A Riemannian approach to strain measures in nonlinear elasticity

Patrizio Neff*[1], Bernhard Eidel[2], Frank Osterbrink[1], and Robert Martin[1]

[1]Chair for Nonlinear Analysis and Modelling, University of Duisburg-Essen,
Thea-Leymann-Str. 9, 45127 Essen
[2]Institute of Mechanics, University of Duisburg-Essen, Universitätsstraße 15, 45141 Essen

May 10, 2013

Abstract

The isotropic Hencky strain energy appears naturally as a distance measure of the deformation gradient to the set $\mathrm{SO}(n)$ of rigid rotations in the canonical left-invariant Riemannian metric on the general linear group $\mathrm{GL}(n)$. Objectivity requires the Riemannian metric to be left-$\mathrm{GL}(n)$-invariant, isotropy requires the Riemannian metric to be right-$\mathrm{O}(n)$-invariant. The latter two conditions are satisfied for a three-parameter family of Riemannian metrics on the tangent space of $\mathrm{GL}(n)$. Surprisingly, the final result is basically independent of the chosen parameters.

In deriving the result, geodesics on $\mathrm{GL}(n)$ have to be parametrized and a novel minimization problem, involving the matrix logarithm for non-symmetric arguments, has to be solved.

1 Introduction

For the deformation gradient $F = \nabla\varphi \in \mathrm{GL}^+(n)$ let $U = \sqrt{F^T F}$ be the symmetric right Biot-stretch tensor. We show that the isotropic Hencky strain energy, defined on the logarithmic strain tensor $\log U$ by

$$W(F) = \mu\,\|\operatorname{dev}\log U\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2 = \mu\,\|\operatorname{dev}\log U\|^2 + \frac{\kappa}{2}(\log\det F)^2\,, \tag{1.1}$$

measures the geodesic distance of F to the group of rotations $\mathrm{SO}(n)$ where $\mathrm{GL}(n)$ is viewed as a Riemannian manifold endowed with a left-invariant metric which is also right $\mathrm{O}(n)$-invariant (isotropic), and where the coefficients μ, $\kappa > 0$ correspond to the shear modulus and the bulk modulus, respectively. Thus we provide yet another characterization of the polar decomposition $F = R\,U$, $R \in \mathrm{SO}(n)$, $U \in \mathrm{PSym}(n)$, since U also provides the minimal euclidean distance to $\mathrm{SO}(n)$, i.e.,

$$\operatorname{dist}^2_{\text{euclid}}(F, \mathrm{SO}(n)) := \min_{Q\in\mathrm{SO}(n)} \operatorname{dist}^2_{\text{euclid}}(F, Q) = \min_{Q\in\mathrm{SO}(n)} \|F - Q\|^2 = \|F - R\|^2 = \|U - \mathbb{1}\|^2\,, \tag{1.2}$$

where the euclidean distance $\operatorname{dist}^2_{\text{euclid}}(X, Y) := \|X - Y\|^2$ is the length of the line segment joining X and Y in $\mathbb{R}^{n^2}$, $\mathbb{1} \in \mathrm{GL}^+(n)$ is the identity and $\|X\| = \sqrt{\operatorname{tr}(X^T X)}$ denotes the Frobenius matrix norm here and henceforth. For both the euclidean and the geodesic distance, the orthogonal factor $R = \operatorname{polar}(F)$ in the polar decomposition of F is the nearest rotation to F.

2 Strain measures in linear and nonlinear elasticity

We consider an elastic body which in a reference configuration occupies the bounded domain $\Omega \subset \mathbb{R}^3$. Deformations of the body are prescribed by mappings $\varphi : \Omega \to \mathbb{R}^3$, where $\varphi(x)$ denotes the deformed position of the material point $x \in \Omega$. Central to elasticity theory is the notion of strain, which is a measure

---

[*] To whom correspondence should be addressed. e-mail: `patrizio.neff@uni-due.de`

of deformation such that vanishing strain implies that the body Ω has been moved rigidly in space. Various such measures exist, e.g. the *Green strain* $\frac{1}{2}(U^2 - \mathbb{1})$, the *generalized Green strain* $\frac{1}{m}(U^m - \mathbb{1})$, where m is a nonzero integer, and the *Hencky (or logarithmic) strain* $\log U$.

In linearized elasticity, one considers $\varphi(x) = x + u(x)$, where $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is the displacement. The classical linearized strain measure is $\varepsilon = \operatorname{sym} \nabla u$. It appears through a matrix-nearness problem in the euclidean distance

$$\operatorname{dist}^2_{\text{euclid}}(\nabla u, \mathfrak{so}(3)) := \min_{W \in \mathfrak{so}(3)} \|\nabla u - W\|^2 = \|\operatorname{sym} \nabla u\|^2 \,, \tag{2.1}$$

where $\mathfrak{so}(3)$ denotes the set of all skew symmetric matrices in $\mathbb{R}^{3\times 3}$. Indeed, $\operatorname{sym}\nabla u$ qualifies as a linearized strain measure: if $\operatorname{dist}^2_{\text{euclid}}(\nabla u, \mathfrak{so}(3)) = 0$ then $u(x) = \widehat{W}.x + \widehat{b}$ is a linearized rigid movement. This is the case since

$$\operatorname{dist}^2_{\text{euclid}}(\nabla u(x), \mathfrak{so}(3)) = 0 \quad \Rightarrow \quad \nabla u(x) = W(x) \in \mathfrak{so}(3) \tag{2.2}$$

and $0 = \operatorname{Curl} \nabla u(x) = \operatorname{Curl} W(x)$ implies that $W(x)$ is constant, see [1]. In nonlinear elasticity theory one assumes that $\nabla\varphi \in \mathrm{GL}^+(3)$ (no self-interpenetration of matter) and may consider the matrix nearness problem

$$\operatorname{dist}^2_{\text{euclid}}(\nabla\varphi, \mathrm{SO}(3)) := \min_{Q \in \mathrm{SO}(3)} \|\nabla\varphi - Q\|^2 = \min_{Q \in \mathrm{SO}(3)} \|Q^T \nabla\varphi - \mathbb{1}\|^2 = \|\sqrt{\nabla\varphi^T \nabla\varphi} - \mathbb{1}\|^2 \,, \tag{2.3}$$

where the last equality is due to (1.2). Indeed, the Biot strain tensor $\sqrt{\nabla\varphi^T\nabla\varphi} - \mathbb{1}$ qualifies as a nonlinear strain measure: if $\operatorname{dist}^2_{\text{euclid}}(\nabla\varphi, \mathrm{SO}(3)) = 0$ then $\varphi(x) = \widehat{Q}.x + \widehat{b}$ is a rigid movement. This is the case since

$$\operatorname{dist}^2_{\text{euclid}}(\nabla\varphi, \mathrm{SO}(3)) = 0 \quad \Rightarrow \quad \nabla\varphi(x) = Q(x) \in \mathrm{SO}(3) \tag{2.4}$$

and $0 = \operatorname{Curl}\nabla\varphi(x) = \operatorname{Curl} Q(x)$ implies that $Q(x)$ is constant, see [1].

In geometrically nonlinear, physically linear isotropic elasticity the formulation of a boundary value problem of place may now be based on minimizing the quadratic Biot strain energy

$$\mathcal{E}(\varphi) = \int_\Omega \mu \,\|\operatorname{dev}[\sqrt{\nabla\varphi^T\nabla\varphi} - \mathbb{1}]\|^2 + \frac{\kappa}{2}\left(\operatorname{tr}\sqrt{\nabla\varphi^T\nabla\varphi} - \mathbb{1}\right)^2 \mathrm{dx} \,, \quad \varphi|_{\Gamma_D} = \varphi_0 \,, \tag{2.5}$$

where $\mu, \kappa > 0$ are the shear modulus and bulk modulus, respectively.

However, since the Euclidean distance in (2.3) is an arbitrary choice, novel approaches in nonlinear elasticity theory aim at putting more geometry (i.e. respecting the group structure of the deformation mappings) into the description of the strain a material endures. In our context, it is now natural to consider a strain measure induced by the geodesic distances stemming from choices for the Riemannian structure respecting also the algebraic group structure of $\mathrm{GL}^+(n)$, which we introduce next.

3 Left invariant Riemannian metrics on $\mathrm{GL}(n)$

Viewing $\mathrm{GL}(n)$ as a Riemannian manifold endowed with a left invariant metric

$$g_H : T_H \,\mathrm{GL}(n) \times T_H \,\mathrm{GL}(n) \to \mathbb{R} : \; g_H(X, Y) = \left\langle H^{-1}X,\, H^{-1}Y \right\rangle \,, \quad H \in \mathrm{GL}(n) \,, \tag{3.1}$$

for a suitable inner product $\langle\cdot,\cdot\rangle$ on the tangent space $T_{\mathbb{1}}\,\mathrm{GL}(n) = \mathfrak{gl}(n) = \mathbb{R}^{n\times n}$ at the identity $\mathbb{1}$, the distance between $F, P \in \mathrm{GL}^+(n)$ can be measured along sufficiently smooth curves. We denote by

$$\mathcal{A} = \{\gamma \in C^0([0,1]; \mathrm{GL}^+(n)) \mid \gamma \text{ piecewise differentiable, } \gamma(0) = F,\, \gamma(1) = P\} \tag{3.2}$$

the admissible set of curves connecting F and P, and by

$$L(\gamma) = \int_0^1 \sqrt{g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))} \,\mathrm{ds} \tag{3.3}$$

the length of $\gamma \in \mathcal{A}$. Then the geodesic distance

$$\operatorname{dist}_{\mathrm{geod}}(F,P) = \inf_{\gamma\in\mathcal{A}} L(\gamma) \tag{3.4}$$

defines a metric on $\mathrm{GL}^+(n)$. While it is generally difficult to explicitly compute this distance or to find length minimizing curves, it can be shown [2, 3] that if the Riemannian metric is defined by an inner product of the form

$$\begin{aligned}
&\langle X,Y\rangle = \langle X,Y\rangle_{\mu,\mu_c,\kappa} := \mu\,\langle \operatorname{dev}\operatorname{sym}X, \operatorname{dev}\operatorname{sym}Y\rangle_{n\times n} + \mu_c\,\langle \operatorname{skew}X, \operatorname{skew}Y\rangle_{n\times n} + \tfrac{\kappa}{2}\operatorname{tr}X\operatorname{tr}Y\,,\\
&\|X\|^2_{\mu,\mu_c,\kappa} := \langle X,X\rangle_{\mu,\mu_c,\kappa} = \mu\,\|\operatorname{dev}\operatorname{sym}X\|^2 + \mu_c\,\|\operatorname{skew}X\|^2 + \tfrac{\kappa}{2}\,[\operatorname{tr}X]^2\,, \qquad \mu,\ \mu_c,\ \kappa>0\,, \qquad (3.5)\\
&\operatorname{dev}X := X - \tfrac{1}{n}\operatorname{tr}X\cdot 1\!\!1\,, \quad \mu_c \text{ denoting the } \textit{spin modulus}\,,
\end{aligned}$$

which is the case if and only if the metric g is right invariant under $\mathrm{O}(n)$ [4], then every geodesic γ connecting F and P is of the form

$$\gamma(t) = F\,\exp(t(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi))\,\exp(t(1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi) \tag{3.6}$$

for some $\xi \in \mathfrak{gl}(n)$, where $\exp : \mathfrak{gl}(n) \to \mathrm{GL}^+(n)$ denotes the matrix exponential, $\operatorname{sym}\xi = \frac{1}{2}(\xi+\xi^T)$ the symmetric part and $\operatorname{skew}\xi = \frac{1}{2}(\xi-\xi^T)$ the skew symmetric part of ξ.

Now, according to the classical Hopf-Rinow theorem of differential geometry, there exists a length minimizing geodesic in $\mathcal{A}$ for all $F,P\in\mathrm{GL}^+(n)$. To obtain such a minimizer γ (and thus the distance $\operatorname{dist}_{\mathrm{geod}}(F,P) = L(\gamma)$), it therefore remains to find $\xi\in\mathfrak{gl}(n)$ with

$$P = \gamma(1) = F\,\exp(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi)\,\exp((1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi)\,. \tag{3.7}$$

The existence of such a ξ is clear from the above.

4 The geodesic distance to $\mathrm{SO}(n)$

Although no closed form solution to (3.7) is known, the equation can be used to obtain a lower bound[1]

$$\operatorname{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) = \min_{Q\in\mathrm{SO}(n)} \operatorname{dist}^2_{\mathrm{geod}}(F,Q) \geq \min_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(QF)\|^2_{\mu,\mu_c,\kappa} \tag{4.1}$$

for the distance of $F\in\mathrm{GL}^+(n)$ to $\mathrm{SO}(n)$, as well as a simple upper bound

$$\begin{aligned}
\operatorname{dist}^2_{\mathrm{geod}}(F,\mathrm{SO}(n)) &\leq \operatorname{dist}^2_{\mathrm{geod}}(F,\ \operatorname{polar}(F))\\
&\leq \|\log(\operatorname{polar}(F)^TF)\|^2_{\mu,\mu_c,\kappa} \;=\; \mu\|\operatorname{dev}\log(U)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2\,,
\end{aligned} \tag{4.2}$$

where $F = RU$, $R = \operatorname{polar}(F)\in\mathrm{SO}(n)$, $U = \sqrt{F^TF}\in\mathrm{PSym}(n)$ denotes the polar decomposition of F. Finally, we can use an extension of a recent optimality result proved by Neff et al. [5]:

Theorem 1. *Let $\|\,.\,\|$ be the Frobenius matrix norm on $\mathfrak{gl}(n)$, $F\in\mathrm{GL}^+(n)$. Then the minimum*

$$\min_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(Q\cdot F)\|^2 = \|\log(\operatorname{polar}(F)^TF)\|^2 = \|\log(\sqrt{F^TF})\|^2 = \|\log(U)\|^2 \tag{4.3}$$

is uniquely attained at $Q = \operatorname{polar}(F)^T$.

A consequence of Theorem 1, combined with (3.5), (4.1) and (4.2), yields our main result [6]:

---

[1] We denote by log the principal matrix logarithm, while the expression Log is used to indicate that the infimum is taken over the whole inverse image under exp, i.e. $\min_{Q\in\mathrm{SO}(n)}\|\operatorname{Log}(QF)\|^2_{\mu,\mu_c,\kappa} = \min\{\|\xi\|^2_{\mu,\mu_c,\kappa} : \xi\in\mathfrak{gl}(n),\ \exp(\xi) = QF\}$.

Theorem 2. *Let g be a left invariant Riemannian metric on* $\mathrm{GL}(n)$ *that is also right invariant under* $\mathrm{O}(n)$*, and let* $F \in \mathrm{GL}^+(n)$*. Then:*

$$\operatorname{dist}^2_{\mathrm{geod}}(F, \mathrm{SO}(n)) = \operatorname{dist}^2_{\mathrm{geod}}(F, \operatorname{polar}(F)) = \mu \|\operatorname{dev}\log(U)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2 \,. \tag{4.4}$$

Thus the geodesic distance of the deformation gradient F to $\mathrm{SO}(n)$ *is the isotropic Hencky strain energy of F. In particular, the result is independent of the spin modulus $\mu_c > 0$.*

Furthermore, for $\mu_c = 0$ (in which case $\operatorname{dist}_{\mathrm{geod}}$ defines only a pseudometric on $\mathrm{GL}^+(n)$), Theorem 2 still holds.

References

[1] P. Neff and I. Münch. Curl bounds Grad on SO(3). *ESAIM: Control, Optimisation and Calculus of Variations*, 14(1):148–159, 2008.

[2] A. Mielke. Finite elastoplasticity, Lie groups and geodesics on SL(d). In P. Newton, P. Holmes, and A. Weinstein, editors, *Geometry, Mechanics, and Dynamics*, pages 61–90. Springer New York, 2002.

[3] P. Neff and R. Martin. Minimal geodesics on GL(n) for left invariant Riemannian metrics which are right invariant under O(n). in preparation, 2013.

[4] R. Bryant. Personal communication, 2013. Mathematical Sciences Research Institute, Berkeley.

[5] P. Neff, Y. Nakatsukasa, and A. Fischle. The unitary polar factor $Q = U_p$ minimizes $\|\operatorname{Log}(Q^*Z)\|^2$ and $\|\operatorname{sym}_* \operatorname{Log}(Q^*Z)\|^2$ in the spectral norm in any dimension and the Frobenius matrix norm in three dimensions. *arXiv:1302.3235*, submitted, 2013.

[6] P. Neff, B. Eidel, F. Osterbrink and R. Martin. The isotropic Hencky strain energy $\|\log U\|^2$ measures the geodesic distance of the deformation gradient $F \in \mathrm{GL}^+(n)$ to $\mathrm{SO}(n)$ in the unique left-invariant Riemannian metric on $\mathrm{GL}^+(n)$ which is also right $\mathrm{O}(n)$-invariant. in preparation, 2013.