \mathbb{R}^{19} \to \mathbb{R}$, where $P(\cdot,\cdot,\cdot)$ is convex. Polyconvexity implies weak lower semicontinuity, quasiconvexity and rank-one convexity, and it implies that the homogeneous solution $\varphi(x) = \overline{F}.\,x,\ x \in \mathbb{R}^3$, is always an energy minimizer to its own Dirichlet boundary conditions.

In fact, polyconvexity is the cornerstone notion for a proof of the existence of minimizers by the direct methods of the calculus of variations for energy functions satisfying no polynomial growth conditions, which is the case in nonlinear elasticity since one has the natural requirement $W(F) \to \infty$ as $\det F \to 0$. Polyconvexity is best understood for isotropic energy functions, but it is not restricted to isotropic response. The polyconvexity condition in the case of space dimension 2 was conclusively discussed by Rosakis [36] and Šilhavý [41–47], while the case of arbitrary spatial dimension was studied by Mielke [27]. The n-dimensional case of the theorem established by Ball [2, page 367] has been reconsidered by Dacorogna and Marcellini [14], Dacorogna and Koshigoe [13] and Dacorogna and Marechal [15]. It was a long-standing open question how to extend the notion of polyconvexity in a meaningful way to anisotropic materials [3]. An answer has been provided in a series of papers [4,17,28,38–40,40].

1.3. Approach of this paper

The main result in this paper is that in plane elastostatics the family of energies $W_{\rm eH}$ given by (1.1) is polyconvex for a suitable choice of parameters $k, \hat{k}$ (Theorem 3.11), satisfies q-growth coercivity for any $1 \leq q < \infty$, (Theorem 4.9) and therefore allows for a complete existence theory (Theorem 5.1). This also confirms the status of the quadratic Hencky energy as a useful approximation in plane elastostatics. Moreover, our family (1.1) of energies admits a unique, stress-free reference configuration $\mathbb{1}$; thus, $\varphi(x) = x$ is the global minimizer for natural boundary conditions in any dimension.

The sufficiency condition for polyconvexity which we use has been discovered by Steigmann [49,50]. Eventually, it is based on a polyconvexity criterion of Ball [2], but it allows one to express polyconvexity directly in terms of the principal isotropic invariants of the right stretch tensor U, namely $i_1 = \operatorname{tr} U, i_2 = \det U$ (see also [6,16,24–26]). As it turns out, in plane elastostatics, Steigmann's criterion is already hidden in another sufficiency criterion for polyconvexity given earlier by Rosakis [37]. However, Steigmann's criterion is clearly not necessary for polyconvexity (see Sect. 2.2).

1.4. Notation

Let us begin with the remark, that although this article is mainly concerned with the planar (two-dimensional) case, we give some of the preliminaries in their more general three-dimensional version. For $a, b \in \mathbb{R}^n$, we let $\langle a, b\rangle_{\mathbb{R}^n}$ denote the scalar product on $\mathbb{R}^n$ with the associated vector norm $\|a\|^2_{\mathbb{R}^n} = \langle a, a\rangle_{\mathbb{R}^n}$. We denote by $\mathbb{R}^{n\times n}$ the set of real $n \times n$ second-order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{R}^{n\times n}$ is given by $\langle X, Y\rangle_{\mathbb{R}^{n\times n}} = \operatorname{tr}(XY^T)$, and thus, the Frobenius tensor norm is $\|X\|^2 = \langle X, X\rangle_{\mathbb{R}^{n\times n}}$. In the following, we do not adopt any summing convention and we omit the subscript $\mathbb{R}^n, \mathbb{R}^{n\times n}$. The identity tensor on $\mathbb{R}^{n\times n}$ will be denoted by $\mathbb{1}$, so that $\operatorname{tr}(X) = \langle X, \mathbb{1}\rangle$. We let $\operatorname{Sym}(n)$ and PSym(n) denote the sets of symmetric and positive definite symmetric tensors, respectively, and adopt the usual abbreviations of Lie-group theory, i.e., $\operatorname{GL}(n) := \{X \in \mathbb{R}^{n\times n} \,|\, \det X \neq 0\}$ is the general linear group, $\operatorname{SL}(n) := \{X \in \operatorname{GL}(n) \,|\, \det X = 1\}, \operatorname{GL}^+(n) := \{X \in \mathbb{R}^{n\times n} \,|\, \det X > 0\}$ is the group of invertible matrices with positive determinant. The superscript ^{T} is used to denote transposition, and $\operatorname{Cof} A = (\det A)A^{-T}$ is the cofactor of $A \in \operatorname{GL}^+(n)$. The set of positive real numbers is denoted by $\mathbb{R}_+ := (0, \infty)$, while $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let us consider $W(F)$ to be the strain energy density function of an elastic material in which F is the deformation gradient from a reference configuration to a configuration in Euclidean n-space; $W(F)$ is measured per unit volume of the reference configuration. The domain of $W(\cdot)$ is $\operatorname{GL}^+(n)$. We denote by $C = F^TF$ the right Cauchy–Green strain tensor, by $B = F F^T$ the left Cauchy–Green (or Finger) strain tensor, by U the right stretch tensor, i.e., the unique element of $\operatorname{PSym}(n)$ for which $U^2 = C$, and by V the left stretch tensor, i.e., the unique element of $\operatorname{PSym}(n)$ for which $V^2 = B$. Here, we are only concerned with rotationally symmetric energy functions (objective and isotropic), i.e., $W(F) = \widehat{W}(Q_1^T\, F\, Q_2)$ for all $F = RU = VR \in \operatorname{GL}^+(n)$, $Q_1, Q_2, R \in \operatorname{SO}(n)$. For vectors $v = (v_1, v_2, v_3)^T \in \mathbb{R}^3$, we define $\operatorname{diag} v = \begin{pmatrix} v_1 & 0 & 0\\ 0 & v_2 & 0\\ 0 & 0 & v_3\end{pmatrix}$, while for a matrix $F = \begin{pmatrix} F_{11} & F_{12} & F_{13}\\ F_{21} & F_{22} & F_{23}\\ F_{31} & F_{32} & F_{33}\end{pmatrix} \in \mathbb{R}^{3\times 3}$ we let $\operatorname{vect} F = (F_{11}, F_{12}, F_{13}, F_{21}, F_{22}, F_{23}, F_{31}, F_{32}, F_{33})^T \in \mathbb{R}^9$.

If the components of the $\mathbb{R}^3$-valued vector field $v = (v_1, v_2, v_3)^T$ are differentiable in the distributional sense, we define

$$\nabla v = \left(\operatorname{grad}^T v_1, \operatorname{grad}^T v_2, \operatorname{grad}^T v_3\right)^T, \tag{1.4}$$

while for a weakly differentiable scalar function $(x_1, x_2, x_3) \mapsto f(x_1, x_2, x_3) \in \mathbb{R}$ the gradient is the column vector

$$\nabla f := \operatorname{grad} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)^T \in \mathbb{R}^3. \tag{1.5}$$

In three dimensions, we consider the singular values (principal stretches) λ_1, λ_2, λ_3 of F, i.e., the eigenvalues of U, and the principal isotropic invariants of U

$$\begin{aligned} i_1 &= \lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}(U), \\ i_2 &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = \operatorname{tr}(\operatorname{Cof} U), \\ i_3 &= \lambda_1\lambda_2\lambda_3 = \det U. \end{aligned} \tag{1.6}$$

Every isotropic and frame-invariant function of F is thus expressible in the form

$$\begin{aligned} W(F) = \widehat{W}(U) = g(\lambda_1, \lambda_2, \lambda_3) = \psi(i_1, i_2, i_3) = \Phi(\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1\lambda_2\lambda_3) \\ = P(F, \operatorname{Cof} F, \det F). \end{aligned}$$

The functions $\widehat{W}, g, \psi$ are uniquely determined by W, while Φ and P are not unique.

We denote by $D^2_\lambda g$ the Hessian matrix of g with respect to the variables $(\lambda_1, \lambda_2, \lambda_3)$, while by $D^2_i \psi$ we denote the Hessian matrix of ψ with respect to the principal invariants (i_1, i_2, i_3). We also consider the third-order tensor $\mathbb{D}^2_\lambda i = (D^2_\lambda i_1 | D^2_\lambda i_2 | D^2_\lambda i_3)$, where $D^2_\lambda i_1$, $D^2_\lambda i_2$ and $D^2_\lambda i_3$ denote the Hessian matrices of i_1, i_2 and i_3 with respect to λ.

2. Preliminary results

2.1. The sum of squared logarithms inequality

In this paper, we also use the sum of squared logarithms inequality recently demonstrated in [5]:

Theorem 2.1. (The sum of squared logarithms inequality in 3D [5])
Let $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3 \in \mathbb{R}_+$ be such that

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &\le \mu_1 + \mu_2 + \mu_3 \,, \\ \lambda_1\,\lambda_2 + \lambda_1\,\lambda_3 + \lambda_2\,\lambda_3 &\le \mu_1\,\mu_2 + \mu_1\,\mu_3 + \mu_2\,\mu_3, \\ \lambda_1\,\lambda_2\,\lambda_3 &= \mu_1\,\mu_2\,\mu_3 \,. \end{aligned} \tag{2.1}$$

Then, the following inequality holds:

$$\log^2 \lambda_1 + \log^2 \lambda_2 + \log^2 \lambda_3 \le \log^2 \mu_1 + \log^2 \mu_2 + \log^2 \mu_3. \tag{2.2}$$

Theorem 2.2. (The sum of squared logarithms inequality in 2D [5]) *Let $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}_+$ be such that $\lambda_1 + \lambda_2 \le \mu_1 + \mu_2,\ \lambda_1\,\lambda_2 = \mu_1\,\mu_2$. Then, the following inequality holds:* $\log^2 \lambda_1 + \log^2 \lambda_2 \le \log^2 \mu_1 + \log^2 \mu_2$.

For the general n-dimensional case, we consider the elementary symmetric polynomials

$$e_k(X_1, X_2, \ldots, X_n) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} X_{j_1} X_{j_2} \ldots X_{j_k}, \qquad k = 1, \ldots, n$$

and we give the conjecture:

Conjecture 2.3. (The sum of squared logarithms inequality in $\mathbb{R}^n_+$, $n \in \mathbb{N}$) *Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_n \in \mathbb{R}_+$ be such that*

$$\begin{aligned} e_k(\lambda_1, \lambda_2, \ldots, \lambda_n) &\le e_k(\mu_1, \mu_2, \ldots, \mu_n), \quad k = 1, \ldots, n-1, \\ e_n(\lambda_1, \lambda_2, \ldots, \lambda_n) &= e_n(\mu_1, \mu_2, \ldots, \mu_n). \end{aligned}$$

Then, the following inequality holds

$$\sum_{k=1}^{n} \log^2 \lambda_k \leq \sum_{k=1}^{n} \log^2 \mu_k.$$

In the next section, we outline the polyconvexity criterion established by Steigmann [50] in terms of the principal invariants (i_1, i_2, i_3) of the right stretch tensor U. Using Steigmann's criterion and the criterion given by Lemma 2.10, we are able to prove the polyconvexity of the exponentiated Hencky energy in plane finite elastostatics.

2.2. Sufficiency criteria for polyconvex strain energies

A function $W(F)$ is polyconvex if and only if it is expressible in the form $W(F) = P(F, \operatorname{Cof} F, \det F)$, where $P(\cdot,\cdot,\cdot)$ is convex. The notion of polyconvexity has been introduced into the framework of elasticity by John Ball in his seminal paper [2]. Various nonlinear issues, results and extensive references are collected in Dacorogna [11]. In general, a function $\Phi(\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1\lambda_2\lambda_3)$ is polyconvex if it is convex, symmetric and monotone increasing (separately) in its first 6 arguments. However, it is known that the monotonicity in the first 6 arguments is not necessary [27]. Since there is no easy way to represent the energy in terms of $(F, \operatorname{Cof} F, \det F)$, we take the detour of the invariant representation. From [50], we have the following result based on the interesting observation that the invariants $i_1 = \operatorname{tr}(U)$, $i_2 = \operatorname{tr}(\operatorname{Cof} U)$, $i_3 = \det U$ are convex with respect to F, $\operatorname{Cof} F$ and $\det F$, respectively (see [50], page 485).

Proposition 2.4. (Steigman's polyconvexity criterion in 3D) *Suppose that*

i) $\psi(i_1, i_2, i_3)$ *is a convex function of* (i_1, i_2, i_3) *jointly,*[1] *and*
ii) $\psi(i_1, i_2, i_3)$ *is a nondecreasing function*[2] *of* i_1 *and* i_2*, separately.*

Then, $W(F) = \psi(i_1, i_2, i_3)$ *is polyconvex.*

In planar elasticity, $U \in \mathbb{R}^{2\times 2}$ and the relevant isotropic principal invariants are

$$i_1 = \lambda_1 + \lambda_2 = \operatorname{tr}(U), \qquad i_2 = \lambda_1\lambda_2 = \det U. \tag{2.3}$$

We have to remark that i_2 from (2.3) does not coincide with i_2 from the three-dimensional case. However, it can be understood from the context which expression for i_2 is used. For planar elasticity, we have the corresponding result [50]:

Proposition 2.5. (Steigman's polyconvexity criterion in 2D) *Suppose that*

i) $\psi(i_1, i_2)$ *is a convex function of* (i_1, i_2) *jointly,*[3] *and*
ii) $\psi(i_1, i_2)$ *is a nondecreasing function of* i_1*.*

Then, $W(F) = \psi(i_1, i_2)$ *is polyconvex.*

[1] The domain in which $\psi(i_1, i_2, i_3)$ is defined is the domain for which $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, i.e., the equation $\lambda^3 - i_1\lambda^2 + i_2\lambda - i_3 = 0$ has three positive real solutions. But this domain is not convex. Therefore, it would be more adequate to say that $\psi(i_1, i_2, i_3)$ is convex in the sense of Busemann, Ewald and Shephard's definition [9], i.e., ψ can be extended to a convex function defined on the convex hull of its domain of definition.

[2] If ψ is differentiable, then this condition means that $\partial_{i_1}\psi(i_1, i_2, i_3) \geq 0, \partial_{i_2}\psi(i_1, i_2, i_3) \geq 0$ for all $(i_1, i_2, i_3) \in \mathbb{R}^3$ for which the equation $\lambda^3 - i_1\lambda^2 + i_2\lambda - i_3 = 0$ has three positive real solutions.

[3] The domain in which $\psi(i_1, i_2)$ is defined is the domain $D(i_1, i_2)$ defined in (2.7), for which $\lambda_1, \lambda_2 > 0$, which is not a convex set. Again, a more appropriate notion of convexity for the function $\psi(i_1, i_2)$ on $D(i_1, i_2)$ is that of Busemann, Ewald and Shephard [9], i.e., that ψ is the restriction to $D(i_1, i_2)$ of a real-valued convex function (in the usual sense) defined on the convex hull of $D(i_1, i_2)$ or, equivalently, that the function ψ can be extended to a convex function defined on the convex hull $CoD(i_1, i_2) = \mathbb{R}^2_+$ of $D(i_1, i_2)$.

Templet and Steigmann's recent claim [51] that these conditions are also necessary for polyconvexity can be easily misinterpreted. Below we present some counterexamples to this point. In fact, formula (41) in [51] does not take care of the possibility that, e.g., the dependence of Φ on F does not have to be transmitted by i_1 alone. For the 3D case, Steigmann showed that the above criterion may be applied to the energy

$$\begin{aligned} W(F) &= a^+(i_1-3)+b^+(i_2-3)+h(i_3) = a^+\langle U-\mathbb{1},\mathbb{1}\rangle + b^+\langle \operatorname{Cof} U-\mathbb{1},\mathbb{1}\rangle + h(\det F) \\ &= a^+(\lambda_1+\lambda_2+\lambda_3)+b^+(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)+h(\det F)-3(a^++b^+), \end{aligned} \tag{2.4}$$

where h is a convex function and $a^+, b^+>0$. The polyconvexity of this energy can also be deduced from a direct application of Ball's theorem [2].

Steigmann's polyconvexity criterion in the planar case [50] is already contained in the paper by Rosakis and Simpson [37] for the choice of the entry parameter $\alpha=-1$. Indeed, Rosakis and Simpson gave sufficient conditions for polyconvexity of $W(\cdot)$ having the form $W(F)=\widetilde{W}(\operatorname{tr}(F^TF),\det F) = \widetilde{W}(\|F\|^2,\det F)$. In the notation of Rosakis and Simpson $I:=\operatorname{tr}(F^TF)=\|F\|^2$, $J:=\det F$, $\mathcal{A}_\alpha=\{(\xi,\eta): \xi\geq 0, \eta\geq 0,\ \xi^2\geq 2\,(1-\alpha)\eta\}$. Let us give the correlations with our notations. Rosakis and Simpson defined the function $\xi:\operatorname{GL}(2)\to\mathbb{R}$ by $\xi_\alpha(F)=\sqrt{\|F\|^2-2\,\alpha\,\det F}, \quad F\in\operatorname{GL}(2)$, and proved (see Lemma 3.1 in [37]) that the function ξ_α is convex[4] for $\alpha\in[-1,1]$. Moreover, they pointed out that $F\in\operatorname{GL}^+(2)\Leftrightarrow(\xi_\alpha,J)\in\mathcal{A}_\alpha$. Let us remark that $J=i_2$ in general and for $\alpha=-1$ the domain $\mathcal{A}_\alpha$ is the domain $\overline{D(i_1,i_2)}$, where $D(i_1,i_2)$ is the domain considered in our further analysis, see (2.7), and $\xi_{-1}=i_1$. The convex hull of $\mathcal{A}_\alpha$ is $\mathbb{R}^2_+$ for $-1\leq\alpha<1$, while for $\alpha=1$, $\mathcal{A}_1=[0,\infty)\times\mathbb{R}_+$ is convex and is exactly the domain considered in our extension (see (3.19)).

Proposition 2.6. (Rosakis and Simpson's early polyconvexity criterion in 2D [37]) *Let $W:\operatorname{GL}^+(2)\mapsto\mathbb{R}$ be isotropic. For each $\alpha\in[-1,1]$ define $\Phi_\alpha(\xi,J)=\widetilde{W}(\xi^2+2\,\alpha\,J,J)$, $(\xi,J)\in\mathcal{A}_\alpha$, and suppose that for some $\alpha\in[-1,1]$,*

i) *Φ_α is convex by extension*[5] *to $\mathcal{A}_1=[0,\infty)\times\mathbb{R}_+$,*
ii) *$\Phi_\alpha(\cdot,J)$ is nondecreasing on $[0,\infty)$ for each $J>0$.*

Then, $W(\cdot)$ is polyconvex.

Rosakis and Simpson [37] already stated that the conditions of the above proposition are not necessary for polyconvexity of isotropic functions. They illustrated this with an example due to Dacorogna et al. [12]. For a complete view, we give this example in the following. The considered function is $W:\operatorname{GL}^+(2)\to\mathbb{R}$ given by $W(F)=\|F\|^4-2(\det F)^2=I^2-2\,J^2, (I,J)\in\mathcal{D}$, where $\mathcal{D}=\{(I,J): I\geq 2\,|J|, J\in\mathbb{R}\}$. Then, $W:\mathbb{R}^{2\times 2}\to\mathbb{R}$, $W(F)=I^2-2J^2$ is convex [12], and hence, its restriction to $\operatorname{GL}^+(2)$ is polyconvex. In this case, we have $\Phi_\alpha(\xi,J)=(\xi^2+2\,\alpha\,J)^2-2\,J^2,\ \forall\,(\xi,J)\in\mathcal{A}_\alpha$. The Hessian matrix of Φ_α fails to be positive semi-definite on $\mathcal{A}_\alpha$ for all $\alpha\in[-1,1]$. Hence, the conditions of Rosakis and Simpson are not necessary. In our notation, we have $W(F)=\langle C,\mathbb{1}\rangle^2-2(\det F)^2=\langle F^TF,\mathbb{1}\rangle^2-2(\det F)^2=\|F\|^4-2(\det F)^2$, which is in fact convex in $F\in\mathbb{R}^{2\times 2}$, while in terms of principal invariants the function $W(F)=\psi(i_1,i_2)= (i_1^2-2\,i_2)^2-2\,i_2^2=i_1^4-4\,i_1^2\,i_2+2\,i_2^2$, does not have a positive semi-definite Hessian on $\mathcal{A}_{-1}=D(i_1,i_2)$.

Another counterexample for this phenomenon, but in the three-dimensional case, is given by the mapping $F\mapsto\|\operatorname{Cof}F\|^2=\|\operatorname{Cof}U\|^2$, which is (obviously) polyconvex (since it is convex in $\operatorname{Cof}F$). This function is rotationally invariant. The eigenvalues of $\operatorname{Cof}U$ are $\lambda_2\lambda_3,\lambda_1\lambda_3,\lambda_1\lambda_2$ and hence $\|\operatorname{Cof}U\|^2=$

[4]First, in [37], it is proved that for $M\in\operatorname{PSym}(n)$, the function $\varphi:\mathbb{R}^n\to\mathbb{R}$, defined by $\varphi(x)=\sqrt{\langle x,M.x\rangle},\ x\in\mathbb{R}^n$ is convex. For $F\in\operatorname{GL}(2)$, we have $\|F\|^2\geq 2\,|\det F|$ because $\|F\|^2-2\,|\det F|=(F_{11}+F_{22})^2+(F_{12}-F_{21})^2$. Hence, the expression $\|F\|^2-2\alpha\det F$ under the radix is, for $\alpha\in[-1,1]$, a quadratic, positive semi-definite function of $F\in\operatorname{GL}(2)$. By means of an isomorphism $F\mapsto\operatorname{vec}(F):=(F_{11},F_{12},F_{21},F_{22})\in\mathbb{R}^4$, the function ξ_α can be expressed as a function of the form $\varphi_\alpha:\mathbb{R}^4\to\mathbb{R}$, defined by $\varphi_\alpha(\operatorname{vec}(F))=\sqrt{\langle\operatorname{vec}(F),M_\alpha.\operatorname{vec}(F)\rangle},\ F\in\operatorname{GL}(2)$, where $M_\alpha=B_\alpha^T\,B_\alpha\in\operatorname{PSym}(4)$ is a positive definite matrix. Thus, $\varphi_\alpha(\operatorname{vec}(F))=\|B_\alpha\operatorname{vec}(F)\|,\ F\in\operatorname{GL}(2)$ and therefore $F\mapsto\varphi_\alpha(\operatorname{vec}(F))$ is convex.

[5]The function Φ_α is well defined in $\mathcal{A}_\alpha$ which is not equal to $\mathcal{A}_1=[0,\infty)\times\mathbb{R}^2_+$ in general.

$(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)^2-2(\lambda_1+\lambda_2+\lambda_3)(\lambda_1\lambda_2\lambda_3)=i_2^2-2\,i_1\,i_3$. The corresponding unique representation function $\varphi(i_1,i_2,i_3)=i_2^2-2i_1i_3$ is a nonconvex function in (i_1,i_2,i_3) and not even convex in a neighborhood of $\mathbb{1}$, i.e., of $(i_1,i_2,i_3)=(3,3,1)$. Although $\|\operatorname{Cof}F\|^2$ is a function of $\operatorname{Cof}F$ only, its representation as a function of the principal invariants of U also contains i_1 and i_3. Furthermore, the resulting function is neither convex in (i_1,i_2,i_3) nor increasing in i_1.

We give in the following some immediate consequence of the Theorems 2.4 and 2.5.

Remark 2.7. If some function ψ satisfies the hypotheses of Theorems 2.4 or 2.5, then e^{ψ} satisfies them as well.

Proof. The proof follows from the monotonicity and convexity of the exponential function. □

Remark 2.8. If ψ fails the hypothesis ii) from Theorems 2.4 or 2.5, then so does e^{ψ}.

Proof. The function log is monotone. Hence, if e^{ψ} were monotone in i_1 (or i_2), $\log(e^{\psi})=\psi$ would be monotone in i_1 (or i_2). □

Note that, however, the convexity condition (i) from Theorems 2.4 or 2.5 can be improved by the exponential function. Polyconvexity is compatible with exponentiating:

Remark 2.9. If a function W is polyconvex, then so is e^{W}.

Proof. According to the definition, W is polyconvex if and only if $W(F)=P(F,\operatorname{Cof}F,\det F)$, where P is convex. But if P is convex, then e^{P} is also convex (the exponential function is convex and monotone), and hence, e^{W} is polyconvex [17,38]. □

2.3. Plane elastostatics

In planar elasticity, the relevant isotropic principal invariants are defined by (2.3). Note again that the meaning of the isotropic invariants of U, namely i_1,i_2, depends on the dimension. Every isotropic and frame-invariant function of $F\in\mathrm{GL}^+(2)$ is expressible in the form

$$W(F)=\widehat{W}(U)=g(\lambda_1,\lambda_2)=\psi(i_1,i_2)=\Phi(\lambda_1,\lambda_2,\lambda_1\lambda_2)=P(F,\det F),\tag{2.5}$$

where the functions $\widehat{W},g,\psi$ are uniquely determined by W, while Φ and P are not unique.

Given the eigenvalues $\lambda_1,\lambda_2\in\mathbb{R}_+$, we can always compute the invariants $i_1,i_2\in\mathbb{R}$. But for given $i_1,i_2\in\mathbb{R}_+$, we cannot always say that the equation $\lambda^2-i_1\,\lambda+i_2=0$ has two different positive solutions λ_1,λ_2, which is the case if and only if $i_1^2-4\,i_2>0$. Our intention is to make the map $(\lambda_1,\lambda_2)\mapsto(i_1,i_2)$ a one-to-one function. For this reason, we define the function

$$i=(i_1,i_2)^T:D(\lambda_1,\lambda_2)\to D(i_1,i_2),\tag{2.6}$$

which maps (λ_1,λ_2) into (i_1,i_2), where (see Fig. 1a, b)

$$D(\lambda_1,\lambda_2)=\{(\lambda_1,\lambda_2)\in\mathbb{R}_+^2:\ \lambda_1>\lambda_2\},\quad D(i_1,i_2)=\{(i_1,i_2)\in\mathbb{R}_+^2:\ i_1^2-4\,i_2>0\}.\tag{2.7}$$

We can also define the function $i(\cdot)$ on the curve $\gamma_1:\ \lambda_1=t,\ \lambda_2=t,\ t\in(0,\infty)$. In this way, $i(\cdot)$ maps the curve γ_1 into the curve $\gamma_2: i_1=t,\ i_2=\frac{t^2}{4},\ t\in(0,\infty)$. The function $i(\cdot)$ is also a one-to-one function on this curve, but it is a C^2-diffeomorphism on the open domain $D(\lambda_1,\lambda_2)$, away from the curve γ_1. Hence, for now we consider the restriction of the function g from (2.5) to the domain $D(\lambda_1,\lambda_2)$, denoted in the following also by g. According to (2.3) and (3.1), the energy $W(F)$ can also be written in terms of (i_1,i_2), i.e., there is $\psi:D(i_1,i_2)\to\mathbb{R}$ such that $\psi=g\circ i^{-1}$. We suppose that the function g is a C^2-function on its domain of definition. Moreover, the function $i=(i_1,i_2)$ is a C^2-diffeomorphism. Using the above notations, the chain rule and according to (1.4) and (1.5), we write

$$\nabla_\lambda g=(\nabla_\lambda i)^T\nabla_i\psi.\tag{2.8}$$

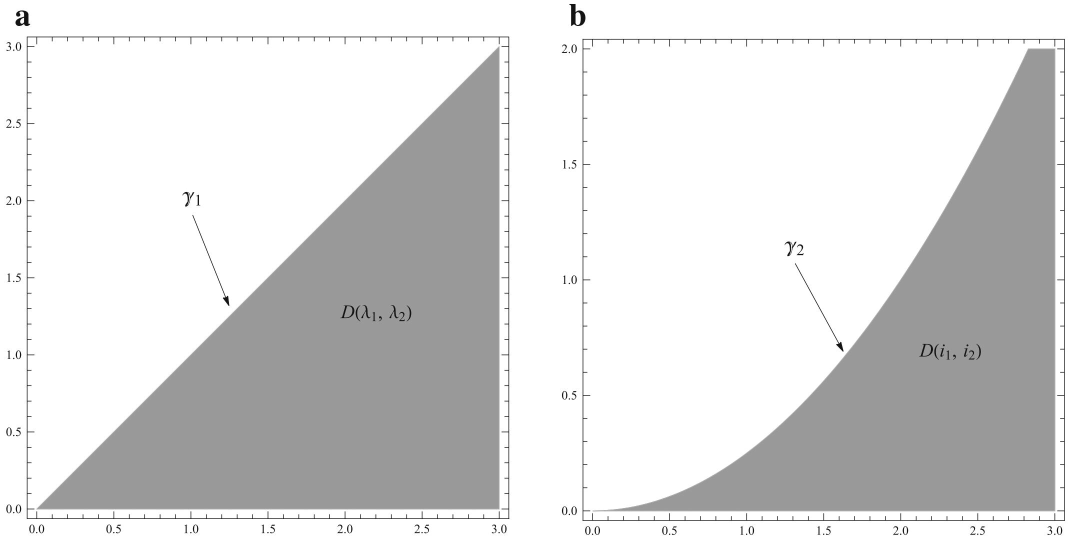

FIG. 1. **a** Domain $D(\lambda_1, \lambda_2)$ of the singular values λ_1, λ_2. **b** Domain $D(i_1, i_2)$ of the admissible values of the isotropic principal invariants i_1, i_2

We know that for $(\lambda_1, \lambda_2) \in D(\lambda_1, \lambda_2)$ the matrix $\nabla_\lambda i = \begin{pmatrix} 1 & 1 \\ \lambda_2 & \lambda_1 \end{pmatrix}$ is invertible and

$$(\nabla_\lambda i)^{-T} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & -\lambda_2 \\ -1 & 1 \end{pmatrix}. \tag{2.9}$$

In 2D, the Hessian matrix of g with respect to the variables (λ_1, λ_2) and the Hessian matrix of ψ with respect to the variables (i_1, i_2) are

$$D^2_\lambda g = \begin{pmatrix} \frac{\partial^2 g}{\partial \lambda_1^2} & \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} \\ \frac{\partial^2 g}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 g}{\partial \lambda_2^2} \end{pmatrix}, \qquad D^2_i \psi = \begin{pmatrix} \frac{\partial^2 \psi}{\partial i_1^2} & \frac{\partial^2 \psi}{\partial i_1 \partial i_2} \\ \frac{\partial^2 \psi}{\partial i_1 \partial i_2} & \frac{\partial^2 \psi}{\partial i_2^2} \end{pmatrix}, \tag{2.10}$$

while $\mathbb{D}^2_\lambda i = (D^2_\lambda i_1 | D^2_\lambda i_2)$. We recall that for a third-order tensor $\mathbb{G}$, we have $\mathbb{G}.Y \in \mathbb{R}^{2\times 2}$ for all $Y \in \mathbb{R}^2$, where $(\mathbb{G}.Y)_{ij} = \sum\limits_{k=1}^{2} \mathbb{G}_{kij} Y_k$. Thus, we can write $\mathbb{D}^2_\lambda i . \nabla_i \psi = \sum_{j=1}^2 \frac{\partial \psi}{\partial i_j} D^2_\lambda i_j$. Using the above notations, we can rewrite the relation

$$\frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} = \frac{\partial^2 \psi}{\partial i_1^2} \frac{\partial i_1}{\partial \lambda_j} \frac{\partial i_1}{\partial \lambda_i} + \frac{\partial^2 \psi}{\partial i_1 \partial i_2} \frac{\partial i_2}{\partial \lambda_j} \frac{\partial i_1}{\partial \lambda_i} + \frac{\partial^2 \psi}{\partial i_1 \partial i_2} \frac{\partial i_2}{\partial \lambda_i} \frac{\partial i_1}{\partial \lambda_j} + \frac{\partial^2 \psi}{\partial i_2^2} \frac{\partial i_2}{\partial \lambda_j} \frac{\partial i_2}{\partial \lambda_i} + \frac{\partial \psi}{\partial i_1} \frac{\partial^2 i_1}{\partial \lambda_j \partial \lambda_i} + \frac{\partial \psi}{\partial i_2} \frac{\partial^2 i_2}{\partial \lambda_j \partial \lambda_i}$$

in the form $D^2_\lambda g = (\nabla_\lambda i)^T D^2_i \psi \, (\nabla_\lambda i) + \mathbb{D}^2_\lambda i . \nabla_i \psi$. Moreover, using (2.8) and the fact that $i : D(\lambda_1, \lambda_2) \to D(i_1, i_2)$ is a C^2-diffeomorphism, i.e., $\det \nabla_\lambda i \neq 0$, we deduce $D^2_\lambda g = (\nabla_\lambda i)^T D^2_i \psi \, (\nabla_\lambda i) + \mathbb{D}^2_\lambda i . [(\nabla_\lambda i)^{-T} \nabla_\lambda g]$, and further $(\nabla_\lambda i)^T D^2_i \psi \, (\nabla_\lambda i) = D^2_\lambda g - \mathbb{D}^2_\lambda i . [(\nabla_\lambda i)^{-T} \nabla_\lambda g]$. In view of the relation

$$\langle (\nabla_\lambda i)^T D^2_i \psi \, (\nabla_\lambda i) \xi, \xi \rangle = \langle D^2_i \psi \, (\nabla_\lambda i) \xi, (\nabla_\lambda i) \xi \rangle, \quad \forall \, \xi \in \mathbb{R}^2, \tag{2.11}$$

it is clear that

$$\begin{aligned} &D^2_i \psi \;\text{ is positive definite in }\; (i_1, i_2) \in D(i_1, i_2) \\ &\qquad \Leftrightarrow D^2_\lambda g - \mathbb{D}^2_\lambda i . [(\nabla_\lambda i)^{-T} \nabla_\lambda g] \;\text{ is positive definite in }\; (\lambda_1, \lambda_2) \in D(\lambda_1, \lambda_2). \end{aligned} \tag{2.12}$$

Hence, we can conclude:

Lemma 2.10. *Let $i=(i_1,i_2)^T: D(\lambda_1,\lambda_2)\subset\mathbb{R}^2\to D(i_1,i_2)\ \subset\mathbb{R}^2$ be the C^2-diffeomorphism defined by (2.6), $\psi: D(i_1,i_2)\to\mathbb{R}$ and $g: D(\lambda_1,\lambda_2)\to\mathbb{R}$ functions of class C^2 on their domain of definition, such that $g(\lambda_1,\lambda_2):=(\psi\circ i)(\lambda_1,\lambda_2)$. Then, $D_i^2\psi$ is positive definite in $D(i_1,i_2)$ (as a function of (i_1,i_2)) if and only if $D_\lambda^2 g-\mathbb{D}_\lambda^2 i.[(\nabla_\lambda i)^{-T}\nabla_\lambda g]$ is positive definite in $D(\lambda_1,\lambda_2)$.*

It is clear that the above lemma holds true in general, for all C^2-diffeomorphisms $i=(i_1,i_2)^T: D(\lambda_1,\lambda_2)\subset\mathbb{R}^2\to D(i_1,i_2)\ \subset\mathbb{R}^2$.

3. Polyconvexity of the exponentiated Hencky energy in plane elastostatics

3.1. Polyconvexity of the isochoric exponentiated Hencky energy in plane elastostatics

In this section, we consider a variant of the exponentiated Hencky energy in plane strain, with isochoric part

$$W_{\text{iso}}(F)=e^{k\,\|\operatorname{dev}_2\log U\|^2}=e^{k\,\|\log\frac{U}{\det U^{1/2}}\|^2}. \tag{3.1}$$

Let us remark again that for small strains the exponentiated Hencky energy reduces to the well-known quadratic Hencky energy:

$$\begin{aligned}W_{\text{eH}}(F)-\left(\frac{\mu}{k}+\frac{\kappa}{2\widehat{k}}\right)&=\underbrace{\frac{\mu}{k}\,e^{k\,\|\operatorname{dev}_n\log U\|^2}+\frac{\kappa}{2\widehat{k}}\,e^{\widehat{k}\,[\operatorname{tr}(\log U)]^2}}_{\text{fully nonlinear elasticity}}-\left(\frac{\mu}{k}+\frac{\kappa}{2\widehat{k}}\right)\\&=\underbrace{\mu\,\|\operatorname{dev}_n\log U\|^2+\frac{\kappa}{2}\,[(\log\det U)]^2}_{\text{materially linear, geometrically nonlinear elasticity}}\quad+\text{h.o.t.}\\&=\underbrace{\mu\,\|\operatorname{dev}_n\operatorname{sym}\nabla u\|^2+\frac{\kappa}{2}\,[\operatorname{tr}(\operatorname{sym}\nabla u)]^2}_{\text{linear elasticity}}+\text{h.o.t.},\end{aligned} \tag{3.2}$$

where $u:\mathbb{R}^n\to\mathbb{R}^n$ is the displacement, $F=\nabla\varphi=\mathbb{1}+\nabla u$ is the gradient of deformation $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ and h.o.t. denotes terms of higher order of $\|\operatorname{dev}_n\log U\|^2$ and $\frac{\kappa}{2}\,[(\log\det U)]^2$.

Coming back to the 2D case, as W is an objective, isotropic tensor function, we can express it as a function of the singular values of F, that is the eigenvalues λ_1,λ_2 of $U=\sqrt{F^TF}$, or of the principal invariants $i_1=\lambda_1+\lambda_2$, $i_2=\lambda_1\lambda_2$, i.e., $W(F)=P(F,\det F)=g(\lambda_1,\lambda_2)=\psi(i_1,i_2)$. For polyconvexity, the representation of the function $W(F)$ in terms of $P(F,\det F)$ is not unique, see (3.1). However, the representations $W(F)=g(\lambda_1,\lambda_2)=\psi(i_1,i_2)$ are unique. This fact is implied by the following lemma:

Lemma 3.1. *Let $k\in\mathbb{R}$ and the matrix $F\in\mathrm{GL}^+(2)$ with singular values λ_1,λ_2. Then,*

$$W(F)=e^{k\,\|\operatorname{dev}_2\log U\|^2}=e^{k\,\|\log\frac{U}{\det U^{1/2}}\|^2}=g(\lambda_1,\lambda_2),\quad\textit{where } g:\mathbb{R}_+^2\to\mathbb{R},\quad g(\lambda_1,\lambda_2):=e^{\frac{k}{2}\left(\log\frac{\lambda_1}{\lambda_2}\right)^2}. \tag{3.3}$$

Proof. The matrix U is positive definite and symmetric and therefore can be assumed, by the spectral representation, to be diagonal, to obtain

$$\begin{aligned}\|\operatorname{dev}_2\log U\|^2&=\|\log U-\frac{1}{2}(\log\lambda_1+\log\lambda_2)\mathbb{1}\|^2=\left\|\begin{pmatrix}\frac{1}{2}\log\lambda_1-\frac{1}{2}\log\lambda_2 & 0\\ 0 & \frac{1}{2}\log\lambda_2-\frac{1}{2}\log\lambda_1\end{pmatrix}\right\|^2\\&=\frac{1}{4}\left[2\,(\log\lambda_1-\log\lambda_2)^2\right]=\frac{1}{2}\left(\log\frac{\lambda_1}{\lambda_2}\right)^2.\end{aligned}$$

□

Remark 3.2. (Nonconvexity of g) Note that the function $g:\mathbb{R}^2_+\to\mathbb{R}$ defined in (3.3) is not convex. We have for the Hessian

$$D^2_\lambda g = k\, e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}\begin{pmatrix}\frac{k\log^2\frac{\lambda_1}{\lambda_2}}{\lambda_1^2}-\frac{\log\frac{\lambda_1}{\lambda_2}}{\lambda_1^2}+\frac{1}{\lambda_1^2} & -\frac{k\log^2\frac{\lambda_1}{\lambda_2}}{\lambda_1\lambda_2}-\frac{1}{\lambda_1\lambda_2}\\ -\frac{k\log^2\frac{\lambda_1}{\lambda_2}}{\lambda_1\lambda_2}-\frac{1}{\lambda_1\lambda_2} & \frac{k\log^2\frac{\lambda_1}{\lambda_2}}{\lambda_2^2}+\frac{\log\frac{\lambda_1}{\lambda_2}}{\lambda_2^2}+\frac{1}{\lambda_2^2}\end{pmatrix} \tag{3.4}$$

and, for all $k\in\mathbb{R}$, $\det D^2_\lambda g = -\frac{k^2\log^2\frac{\lambda_1}{\lambda_2}\, e^{k\log^2\frac{\lambda_1}{\lambda_2}}}{\lambda_1^2\lambda_2^2}\le 0$, for all $\lambda_1,\lambda_2>0$. The function g is, therefore, not a convex function in λ_1,λ_2. By a general theorem [2, Theorem 5.1], this implies that $e^{k\,\|\mathrm{dev}_2\log U\|^2}$ is not convex as a function of $U\in\mathrm{Psym}(2)$. Thus, the nonconvexity of g allows us to conclude that W cannot be a convex function of F [16].

In the following, we can assume without loss of generality (by the symmetry of $\log^2\frac{\lambda_1}{\lambda_2}$ under inversion), that $\lambda_1\ge\lambda_2$.

Proposition 3.3. *The map*

$$\psi: D(i_1,i_2)\to\mathbb{R}_+,\quad \psi(i_1,i_2)=e^{\frac{k}{2}\log^2\frac{i_1+\sqrt{i_1^2-4i_2}}{i_1-\sqrt{i_1^2-4i_2}}} \tag{3.5}$$

has a positive definite Hessian matrix $D^2\psi$ in the domain $D(i_1,i_2)$, as a function of (i_1,i_2), if and only if $k\ge\dfrac{1}{3}$.

Proof. To prove this result, we will use the criterion given by Lemma 2.10. Let us remark that

$$\mathbb{D}^2_\lambda i.[(\nabla_\lambda i)^{-T}\nabla_\lambda g] = -k\, e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}\left(\log\frac{\lambda_1}{\lambda_2}\right)\frac{1}{(\lambda_1-\lambda_2)}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix} \tag{3.6}$$

and

$$\det\left(\mathbb{D}^2_\lambda i.[(\nabla_\lambda i)^{-T}\nabla_\lambda g]\right) = k^2 e^{k\log^2\frac{\lambda_1}{\lambda_2}}\left(\log^2\frac{\lambda_1}{\lambda_2}\right)\frac{1}{(\lambda_1-\lambda_2)^2}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)^2\ge 0. \tag{3.7}$$

In order to justify the above relations, we outline the following calculations:

$$D^2_\lambda i_1=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad D^2_\lambda i_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \nabla_\lambda g = k\, e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}\log\frac{\lambda_1}{\lambda_2}\begin{pmatrix}\frac{1}{\lambda_1}\\ -\frac{1}{\lambda_2}\end{pmatrix}, \tag{3.8}$$

$$\nabla_\lambda i=\begin{pmatrix}1&1\\ \lambda_2&\lambda_1\end{pmatrix},\quad (\nabla_\lambda i)^{-T}=\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix}\lambda_1&-\lambda_2\\-1&1\end{pmatrix},$$

$$(\nabla_\lambda i)^{-T}\nabla_\lambda g = k\, e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}\left(\log\frac{\lambda_1}{\lambda_2}\right)\frac{1}{(\lambda_1-\lambda_2)}\begin{pmatrix}2\\ -\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\end{pmatrix}.$$

In view of (3.4) and (3.6), we have

$$\begin{aligned}&D^2_\lambda g-\mathbb{D}^2_\lambda i.[(\nabla_\lambda i)^{-T}\nabla_\lambda g] \\ &= k\, e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}\begin{pmatrix}\frac{k\log^2\frac{\lambda_1}{\lambda_2}-\log\frac{\lambda_1}{\lambda_2}+1}{\lambda_1^2} & -\frac{k(\lambda_1-\lambda_2)\log^2\frac{\lambda_1}{\lambda_2}-(\lambda_1+\lambda_2)\log\frac{\lambda_1}{\lambda_2}+\lambda_1-\lambda_2}{\lambda_1(\lambda_1-\lambda_2)\lambda_2}\\ -\frac{k(\lambda_1-\lambda_2)\log^2\frac{\lambda_1}{\lambda_2}-(\lambda_1+\lambda_2)\log\frac{\lambda_1}{\lambda_2}+\lambda_1-\lambda_2}{\lambda_1(\lambda_1-\lambda_2)\lambda_2} & \frac{k\log^2\frac{\lambda_1}{\lambda_2}+\log\frac{\lambda_1}{\lambda_2}+1}{\lambda_2^2}\end{pmatrix}.\end{aligned} \tag{3.9}$$

First, let us study the sign of the (1,1)-entry $\widetilde{g}(\lambda_1,\lambda_2):=e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}\frac{1}{\lambda_1^2}\left[k\log^2\frac{\lambda_1}{\lambda_2}-\log\frac{\lambda_1}{\lambda_2}+1\right]$ of the above matrix, which is related to the Hessian matrix of $\psi(i_1,i_2)$. We introduce the function $r:[0,\infty)\to\mathbb{R}$ given by $r(t)=k\,t^2-t+1$. It is clear that if $k>\dfrac{1}{4}$, then $r(t)=k\,t^2-t+1>\left(\frac{1}{2}t-1\right)^2\ge 0$, for all $t\in\mathbb{R}$.

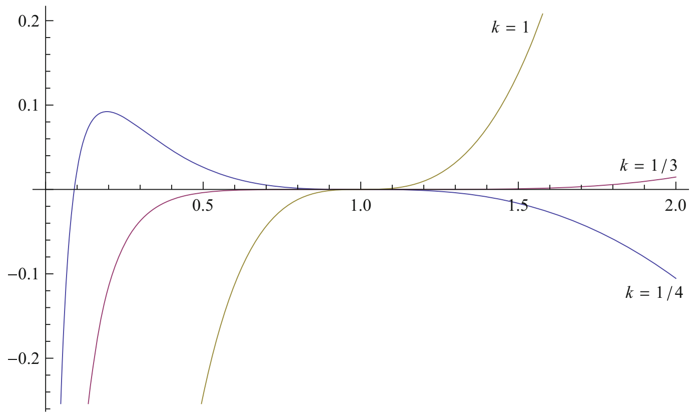

FIG. 2. $\widehat{r}(t)$ for different values of k

Moreover, if $r(t)>0$ for all $t\in[0,\infty)$, then $k>\frac{1}{4}=\max\limits_{t\in[0,\infty)}\left\{\frac{t-1}{t^2}\right\}$. Thus, $r(t)>0$ for all $t\in[0,\infty)$ if and only if[6] $k>\dfrac{1}{4}$. In consequence, we deduce

$$\widetilde{g}(\lambda_1,\lambda_2)=k\,e^{\frac{k}{2}\log^2\frac{\lambda_1}{\lambda_2}}\frac{1}{\lambda_1^2}\,r\left(\log\frac{\lambda_1}{\lambda_2}\right)>0\quad\text{for all } \lambda_1\geq\lambda_2\in\mathbb{R}^+\quad\text{if and only if}\quad k>\frac{1}{4}. \tag{3.10}$$

On the other hand,

$$\begin{aligned}&\det[D^2_\lambda g-\mathbb{D}^2_\lambda i.[(\nabla_\lambda i)^{-T}\nabla_\lambda g]] \\ &\quad=\frac{2k^2e^{k\log^2\frac{\lambda_1}{\lambda_2}}}{\lambda_1^2\lambda_2^2(\lambda_1-\lambda_2)^2}\left(\log\frac{\lambda_1}{\lambda_2}\right)\left[k\big(\lambda_1^2-\lambda_2^2\big)\log^2\frac{\lambda_1}{\lambda_2}-\big(\lambda_1^2+\lambda_2^2\big)\log\frac{\lambda_1}{\lambda_2}+\big(\lambda_1^2-\lambda_2^2\big)\right].\end{aligned} \tag{3.11}$$

Hence,

$$\begin{aligned}&\det[D^2_\lambda g-\mathbb{D}^2_\lambda i.[(\nabla_\lambda i)^{-T}\nabla_\lambda g]]>0\quad\forall\,(\lambda_1,\lambda_2)\in D(\lambda_1,\lambda_2) \\ &\quad\Leftrightarrow\left(\log\frac{\lambda_1}{\lambda_2}\right)\left[k\big(\lambda_1^2-\lambda_2^2\big)\log^2\frac{\lambda_1}{\lambda_2}-\big(\lambda_1^2+\lambda_2^2\big)\log\frac{\lambda_1}{\lambda_2}+\big(\lambda_1^2-\lambda_2^2\big)\right]>0\quad\forall\,(\lambda_1,\lambda_2)\in D(\lambda_1,\lambda_2).\end{aligned} \tag{3.12}$$

In the following, we will prove that for all $\lambda_1>\lambda_2>0$, $\left(\log\frac{\lambda_1}{\lambda_2}\right)\lambda_2^2\,\widehat{r}\left(\frac{\lambda_1}{\lambda_2}\right)>0$, where the function $\widehat{r}:(0,\infty)\to\mathbb{R}$ is defined by $\widehat{r}(t):=k\,\big(t^2-1\big)\log^2 t-\big(t^2+1\big)\log t+\big(t^2-1\big)$. To this aim, we prove that $\widehat{r}(t)>0$, for all $t\in(1,\infty)$ if and only if $k\geq\dfrac{1}{3}$.

The first derivative of $\widehat{r}$ is given by $\widehat{r}'(t)=\frac{k}{2}\left(4\,t\,\log^2 t+4\,t\,\log t-\frac{4\,\log t}{t}\right)+t-\frac{1}{t}-2\,t\,\log t$, the second derivative by $\widehat{r}''(t)=\frac{k}{2}\left(\frac{4\,(t^2-1)}{t^2}+4\,\left(\frac{1}{t^2}+3\right)\log t+4\,\log^2 t\right)-\frac{t^2-1}{t^2}-2\,\log t$, and $\widehat{r}'''(t)=2\,\frac{(3\,k-1)\left(t^2+1\right)+2\,k\,\left(t^2-1\right)\log t}{t^3}$. We also have that $\widehat{r}'(1)=0$ and $\widehat{r}''(1)=0$. It is easy to see that if $k\geq\frac{1}{3}$, then $(3\,k-1)\,\big(t^2+1\big)+2\,k\,\big(t^2-1\big)\log t>0$, for all $t\in(1,\infty)$.

This means that $\widehat{r}''(\cdot)$ is a monotone increasing function, which implies $\widehat{r}''(t)>\widehat{r}''(1)=0$ if $t>1$. This implies that $\widehat{r}'(\cdot)$ is monotone increasing on $(1,\infty)$, i.e., $\widehat{r}'(t)>\widehat{r}(1)=0$ if $t>1$. Hence, $\widehat{r}'(t)>0$ for all $t\in(1,\infty)$, i.e., $\widehat{r}$ is monotone increasing. In conclusion, if $k\geq\dfrac{1}{3}$, then $\widehat{r}$ is monotone increasing

[6] In fact, $k\,t^2-t+1>0$ for all $t\in\mathbb{R}$ if and only if $k>\dfrac{1}{4}$.

and convex on $(1,\infty)$, and $\widehat{r}'(1)=0=\widehat{r}(1)$ (see Fig. 2). Hence, we have proved that $\widehat{r}(t)>0$ for all $t\in(1,\infty)$ if $k\geq\dfrac{1}{3}$. In fact $\widehat{r}'''(t)\geq 0$ for all $t\in(0,\infty)$ if and only if

$$k\geq\frac{1}{3}=\sup_{t\in(1,\infty)}\left\{\frac{t^2+1}{3\,t^2+2\,t^2\log t-2\,\log t+3}\right\}. \tag{3.13}$$

This completes the proof. □

It is possible to have a direct proof of the positive definiteness of the Hessian matrix $D^2\psi$ in the domain $D(i_1,i_2)$, but this direct method leads to complicated calculations in the three-dimensional case.

Remark 3.4. Assuming $\lambda_1>\lambda_2$, with the help of the substitution $t=\dfrac{\lambda_1}{\lambda_2}$ and the choice $k=2$ in (3.3), we obtain the function $s:(1,\infty)\to\mathbb{R}$, $s(t)=e^{(\log t)^2}$. The function $s(\cdot)$ is convex and monotone increasing in t, for $t\in(1,\infty)$. However,

i) $(\lambda_1,\lambda_2)\mapsto t=\dfrac{\lambda_1}{\lambda_2}$ is not convex as a function of (λ_1,λ_2);

ii) $(i_1,i_2)\mapsto t=\dfrac{\lambda_1}{\lambda_2}=\dfrac{i_1+\sqrt{i_1^2-4i_2}}{i_1-\sqrt{i_1^2-4i_2}}$ is not convex as a function of the two invariants (i_1,i_2).

It seems, therefore, that the conclusion of convexity of the map ψ defined by (3.5) cannot simply be inferred from the composition of a convex mapping with the convex and nondecreasing mapping $s:(1,\infty)\to\mathbb{R}$, $s(t)=e^{(\log t)^2}$.

In the following, we prove that the function ψ considered in Proposition 3.3 is convex on $D(i_1,i_2)$ in the sense of Busemann, Ewald and Shephard's definition [9], i.e., ψ is the restriction to $D(i_1,i_2)$ of a real-valued convex function (in the usual sense) defined on the convex hull of $D(i_1,i_2)$; equivalently, the function ψ can be extended to a convex function defined on the convex hull $CoD(i_1,i_2)=\mathbb{R}^2_+$ of $D(i_1,i_2)$ [37]. From [9], we have:

Theorem 3.5. (Busemann, Ewald and Shephard [9, page 6]) *A function ϕ defined on an arbitrary set $M\subset\mathbb{R}^n$ is convex if and only if it is bounded linearly below and the inequality*

$$\phi(x)\leq\sum_{i=1}^{r}\mu_i\,\phi(x_i),\quad 1\leq r<\infty \tag{3.14}$$

holds for all $x_1,x_2,\dots,x_r\in M$, $\sum_{i=1}^r\mu_i=1$ and $x=\sum_{i=1}^r\mu_i x_i$ lying in M. The convex extension of ϕ to the convex hull of M is $\widehat{\phi}(x)=\inf\{\sum_{i=1}^r\mu_i\,\phi(x_i)\ :\ x=\sum_{i=1}^r\mu_i x_i,\ \sum_{i=1}^r\mu_i=1,\ 1\leq r<\infty\}$.

As we already mentioned in the previous section, according to the definition (2.3), we can extend the function $i=(i_1,i_2)$ on the curve γ_2 keeping its one-to-one property. In the following, we denote by $\widetilde{i}(\cdot,\cdot)$ the extension of $i(\cdot,\cdot)$ to the domain $D(i_1,i_2)\cup\gamma_2$. In fact we can extend the function $i=(i_1,i_2)$, preserving the definition (2.3), in all $\mathbb{R}^2_+$, which is the convex hull of $D(i_1,i_2)$, but it does not remain a one-to-one function, and also we do not have a mechanical interpretation for this choice. However, we intend to construct an energy function $\widehat{\psi}:\mathbb{R}^2_+\to\mathbb{R}^2_+$ which is convex in all $\mathbb{R}^2_+$, the convex hull of $D(i_1,i_2)$, using the above results.

First, we extend the function ψ to $D(i_1,i_2)\cup\gamma_2$ by

$$\widetilde{\psi}(i_1,i_2)=\begin{cases}\psi(i_1,i_2) & \text{if}\quad (i_1,i_2)\in D(i_1,i_2),\\ 1 & \text{if}\quad (i_1,i_2)\in\gamma_2.\end{cases} \tag{3.15}$$

The function $\widetilde{\psi}$ preserves the continuity property of ψ. One can see this fact more clearly, by using that

$$i^{-1}(i_1,i_2)=\left(\frac{i_1+\sqrt{i_1^2-4i_2}}{2},\frac{i_1-\sqrt{i_1^2-4i_2}}{2}\right)\in D(\lambda_1,\lambda_2)\cup\gamma_1,\quad \text{for all}\quad (i_1,i_2)\in D(i_1,i_2)\cup\gamma_2 \tag{3.16}$$

and the definition of $g(\cdot,\cdot)$. Hence, we have

$$\widetilde{\psi}(i_1,i_2)=e^{\frac{k}{2}\log^2\frac{i_1+\sqrt{i_1^2-4i_2}}{i_1-\sqrt{i_1^2-4i_2}}}\quad \text{for all}\quad (i_1,i_2)\in D(i_1,i_2)\cup\gamma_2 \tag{3.17}$$

and the continuity of $\widetilde{\psi}(\cdot,\cdot)$ follows.

The function $\widetilde{\psi}(\cdot,\cdot)$ satisfies the condition (3.14) from Theorem 3.5 if and only if $k\geq\dfrac{1}{3}$ because for these values of k it is convex in every convex open domain $\omega\subset D(i_1,i_2)\cup\gamma_2$ and it is a continuous function. It is bounded linearly from below by 0. On the other hand, from the definition of $\widetilde{\psi}$, we have

$$\min\left\{\sum_{i=1}^r\mu_i\widetilde{\psi}(x_i)\ :\ x=\sum_{i=1}^r\mu_i x_i,\ \sum_{i=1}^r\mu_i=1,\ 1\leq r<\infty\right\}=1. \tag{3.18}$$

Hence, we conclude:

Proposition 3.6. *The elastic energy $\widehat{\psi}:[0,\infty)\times\mathbb{R}_+\to\mathbb{R}_+$ defined by*

$$\widehat{\psi}(i_1,i_2)=\begin{cases} e^{\frac{k}{2}\log^2\frac{i_1+\sqrt{i_1^2-4i_2}}{i_1-\sqrt{i_1^2-4i_2}}} & \text{if}\quad (i_1,i_2)\in D(i_1,i_2)\cup\gamma_2,\\ 1 & \text{if}\quad (i_1,i_2)\in([0,\infty)\times\mathbb{R}_+)\backslash(D(i_1,i_2)\cup\gamma_2)\end{cases} \tag{3.19}$$

is convex if and only if $k\geq\dfrac{1}{3}$.

Using the sum of squared logarithms inequality given by Propositions 2.1 and 2.2, we obtain the conditions ii) from Propositions 2.4 and 2.5. Therefore, we get:

Proposition 3.7. (The exponentiated sum of squared logarithms inequality and monotonicity)
The function $F\mapsto e^{k\,\|\,\mathrm{dev}_n\log U\|^2}$, $k\geq 0$ is separately monotone in i_1,i_2 for $n=3$ and monotone in i_1 for $n=2$.

Proof. In this proof, we will restrict ourselves to the case $n=3$ and show slightly more than separate monotonicity. To this aim, let $i_1=\lambda_1+\lambda_2+\lambda_3$, $i_2=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3$, $i_3=\lambda_1\lambda_2\lambda_3$ and $\widehat{\lambda}_1,\widehat{\lambda}_2,\widehat{\lambda}_3$ analogously corresponding to $\widehat{i}_1$, $\widehat{i}_2$, $\widehat{i}_3$ be given in such a way that $i_1\leq\widehat{i}_1$, $i_2\leq\widehat{i}_2$ and $i_3=\widehat{i}_3$. Then, these inequalities coincide with those from (2.1) and Theorem 2.1 and monotonicity of the exponential function yield $e^{k(\log^2\lambda_1+\log^2\lambda_2+\log^2\lambda_3)}\leq e^{k(\log^2\widehat{\lambda}_1+\log^2\widehat{\lambda}_2+\log^2\widehat{\lambda}_3)}$. The proof of the proposition follows from the equality

$$e^{k\,\|\,\mathrm{dev}_n\log U\|^2}=e^{k\,\|\log U\|^2-\frac{k}{n}\,\mathrm{tr}[\log U]^2}=e^{k\,\|\log U\|^2}\cdot e^{-\frac{k}{n}\log^2(\det U)},$$

where we have shown the monotonicity of the first factor by the sum of squared logarithms inequality and the second factor is independent of i_1,i_2. (And independent of i_1 in the analogous proof for $n=2$.) □

This holds for dimensions $n=2$ and $n=3$ and indeed in any dimension, in which the sum of squared logarithms inequality holds, see Conjecture 2.3. Therefore, $\widehat{\psi}$ satisfies the criterion of Steigmann from Lemma 2.5, and in consequence, we have our main result:

Proposition 3.8. *The map $W:\mathrm{GL}^+(2)\to\mathbb{R}_+$ defined by $W(F)=e^{k\,\|\mathrm{dev}_2\log U\|^2}$ is polyconvex for $k\geq\dfrac{1}{3}$.*

3.2. Polyconvexity of the volumetric response $F \mapsto e^{\hat{k}(\log \det F)^m}$ in arbitrary dimensions

In a previous work [32, Proposition 5.11], we have established the following lemma:

Lemma 3.9. *The function $t \mapsto e^{\widehat{k}\,(\log t)^m}$ is convex if and only if $\widehat{k} \geq \frac{1}{m^{(m+1)}}$.*

This implies:

Proposition 3.10. *For $m \in \mathbb{N}$ the function $F \mapsto e^{\widehat{k}\,(\log \det F)^m}$, $F \in \mathrm{GL}^+(n)$, is polyconvex for $\widehat{k} \geq \frac{1}{m^{(m+1)}}$. Explicitly evaluating this condition in the case of $m = 2$, we arrive at $\widehat{k} \geq \frac{1}{8}$.*

3.3. The main polyconvexity statement

In view of the results established in Sects. 3.1 and 3.2, we conclude that:

Theorem 3.11. *The functions $W_{\rm eH} : \mathbb{R}^{n\times n} \to \mathbb{R}$ from the family of exponentiated Hencky type energies*

$$W_{\rm eH}(F) = W_{\rm iso}\left(\frac{F}{\det F^{\frac{1}{n}}}\right) + W_{\rm vol}(\det F^{\frac{1}{n}}\,\mathbb{1}) = \begin{cases} \dfrac{\mu}{k}\, e^{k\,\|\mathrm{dev}_n \log U\|^2} + \dfrac{\kappa}{2\widehat{k}}\, e^{\widehat{k}\,[(\log\det U)]^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0, \end{cases} \tag{3.20}$$

are polyconvex for $n = 2$, $\mu > 0, \kappa > 0$, $k \geq \dfrac{1}{3}$ and $\widehat{k} \geq \dfrac{1}{8}$.

4. Unconditional coercivity: coercivity for every exponent $1 \leq q < \infty$

We start the analysis of coercivity problem by considering the simple one-dimensional case:

$$e^{(\log t)^2} = (e^{\log t})^{\log t} = t^{\log t}. \tag{4.1}$$

Since for some particular choices (see Fig. 3) we see that for large values of t, the function $e^{(\log t)^2}$ dominates $|t-1|^\alpha$, for arbitrary $\alpha > 0$. However, $(\log t)^2$ alone does not satisfy any growth condition of this type. This motivates:

Lemma 4.1. (Unconditional coercivity) *For all $\alpha > 0$, there exists $K > 0$ such that for all $t > 0$*

$$e^{(\log t)^2} \geq K\,|t-1|^\alpha. \tag{4.2}$$

Proof. First, we consider "large" values of t, use the substitution $s = \log t$ and observe that

$$\widehat{K} = \inf_{t>1} \frac{e^{\log^2 t}}{(t-1)^\alpha} = \inf_{s>0} \frac{e^{s^2}}{(e^s-1)^\alpha} \geq \inf_{s>0} \frac{e^{s^2}}{e^{\alpha s}} = \inf_{s>0} e^{s^2-\alpha s}$$

is positive, because $\inf\limits_{s>0}(s^2-\alpha s) > -\infty$. Secondly, $\inf_{t\in(0,1)} \frac{e^{\log^2 t}}{(1-t)^\alpha} \geq 1 > 0$. Hence, the claim follows upon the choice $K = \min\{\widehat{K}, 1\}$. □

Corollary 4.2. *For all $\alpha, \beta > 0$, there is $K > 0$ such that for all $t > 0$*

$$e^{\beta(\log t)^2} \geq K\,|t-1|^{\alpha\beta}.$$

Lemma 4.3. *Let $a > 0$ and $\gamma > 0$. Then, there exist positive constants $C, \widetilde{K}$ such that for all $t \in (0,a)$:*

$$C\sqrt{s+t}^{\,\gamma} - \widetilde{K} \leq \sqrt{s}^{\,\gamma}. \tag{4.3}$$

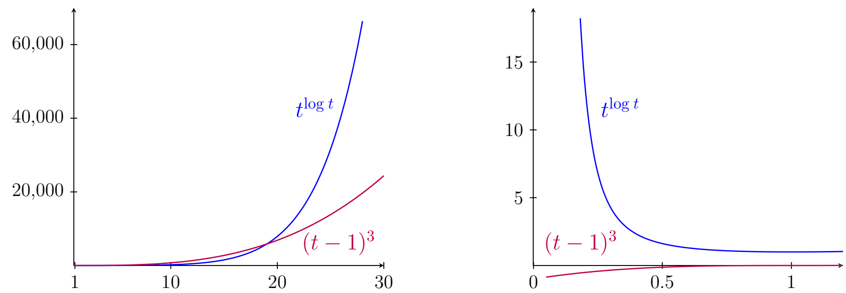

FIG. 3. Function $e^{(\log t)^2}$ dominates $|t-1|^3$ on $(1,\infty)$ as well as on $(0,1)$

Proof. Let us first consider $\gamma<2$. In this case, by the equivalence of norms on $\mathbb{R}^2$, there exists a positive constant C such that $(s+t)^{\frac{\gamma}{2}} = [\big(s^{\frac{\gamma}{2}}\big)^{\frac{2}{\gamma}} + \big(t^{\frac{\gamma}{2}}\big)^{\frac{2}{\gamma}}]^{\frac{\gamma}{2}} \le C\left(|s|^{\frac{\gamma}{2}}+|t|^{\frac{\gamma}{2}}\right)$. Since $t\in(0,a)$, the inequality (4.3) follows. On the other hand, for $\gamma\ge 2$, the inequality (4.3) is a direct consequence of the inequality $(s+t)^{\frac{\gamma}{2}} \le 2^{\frac{\gamma}{2}-1}\left(s^{\frac{\gamma}{2}}+t^{\frac{\gamma}{2}}\right)$. □

Moreover, from Lemma 4.4, we see that there cannot be any polynomial upper bound $C(1+\|F\|^q)\ge W(F)$.

Lemma 4.4. *Let $\alpha,\beta>0$. Then, there are constants $K_1,K_2>0$ such that for all $\lambda_1,\ \lambda_2\in\mathbb{R}_+$*

$$e^{\beta(\log^2\lambda_1+\log^2\lambda_2)} \ge K_1\Big((\lambda_1-1)^2+(\lambda_2-1)^2\Big)^{\frac{\alpha\beta}{2}} - K_2$$

Proof. For $\lambda_1,\lambda_2\in(0,3]$, $e^{\beta(\log^2\lambda_1+\log^2\lambda_2)}\ge 0$ and the claim follows, even for arbitrary large K_1, by setting

$$K_2 := \sup_{(\lambda_1,\lambda_2)\in[0,3]\times[0,3]} \left\{K_1((\lambda_1-1)^2+(\lambda_2-1)^2)^{\frac{\alpha\beta}{2}}\right\},$$

which is finite by continuity of $\lambda\mapsto K_1((\lambda_1-1)^2+(\lambda_2-1)^2)^{\frac{\alpha\beta}{2}}$ and compactness of $[0,3]\times[0,3]$.

For $\lambda_1,\lambda_2\in[3,\infty)$ note that

$$(\lambda_1-1)^2(\lambda_2-1)^2 = (\lambda_1-1)^2\frac{(\lambda_2-1)^2}{2} + \frac{(\lambda_1-1)^2}{2}(\lambda_2-1)^2 \ge (\lambda_1-1)^2+(\lambda_2-1)^2$$

and hence

$$[(\lambda_1-1)(\lambda_2-1)]^{\alpha\beta} \ge ((\lambda_1-1)^2+(\lambda_2-1)^2)^{\frac{\alpha\beta}{2}}. \tag{4.4}$$

Using Corollary 4.2, we obtain $K>0$ fulfilling

$$e^{\beta(\log^2\lambda_1+\log^2\lambda_2)} = e^{\beta\log^2\lambda_1}e^{\beta\log^2\lambda_2} \ge K^2(\lambda_1-1)^{\alpha\beta}(\lambda_2-1)^{\alpha\beta} \overset{(4.4)}{\ge} K^2((\lambda_1-1)^2+(\lambda_2-1)^2)^{\frac{\alpha\beta}{2}}.$$

Now let us consider the last possible case: $\lambda_1\ge 3,\ \lambda_2\in(0,3)$. Then, Corollary 4.2 and Lemma 4.3 with $s=(\lambda_1-1)^2$, $\gamma=\alpha\beta$, $t=(\lambda_2-1)^2$ and $a=4$ yield $K,C,\widetilde{K}>0$ such that

$$e^{\beta\log^2\lambda_1+\beta\log^2\lambda_2} \ge K(\lambda_1-1)^{\alpha\beta}\underbrace{e^{\beta\log^2\lambda_2}}_{\ge 1} \ge KC((\lambda_1-1)^2+(\lambda_2-1)^2)^{\frac{\alpha\beta}{2}} - K\widetilde{K}.$$

Finally, we choose the smallest K_1 and largest K_2 required by any of these individual cases. □

Remark 4.5. The same can be done in dimension $n=3$ or higher. For larger n, the domain must be split into the single cases in a different way, replacing 3 as the separating number (for $n=2$, indeed $\sqrt{2}+1$ would have sufficed) and Lemma 4.3 must be applied several times.

Theorem 4.6. *Regardless of dimension $n \in \mathbb{N}$ and $\beta > 0$, $e^{\beta \| \log U \|^2}$ is* **unconditionally coercive***, in the sense that for arbitrary $\alpha > 0$ there are constants $K_1, K_2 > 0$ such that*

$$e^{\beta \| \log U \|^2} \geq K_1 \| U - \mathbb{1} \|^{\alpha \beta} - K_2. \tag{4.5}$$

Proof. Unitarily diagonalizing the symmetric positive definite matrix U equivalently transforms equation (4.5) into

$$e^{\beta \sum_{i=1}^{n} \log^2 \lambda_i} \geq K_1 \left(\sum_{i=1}^{n} (\lambda_i - 1)^2 \right)^{\frac{\alpha\beta}{2}} - K_2.$$

Here, we can apply Lemma 4.4 (or Remark 4.5 for $n > 2$). $\square$

Remark 4.7. Let $n = 2, k > 0$ and consider $e^{k \, \| \operatorname{dev}_2 \log U \|^2}$. This energy is not coercive in the following sense: Neither are there constants $K_1, K_2, \alpha > 0$ such that

$$e^{k \, \| \operatorname{dev}_2 \log U \|^2} \geq K_1 \| U - \mathbb{1} \|^{\alpha k} - K_2, \tag{4.6}$$

nor do there exist $K_1, K_2, \alpha > 0$ such that

$$e^{k \| \operatorname{dev}_2 \log U \|^2} \geq K_1 \| \operatorname{dev}_2 U \|^{\alpha k} - K_2. \tag{4.7}$$

Proof. Suppose there were K_1, K_2, α satisfying (4.6), i.e., for all $\lambda_1, \lambda_2 > 0$:

$$e^{\frac{k}{2} \log^2 \frac{\lambda_1}{\lambda_2}} \geq K_1 ((\lambda_1 - 1)^2 + (\lambda_2 - 1)^2)^{\frac{\alpha k}{2}} - K_2.$$

Choose $\lambda_1 = \lambda_2 = N + 1$. This would lead to $1 \geq K_1 (2N^2)^{\frac{\alpha k}{2}} - K_2 \to \infty$. In the same manner, (4.7) corresponds to

$$e^{\frac{k}{2} \log^2 \frac{\lambda_1}{\lambda_2}} \geq \frac{K_1}{2^{\frac{\alpha k}{2}}} |\lambda_1 - \lambda_2|^{\alpha k} - K_2 \, .$$

Choose $\lambda_2 = \frac{\lambda_1}{2} = N$ to obtain a contradiction by $e^{\frac{k}{2} \log^2 2} \geq \frac{K_1}{2^{\frac{\alpha k}{2}}} N^{\alpha k} - K_2 \to \infty$. $\square$

However, we have the following results which will finally lead to the coercivity of $W(U)$:

Lemma 4.8. *Assume $\mu > 0$, $\kappa > 0$. For arbitrary dimension $n \in \mathbb{N}$ and $k, \widehat{k} > 0$, and for arbitrary $\alpha_1, \alpha_2 > 0$, there are constants $C_1, C_2, C_3 > 0$ such that for any $U \in PSym(n)$*

$$\widehat{W}_{\text{eH}}(U) = \frac{\mu}{k} \, e^{k \, \| \operatorname{dev}_n \log U \|^2} + \frac{\kappa}{2\widehat{k}} \, e^{\widehat{k} \, |\operatorname{tr}(\log U)|^2} \geq C_1 \left\| \frac{U}{\det U^{1/n}} - \mathbb{1} \right\|^{\alpha_1 k} + C_2 |\det U - 1|^{\alpha_2 \widehat{k}} - C_3. \tag{4.8}$$

Proof. Let us repeat that

$$\operatorname{dev}_n \log U = \log U - \frac{1}{n} \operatorname{tr}(\log U) \cdot \mathbb{1} = \log U - \frac{1}{n} \, \log(\det U) \cdot \mathbb{1} = \log \frac{U}{\det U^{1/n}}. \tag{4.9}$$

Hence, using (4.5) we know that for arbitrary $\alpha_1 > 0$ there are constants $K_1, K_3 > 0$ such that

$$e^{k \, \| \operatorname{dev}_n \log U \|^2} = e^{k \, | \log \frac{U}{\det U^{1/n}} |^2} \geq K_1 \left\| \frac{U}{\det U^{1/n}} - \mathbb{1} \right\|^{\alpha_1 k} - K_3. \tag{4.10}$$

On the other hand, using Corollary 4.2 we obtain that for arbitrary $\alpha_2 > 0$ there is the constant $K_2 > 0$ such that

$$e^{\widehat{k} |\operatorname{tr}(\log U)|^2} \geq K_2 \, | \det U - 1 |^{\alpha_2 \widehat{k}}. \tag{4.11}$$

With the choices $C_1 = \frac{\mu}{k} \, K_1, C_2 = \frac{\kappa}{2\widehat{k}} \, K_2, C_3 = \frac{\mu}{k} \, K_3$, the proof is complete. $\square$

Using a technique similar to that used in [17], we obtain:

Theorem 4.9. *Assume $\mu > 0$, $\kappa > 0$. Regardless of dimension $n \in \mathbb{N}$ and $k, \widehat{k} > 0$, and for arbitrary $q \geq 1$, there are the constants K_1, $K_2 > 0$ such that for all $U \in PSym(n)$*

$$\widehat{W}_{\rm eH}(U) = \frac{\mu}{k}\, e^{k\,\|\operatorname{dev}_n \log U\|^2} + \frac{\kappa}{2\widehat{k}}\, e^{\widehat{k}\,|\operatorname{tr}(\log U)|^2} \geq K_1 \|U - 1\!\!1\|^q - K_2. \tag{4.12}$$

Proof. Using the inequality $|a+b|^q \leq 2^{q-1}\left(|a|^q + |b|^q\right)$ for all $a, b > 0$, and $q \geq 1$, we deduce

$$\begin{aligned}\|U - 1\!\!1\|^{\,q} &= \left\|\left(\frac{U}{\det U^{1/n}} - 1\!\!1\right)\det U^{1/n} + \det U^{1/n}\cdot 1\!\!1 - 1\!\!1\right\|^q \\ &\leq \left[\left\|\frac{U}{\det U^{1/n}} - 1\!\!1\right\|\,|\det U|^{1/n} + n|\det U^{1/n} - 1|\right]^q \\ &\leq 2^{q-1}\left[\left\|\frac{U}{\det U^{1/n}} - 1\!\!1\right\|^q\,|\det U|^{q/n} + n^q|\det U^{1/n} - 1|^q\right].\end{aligned} \tag{4.13}$$

Young's inequality leads to

$$\begin{aligned}\|U - 1\!\!1\|^{\,q} &\leq 2^{q-1}\left[\left\|\frac{U}{\det U^{1/n}} - 1\!\!1\right\|^q\,|\det U|^{q/n} + n^q|\det U^{1/n} - 1|^q\right] \\ &\leq 2^{q-1}\left[\frac{1}{2}\left\|\frac{U}{\det U^{1/n}} - 1\!\!1\right\|^{2q} + \frac{1}{2}|\det U|^{2q/n} + \frac{n^q}{2}|\det U^{1/n} - 1|^{2q} + \frac{n^q}{2}\right],\end{aligned} \tag{4.14}$$

which entails

$$\begin{aligned}\|U - 1\!\!1\|^{\,q} &\leq 2^{q-1}\left[\frac{1}{2}\|\frac{U}{\det U^{1/n}} - 1\!\!1\|^{2q} + \frac{1}{2}(|\det U^{1/n} - 1| + 1)^{2q} + \frac{n^q}{2}|\det U^{1/n} - 1|^{2q} + \frac{n^q}{2}\right] \\ &\leq 2^{q-2}\left[\|\frac{U}{\det U^{1/n}} - 1\!\!1\|^{2q} + 2^{2q-1}(|\det U^{1/n} - 1|^{2q} + 1) + n^q|\det U^{1/n} - 1|^{2q} + n^q\right] \\ &= 2^{q-2}\left[\|\frac{U}{\det U^{1/n}} - 1\!\!1\|^{2q} + \left(n^q + 2^{2q-1}\right)|\det U^{1/n} - 1|^{2q} + n^q + 2^{2q-1}\right].\end{aligned} \tag{4.15}$$

Let $C_1, C_2, C_3 > 0$ be as provided upon an application of Lemma 4.8 with the choices of $\alpha_1 = 2q/k$, $\alpha_2 = 2q/\widehat{k}$, and define $A_1 = \max\{\frac{2^{q-2}}{C_1}, \frac{2^{q-2}n^q + 2^{3q-3}}{C_2}\}$ and $A_2 = 2^{q-2}n^q + 2^{3q-3}$. Then, (4.15) leads to

$$\|U - 1\!\!1\|^{\,q} \leq A_1\left[C_1\|\frac{U}{\det U^{1/n}} - 1\!\!1\|^{\alpha_1 k} + C_2|\det U^{1/n} - 1|^{\alpha_2 \widehat{k}}\right] + A_2, \tag{4.16}$$

and thus, by definition of C_1, C_2, C_3, the inequality given by Lemma 4.8 can be used to deduce

$$\|U - 1\!\!1\|^{\,q} \leq A_1\left[\frac{\mu}{k}\, e^{k\,\|\operatorname{dev}_n \log U\|^2} + \frac{\kappa}{2\widehat{k}}\, e^{\widehat{k}\,|\operatorname{tr}(\log U)|^2} + C_3\right] + A_2, \tag{4.17}$$

and further

$$A_1^{-1}\|U - 1\!\!1\|^q - C_3 - A_1^{-1}A_2 \leq \frac{\mu}{k}\, e^{k\,\|\operatorname{dev}_n \log U\|^2} + \frac{\kappa}{2\widehat{k}}\, e^{\widehat{k}|\operatorname{tr}(\log U)|^2}. \tag{4.18}$$

Choosing $K_1 = A_1^{-1}$ and $K_2 = C_3 + A_1^{-1}A_2$, we obtain the inequality (4.12), and the proof is complete. □

Definition 4.10. (Coercivity) Let $I(\varphi)$ be the elastic stored energy functional depending on the deformation $\varphi(x,t)$. We say that I is q-coercive (for $q \geq 1$) whenever for all $K > 0$ there is some $\widetilde{K} > 0$ such that for any $\varphi \in W^{1,q}(\Omega, \mathbb{R}^n)$

$$I(\varphi) \leq K \Rightarrow \|\nabla\varphi\|_{L^q(\Omega)} \leq \widetilde{K}. \tag{4.19}$$

A direct consequence of Theorem 4.9 is the following result:

Theorem 4.11. *Assume for the elastic moduli $\mu > 0,\ \kappa > 0$ and $k > 0,\ \widehat{k} > 0$. Consider the energy*

$$I(\varphi) = \int_\Omega W_{_{\rm eH}}(\nabla\varphi(x))\,{\rm d}x \tag{4.20}$$

where $W_{_{\rm eH}}(F) = \widehat{W}_{_{\rm eH}}(U) = \frac{\mu}{k}\,e^{k\,\|\,{\rm dev}_2 \log U\|^2} + \frac{\kappa}{2\,\widehat{k}}\,e^{\widehat{k}\,|{\rm tr}(\log U)|^2}$. Then, $I(\varphi)$ is q-coercive for any $1 \le q < \infty$.

5. The static problem in the planar case

5.1. Formulation of the static problem in the planar case

The static problem in the planar case consists in finding the solution φ of the equilibrium equation

$$0 = {\rm Div}\, S_1(\nabla\varphi) \qquad \text{in} \qquad \Omega \subset \mathbb{R}^2, \tag{5.1}$$

where the first Piola–Kirchhoff stress tensor corresponding to the energy $W_{_{\rm eH}}(F)$ is given by the constitutive equation

$$S_1(F) = \left[2\mu\, e^{k\,\|\,{\rm dev}_2 \log\, U\|^2}\cdot {\rm dev}_2 \log\, U + \kappa\, e^{\widehat{k}\,[{\rm tr}(\log U)]^2}\, {\rm tr}(\log U)\cdot 1\!\!1\right] F^{-T}, \qquad x \in \overline{\Omega}, \tag{5.2}$$

with $F = \nabla\varphi,\ U = \sqrt{F^T F}$. The above system of equations is supplemented, in the case of the mixed problem, by the boundary conditions

$$\varphi(x) = \widehat{\varphi}_i(x) \qquad \text{on} \qquad \Gamma_D, \qquad S_1(x).\, n = \widehat{s}_1(x) \qquad \text{on} \qquad \Gamma_N,$$

where Γ_D, Γ_N are subsets of the boundary $\partial\Omega$, so that $\Gamma_D \cup \overline{\Gamma}_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, n is the unit outward normal to the boundary and $\widehat{\varphi}_i, \widehat{s}_1$ are prescribed fields.

5.2. Existence of minimizers in plane elastostatics

In plane elastostatics, having proved the coercivity and polyconvexity of the energy $W(U)$, it is a standard matter to prove the existence of a minimizer.

Theorem 5.1. (Existence of minimizers) *Let the reference configuration $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let Γ_D be a nonempty and relatively open part of the boundary $\partial\Omega$. Assume that $I(\varphi) = \int_\Omega W_{eH}(\nabla\varphi(x)){\rm d}x$ where $W_{_{\rm eH}}(F) = \widehat{W}_{_{\rm eH}}(U) = \frac{\mu}{k}\,e^{k\,\|\,{\rm dev}_2 \log U\|^2} + \frac{\kappa}{2\widehat{k}}\,e^{\widehat{k}\,|{\rm tr}(\log U)|^2}$. Let $\varphi_0 \in W^{1,q}(\Omega),\ q \ge 1$ be given with $I(\varphi_0) < \infty$ and $\mu > 0,\ \kappa > 0,\ k > \frac{1}{3}$ and $\widehat{k} > \frac{1}{8}$. Then, the problem*

$$\min\left\{ I(\varphi) = \int_\Omega W_{_{\rm eH}}(\nabla\varphi(x)){\rm d}x,\ \varphi = \varphi_0 \quad \text{on } \Gamma_D \subset \partial\Omega, \quad \varphi \in W^{1,q}(\Omega)\right\} \tag{5.3}$$

admits at least one solution φ. Moreover, $\varphi \in W^{1,p}(\Omega)$ for all $p \ge 1$.

Remark 5.2. Formally, this solution corresponds to a solution of the boundary value problem formulated in Sect. 5.1. However, the minimizing property of φ alone is not sufficient to show that the Euler–Lagrange equation (5.1) is satisfied by φ in a weak sense: Since we do not know whether $\det \nabla\varphi \ge c > 0$, it is not clear whether the energy functional is Frechét differentiable at the minimizer.

Remark 5.3. While the parameters $\mu,\ \kappa > 0$ are already uniquely determined from the infinitesimal material response, $k, \widehat{k} > 0$ can be used to fit some nonlinear aspects of the response. This will be done in a future contribution.

6. The three-dimensional case: $F \mapsto e^{k\,\|\,\mathrm{dev}_3 \log U\|^2}$

The 3D-case is, as usual, much more involved. As was previously shown [32], the exponentiated Hencky energy

$$F \mapsto \frac{\mu}{k}\, e^{k\,\|\,\mathrm{dev}_n \log U\|^2}, \quad k > \frac{3}{16},\ \mu > 0$$

in dimension $n = 3$ is not rank-one convex and therefore not polyconvex. However, numerical results strongly suggest that $W_{\rm eH}$ is, in fact, rank-one convex on a cone of the form

$$\mathcal{E} = \{U \in \mathrm{PSym}(3)\,\big|\, \|\,\mathrm{dev}_3 \log U\|^2 < \frac{2}{3}\,\widetilde{\boldsymbol{\sigma}}_{\mathbf{y}}^2\,\}.$$

with $\widetilde{\boldsymbol{\sigma}}_{\mathbf{y}} \gg 1$. This convexity property is of particular interest in the theory of plasticity, since the loss of rank-one convexity occurs only for strains which induce permanent deformations. We will discuss the possible application of the exponentiated Hencky energy in plasticity theory in the near future [31].

7. Summary and open problems

To summarize, in the present paper,

- We have applied Steigmann's polyconvexity condition and proved that the planar exponentiated Hencky-strain energy function

$$F \mapsto W_{\rm eH}(F) := \widehat{W}_{\rm eH}(U) := \begin{cases} \frac{\mu}{k}\, e^{k\,\|\,\mathrm{dev}_2 \log U\|^2} + \frac{\kappa}{2\widehat{k}}\, e^{\widehat{k}\,(\mathrm{tr}(\log U)^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \le 0 \end{cases} \tag{7.1}$$

  is **polyconvex** for $\mu > 0, \kappa > 0,\ k \ge \dfrac{1}{3}$ and $\widehat{k} \ge \dfrac{1}{8}$.
- We have shown that the exponentiated volumetric energy function

$$F \mapsto \frac{\kappa}{2\widehat{k}}\, e^{\widehat{k}\,(\mathrm{tr}(\log U))^m}, \quad F \in \mathrm{GL}^+(n) \tag{7.2}$$

  is **polyconvex** w.r.t F for $\widehat{k} \ge \frac{1}{m^{(m+1)}}$.
- We have proven that, regardless of dimension $n \in \mathbb{N}$ and $k, \widehat{k} > 0$, the energies of the family $F \mapsto W_{\rm eH}(F)$ satisfy q-growth coercivity for any $1 \le q < \infty$ and therefore allow in the planar case $n = 2$ for a complete existence theory based on the direct methods of the calculus of variations.

Using the terminology from [30,33], in the present paper, we have shown polyconvexity of

$$W_{\rm eH}(F) := \frac{\mu}{k}\, e^{k\,\mathrm{dist}^2_{\mathrm{geod},\mathrm{SL}(2)}\left(\frac{F}{\det F^{1/2}},\,\mathrm{SO}(2)\right)} + \frac{\kappa}{2\widehat{k}}\, e^{\widehat{k}\,\mathrm{dist}^2_{\mathrm{geod},\mathbb{R}_+\cdot 1\!\!1}\left(\det F^{1/2}\cdot 1\!\!1,\,1\!\!1\right)}, \tag{7.3}$$

and we have proved the existence of the solution of the corresponding minimization problem.

In the first part [32] of this paper, we have shown rank-one convexity for $k \ge \frac{1}{4}$. Here, we have obtained polyconvexity for $k \ge \frac{1}{3}$. Hence, a first open problem is to investigate the gap $\frac{1}{3} > k \ge \frac{1}{4}$.

Results obtained by Pipkin [35], concerning convexity conditions when F is a 3×2 matrix, may be used to extend our polyconvexity results to membrane theory. The associated stretch tensor is $U = \sqrt{F^T F}$, which is still a 2×2 matrix, just as in the case of plane strain considered here. The results of [35] ensure that polyconvexity with respect to 2×2 deformation gradients—established here for the family $W_{\rm eH}$—yield polyconvexity of the same energy with respect to the 3×2 deformation gradients of membrane theory [48], provided that the first Piola–Kirchhoff stress S_1 (the right-hand side of Eq. (5.2)) is positive semi-definite. The latter restriction is necessary for rank-one convexity (and hence also for polyconvexity) when F is a 3×2 matrix. However, this is not enough to yield the existence of minimizers, even in the presence of coercivity, because the restriction on F, required for a positive semi-definite stress, cannot be guaranteed *a priori.*

Acknowledgments

We would like to thank Prof. Mircea Bîrsan (University of Duisburg-Essen) for indicating to us a technical point and showing that $e^{\|\log U\|^2}$ does not satisfy the Steigmann's sufficiency criterion in 3D, Prof. Bernard Dacorogna (EPFL-Lausanne) for sending us reference [15] and Prof. Miroslav Šilhavý (Academy of Sciences of the Czech Republic, Prague) for comments on rank-one convexity and polyconvexity.

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(Received: August 17, 2014; revised: December 15, 2014)