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ESSEN

A polyconvex extension of the logarithmic Hencky strain energy

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1 Non-ellipticity of the Hencky energy

It has been known for a while that the *quadratic isotropic Hencky energy* W_{H} with [6]

$$W_{\mathrm{H}}(F) = \mu\,\|\mathrm{dev}_n \log U\|^2 + \frac{\kappa}{2}\,[\mathrm{tr}(\log U)]^2\,, \qquad (1)$$

which is based on the logarithmic strain measures [9, 5]

$$\|\mathrm{dev}_n \log U\|^2\,, \quad \|\log U\|^2 \quad \text{and} \quad [\mathrm{tr}(\log U)]^2\,, \quad (2)$$

where $U = \sqrt{F^T F}$ denotes the Biot stretch tensor corresponding to the deformation gradient F, is not overall rank-one convex [8, 3]. Rank-one convexity (or Legendre-Hadamard ellipticity) is a necessary condition for polyconvexity, which, in turn, is an essential requirement for the applicability of existence proofs based on the direct methods of the calculus of variations [1]. This shortcoming raises some concern regarding the suitability of the Hencky model in finite element methods, although (rather large) ellipticity domains of the Hencky energy have been determined explicitly [3, 4].

Moreover [7], for $n \geq 3$, there exists no strictly monotone function $\Psi\colon [0,\infty) \to \mathbb{R}$ such that either of the energy functions $W\colon \mathrm{GL}^+(n) \to \mathbb{R}$ with

$$W(F) = \Psi(\|\log U\|^2) \quad \text{or} \quad W(F) = \Psi(\|\mathrm{dev}_n \log U\|^2)$$

is elliptic. If Ψ is additionally twice-differentiable, then there exists no smooth function $W_{\mathrm{vol}}\colon (0,\infty) \to \mathbb{R}$ such that the energy $W\colon \mathrm{GL}^+(n) \to \mathbb{R}$ with

$$W(F) = \Psi(\|\mathrm{dev}_n \log U\|^2) + W_{\mathrm{vol}}(\det F)$$

is Legendre-Hadamard elliptic.

2 A polyconvex extension of the classical Hencky energy

In order to find an elliptic energy function which approximates (or, better yet, is identical to) the Hencky strain energy in the small-strain range, we adapt an approach by Ball, Muite, Schryvers and Tirry [2] to construct a polyconvex (and thus rank-one convex) extension of the quadratic-logarithmic Hencky energy (1) and, more generally, for suitable energy expressions of the Valanis-Landel type. In addition, the extension of the Hencky energy considered here is (unconditionally) coercive, which implies an immediate applicability of the direct methods of the calculus of variations to prove the existence of energy minimizers under appropriate boundary conditions.

Lemma. *For $\gamma \leq 1$, let*

$$\mathcal{S}_\gamma := \{F \in \mathrm{GL}^+(n) \mid e^{\gamma-1} < \lambda < e^\gamma \text{ for each singular value } \lambda \text{ of } F\}\,.$$

Then the function

$$W\colon \mathcal{S}_\gamma \to \mathbb{R}\,, \quad W(F) = \|\log \sqrt{F^T F}\|^2 = \sum_{i=1}^n \ln^2(\lambda_i)$$

has a polyconvex extension $\widetilde{W}_\gamma\colon \mathrm{GL}^+(n) \to \mathbb{R}$ to $\mathrm{GL}^+(n)$, which is given by

$$\widetilde{W}_\gamma(F) = \sum_{i=1}^n \varphi_\gamma(\lambda_i) \; - (2-2\gamma)\,\ln(\det F)\,,$$

where

$$\varphi_\gamma(\lambda) = \begin{cases} -(\gamma-1)^2 & : \lambda \leq e^{\gamma-1}\,, \\ \ln^2(\lambda) + (2-2\gamma)\ln(\lambda) & : e^{\gamma-1} < \lambda < e^\gamma\,, \\ -\gamma^2 + 2\gamma + \frac{2}{e^\gamma}(e^{\lambda - e^\gamma} - 1) & : e^\gamma \leq \lambda\,. \end{cases}$$

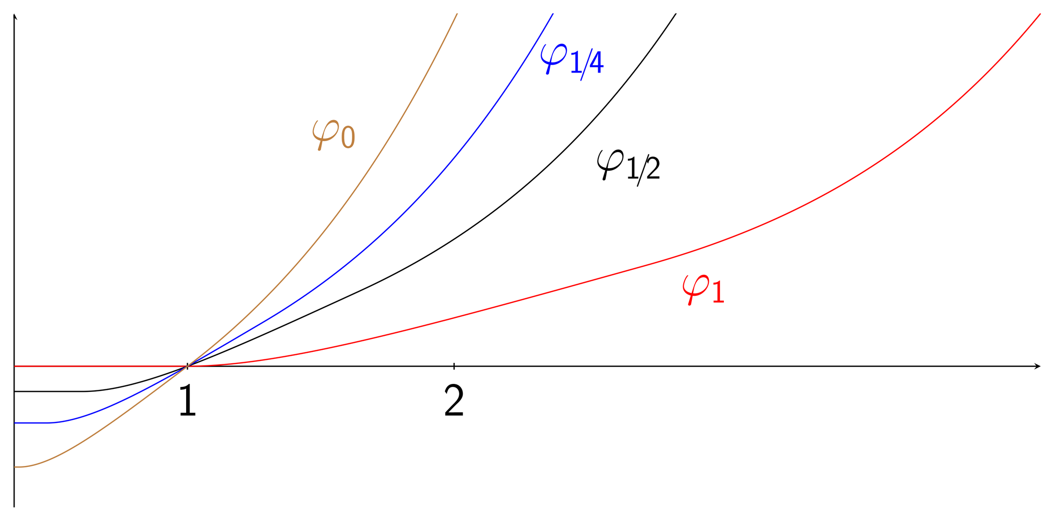

Figure 1: The function φ_γ for different values of γ.

Proposition. *Let W_H denote the quadratic Hencky energy, given by*

$$W_H(F) = \mu\,\|\log\sqrt{F^T F}\|^2 + \frac{\Lambda}{2}\,[\mathrm{tr}(\log\sqrt{F^T F})]^2\,,$$

where μ is the shear modulus and Λ is the first Lamé parameter. If $\Lambda \geq 0$, then the restriction of W_H to the set

$$\mathcal{S}_{1/3} = \{F \in \mathrm{GL}^+(n) \mid e^{-2/3} < \lambda < e^{1/3} \text{ for each singular value } \lambda \text{ of } F\}$$

has a polyconvex extension to $\mathrm{GL}^+(n)$.

The energy $\widetilde{W}_{\mathrm{H}}$ is polyconvex as well as coercive and bounded below, allowing for a direct application of Ball's classical result on the existence of minimizers [1].

Proposition. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, Γ_D be a non-empty and relatively open part of the boundary $\partial\Omega$ and $\varphi_0 \in W^{1,q}(\Omega)$ for some $q > 1$ such that $\int_\Omega \widetilde{W}_{\mathrm{H}}(\nabla\varphi_0(x))\,\mathrm{d}x < \infty$. Then the minimization problem*

$$\min_{\substack{\varphi \in W^{1,p}(\Omega) \\ \varphi|_{\Gamma_D} = \varphi_0}} \int_\Omega \widetilde{W}_{\mathrm{H}}(\nabla\varphi(x))\,\mathrm{d}x$$

admits at least one solution $\widehat{\varphi} \in W^{1,p}(\Omega)$.

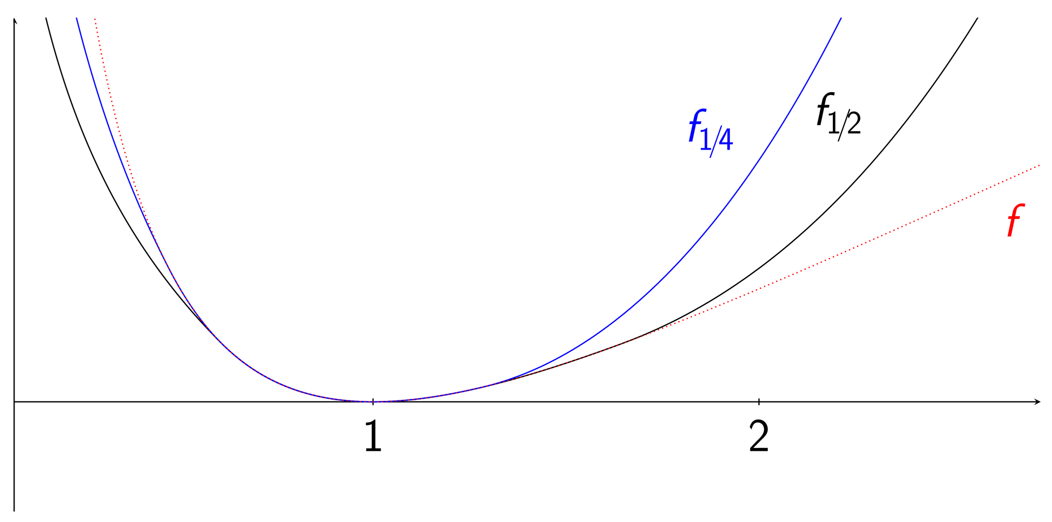

Figure 2: The function $f_\gamma\colon \lambda \mapsto \widetilde{W}_\gamma(\lambda\mathbb{1})$ compared to the mapping $f\colon \lambda \mapsto W_{\mathrm{H}}(\lambda\mathbb{1})$ with $\mu = 1$ and $\Lambda = 0$; note the singularity at $\lambda = 0$.

3 Valanis-Landel energies

We can apply the same extension method to the more general case of *Valanis-Landel type* energy functions, i.e. to functions of the form

$$W_{\mathrm{VL}}\colon \mathrm{GL}^+(n) \to \mathbb{R}\,, \quad W_{\mathrm{VL}}(F) = \sum_{i=1}^n w(\lambda_i)$$

with a scalar function $w\colon (0,\infty) \to \mathbb{R}$. Functions of this type were suggested by Valanis and Landel [11] as a general hyperelastic model for *incompressible* materials, but are often coupled additively with volumetric energy terms in order to obtain elastic models for compressible materials (including the quadratic Hencky energy W_{H} as well as Ogden's classical material model [10]). Note that the energy W_{VL} can only be compatible with linear elasticity at the identity $\mathbb{1}$ if $w(1) = 0$, $w'(1) = 0$ and $w''(1) > 0$; the latter two conditions represent the requirements of a stress-free reference configuration and ellipticity at $\mathbb{1}$, respectively.

Proposition. *Let $w \in C^2((0,\infty))$ such that $w'(1) = 0$ and $w''(1) > 0$. Then the function*

$$W_{\mathrm{VL}}\colon \mathrm{GL}^+(n) \to \mathbb{R}\,, \quad W_{\mathrm{VL}}(F) = \sum_{i=1}^n w(\lambda_i)$$

has a polyconvex extension from a neighborhood of the identity $F = \mathbb{1}$ to $\mathrm{GL}^+(n)$.

4 References

[1] J. M. Ball. "Convexity conditions and existence theorems in nonlinear elasticity". *Archive for Rational Mechanics and Analysis* 63.4 (1976). Pp. 337–403.

[2] J. M. Ball, B. Muite, D. Schryvers and W. Tirry. "Stress-induced interfaces between austenite and untwinned martensite". *unpublished* (2009). excerpt provided by Sir John Ball.

[3] O. T. Bruhns, H. Xiao and A. Mayers. "Constitutive inequalities for an isotropic elastic strain energy function based on Hencky's logarithmic strain tensor". *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 457 (2001). Pp. 2207–2226.

[4] I.-D. Ghiba, P. Neff and R. J. Martin. "An ellipticity domain for the distortional Hencky logarithmic strain energy". *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 471.2184 (2015). DOI: 10.1098/rspa.2015.0510.

[5] I.-D. Ghiba, P. Neff and M. Šilhavỳ. "The exponentiated Hencky-logarithmic strain energy. Improvement of planar polyconvexity". *International Journal of Non-Linear Mechanics* 71 (2015). Pp. 48–51. DOI: 10.1016/j.ijnonlinmec.2015.01.009.

[6] H. Hencky. "Welche Umstände bedingen die Verfestigung bei der bildsamen Verformung von festen isotropen Körpern?" *Zeitschrift für Physik* 55 (1929). Pp. 145–155 available at www.uni-due.de/imperia/md/content/mathematik/ag_neff/hencky1929.pdf.

[7] R. J. Martin, I.-D. Ghiba and P. Neff. "A non-ellipticity result, or the impossible taming of the logarithmic strain measure". *to appear in the International Journal of Non-Linear Mechanics* (2017). available at arXiv:1712.04846.

[8] P. Neff. *Mathematische Analyse multiplikativer Viskoplastizität. Ph.D. Thesis, Technische Universität Darmstadt.* available at http://www.uni-due.de/~hm0014/Download_files/neffdiss.ps. Aachen: Shaker Verlag, 2000.

[9] P. Neff, B. Eidel and R. J. Martin. "Geometry of logarithmic strain measures in solid mechanics". *Archive for Rational Mechanics and Analysis* 222.2 (2016). Pp. 507–572 available at arXiv:1505.02203. DOI: 10.1007/s00205-016-1007-x.

[10] R. W. Ogden. "Large deformation isotropic elasticity-on the correlation of theory and experiment for incompressible rubberlike solids". *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 326.1567 (1972). Pp. 565–584.

[11] K. C. Valanis and R. F. Landel. "The strain-energy Function of a hyperelastic material in terms of the extension ratios". *Journal of Applied Physics* 38.7 (1967). Pp. 2997–3002.

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