The sum of squared logarithms conjecture

by Patrizio Neff

I offer

one ounce of fine gold ($\simeq$ 1090 € as of 5/2015)

for the first proof of (or a counterexample to) the following conjecture which has been puzzling me for some time:

Conjecture (Sum of squared logarithms inequality). *For all natural numbers* n *and positive numbers* $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n > 0$ *such that*

$$\sum_{i_1<\ldots<i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \leq \sum_{i_1<\ldots<i_k} y_{i_1} y_{i_2} \ldots y_{i_k} \qquad \text{for all } k \in \{1, \ldots, n-1\}$$

and

$$x_1\, x_2 \ldots x_n = y_1\, y_2 \ldots y_n\,,$$

it follows

$$\sum_{i=1}^{n} (\log x_i)^2 \leq \sum_{i=1}^{n} (\log y_i)^2\,.$$

Replacing the assumption $x_1\, x_2 \ldots x_n = y_1\, y_2\, \ldots\, y_n$ by $x_1\, x_2 \ldots x_n \leq y_1\, y_2\, \ldots\, y_n$ easily admits counterexamples.

The conjecture is most likely true for all $n \in \mathbb{N}$ as numerical sampling suggests, while currently a proof is available only for $n \in \{1, 2, 3, 4\}$, see [1, 2, 9]. The sum of squared logarithms inequality finds important applications in matrix analysis and nonlinear elasticity [1, 3, 4, 5, 6, 7, 8]. The case $n = 3$, which has been shown in Bîrsan, Neff and Lankeit [1], reads that for $x_1, x_2, x_3, y_1, y_2, y_3 > 0$,

$$\begin{aligned} x_1 + x_2 + x_3 &\leq y_1 + y_2 + y_3 \\ x_1\, x_2 + x_1\, x_3 + x_2\, x_3 &\leq y_1\, y_2 + y_1\, y_3 + y_2\, y_3 \\ x_1\, x_2\, x_3 &= y_1\, y_2\, y_3 \end{aligned}$$

implies

$$(\log x_1)^2 + (\log x_2)^2 + (\log x_3)^2 \leq (\log y_1)^2 + (\log y_2)^2 + (\log y_3)^2\,.$$

How to get the gold: The successful candidate will have to give a talk in the analysis seminar at the faculty of mathematics of the University of Duisburg-Essen, explaining all details and answering to all questions until a complete understanding of the proof is obtained. The result must have successfully been published in a refereed international and respected journal. In addition, the correctness of the result will be judged by an ad hoc dean's commission including two professors from our Analysis and Algebra groups, respectively. The relevant date is the day of submission to arXiv.

If the solution to the above problem is non-trivial, the subject could lend itself for a PhD-project under my supervision, in which case the positive proof of the conjecture will constitute the main part of the dissertation.

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https://www.uni-due.de/mathematik/ag_neff/log_conjecture

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