Faltings' construction of the moduli space of vector bundles on a smooth projective curve

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Abstract. In 1993 Faltings gave a construction of the moduli space of semistable vector bundles on a smooth projective curve X over an algebraically closed field k. This construction was presented by the author in the *German-Spanish Workshop on Moduli Spaces of Vector Bundles* at the University of Essen in February 2007.

To ease notation and to simplify the necessary proofs only the case of rank two vector bundles with determinant isomorphic to ω_X was considered.

These notes give a self-contained introduction to the moduli spaces of vector bundles and the generalized Θ divisor.

Keywords: moduli space, vector bundles on a curve, generalized Theta divisor.
MSC: (MSC 2000) 14H60, 14D20, 14F05.

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Contents

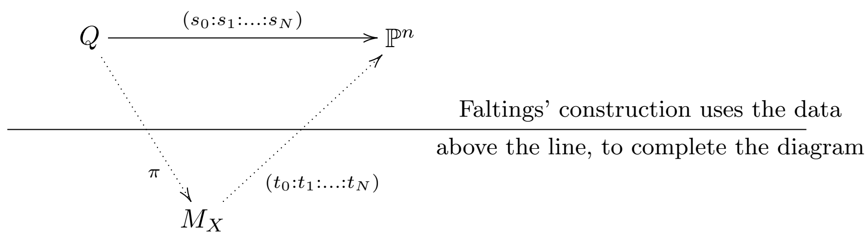
1. Outline of the construction

In his paper [8] Faltings gives a "GIT-free" construction of the moduli space M_X of semistable vector bundles on a smooth projective curve X. This moduli space was constructed using Mumford's *Geometric Invariant Theory* (see [22]). By construction M_X is a projective variety when we fix the rank and the degree of the vector bundles. Thus, we have an ample line bundle $\mathcal{L}$ on M_X.
We next present the central idea of Faltings' construction. The usual GIT construction of M_X yields the following data:

- (i) a quasi projective scheme Q
- (ii) a surjection $\pi : Q \to M_X$ with connected fibers
- (iii) a ample line bundle $\mathcal{L}$ on M_X
- (iv) global sections $\{s_i \in H^0(M_X, \mathcal{L})\}_{i=0,\dots,N}$ defining a finite morphism $s : M_X \to \mathbb{P}^N$

The idea is to recover these data without assuming a priori that M_X exists. We proceed as follows:

1. We give a vector bundle $\mathcal{E}$ on $\mathbb{P}(V) \times X$ parameterizing all semistable E of rank two and determinant ω_X on the curve X.
2. We construct a line bundle $\mathcal{O}_{\mathbb{P}(V)}(\Theta)$ on $\mathbb{P}(V)$ and global sections s_L of this line bundle.
3. We identify the open subset $Q \subset \mathbb{P}(V)$ where the global sections $\{s_L\}$ generate $\mathcal{O}_{\mathbb{P}(V)}(\Theta)$ with the locus parametrizing semistable vector bundles.
4. We show that the image of $s : Q \to \mathbb{P}^N$ defined by the global section $\{s_L\}$ has proper image.
5. We analyze the connected fibers of $s : Q \to \mathbb{P}^N$.



A sketch of the construction

To simplify the presentation, we discuss this construction only for vector bundles of rank two and determinant ω_X. We indicate in 8.3 the generalization to arbitrary ranks and degrees.

2. Background and notation

2.1. Notation

We fix a smooth projective curve X defined over an algebraically closed field k. As usual the genus of X is denoted by g. For coherent sheaves F and E on X we use the following notations.

k	-	algebraically closed field
X	-	smooth projective curve over k
g	-	the genus of X
ω_X	-	the dualizing sheaf $\omega_X = \Omega_{X/k}$
$\mathrm{tors}(E)$	-	the torsion subsheaf of E
$E^\vee$	-	the dual sheaf $E^\vee = \mathcal{H}om(E, \mathcal{O}_X)$
$\mathrm{rk}(E)$	-	the rank of E
$\deg(E)$	-	the degree of E
$\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$	-	the slope of E
$h^i(E)$	-	the dimension of the k-vector space $H^i(E) = H^i(X, E)$
$\chi(E)$	-	the Euler characteristic of E $\chi(E) = h^0(E) - h^1(E)$, and by the Riemann-Roch theorem $\chi(E) = \deg(E) - \mathrm{rk}(E)(g-1)$
$\mathrm{hom}(E, F)$	-	the dimension of the k-vector space $\mathrm{Hom}(E, F)$
$\mathrm{ext}^1(E, F)$	-	the dimension of the k-vector space $\mathrm{Ext}^1(E, F)$
$\chi(E, F)$	-	the difference $\mathrm{hom}(E, F) - \mathrm{ext}^1(E, F)$

If $\mathrm{tors}(E) = 0$ we say that E is torsion free, or a vector bundle. In general a torsion free sheaf is not a vector bundle, this is a special feature for coherent sheaves on smooth curves.

When classifying vector bundles we have two numerical invariants, the rank and the degree. These invariants do not vary in flat families. Thus, it is natural to classify vector bundles with given rank and degree. Classifying means to answer the following questions:

1. Do vector bundles of given rank r and degree d exist?
2. Do they form a family?
3. What is the dimension of this family?
4. Is this family irreducible?

The answer to question 1 is yes. To understand and answer question 2 for the case of rank two bundles with determinant ω_X, we consider the following functor from the category $(\underline{k-\mathrm{schemes}})$ of schemes over k to the category $(\underline{\mathrm{sets}})$ of sets

$$\mathrm{SU}_X(2, \omega_X) : (\underline{k-\mathrm{schemes}}) \to (\underline{\mathrm{sets}}) \qquad S \mapsto \mathrm{SU}_X(2, \omega_X)(S)$$

Here $\mathrm{SU}_X(2,\omega_X)(S)$ stands for the set

$$\mathrm{SU}_X(2,\omega_X)(S) := \left\{ \begin{array}{l} \text{rank two vector bundles } \mathcal{E} \text{ on } S \times X \\ \text{such that for each } s \in S(k) \text{ the bundle} \\ \mathcal{E}_s = \mathcal{E}_S|_{\{s\}\times X} \text{ is semistable and we} \\ \text{have an isomorphism} \\ \qquad \det(\mathcal{E}) \cong \mathrm{pr}_S^* L_S \otimes \mathrm{pr}_X^* \omega_X \\ \text{for some line bundle } L_S \in \mathrm{Pic}(S). \end{array} \right\} \Big/ \sim .$$

Here $\sim$ denotes an equivalence relation we discuss in 2.2. It turns out that $\sim$ should be chosen suitably, and we should restrict to S-equivalence classes of (semi)stable vector bundles. Now this functor cannot be represented by a scheme. That means the answer to the second question is no. However there exists a coarse moduli space which is also called SU_X which is an irreducible projective variety of dimension $3g-3$. We interpret this coarse moduli space as an affirmative answer to the second question. This also answers questions 3 and 4.

2.2. The Picard torus and the Poincaré line bundle

Fix an integer $d \in \mathbb{Z}$. We consider the Picard functor

$$\mathrm{Pic}_X^d : (\underline{k-\text{schemes}}) \to (\underline{\text{sets}}), \qquad S \mapsto \mathrm{Pic}_X^d(S)$$

where $\mathrm{Pic}_X^d(S)$ is the set of all line bundles $\mathcal{L}$ on $S \times X$ which have degree d on each fiber over S modulo a certain equivalence relation. Let us explain what the correct equivalence relation between two such line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ is next.

symbol	name	explanation
$\sim_1$	global equivalence	there exists an isomorphism $\mathcal{L}_1 \xrightarrow{\psi} \mathcal{L}_2$.
$\sim_2$	twist equivalence	there exists a line bundle M on S and an isomorphism $\mathcal{L}_1 \xrightarrow{\psi} \mathcal{L}_2 \otimes \mathrm{pr}_S^* M$.
$\sim_3$	local equivalence	there exists an open covering $S = \bigcup_i S_i$ and isomorphisms $\mathcal{L}_1\|_{S_i \times X} \xrightarrow{\psi_i} \mathcal{L}_2\|_{S_i \times X}$.
$\sim_4$	fiber-wise equivalence	for all points $s \in S(k)$ we have an isomorphism $\psi_s : \mathcal{L}_1\|_{\{s\}\times X} \to \mathcal{L}_2\|_{\{s\}\times X}$.

There are some implications among these relations, which also hold when we consider vector bundles on $S \times X$:

$\sim_1 \Longrightarrow \sim_2$ The strongest equivalence relation is $\sim_1$ and it implies the others.

$\sim_2 \Longrightarrow \sim_3$ Since for any line bundle M on S we have a covering $S = \bigcup_i S_i$ such that $M|_{S_i} \cong \mathcal{O}_{S_i}$, we deduce that $\mathcal{L}_1 \sim_2 \mathcal{L}_2$ implies $\mathcal{L}_1 \sim_3 \mathcal{L}_2$.

$\sim_3 \Longrightarrow \sim_4$ This implication is obvious.

If there are nontrivial line bundles M on S, then we conclude that $\sim_2$ does not imply $\sim_1$.

To see that $\sim_4$ does not imply $\sim_3$, take $S = \mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$ and $\mathcal{L}$ a nontrivial deformation. If we were to restrict ourselves to reduced schemes (which we don't do) then $\sim_4$ would imply $\sim_3$.

Since the local isomorphisms of line bundles form the glueing data for a line bundle on S we can conclude $\sim_3 \implies \sim_2$. It is this equivalence which we use for the definition of Pic^d_X and which guarantees that Pic^d_X is a scheme (or more precisely: is a functor which is represented by a scheme which is (usual abuse of notation) denoted by Pic^d_X).
The equivalence of the relations $\sim_2$ and $\sim_3$ comes from the following facts

1. If L_1 and L_2 are two line bundles of the same degree on X, then we have
$$L_1 \cong L_2 \iff \mathrm{Hom}(L_1, L_2) \neq 0.$$
2. For any line bundle L on X we have $\mathrm{Hom}(L, L) = k$, and $\mathrm{Isom}(L, L) = k^*$.

These properties do not hold for vector bundles of rank two or greater. However, restricting to stable bundles we regain these two properties.
To guarantee that the reader is really used to the concept of stability we repeat the basic properties around this concept in the next subsection.

2.3. Stability

For a sheaf E we define its slope $\mu(E)$ to be the quotient $\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$ of degree and rank. For a nonzero torsion sheaf we define its slope to be ∞, and for the zero sheaf we define $\mu(0) = -\infty$. Now we come to the definition:

Definition 2.1. A sheaf E on X is *stable* if for all proper subsheaves $E' \subsetneq E$ we have the inequality $\mu(E') < \mu(E)$.
A sheaf E on X is *semistable*, if for all subsheaves $E' \subset E$ we have the inequality $\mu(E') \leq \mu(E)$.

In their book [17] Huybrechts and Lehn introduced the following convention, which has become standard by now: if in a statement *(semi)stable* and $(\leq)$ both appear, then this statement stands for two, one with *stable* and strict inequality $<$ and the other with *semistable* and the mild inequality $\leq$. As a final test for the reader, and to show the use of this convention, we repeat the above

Definition 2.2. A sheaf E on X is *(semi)stable*, if for all proper subsheaves $E' \subsetneq E$ we have the inequality $\mu(E')\ (\leq)\ \mu(E)$.

Remark 2.3. A torsion sheaf T is by definition always semistable. It is stable, if the only proper subsheaf is the zero sheaf, that is when T is of length one which means (remember, that we work over an algebraically closed field) T is isomorphic to the skyscraper sheaf $k(P)$ for a closed point $P \in X(k)$.
If E is a sheaf of positive rank, then semistability implies that the torsion subsheaf of E is 0, which means E is a vector bundle.
If E is not semistable, then there exists a subsheaf $E' \subset E$ with $\mu(E') > \mu(E)$. Such a sheaf E' is called a *destabilizing* sheaf.

From the Riemann-Roch theorem for curves we deduce the inequality

$$\mu(E) = (g-1) + \frac{\chi(E)}{\mathrm{rk}(E)} \leq (g-1) + \frac{h^0(E)}{\mathrm{rk}(E)} \leq (g-1) + h^0(E)$$

for sheaves E of rank $\mathrm{rk}(E) \geq 1$. This inequality gives also an upper bound for the slope of all subsheaves of E provided that E is torsion free. Thus, there is a maximal slope for all subsheaves. A subsheaf $E_1 \subset E$ of this maximal slope with the maximal possible rank is called the *maximal destabilizing subsheaf* of E. It is easy to check that E_1 is semistable and unique. Considering the maximal destabilizing subsheaf of E/E_1 we derive inductively the existence of the

Harder-Narasimhan filtration. For any sheaf E there exists a unique filtration $0 \subset \mathrm{tors}(E) = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_l = E$ with the property that E_k/E_{k-1} is semistable and $\mu(E_{k+1}/E_k) < \mu(E_k/E_{k-1})$ (see [27]).

For semistable sheaves which are not stable we can refine the above. This is the

Jordan-Hölder filtration. If E is a semistable bundle then there exists a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_l = E$ where E_k/E_{k-1} is stable of slope $\mu(E_k/E_{k-1}) = \mu(E)$. Here we have no uniqueness (think of the direct sum of two stable objects of the same slope). However, the graded object $\mathrm{gr}(E) := \bigoplus_{k=1}^{l} E_k/E_{k-1}$ is unique.

S-equivalence. Two semistable sheaves E_1 and E_2 are defined to be S-equivalent when their graded objects $\mathrm{gr}(E_1)$ and $\mathrm{gr}(E_2)$ are isomorphic.

2.4. Properties of vector bundles on algebraic curves

Here we repeat basic properties of vector bundles on a smooth projective curve X of genus g over a field k. We sketch the proofs or give a reference.

Theorem 2.4. (The Riemann-Roch theorem and Serre duality on curves)
We have for the Euler characteristic $\chi(E)$ of the bundle E

$$\chi(E) = h^0(E) - h^1(E) = \deg(E) - \mathrm{rk}(E)(g-1)\,.$$

The $h^1(E)$-dimensional vector space $H^1(E)$ is dual to $H^0(\omega_X \otimes E^\vee)$.

For a proof see §IV.1 in Hartshorne's book [10]. We deduce the next result.

Corollary 2.5. *For a vector bundle E we have the implications*

$$\begin{array}{lcl} \chi(E) > 0 & \Longrightarrow & h^0(E) \geq \chi(E) > 0 \\ \chi(E) < 0 & \Longrightarrow & h^1(E) \geq -\chi(E) > 0 \\ \chi(E) < 0 & \Longrightarrow & h^0(\omega_X \otimes E^\vee) \geq -\chi(E) > 0\,. \end{array}$$

Proposition 2.6. (Some basic properties of vector bundles)
Let E and F be two vector bundles on a smooth projective curve X of genus g

defined over an algebraically closed field k.

(i) $\mu(E \otimes F) = \mu(E) + \mu(F)$.

(ii) *E is (semi)stable if and only if, for all surjections $E \to E''$, we have $\mu(E)(\leq)\mu(E'')$.*

(iii) *E is (semi)stable $\iff$ $E^\vee$ is (semi)stable.*

(iv) *If E is semistable of slope $\mu(E) < 0$, then $h^0(E) = 0$.*

(v) *If M is a line bundle on X, then E is (semi)stable $\iff$ $E \otimes M$ is (semi)stable.*

(vi) *If E is semistable and $\mu(E) > 2g-2$, then $h^1(E) = 0$.*

(vii) *If E is semistable and $\mu(E) > 2g-1$, then E is globally generated.*

(viii) *If E is globally generated, then there exists a short exact sequence*

$$0 \to \det(E)^{-1} \to \mathcal{O}_X^{\mathrm{rk}(E)+1} \to E \to 0\,.$$

(ix) *If E is globally generated, then there exists a short exact sequence*

$$0 \to \mathcal{O}_X^{\mathrm{rk}(E)-1} \to E \to \det(E) \to 0\,.$$

(x) *If E and F are semistable with $\mu(E) > \mu(F)$, then* $\mathrm{Hom}(E,F) = 0$.

(xi) *If E and F are stable with $\mu(E) = \mu(F)$, then* $\mathrm{Hom}(E,F) = 0$ *or* $E \cong F$.

(xii) *For E stable we have $Hom(E,E) = \{\lambda \cdot \mathrm{id}_E \mid \lambda \in k\}$.*

Proof. (i) Computing the rank and the degree of the tensor product yields this formula.
(ii) This is just the observation that we can reconstruct an epimorphism $\pi : E \to E''$ by knowing its kernel, and an injection $\iota : E' \to E$ can be reconstructed from the cokernel. Elementary manipulations of inequalities give for a short exact sequence $0 \to E' \to E \to E'' \to 0$ that $\mu(E')(\leq)\mu(E) \iff \mu(E)(\leq)\mu(E'')$.
(iii) Passing to the dual interchanges injections to E with surjections from E. Since both can be used to express (semi)stability the statement follows.
(iv) Indeed, any section $s \in H^0(E)$ with $s \neq 0$ defines an injection $\mathcal{O}_X \xrightarrow{s} E$.
(v) Twisting with M is an equivalence on the category of coherent sheaves.
(vi) Semistability of E is equivalent to the semistability of $\omega_X \otimes E^\vee$ by (iii) and (v). Now the statement follows from $\mu(\omega_X \otimes E^\vee) = 2g-2-\mu(E)$, the equality $h^1(E) = h^0(\omega_X \otimes E^\vee)$ and (iv).
(vii) For any point $P \in X(k)$ we have $\mu(E(-P)) = \mu(E) - 1 > 2g-2$. Thus, $H^1(E(-P)) = 0$ by (vi). Thus, applying the functor H^0 to the short exact sequence $0 \to E(-P) \to E \to E \otimes k(P) \to 0$ yields that E is generated by global sections at the point P.
(viii) If E is globally generated, then $H^0(E) \otimes \mathcal{O}_X \xrightarrow{\alpha} E$ is surjective. If $h^0(E) = \mathrm{rk}(E) + 1$, then we are done. If $h^0(E) = \mathrm{rk}(E)$, then α is an isomorphism. We consider the morphism $\mathcal{O}_X \oplus H^0(E) \otimes \mathcal{O}_X \xrightarrow{0 \oplus \alpha} E$; this has kernel $\mathcal{O}_X$ which is the determinant of E. Thus, we have to discuss only the case when $h^0(E) > \mathrm{rk}(E)+1$. The idea is to consider the Grassmannian G of $(\mathrm{rk}(E)+1)$-dimensional subspaces

of $H^0(E)$. For any point $P \in X(k)$, the $(\mathrm{rk}(E)+1)$-dimensional subspaces of G which do not surject to $E \otimes k(P)$ form a codimension two subvariety of G. Since a one dimensional family of codimension two subvarieties cannot cover G, a general subspace $V \subset H^0(E)$ of dimension $\mathrm{rk}(E)+1$ gives a surjection $V \otimes \mathcal{O}_X \xrightarrow{\beta} E$. The kernel of β is locally free of rank one, a line bundle. Considering the determinant we obtain $\ker\beta = \det(\ker\beta) = \det(V \otimes \mathcal{O}_X) \otimes \det(E)^{-1}$.
(ix) Here the proof works similarly to (viii). We show that the $(\mathrm{rk}(E)-1)$-dimensional subspaces V of $H^0(E)$ which do not give a $(\mathrm{rk}(E)-1)$-dimensional subspace of $E \otimes k(P)$ under the composition $V \otimes \mathcal{O}_X \to H^0(E) \otimes \mathcal{O}_X \to E \to E \otimes k(P)$ form a codimension two subvariety in the Grassmannian of $(\mathrm{rk}(E)-1)$-dimensional subspaces of $H^0(E)$. Thus, for a general $V \subset H^0(E)$ of dimension $\mathrm{rk}(E)-1$ we obtain a subbundle $V \otimes \mathcal{O}_X \to E$. The cokernel is a line bundle. Considering the determinant of the cokernel we obtain from $\det(V \otimes \mathcal{O}_X) \cong \mathcal{O}_X$, the asserted short exact sequence.
(x) For a nontrivial morphism $\alpha : E \to F$ we consider its image. Semistability of F implies $\mu(\mathrm{im}\alpha) \le \mu(F)$. Semistability of E implies by (ii) that $\mu(E) \le \mu(\mathrm{im}\alpha)$.
(xi) Suppose $\mathrm{Hom}(E,F) \neq 0$. As in the proof of (x) we consider the image $\mathrm{im}\alpha$ of a nontrivial $\alpha : E \to F$. We deduce $\mu(E) \le \mu(\mathrm{im}\alpha) \le \mu(F)$. Since $\mu(E) = \mu(F)$ and equality holds only when $E \cong \mathrm{im}\alpha \cong F$ is fulfilled, we conclude the statement.
(xii) Suppose that there exists an endomorphism $\beta \in \mathrm{End}(E)$ which is not a scalar multiple of the identity. Then for some $P \in X(k)$ we have the restricted morphism $\beta_P \in \mathrm{End}(E \otimes k(P))$ is not a scalar multiple of the identity. Take an eigenvalue $\lambda \in k$ of β_P. Then $\beta - \lambda \cdot \mathrm{id} \in \mathrm{End}(E)$ is not surjective. However by (xi) this endomorphism is an isomorphism, a contradiction. □

The next proposition is as elementary as each statement in the preceding one. However, here a new criterion for semistability pops up for the first time which we will investigate later on.

Proposition 2.7. *Let X be a smooth projective curve, and E and F two coherent sheaves with $F \neq 0$. If $H^*(E \otimes F) = 0$, i.e. $h^0(E \otimes F) = 0 = h^1(E \otimes F)$, then E is semistable.*

Proof. Assume we have two sheaves E and F with $H^*(E \otimes F) = 0$. We may assume that $E \neq 0$. Further we remark that $\mathrm{tors}(F) \neq 0$ implies E and F are torsion sheaves with disjoint support. Hence we assume that F is torsion free, this implies F is a vector bundle and from $H^0(E \otimes F) = 0$ we conclude that E is torsion free too. From $H^*(E \otimes F) = 0$ we deduce that $\chi(E \otimes F) = 0$, or $\mu(E) + \mu(F) = g - 1$. Suppose $E' \subset E$ is a subsheaf with $\mu(E') > \mu(E)$. From $\mu(E') + \mu(F) > g - 1$ we deduce that $h^0(E' \otimes F) > 0$. However, $H^0(E' \otimes F)$ is a subspace of $H^0(E \otimes F) = 0$, a contradiction. □

3. A nice over-parameterizing family

Our aim is to find a nice family parameterizing all semistable vector bundles on a smooth projective curve X of genus g. We concentrate on the case of rank two vector bundles of determinant isomorphic to ω_X. Fix a line bundle L of degree $g+1$.

Proposition 3.1. *For a semistable vector bundle E with $\mathrm{rk}(E) = 2$ and $\det(E) \cong \omega_X$ we have a short exact sequence*

$$0 \to L^{-1} \to E \to L \otimes \omega_X \to 0\,.$$

Proof. The vector bundle $E \otimes L$ is semistable by Proposition 2.6 (v) of slope $\mu(E \otimes L) = 2g$. By Proposition 2.6 (vii) $E \otimes L$ is globally generated and appears by 2.6 (ix) in a short exact sequence

$$0 \to \mathcal{O}_X \to E \otimes L \to \det(E \otimes L) \to 0\,.$$

Twisting this short exact sequence with L^{-1} we obtain the result. □

Thus, all semistable vector bundles E of rank two with $\det(E) \cong \omega_X$ can be parameterized by extensions in the vector space $\mathrm{Ext}^1(L \otimes \omega_X, L^{-1})$. Clearly not all these extensions parameterize semistable vector bundles. To construct a universal family over $(V \setminus 0)/k^*$ we consider the vector space

$$V := H^1(L^{-2} \otimes \omega_X^{-1})^\vee = \mathrm{Ext}^1(L \otimes \omega_X, L^{-1})^\vee\,.$$

When writing $\mathbb{P}(V)$ we follow Grothendieck's definition that $\mathbb{P}(V)$ is the space of linear hyperplanes in V. With our settings a k-valued point in $\mathbb{P}(V)$ corresponds to a one dimensional linear subspace in $\mathrm{Ext}^1(L \otimes \omega_X, L^{-1})$. We consider the product space $\mathbb{P}(V) \times X$ with the projections

$$\mathbb{P}(V) \xleftarrow{\;p\;} \mathbb{P}(V) \times X \xrightarrow{\;q\;} X\,.$$

On $\mathbb{P}(V) \times X$ we have a canonical extension class in

$$\begin{aligned} &\mathrm{Ext}^1(p^*\mathcal{O}_{\mathbb{P}(V)}(-1) \otimes q^*(L \otimes \omega_X), q^*L^{-1}) = \\ &\quad = H^1(\mathbb{P}(V) \times X, p^*\mathcal{O}_{\mathbb{P}(V)}(1) \otimes q^*(L^{-2} \otimes \omega_X^{-1})) \\ &\quad = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) \otimes H^1(X, L^{-2} \otimes \omega_X^{-1}) \\ &\quad = V \otimes H^1(L^{-2} \otimes \omega_X^{-1}) \\ &\quad = \mathrm{End}(V) \end{aligned}$$

corresponding to the identity $\mathrm{id}_V \in \mathrm{End}(V)$. This way we obtain the

Lemma 3.2. *There exists a universal extension sequence on $\mathbb{P}(V) \times X$, namely*

$$0 \to q^*L^{-1} \to \mathcal{E} \to p^*\mathcal{O}_{\mathbb{P}(V)}(-1) \otimes q^*(L \otimes \omega_X) \to 0\,.$$

Next we want to investigate, how over-parameterizing the family $\mathcal{E}$ is. To do so, we introduce the following notation: For a point $e \in \mathbb{P}(V)(k)$ we denote by $\mathcal{E}_e$ the sheaf $\mathcal{E}|_{\{e\} \times X}$ considered as a sheaf on X. To understand how many extensions parameterize a sheaf isomorphic to a given sheaf E (of rank two and determinant ω_X) we define the two sets

$$\begin{array}{rcl} \mathrm{Hom}^{\max}(L^{-1}, E) & := & \{\varphi : L^{-1} \to E \text{ which have a line bundle as cokernel}\} \\ \mathbb{P}(V)_E & := & \{e \in \mathbb{P}(V)(k) \mid e \leftrightarrow (L^{-1} \to \mathcal{E}_e \to L \otimes \omega_X) \text{ with } E \cong \mathcal{E}_e\}. \end{array}$$

Proposition 3.3. *For a stable bundle E of rank two and determinant* $\det(E) \cong \omega_X$ *we we have a 1-1 correspondence*

$$\mathrm{Hom}^{\max}(L^{-1}, E)/k^* \xleftrightarrow{1-1} \mathbb{P}(V)_E \ .$$

In particular, we see that $\mathbb{P}(V)_E$ is an irreducible quasi projective variety.

Proof. Suppose $e \in \mathbb{P}(V)_E$. Then there is, up to scalar multiple, only one isomorphism $\mathcal{E}_e \cong E$ by Proposition 2.6 (xii). Thus, the map $L^{-1} \to \mathcal{E}_e$ determines a unique subbundle isomorphic to L^{-1}. On the other hand for a subbundle $L^{-1} \to E$ considering the determinant gives that the cokernel is isomorphic to $L \otimes \omega_X$. The rest is exercise A3.26 (pages 645–647) in Eisenbud's book [7]. □

Our family $\mathcal{E}$ parameterized by $\mathbb{P}(V)$ induces locally any other family. The precise statement for this is the content of the following

Proposition 3.4. *Let $\mathcal{E}_S$ be a family of semistable bundles on X of rank two and determinant ω_X on $S \times X$, then there exists a covering $S = \bigcup_i S_i$ of S and morphisms $\varphi_i : S_i \to \mathbb{P}(V)$ such that we have isomorphisms*

$$\mathcal{E}_{S_i} = \mathcal{E}|_{S_i \times X} \cong (\varphi_i \times \mathrm{id}_X)^* \mathcal{E} \,.$$

Proof. We may assume that S is affine. We consider the morphisms

$$S \xleftarrow{p_S} S \times X \xrightarrow{q_S} X \ .$$

Since $\mathcal{E}_s$ is semistable of slope $g-1$ for all points $s \in S$, the sheaf $\mathcal{E}_s \otimes L$ is semistable of slope $2g$ for all $s \in S$. Therefore the vector bundle $\mathcal{E}_S \otimes q_S^* L$ has no higher direct image with respect to p_S. Take a closed point $s \in S(k)$. Having in mind that S was affine we obtain a surjection

$$(p_S)_*(\mathcal{E}_S \otimes q_S^* L) \xrightarrow{\beta} (p_S)_*(\mathcal{E}_S \otimes q_S^* L) \otimes k(s) = H^0(\mathcal{E}_s \otimes L) \ .$$

By Proposition 2.6 (vii) and (ix) we may find a section $m_s \in H^0(\mathcal{E}_s \otimes L)$ such that $\mathcal{O}_X \stackrel{m_s}{\to} \mathcal{E}_s \otimes L$ defines a subbundle. Since β is surjective, this section m_s can be lifted to a section $m_S : \mathcal{O}_{S \times X} \to \mathcal{E}_S \otimes q_S^* L$. After passing to a smaller neighborhood of s in S we may assume m_S defines a line subbundle. The cokernel is fiber-wise (with respect to p_S) isomorphic to $L^2 \otimes \omega_X$. Thus, after twisting with $q_S^* L^{-1}$ we obtain a short exact sequence

$$0 \to q_S^* L^{-1} \to \mathcal{E}_S \to p_S^* M \otimes q_S^*(L \otimes \omega_X) \to 0 \,,$$

where M is the line bundle $p_{S*}\mathcal{H}om(q_S^*(L^2 \otimes \omega_X), \mathrm{coker}(m_S))$. Thus, there exists a morphism $S \to \mathbb{P}(V)$ as indicated. □

Remark So far we have only seen that $\mathbb{P}(V)$ parameterizes all semistable bundles of rank two with determinant ω_X. It parameterizes also some unstable bundles. However, these unstable bundles are contained in a closed subscheme of positive codimension. Or formulated differently: the points $s \in \mathbb{P}(V)$ which correspond to semistable bundles form a dense Zariski open subset. If follows from GIT that for any parameter space the set of points corresponding to stable (or semistable) objects forms a Zariski open subset. See Remark 3 after Lemma 4.2.

4. The generalized Θ-divisor

We consider our family $\mathcal{E}$ of vector bundles on X parameterized by $\mathbb{P}(V)$ as obtained in the preceding section. We construct the generalized Θ-line bundle on $\mathbb{P}(V)$ next. If we proceeded formally, then everything would be very easy: since $\mathrm{Pic}(\mathbb{P}(V)) \cong \mathbb{Z}$ we just have to tell which $a \in \mathbb{Z}$ corresponds to the generalized Θ-divisor. We are not only giving the construction of $\mathcal{O}_{\mathbb{P}(V)}(\Theta)$ but we give invariant sections in this line bundle. However, we finally compute the number $a \in \mathbb{Z}$ for the sake of completeness.
The construction of the generalized Θ-divisor follows Drezet and Narasimhan's work [6].

4.1. The line bundle $\mathcal{O}_{\mathbb{P}(V)}(R \cdot \Theta)$

We consider the morphisms

$$\mathbb{P}(V) \xleftarrow{p} \mathbb{P}(V) \times X \xrightarrow{q} X$$

and the universal vector bundle $\mathcal{E}$ on $\mathbb{P}(V) \times X$ appearing in the short exact sequence of Lemma 3.2.
We fix a line bundle $A \in \mathrm{Pic}^{-2g}(X)$ of degree $-2g$. Since $\deg(A \otimes L \otimes \omega_X) = g-1$ we may assume that $h^0(A \otimes L \otimes \omega_X) = 0$. Now we define two vector bundles

$$A_{R,0} := A^{\oplus(R+1)} \quad \text{and} \quad A_{R,1} := A^{\otimes(R+1)}\,.$$

Computing the ranks and determinants of these two bundles we obtain the following equality in the Grothendieck group $K(X)$ of the curve X:

$$R[\mathcal{O}_X] = [A_{R,0}] - [A_{R,1}]\,.$$

The short exact sequence of Lemma 3.2 and $h^0(A \otimes L \otimes \omega_X) = 0$ gives that for $i \in \{0,1\}$ we have $p_*(\mathcal{E} \otimes q^* A_{R,i}) = 0$. Thus, we have two vector bundles

$$B_{R,0} := R^1 p_*(\mathcal{E} \otimes q^* A_{R,0}) \quad \text{and} \quad B_{R,1} := R^1 p_*(\mathcal{E} \otimes q^* A_{R,1})\,.$$

on $\mathbb{P}(V)$ of rank $4g(R+1)$. Now we have everything at our disposal to define the generalized Θ-divisor as the Cartier divisor associated to the Θ-line bundle

$$\mathcal{O}_{\mathbb{P}(V)}(R\cdot\Theta) := \det(B_{R,1})^{-1} \otimes \det(B_{R,0}) = \left(\Lambda^{4g(R+1)} B_{R,1}\right)^{-1} \otimes \left(\Lambda^{4g(R+1)} B_{R,0}\right)\,.$$

4.2. The invariant sections

Consider a nontrivial morphism $\alpha \in \mathrm{Hom}(A_{R,1}, A_{R,0})$. This morphism must be injective and defines a short exact sequence

$$0 \to A_{R,1} \xrightarrow{\alpha} A_{R,0} \to F(\alpha) \to 0\,,$$

where $F(\alpha)$ is the cokernel. In $K(X)$ the sheaf $F(\alpha)$ equals $\mathcal{O}_X^{\oplus R}$. In particular $\mu(F(\alpha)) = 0$ and for any sheaf $\mathcal{E}_s$ parameterized by $s \in \mathbb{P}(V)$ we have $H^*(X, \mathcal{E}_s \otimes$

$F(\alpha)) = 0 \iff H^1(X, \mathcal{E}_s \otimes F(\alpha)) = 0$. Applying the functor $R^* p_*(\mathcal{E} \otimes q^* \_\_)$ to the previous short exact sequence we obtain the long cohomology sequence

$$0 \longrightarrow p_*(\mathcal{E} \otimes q^* F(\alpha)) \longrightarrow B_{R,1} \xrightarrow{R^1\alpha} B_{R,0} \longrightarrow R^1 p_*(\mathcal{E} \otimes q^* F(\alpha)) \longrightarrow 0\,.$$

We obtain a section $\theta_\alpha \in \mathcal{O}_{\mathbb{P}(V)}(R \cdot \Theta)$ by passing to the top exterior power of $R^1\alpha$, in short: $\theta_\alpha := \Lambda^{4g(R+1)}(R^1\alpha) \in \mathrm{Hom}(\Lambda^{4g(R+1)} B_{R,1}, \Lambda^{4g(R+1)} B_{R,0}) = \Gamma(\mathcal{O}_{\mathbb{P}(V)}(R \cdot \Theta))$.

Proposition 4.1. *The global section $\theta_\alpha \in \Gamma(\mathcal{O}_{\mathbb{P}(V)}(R \cdot \Theta))$ vanishes in*

$$V(\theta_\alpha) = \{s \in \mathbb{P}(V) \,|\, H^*(\mathcal{E}_s \otimes F(\alpha)) \neq 0\}\,.$$

Proof. We remark that the morphism $R^1\alpha$ is surjective at the point $s \in \mathbb{P}(V)$ iff $H^1(\mathcal{E}_s \otimes F(\alpha)) = 0$. The last equality implies $H^0(\mathcal{E}_s \otimes F(\alpha)) = 0$ because $\chi(\mathcal{E}_s \otimes F(\alpha)) = 0$. Since $R^1\alpha \otimes k(s)$ is a morphism of vector spaces of the same dimension we have $R^1\alpha$ is surjective iff $\det(R^1(\alpha)) \neq 0$. □

Now let us investigate which cokernels $F(\alpha)$ yield nontrivial sections of the generalized Θ-bundle. Suppose that $F(\alpha)$ is not semistable. If $H^*(\mathcal{E}_s \otimes F) = 0$ for some $s \in \mathbb{P}(V)(k)$, then F is semistable by Proposition 2.7, a contradiction. Hence, we are interested only in those α with semistable cokernel $F(\alpha)$. On the other hand, let F be a semistable rank R vector bundle of determinant $\mathcal{O}_X$. Then $F \otimes A^{-1}$ is globally generated by Proposition 2.6 (vii). So there exists a short exact sequence $0 \to \det(F \otimes A^{-1})^{-1} \to \mathcal{O}_X^{\oplus(R+1)} \to F \otimes A^{-1} \to 0$ by (viii) of 2.6. Twisting this short exact sequence with A yields the next result:

Lemma 4.2. *If F is semistable of rank R and* $\det(F) \cong \mathcal{O}_X$*, then there exists a morphism $A_{R,1} \xrightarrow{\alpha} A_{R,0}$ with* $\mathrm{coker}(\alpha) = F(\alpha) \cong F$*. For $F(\alpha)$ not semistable we have $\theta_\alpha = 0$.*

Remark 1. It is very tempting to write θ_F instead of $\theta_{F(\alpha)}$ or θ_α. Doing this, the formula of Proposition 4.1 reads

$$V(\theta_F) = \{s \in \mathbb{P}(V) \,|\, H^*(\mathcal{E}_s \otimes F) \neq 0\}\,.$$

The author yielded to this temptation. However, we have in mind that we have to choose a framing of F (that is a short exact sequence $0 \to A_{R,1} \to A_{R,0} \to F \to 0$) to obtain the section θ_F. The vanishing locus is by 4.1 not affected by this choice.

Remark 2. On the reduced variety $\mathbb{P}(V)$ a reduced divisor D is given by the set of underlying points. Therefore, very sloppily one could define

$$\Theta_R := \{s \in \mathbb{P}(V) \,|\, H^*(\mathcal{E}_s \otimes F_R) \neq 0\}$$

for a fixed vector bundle F_R of rank R and determinant $\mathcal{O}_X$. If we had done so, then we could define $\mathcal{O}_{\mathbb{P}(V)}(R\cdot\Theta) := \mathcal{O}_{\mathbb{P}(V)}(\Theta_{F_R})$. However, it is not clear why F_R gives a reduced divisor of $\mathcal{O}_{\mathbb{P}(V)}(R\cdot\Theta)$.

Remark 3. If $R = 1$ then there exists only one semistable sheaf of rank R of determinant $\mathcal{O}_X$, namely $\mathcal{O}_X$ so the definition of Θ is independent of F_1 and reads

$$\Theta := \{s \in \mathbb{P}(V) \mid H^*(\mathcal{E}_s) \neq 0\}\,.$$

Since there are line bundles M of degree $g-1$ with $h^0(M) = 0$, we deduce that $E := M \oplus (\omega_X \otimes M^{-1})$ is a rank two vector bundle of determinant ω_X which satisfies $H^*(E) = 0$. So not all points $s \in \mathbb{P}(V)$ are in Θ. By Proposition 2.7 the complement of Θ is a dense open subset which parameterizes semistable vector bundles on X.

Remark 4. If $\mathcal{E}_S$ is a family of semistable sheaves of rank two and determinant ω_X on $S \times X$, then we can copy the above definition to define the generalized Θ divisor $\mathcal{O}_S(R\cdot\Theta)$ on S. The same holds for the section θ_α.

4.3. The multiplicative structure

So far we have defined $\mathcal{O}(R\cdot\Theta)$ for each $R \in \mathbb{N}$ individually. Next we show the expected equality that $\mathcal{O}(R\cdot\Theta) \cong \mathcal{O}(\Theta)^{\otimes R}$. We start with two non trivial morphisms $A_{R,1} \xrightarrow{\alpha} A_{R,0}$ and $A_{R',1} \xrightarrow{\alpha'} A_{R',0}$ defining two sections $\theta_\alpha \in \Gamma(\mathcal{O}(R\cdot\Theta))$ and $\theta_{\alpha'} \in \Gamma(\mathcal{O}(R'\cdot\Theta))$ respectively.
If we choose a general surjection $A_{R,0} \oplus A_{R',0} \xrightarrow{\pi} A$, then the composition $A_{R,1} \oplus A_{R',1} \to A_{R,0} \oplus A_{R',0} \to A$ will also be a surjection. To see this, we remark that $A_{R,1} \oplus A_{R',1} \to A_{R,0} \oplus A_{R',0}$ is a rank two vector subbundle, and we have an isomorphism $\mathrm{Hom}(A_{R,0} \oplus A_{R',0}, A) \cong \mathrm{Hom}(k^{R+R'+2}, k)$. Now for any point $P \in X(k)$ the morphisms $A_{R,0} \oplus A_{R',0} \to A$ for which the induced morphism $A_{R,1} \oplus A_{R',1} \to A$ is not surjective at P form a linear subspace V_P of codimension two in $\mathrm{Hom}(k^{R+R'+2}, k)$. The one dimensional family $\{V_P\}_{P\in X(k)}$ of those linear subspaces is contained in a divisor in $\mathrm{Hom}(k^{R+R'+2}, k)$. This proves the above claim. Now we take a π outside this divisor.
The kernel of the surjection $\pi|_{A_{R,1}\oplus A_{R',1}} : A_{R,1} \oplus A_{R',1} \to A$ is a vector bundle of rank one and determinant $A^{\otimes(R+R'+1)}$. Thus, it is isomorphic to $A_{R+R',1}$. From these two surjections we obtain therefore the commutative diagram with exact

rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A_{R+R',1} & \longrightarrow & A_{R,1} \oplus A_{R',1} & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \alpha\oplus\alpha'} & & \| & & \\
0 & \longrightarrow & A_{R+R',0} & \longrightarrow & A_{R,0} \oplus A_{R',0} & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & F(\tilde{\alpha}) & = & F(\alpha) \oplus F(\alpha') & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Applying the functor $R^1p_*(\mathcal{E}\otimes q^*\_\_)$ to this diagram yields an isomorphism $\mathcal{O}((R+R')\Theta) \cong \mathcal{O}(R\cdot\Theta)\otimes\mathcal{O}(R'\cdot\Theta)$ identifying the section $\theta_\alpha\cdot\theta_{\alpha'}$ with $\theta_{\tilde{\alpha}}$. Thus we obtain

Lemma 4.3. *We have isomorphisms* $\mathcal{O}((R+R')\cdot\Theta) \cong \mathcal{O}(R\cdot\Theta)\otimes\mathcal{O}(R'\cdot\Theta)$. *Under these isomorphisms the global sections* $\theta_\alpha\cdot\theta_{\alpha'}$ *and* $\theta_{\tilde{\alpha}}$ *are identified. This also reads* $\theta_{F\oplus F'} = \theta_F\cdot\theta_{F'}$.

We finish this section with an aside which gives a useless description of the line bundles $\mathcal{O}_{\mathbb{P}(V)}(R\cdot\Theta)$ on $\mathbb{P}(V)$.

Proposition 4.4. *On* $\mathbb{P}(V)$ *with ample generator* $\mathcal{O}_{\mathbb{P}(V)}(1)$ *of the Picard group* $\mathrm{Pic}(\mathbb{P}(V))$ *we have an isomorphism* $\mathcal{O}(R\cdot\Theta)\cong\mathcal{O}_{\mathbb{P}(V)}(2gR)$.

Proof. Consider the push forward Rp_* of the universal short exact sequence of Lemma 3.2 to obtain the long exact cohomology sequence on $\mathbb{P}(V)$.

$$0\to p_*\mathcal{E}\to\mathcal{O}_{\mathbb{P}(V)}(-1)\otimes H^0(X,L\otimes\omega_X)\xrightarrow{\gamma}\mathcal{O}_{\mathbb{P}(V)}\otimes H^1(X,L^{-1})\to R^1p_*\mathcal{E}\to 0\,.$$

The complex $A_{1,1}\xrightarrow{\alpha}A_{1,0}$ equals $\mathcal{O}_X$ in the Grothendieck group $K(X)$. Thus we can replace $R^ip_*\mathcal{E}$ in the above exact sequence by $R^ip_*(\mathcal{E}\otimes q^*F(\alpha))$. Thus this exact sequence, and the one before Proposition 4.1 yield $\mathcal{O}_{\mathbb{P}(V)}(\Theta)=\mathcal{O}(1)^{\otimes h^0(X,L\otimes\omega_X)}$. Since $h^0(X,L\otimes\omega_X)=2g$ we are done. $\square$

5. Raynaud's vanishing result for rank two bundles

In 1982 Raynaud published in [26] the following result:

Theorem 5.1. *(***Proposition 1.6.2 in** [26]*)*
Let E be a rank two vector bundle of degree $2g-2$ on a smooth projective curve X. We have an equivalence:

$$E \text{ is semistable} \iff \text{there exists } L \in \operatorname{Pic}^0(X) \text{ with } H^*(E \otimes L) = 0\,.$$

We remark that $H^*(E \otimes L) = 0$ implies the semistability of E, as we have seen in Proposition 2.7. Thus, we have to show the implication $\implies$ only. We give different proofs of this result which also catch a glimpse of the generalizations of Le Potier (see [20]) and Popa (see [25]) of this result to the case of higher ranks. But first things first:

5.1. The case of genus zero and one

For the case of genus zero and one we can without further difficulties prove the following generalization of Theorem 5.1

Proposition 5.2. *Let X be a smooth projective curve of genus $g \in \{0,1\}$. For a rank r vector bundle of slope $g-1$ we have the equivalence:*

$$E \text{ is semistable} \iff \text{there exists } L \in \operatorname{Pic}^0(X) \text{ with } H^*(E \otimes L) = 0\,.$$

Proof for genus zero. If E is semistable, then there exists no global section of E because the slope of the semistable bundle $\mathcal{O}_X$ is greater than the slope of E. We have $\chi(E) = 0$ and we conclude $h^0(E) = 0 = h^1(E)$. In short, we have $H^*(E \otimes \mathcal{O}_X) = 0$ as claimed. □

Proof for genus one. Here we proceed inductively by proving the statement: For E semistable of rank r and slope zero, there are at most r line bundles $L \in \operatorname{Pic}^0(X)$ with $H^*(E \otimes L) \neq 0$. For rank $r = 1$, E is a line bundle of degree zero. Thus for all $L \ncong E^\vee$ we have $H^*(E \otimes L) = 0$.
Now we assume the statement holds for $r-1$ and take a semistable E of slope zero and rank r. Suppose we have $H^*(E \otimes L_r) \neq 0$ for a line bundle L_r of degree zero. We obtain a nontrivial morphism $L_r^\vee \to E$. The cokernel we denote by E''. It is a sheaf of rank $r-1$ and degree zero. If E'' were not semistable, then we would have a surjection $E'' \to F$ to a sheaf of negative degree. However, the composition $E \to E'' \to F$ contradicts the semistability of E. Thus, E'' is semistable too and we have a short exact sequence

$$0 \to L_r^\vee \to E \to E'' \to 0$$

of semistable sheaves of slope zero. If a line bundle L of degree zero is not isomorphic to L_r, then we have $H^*(E \otimes L) = H^*(E'' \otimes L)$ and conclude by induction. □

5.2. Preparations for the proof of 5.1

We collect here some facts which will be used in the sequel. The first fact is just an observation which allows us to concentrate on stable bundles for the proof of 5.1. The next is a detail we will need later on.

Lemma 5.3. *If E is a semistable but not stable vector bundle of rank two and degree $2g-2$, then there exist line bundles $L \in \mathrm{Pic}^0(X)$ with $H^*(E \otimes L) = 0$.*

Proof. Since E is not stable we have a short exact sequence

$$0 \to L_1 \to E \to L_2 \to 0\,,$$

where the L_i are line bundles of degree $g-1$. Thus, they define two Θ-divisors in $\mathrm{Pic}^0(X)$, namely the divisors

$$\Theta_i = \{L \in \mathrm{Pic}^0(X) | H^*(L \otimes L_i) \neq 0\}\,.$$

If L is any line bundle of degree zero not contained in $\Theta_1 \cup \Theta_2$, then we have $H^*(L \otimes L_i) = 0$ and the above short exact sequence yields $H^*(L \otimes E) = 0$. □

Lemma 5.4. *Let E be a rank two vector bundle, and L be a line bundle of degree d. Suppose that $\hom(L, E) \geq 2$ and $\deg(E) \neq 2d$ holds. Under these assumptions there exists $\alpha : L \to E$ such that the cokernel $\mathrm{coker}(\alpha)$ has torsion different from zero.*

Proof. Take two linearly independent morphisms $\beta_1, \beta_2 \in \mathrm{Hom}(L, E)$. We consider the resulting morphism $\beta = \beta_1 \oplus \beta_2 : L \oplus L \to E$. The morphism β cannot be surjective, because we assumed $\deg(E) \neq \deg(L \oplus L)$. Thus, we may take a geometric point P in the support of $\mathrm{coker}(\beta)$. It follows that the composite morphism $L \oplus L \xrightarrow{\beta} E \to E \otimes k(P) \cong k(P) \oplus k(P)$ is not surjective. Thus, for a suitable nontrivial linear combination $\alpha = \lambda_1\beta_1 + \lambda_2\beta_2$ we have $\alpha \otimes k(P) = 0$. □

5.3. A proof for genus two using the rigidity theorem

Let us start with the following result about proper morphisms (see [5]).

Theorem 5.5. (Rigidity theorem)
We consider the following morphisms of varieties

$$Y \xleftarrow{f} X \xrightarrow{g} Z$$

where f is proper with connected fibers. If for one point $y_0 \in Y$ the inverse image $f^{-1}(y_0)$ is mapped to a point under g, then there exists an open neighborhood U of y_0 in Y such that for all $y \in U$ the image $g(f^{-1}(y))$ is a point.

Proof. Take an affine open subset $\mathrm{Spec}(A) \subset Z$ which contains the point $g(f^{-1}(y_0))$. Then $X' = g^{-1}\mathrm{Spec}(A)$ is open and contains the fiber over y_0. Thus the complement $X'' = X \setminus X'$ is closed and maps under the proper morphism f to some closed set $Y'' = f(X'')$. By construction y_0 is contained in the open subset $U = Y \setminus Y''$. For $y \in U$ we have by construction that $f^{-1}(y)$ is contained in X' and by assumption $f^{-1}(y)$ is proper and connected. Thus $g(f^{-1}(y)) \subset \mathrm{Spec}(A)$ is proper and connected. Hence, $g(f^{-1}(y))$ is a point. □

Proof of Theorem 5.1 **for curves of genus two**
Let E be a semistable vector bundle of rank two and degree two on a smooth projective curve X of genus two. We want to show that there exist line bundles $L \in \mathrm{Pic}^0(X)$ with $H^*(E \otimes L) = 0$. We may assume that E is stable by Lemma 5.3. If for some $L \in \mathrm{Pic}^0(X)$ we have $\mathrm{Hom}(L, E) = 0$, then $H^*(E \otimes L^\vee) = 0$ and we are done. Thus, we may assume by Lemma 5.4 that $\hom(L, E) = 1$ for all $L \in \mathrm{Pic}^0(X)$. Indeed, if $\alpha : L \to E$ had cokernel with torsion supported at P, then α would give rise to some $\tilde{\alpha} : L(P) \to E$. This contradicts the stability of E. Thus, we have morphisms $\alpha_L : L \to E$, they are unique up to scalar multiplication, and their image is a line subbundle.
This way we obtain for a point $P \in X(k)$ the morphism

$$\beta_P : \mathrm{Pic}^0(X) \to \mathbb{P}(E \otimes k(P)) \cong \mathbb{P}^1 \qquad L \mapsto (\ L \otimes k(P) \xrightarrow{\alpha_L \otimes k(P)} E \otimes k(P)\ )\,.$$

This morphism is just the specialization of the X-morphism β over $P \in X$.

$$\begin{array}{ccc} \mathrm{Pic}^0(X) \times X & \xrightarrow{\quad\beta\quad} & \mathbb{P}(E) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

We take a proper curve $C \subset \mathrm{Pic}^0(X)$ which is contracted to a point under β_P which exists for dimensional reasons. Now we apply the rigidity theorem to the morphism

$$X \xleftarrow{\mathrm{pr}_2} C \times X \xrightarrow{\beta} \mathbb{P}(E)\ .$$

By definition the fiber of pr_2 over P is contracted to a point by β. Thus, by the rigidity theorem almost all fibers are contracted to a point by β. Thus, all line bundles parameterized by C describe the same line subbundle of E which is absurd. □

5.4. A proof based on Clifford's theorem

For the sake of completeness we repeat here Clifford's theorem which will be the main ingredient in our proof of Theorem 5.1. For a proof we refer to Hartshorne's book [10, Theorem IV.5.4].

Theorem 5.6. (Clifford's theorem) *Let L be a line bundle on a smooth projective curve X with $0 \leq \deg(L) \leq 2g-2$. Then we have the estimate $h^0(L) \leq \frac{\deg(L)}{2} + 1$, and equality holds only for a finite number of line bundles.*

Proof of Theorem 5.1 **for curves of genus** $g \geq 2$
We fix a stable vector bundle E of rank two and degree $2g-2$. Let us assume that for all line bundles L of degree zero we have $\mathrm{Hom}(L, E) \neq 0$. We consider the following schemes $\{B^d\}_{d \in \mathbb{N}}$ with the reduced subscheme structure. Indeed, we are

computing dimensions in this proof only, so the underlying scheme structure is of no interest for us.

$$B^d := \left\{ (L \xrightarrow{\alpha} M) \text{ with } \begin{array}{l} L \in \operatorname{Pic}^0(X) \text{ and } M \in \operatorname{Pic}^d(X) \\ \alpha \in (\operatorname{Hom}(L,M) \setminus 0)/k^* \\ \text{and } \operatorname{Hom}(M,E) \neq 0 \end{array} \right\}.$$

Since E is stable we have $B^d = \emptyset$ for $d \geq g-1$. We consider the natural projection $B^d \xrightarrow{\beta_d} \operatorname{Pic}^0(X)$ which maps a triple $(L \xrightarrow{\alpha} M)$ to L. Suppose $d > 0$ and $(L \xrightarrow{\alpha} M) \in B^d$. For $P \in \operatorname{supp}(\operatorname{coker}(\alpha))$, we have that $(L \xrightarrow{\alpha} M(-P)) \in B^{d-1}$. Thus, we have inclusions $\operatorname{im}(\beta_d) \subseteq \operatorname{im}(\beta_{d-1}) \subseteq \ldots \subseteq \operatorname{im}(\beta_0)$. If $\operatorname{im}(\beta_0) \subsetneq \operatorname{Pic}^0(X)$, then we have for any line bundle L not contained in $\operatorname{im}(\beta_0)$ that $\operatorname{Hom}(L,E) = 0$ which contradicts our assumption. Let now d be the maximal integer, such that $\operatorname{im}(\beta_d) = \operatorname{Pic}^0(X)$. We consider the open set $U_d = \operatorname{Pic}^0(X) \setminus \operatorname{im}(\beta^{d+1})$. Take a triple $(L \xrightarrow{\alpha} M)$ with $L \in U_d$. Next we show that each nontrivial morphism $M \to E$ has torsion free cokernel. If $M \to E$ has cokernel with torsion supported in P, then we obtain a nonzero morphism $M(P) \to E$. However, from the composition $L \to M \to M(P)$ we deduce that $L \in \operatorname{im}(\beta_{d+1})$, so M is a line subbundle and $\hom(M,E) = 1$ by Lemma 5.4.

Now we consider the projection $B^d \xrightarrow{\alpha_d} \operatorname{Pic}^d(X)$ assigning $(L \xrightarrow{\alpha} M) \mapsto M$. Since α_d is a $X^{(d)}$-bundle, we obtain that $\alpha_d(\beta_d^{-1}(U_d))$ is a family of line bundles M of degree d of dimension at least $g-d$. We have an inclusion

$$\alpha_d(\beta_d^{-1}(U_d)) \subseteq \{M \in \operatorname{Pic}^d(X) \mid \hom(M,E) = 1 \text{ and } M \to E \text{ a line subbundle } \}.$$

It follows that the Quot scheme of quotients of E of degree $2g-2-d$ is at least $g-d$ dimensional in a neighborhood of the point $[E \to E/M]$. Thus, the tangent space $\operatorname{Hom}(M, E/M)$ of the Quot scheme is at least of that dimension. However $\operatorname{Hom}(M,E/M) := H^0((E/M) \otimes M^\vee) = H^0(\det(E) \otimes M^{\otimes -2})$. By Clifford's theorem the dimension is at most $g-d$ and this equality can hold only for finitely many M. Since $g-d$ is positive, this contradicts our assumption that $\operatorname{Hom}(L,E) \neq 0$ for all $L \in \operatorname{Pic}^0(X)$, and we are done. □

5.5. Generalizations and consequences

First of all we deduce two consequences from Theorem 5.1.

Corollary 5.7. *Let $r \geq 2$ be an integer. We consider a family $\mathcal{E}$ on $S \times X$ of rank two vector bundles on X of degree $2g-2$. The base points of the linear subsystem of $\mathcal{O}_S(r \cdot \Theta)$ given by the sections θ_F where F runs through all rank r vector bundles of trivial determinant on X is the set of all points $s \in S$ such that $\mathcal{E}_s$ is not semistable.*

Proof. We know by Theorem 5.1 that points $s \in S$ which are not base points parameterize semistable bundles $\mathcal{E}_s$. Suppose now that $E = \mathcal{E}_s$ is semistable. We have to find a rank r bundle F with trivial determinant such that $H^*(E \otimes F) = 0$. We find this by setting $F := L_1 \oplus L_2 \oplus \ldots \oplus L_r$. □

Proposition 5.8. *Let X be a smooth projective curve. For a family $\mathcal{E}$ over $S \times X$ of rank two vector bundles of degree $2g-2$ on X the set $S^{\mathrm{ss}} := \{s \in S \mid \mathcal{E}_s \text{ semistable}\}$ is an open subset of X.*

Proof. We consider the morphisms $S \xleftarrow{p} S \times X \xrightarrow{q} X$. Suppose $s \in S^{\mathrm{ss}}$, then there exists by Theorem 5.1 a line bundle L such that $H^*(X, \mathcal{E}_s \otimes L) = 0$. Thus, the coherent sheaf $R^1p_*(\mathcal{E} \otimes q^*L)$ is zero at s. So there exists an open $U \subset S$ containing s such that $R^1p_*(\mathcal{E} \otimes q^*L)$ is zero on U. By base change we have $H^1(X, \mathcal{E}_t \otimes L) = 0$ for all $t \in U$. From Riemann-Roch we deduce $H^*(X, \mathcal{E}_t \otimes L) = 0$ for $t \in U$. Theorem 5.1 tells us that all these $\mathcal{E}_t$ are semistable. □

The next results give us further equivalent conditions for semistability and show that these results generalize to sheaves E of arbitrary rank and degree. However, these results will not be used in the sequel. So the reader may skip to the next section. The following result holds:

Theorem 5.9. *Let X be a smooth projective curve of genus g. Then there exists a vector bundle P_{2g+1} on X such that for sheaves E of rank two and degree $2g-2$ on X the following conditions are equivalent:*
- *(i)* *E is semistable.*
- *(ii)* $\mathrm{Hom}(L, E) = 0$ *for a line bundle L of degree zero.*
- *(iii)* $H^*(E \otimes M) = 0$ *for a line bundle M of degree zero.*
- *(iv)* $\mathrm{Hom}(P_{2g+1}, E) = 0$.

We have proved the equivalence of (*i*)–(*iii*) in Theorem 5.1. Considering the Fourier-Mukai transform on the pair Jacobian and Picard torus of X the condition (iv) is deduced (see [12] for details and a proof).

Theorem 5.10. *Let X be a smooth projective curve of genus g. Furthermore two integers $r \geq 1$ and d are given. Then there exist integers R and D, and a vector bundle $P_{r,d}$ depending on r, d and g, such that for all vector bundles E of rank r and degree d the following are equivalent:*
- *(i)* *E is semistable.*
- *(ii)* $H^*(E \otimes F) = 0$ *for a sheaf F of rank R and degree D.*
- *(iii)* $\mathrm{Hom}(P_{r,d}, E) = 0$.

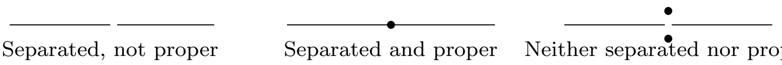
The equivalence of (i) and (ii) is shown in [25]. The equivalence of (ii) and (iii) is essentially linear algebra and carried out in [13].

Remark. It follows that the construction, which we present here for the rank two case, allows an obvious generalization to the case of arbitrary rank and degree. However, in our case we can take line bundles as the parameters for our generalized Θ-divisors which is very convenient.

6. Semistable limits

6.1. Limits of vector bundles

We repeat here the definition of separatedness for noetherian schemes over an algebraically closed field. The concept of a limit is formalized in algebraic geometry by the concept of discrete valuation domain R. We have $\mathrm{Spec}(R) = \{\eta, 0\}$ where η denotes the generic point and 0 denotes the closed point. The standard example is of course the localization of a curve in a smooth point. The closed point 0 is the limit of the generic point η.
The Hausdorff separation axiom from topology is too strong for the Zariski topology and is replaced by the concept of separatedness:
X is separated if for all maps $\psi : \{\eta\} \to X$ there exists at most one extension $\tilde{\psi} : \mathrm{Spec}(R) \to X$.
If there exists a unique extension $\tilde{\psi} : \mathrm{Spec}(R) \to X$, then X is proper. The usual picture is the following:



Separated, not proper — Separated and proper — Neither separated nor proper

Morphisms $S \to M_X$ to the (potential) moduli space M_X of vector bundles on X should correspond to families $\mathcal{E}_S$ of vector bundles on $S \times X$. Thus, we take a DVR R with $\mathrm{Spec}(R) = \{\eta, 0\}$ a vector bundle $\mathcal{E}_\eta$ on $\{\eta\} \times X$. The first thing we need it the following lemma which does not generalize to higher dimensional varieties or singular curves.

Lemma 6.1. *Let X be a smooth projective curve. Any family $\mathcal{E}_\eta$ of vector bundles on $\{\eta\} \times X$ can be extended to a family $\mathcal{E}_R$ on $\mathrm{Spec}(R) \times X$.*

Proof. Set $r := \mathrm{rk}(\mathcal{E}_\eta)$. We consider the morphisms

$$\begin{array}{ccccc}
\{\eta\} & \xleftarrow{p} & X_\eta := \{\eta\} \times X & \xrightarrow{q} & \\
\downarrow & & \downarrow & & \searrow \\
\mathrm{Spec}(R) & \xleftarrow{p} & X_R := \mathrm{Spec}(R) \times X & \xrightarrow{q} & X\,.
\end{array}$$

We may replace $\mathcal{E}_\eta$ by $\mathcal{E}_\eta \otimes q^*L$. Thus, we may assume $\mathcal{E}_\eta$ is globally generated and its determinant is of degree at least $2g$. On $\{\eta\} \times X$ we have by Proposition 2.6 a short exact sequence

$$0 \to \mathcal{O}_{X_\eta}^{\oplus r-1} \to \mathcal{E}_\eta \to \det(\mathcal{E}_\eta) \to 0\,.$$

The properness of the Picard functor $\mathrm{Pic}(X)$ guarantees that there exists an extension (a unique) $\mathcal{L}_R$ of the line bundle $\det(\mathcal{E}_\eta)$. On the other hand $\mathcal{O}_{X_R}^{\oplus r-1}$ is an extension of $\mathcal{O}_{X_\eta}^{\oplus r-1}$ to X_R.
The extension $\mathcal{E}_\eta$ is given by a $k(\eta)$-valued section α_η of the coherent sheaf $R^1p_*(\mathcal{H}om(\mathcal{L}_R, \mathcal{O}_{X_R}^{\oplus r-1}))$. Changing α_η by an element of $k(\eta)^*$ does not affect the isomorphism class of $\mathcal{E}_\eta$. This way we can obtain an R-valued section α_R of this

sheaf which gives α_η at the point η. Now since $H^0(R^1p_*(\mathcal{H}om(\mathcal{L}_R, \mathcal{O}_{X_R}^{\oplus r-1}))) = \mathrm{Ext}^1(\mathcal{L}_R, \mathcal{O}_{X_R}^{\oplus r-1})$ we are done. □

6.2. Changing limits — elementary transformations

Now let $\mathcal{E}_R$ be a vector bundle on X_R. The restriction $\mathcal{E}_0$ of $\mathcal{E}_R$ to the closed fiber $X_0 = p^{-1}(0)$ is sometimes called the limit of $\mathcal{E}_\eta = \mathcal{E}_R|_{X_\eta}$. Since X_0 is a Cartier divisor on X_R the structure sheaf $\mathcal{O}_{X_0}$ considered as an $\mathcal{O}_{X_R}$-module is of projective dimension one. Thus, every X_0-vector bundle F_0 is as a sheaf on X_R of projective dimension one. This observation allows the following construction:

Construction: Elementary transformation
Let $\mathcal{E}_0$ be the restriction of the vector bundle $\mathcal{E}_R$ to X_0. Furthermore, let $\mathcal{E}_0 \stackrel{\pi_0}{\to} F_0$ be a surjection of vector bundles on X_0. Composing we obtain a surjection $\mathcal{E}_R \stackrel{\pi}{\to} F_0$ of sheaves on X_R. The kernel of π is of projective dimension zero. Thus, $\ker(\pi)$ is a vector bundle. This vector bundle $\mathcal{E}'_R$ is called the elementary transformation of $\mathcal{E}_R$ along F_0. The bundle $\mathcal{E}'_R$ appears in a short exact sequence

$$0 \to \mathcal{E}'_R \to \mathcal{E}_R \stackrel{\pi}{\to} F_0 \to 0\,.$$

When restricting this short exact sequence to X_0 and denoting the restriction of $\mathcal{E}'_R$ to X_0 by $\mathcal{E}'_0$ we obtain the exact sequence

$$0 \to \left(\mathrm{Tor}_1^{\mathcal{O}_{X_R}}(\mathcal{O}_{X_0}, F_0) = F_0\right) \to \mathcal{E}'_0 \to \mathcal{E}_0 \stackrel{\pi_0}{\to} F_0 \to 0\,.$$

Obviously the sheaves $\mathcal{E}_\eta$ and $\mathcal{E}'_\eta$ coincide. Summing up we have the next result:

Proposition 6.2. *Let $\mathcal{E}'_R$ be the elementary transformation of $\mathcal{E}_R$ along F_0 then we have an isomorphism over X_η: $\mathcal{E}'_\eta \cong \mathcal{E}_\eta$.*
Over the special fiber X_0 we have two short exact sequences

$$0 \to F_0 \to \mathcal{E}'_0 \to \ker(\pi_0) \to 0 \quad \textit{and} \quad 0 \to \ker(\pi_0) \to \mathcal{E}_0 \to F_0 \to 0\,.$$

In short we may say: Elementary transformations along F_0 do not change the bundle over the generic point, they transform the quotient F_0 of $\mathcal{E}_0$ to a subsheaf of $\mathcal{E}'_0$.

6.3. Example: Limits are not uniquely determined

The following example shows that there are infinitely many possible limits for a vector bundle $\mathcal{E}_\eta$. The example is intended to show that it makes sense to restrict to a nice class of bundles to avoid this plenitude of limits.

Proposition 6.3. *Let $\mathcal{E}_R$ be a rank two vector bundle on X_R such that* $\det(\mathcal{E}_0) \cong \mathcal{O}_{X_R}$ *and $\mathcal{E}_0$ is semistable. Then for any X-line bundle L of degree* $\deg(L) \geq 2g$ *there exists a elementary transformation $\mathcal{E}'_R$ of $\mathcal{E}_R$ such that $\mathcal{E}'_R \cong L \oplus L^{-1}$.*

Proof. By Proposition 2.6 $\mathcal{E}_0 \otimes L$ is globally generated, and there exists by part (ix) a surjection $\mathcal{E}_0 \otimes L \to L^{\otimes 2}$. Twisting with L^{-1} we obtain a short exact sequence

$$0 \to L^{-1} \to \mathcal{E}_0 \stackrel{\pi_0}{\to} L \to 0\,.$$

Applying an elementary transformation of $\mathcal{E}_R$ along L we obtain by Proposition 6.2 a vector bundle $\mathcal{E}'_R$ with $\mathcal{E}'_\eta \cong \mathcal{E}_\eta$ and a short exact sequence

$$0 \to L \to \mathcal{E}'_0 \to L^{-1} \to 0\,.$$

However, $\mathrm{Ext}^1(L^{-1}, L) = H^1(L^{\otimes 2}) = 0$ which implies that $\mathcal{E}'_0 \cong L \oplus L^{-1}$. □

6.4. Semistable limits exist

The next result of Langton (see [18]) tells us that a semistable vector bundle $\mathcal{E}_\eta$ on X_η can be extended to a bundle $\mathcal{E}_R$ on X_R such that the restriction $\mathcal{E}_0$ of $\mathcal{E}_R$ to the special fiber X_0 is also semistable. The proof uses that we can *stabilize* a given extension by elementary transformations. The new idea is that we can control the number of these elementary transformations by the badness, a number introduced in [11] by the author. Please remember that Langton's theorem (as well as the proof below) holds for arbitrary rank and determinant.

Theorem 6.4. (Langton's theorem on the existence of semistable limits) *Let $\mathcal{E}_\eta$ be a semistable rank two vector bundle with determinant $q^*\omega_X$. Then there exists a vector bundle $\mathcal{E}_R$ on X_R such that $\mathcal{E}_R|_{X_\eta} \cong \mathcal{E}_\eta$ and $\mathcal{E}_0 = \mathcal{E}_R|_{X_0}$ is semistable.*

Proof. By Lemma 6.1 there exist extensions $\mathcal{E} = \mathcal{E}_R$ of $\mathcal{E}_\eta$ to X_R. We take such an extension $\mathcal{E}$ and assign it an integer $\mathrm{bad}(\mathcal{E})$ - the badness of $\mathcal{E}$.

$$\mathrm{bad}(\mathcal{E}) := \min_{L \in \mathrm{Pic}^0(X)} \{\mathrm{length}(R^1p_*(\mathcal{E} \otimes q^*L))\}\,.$$

By Theorem 5.1 the semistability of $\mathcal{E}_\eta$ implies that there exist $L \in \mathrm{Pic}^0(X)$ such that the sheaf $R^1p_*(\mathcal{E} \otimes q^*L)$ is zero at η. We conclude that the badness is well defined. By base change and Theorem 5.1 we deduce the equivalence

$$\mathcal{E}_0 \text{ is semistable} \iff \mathrm{bad}(\mathcal{E}) = 0.$$

It is enough to show the following statement: If $\mathcal{E}$ is an extension with positive badness, then there exists an elementary transformation $\mathcal{E}'$ of $\mathcal{E}$ such that $\mathrm{bad}(\mathcal{E}') < \mathrm{bad}(\mathcal{E})$. Let us show this. Since $\mathcal{E}_0$ is not semistable, there exists a line bundle quotient $\mathcal{E}_0 \to F$ on X_0 with $\deg(F) < g - 1$. This implies $\chi(F \otimes L) = \chi(F) < 0$ for all $L \in \mathrm{Pic}^0(X)$. We consider the elementary transformation $\mathcal{E}'$ of $\mathcal{E}$ along F:

$$0 \to \mathcal{E}' \to \mathcal{E} \to F \to 0\,.$$

Next we choose a line bundle $L \in \mathrm{Pic}^0(X)$ such that $\mathrm{bad}(\mathcal{E}) = \mathrm{length}(R^1p_*(\mathcal{E} \otimes q^*L))$. This implies that $p_*(\mathcal{E} \otimes q^*L) = 0$ because it is torsion free and zero at η. From the above short exact sequence we obtain

$$0 \to p_*(F \otimes q^*L) \to R^1p_*(\mathcal{E}' \otimes q^*L) \to R^1p_*(\mathcal{E} \otimes q^*L) \to R^1p_*(F \otimes q^*L) \to 0\,.$$

We have $p_*(F \otimes q^*L) = H^0(F \otimes L)$ and $R^1p_*(F \otimes q^*L) = H^1(F \otimes L)$. Thus, both are sheaves of finite length on R. We deduce that $R^1p_*(\mathcal{E}' \otimes q^*L)$ is also a sheaf of finite length which we compute to be

$$\mathrm{length}(R^1p_*(\mathcal{E}' \otimes q^*L)) = \mathrm{length}(R^1p_*(\mathcal{E} \otimes q^*L)) + h^0(F \otimes L) - h^1(F \otimes L)\,.$$

Having in mind that $\chi(F \otimes L)$ is negative, we are done. □

6.5. Semistable limits are almost uniquely determined

We have seen in Theorem 6.4 that semistable limits exist. However the example of Proposition 6.3 shows that we should not hope for a unique limit. One might hope that semistable limits are unique. Unfortunately this is not the case. Let us illustrate it:
Let L be a line bundle of degree $g-1$ which is not isomorphic to $\omega_X \otimes L^{-1}$. The vector space $\mathrm{Ext}^1(L, \omega_X \otimes L^{-1})$ is of dimension $g-1$. If $g \geq 2$, then there exist nontrivial extensions fitting in short exact sequences

$$0 \to \omega_X \otimes L^{-1} \to E \to L \to 0\,.$$

Since $\deg(\omega_X \otimes L^{-1}) = \deg(L) = g-1$, the vector bundle E is semistable but not stable. Taking the pull back $\mathcal{E} = q^*E$ we obtain a sheaf on X_R which gives E when restricted to X_0. The short exact sequence above allows an elementary transformation of $\mathcal{E}$ along L. This way we obtain (by Proposition 6.2) a sheaf $\mathcal{E}'$ on X_R with restriction $E' = \mathcal{E}'|_{X_0}$ fitting into an exact sequence

$$0 \to L \to E' \to \omega_X \otimes L^{-1} \to 0\,.$$

Since the extension was not trivial E' and E are both semistable but not isomorphic. However, both appear as a limit of $\mathcal{E}_\eta = \mathcal{E}|_{X_\eta} \cong \mathcal{E}'|_{X_\eta}$.
We note, that both sheaves are S-equivalent because

$$\mathrm{gr}(E) = \mathrm{gr}(E') = L \oplus \left(\omega_X \otimes L^{-1}\right).$$

Thus, we can only hope that the S-equivalence class of a limit is unique. This is generally the case and is shown in our case now.

Proposition 6.5. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two rank two vector bundles on X_R such that $\mathcal{E}_\eta \cong \mathcal{E}'_\eta$. If the restrictions $\mathcal{E}_0$ and $\mathcal{E}'_0$ to X_0 are semistable, then they are S-equivalent.*

Proof. We consider the vector bundle $\mathcal{E}' \otimes \mathcal{E}^\vee \otimes q^*\omega_X$. When applying R^1p_* to it, we obtain by base change a sheaf whose restriction to the generic point η equals $H^1(X_\eta, \mathcal{E}'_\eta \otimes \mathcal{E}_\eta^\vee \otimes q^*\omega_X)$. This is the Serre dual of $\mathrm{Hom}(\mathcal{E}'_\eta, \mathcal{E}_\eta)$. Since both sheaves are isomorphic there exist non trivial homomorphisms and we conclude, that $R^1p_*(\mathcal{E}' \otimes \mathcal{E}^\vee \otimes q^*\omega_X) \neq 0$. So the specialization of this sheaf to the special point $0 \in \mathrm{Spec}(R)$ is not zero. Again by base change and Serre duality we conclude $\mathrm{Hom}(\mathcal{E}'_0, \mathcal{E}) \neq 0$. We obtain a morphism $\varphi : \mathcal{E}'_0 \to \mathcal{E}$ between two semistable rank two vector bundles of the same determinant. If φ is an isomorphism, then we are done. Thus, we may assume $\mathrm{rk}(\ker(\varphi)) = 1$. The image $M = \mathrm{im}(\varphi)$ of φ is at the same time quotient and subbundle of semistable bundles with slope $g-1$. Thus, M is a degree $g-1$ line bundle. Considering the determinant we conclude $\ker(\varphi) \cong \mathrm{coker}(\varphi)$. This finishes the proof because the graded objects are given by $\mathrm{gr}(\mathcal{E}'_0) = \ker(\varphi) \oplus M$, and $\mathrm{gr}(\mathcal{E}_0) = M \oplus \mathrm{coker}(\varphi)$. □

S-equivalence prevents us from non-separated moduli functors

So by passing to the moduli functor of S-equivalence classes, we obtain a proper moduli functor. That is there exist semistable extensions of $\mathcal{E}_\eta$ and all those extensions are in the same S-equivalence class. Note, that S-equivalence classes are required to circumvent a non-separated moduli functor. It causes several problems because we don't have a universal object.

However, for a stable vector bundle E the S-equivalence class of E is the class of vector bundles isomorphic to E. Thus, on a dense open subset of the moduli space there is no difference between S-equivalence and isomorphism, whereas for strictly semistable bundles it makes a difference.

7. Positivity

7.1. Notation and preliminaries

For this section we fix the following objects:

X	-	our smooth projective curve of genus $g_X \geq 2$ over $k = \bar{k}$
C	-	a smooth projective curve over k of genus g_C
p, q	-	the natural projections $C \xleftarrow{p} C \times X \xrightarrow{q} X$
$A \boxtimes B$	-	a shorthand for the tensor product $p^*A \otimes q^*B$
$\mathcal{E}$	-	a rank two vector bundle on $C \times X$
$\mathcal{E}_c$	-	for a point $c \in C(k)$ the X-vector bundle $q_*(\mathcal{E} \otimes p^*k(c))$, that is the vector bundle parameterized by the point c
$\mathcal{E}_x$	-	for $x \in X(k)$ the vector bundle $p_*(\mathcal{E} \otimes q^*k(x))$ on C

As a warm up we consider a well-known result (see Corollary 7.3). Apart from the obvious proof via Proposition 7.1, we deduce this result from Proposition 7.2 which allows a generalization to vector bundles.

Proposition 7.1. *Let $\mathcal{L}$ be a line bundle on $C \times X$ with $\mathcal{L}_c \cong \mathcal{L}_{c'}$ for all points $c, c' \in C(k)$. Then there exist line bundles M and N on C and X respectively, such that $\mathcal{L} \cong M \boxtimes N$.*

Proof. Let $N = \mathcal{L}_c$ for a point $c \in C$. Since $R^1p_*(\mathcal{L} \otimes q^*N^{-1})$ is a vector bundle, it follows that $p_*(\mathcal{L} \otimes q^*N^{-1})$ also commutes with base change (see [23, page 50,Corollary 2]). Thus, $M := p_*(\mathcal{L} \otimes q^*N^{-1})$ is a line bundle on C. The composition morphism $p^*M \to p^*(p_*(\mathcal{L} \otimes q^*N^{-1})) \to \mathcal{L} \otimes q^*N^{-1}$ is an isomorphism on all fibers of p. Thus, it is an isomorphism on $C \times X$. $\square$

Proposition 7.2. *Suppose $\mathcal{L}$ is a line bundle on $C \times X$. The degrees $d_1 := \deg(\mathcal{L}_x)$ and $d_2 := \deg(\mathcal{L}_c)$ of the line bundles parameterized by X and C do not depend on the choice of the points $x \in X(k)$ and $c \in C(k)$. We obtain two morphisms*

$$\varphi_1 : X \xrightarrow{x \mapsto \mathcal{L}_x} \mathrm{Pic}^{d_1}(C) \quad \text{and} \quad \varphi_2 : C \xrightarrow{c \mapsto \mathcal{L}_c} \mathrm{Pic}^{d_2}(X)\,.$$

For the principal polarizations Θ_1 and Θ_2 on $\mathrm{Pic}^{d_1}(C)$ and $\mathrm{Pic}^{d_2}(X)$, the degrees of X in $\mathrm{Pic}^{d_1}(C)$ and C in $\mathrm{Pic}^{d_2}(X)$ coincide, that is

$$\deg_X(\varphi_1^*\mathcal{O}_{\mathrm{Pic}^{d_1}(C)}(\Theta_1)) = \deg_C(\varphi_2^*\mathcal{O}_{\mathrm{Pic}^{d_2}(X)}(\Theta_2))\,.$$

Proof. First we remark that the theorem is invariant under twisting L with line bundles of type p^*M or p^*N. Therefore, we may assume $d_1 = g_C - 1$ and $d_2 = g_X - 1$. On $\mathrm{Pic}^{g_C-1}(C)$ the Θ-line bundle is the determinant of cohomology, that is:

$$\begin{aligned}\deg_X(\varphi_1^*\mathcal{O}_{\mathrm{Pic}^{d_1}(C)}(\Theta_1)) &= -\deg(R^*p_*(\mathcal{L})) \quad \text{this we can rewrite as}\\ &= -\int_X \mathrm{ch}(R^*p_*(\mathcal{L}))\\ &\quad \text{using Grothendieck-Riemann-Roch we obtain}\\ &= -\int_{C\times X}(\mathrm{ch}(\mathcal{L})\cdot p^*\mathrm{Td}(C))\\ &= -\int_{C\times X}(1+c_1(\mathcal{L})+\tfrac{c_1^2(\mathcal{L})}{2})(1-\tfrac{p^*\omega_C}{2})\\ &= \tfrac{1}{2}\int_{C\times X}\left(p^*\omega_C.c_1(\mathcal{L})-c_1^2(\mathcal{L})\right)\ .\end{aligned}$$

Since $p^*\omega_C$ is numerically equivalent to $2g_C-2$ fibers of p and the intersection of $c_1(\mathcal{L})$ with a fiber of p is the degree $d_2=g_X-1$, we conclude that

$$\deg_X(\varphi_1^*\mathcal{O}_{\mathrm{Pic}^{d_1}(C)}(\Theta_1)) = (g_C-1)(g_X-1)-\frac{1}{2}\int_{C\times X}c_1^2(\mathcal{L})\,.$$

By symmetry the right hand side is also the degree of $\varphi_2^*\mathcal{O}_{\mathrm{Pic}^{d_2}(X)}(\Theta_2)$. □

Corollary 7.3. *Let $\mathcal{L}$ be a line bundle on $C\times X$. If for any two points $c,c'\in C(k)$ the line bundles $\mathcal{L}_c$ and $\mathcal{L}_{c'}$ are isomorphic, then for all points $x,x'\in X(k)$ the line bundles $\mathcal{L}_x$ and $\mathcal{L}_{x'}$ are isomorphic.*

Proof. Indeed, this result can be obviously deduced from Proposition 7.1. In order to deduce it from Proposition 7.2, we consider the maps $\varphi_1:X\to\mathrm{Pic}^{d_1}(C)$ and $\varphi_2:C\to\mathrm{Pic}^{d_2}(X)$ from Proposition 7.2. By assumption C is mapped to a point. Hence $\deg_C(\varphi_2^*\Theta_2)=0$. By Proposition 7.2 this implies $\deg_X(\varphi_1^*\Theta_1)=0$. Since the Theta divisor Θ_1 is ample X is also mapped to a point. □

We assume $\det(\mathcal{E}_c)\cong\omega_X$ for all $c\in C(k)$. This implies that $\det(\mathcal{E})=M\boxtimes\omega_X$ by Proposition 7.1. Suppose that $\mathcal{E}_c$ is semistable. By Theorem 5.1 there exists an $L\in\mathrm{Pic}^0(X)(k)$ such that $H^*(X,\mathcal{E}_c\otimes L)=0$. By base change this implies that $R^1p_*(\mathcal{E}\otimes q^*L)$ is not supported in c. Thus, it is a torsion sheaf of finite length. $\mathcal{E}_c$ is therefore semistable for all $c\notin\mathrm{supp}R^1p_*(\mathcal{E}\otimes q^*L)$. As seen in Theorem 6.4 we can elementary transform $\mathcal{E}$ to become semistable in the remaining points.
We will see C as a parameter curve mapping into the moduli space M_X (which we have not constructed so far). The aim of this section is to show that the intersection number of C with the Θ-divisor is not negative, and that this intersection number is zero iff all bundles parameterized by C are S-equivalent.
It will be convenient for us that the vector bundles $\mathcal{E}_x$ on C will have degree $2g_C-2$. If the degree of $\mathcal{E}_x$ is an even number, then this is obtained by twisting $\mathcal{E}$ with a line bundle p^*M where M is of degree $g_C-1-\frac{\deg(\mathcal{E}_x)}{2}$. In case $\deg(\mathcal{E}_x)$ is odd, we replace C by a irreducible curve $\psi:C'\xrightarrow{2:1}C$ and $\mathcal{E}$ by $\mathcal{E}':=(\psi\times\mathrm{id}_X)^*\mathcal{E}$. This way we obtain the same bundles on X doubly parameterized by C' and an even degree of $\mathcal{E}'_x$.
Summing up we have the following data

$\det(\mathcal{E}_c)\cong\omega_X$	for all $c\in C(k)$,
$\mathcal{E}_c$ is semistable	for all $c\in C(k)$, and
$\deg(\mathcal{E}_x)=2g_C-2$.	

7.2. Positivity

Global sections of $\mathcal{O}(\Theta)$

The Θ-line bundle on C is now the inverse of the determinant of cohomology. Since vector bundles $\mathcal{E}_c$ and $\mathcal{E}_{c'}$ of the same rank and isomorphic determinant coincide in the Grothendieck group $K(X)$, we see that $\det(R^*p_*(\mathcal{E}\otimes q^*L))^{-1}$ does not depend on the choice of $L \in \mathrm{Pic}^0(X)$. This line bundle is the generalized Θ-line bundle for the family $\mathcal{E}$ of vector bundles on X parameterized by C and therefore denoted by $\mathcal{O}_C(\Theta_C)$. See Remark 4 after Proposition 4.1.

Proposition 7.4. *The Θ-line bundle $\mathcal{O}_C(\Theta_C)$ is base point free. In particular, we have* $\deg(\mathcal{O}_C(\Theta_C)) \geq 0$.

Proof. We have seen in Theorem 5.1 that there exists for each $c \in C(k)$ a line bundle $L \in \mathrm{Pic}^0(X)$ such that $H^*(\mathcal{E}_c \otimes L) = 0$. This means that the section of $\mathcal{O}_C(\Theta_C)$ corresponding to L does not vanish at c by Proposition 4.1. □

7.3. The case of $\deg(\mathcal{O}_C(\Theta_C)) = 0$

Here we prove a special case of Theorem I.4 from Faltings' article [8]. The proof mainly follows the idea given there.

Theorem 7.5. *We have an equivalence of* $\deg(\mathcal{O}_C(\Theta_C)) = 0$, *and* $\mathcal{E}_c \sim_S \mathcal{E}_{c'}$ *for all points* $c, c' \in C(k)$. *In this case the sheaves* $\mathcal{E}_x$ *and* $\mathcal{E}_{x'}$ *are isomorphic for all* $x, x' \in X(k)$.

The proof of Theorem 7.5 follows directly from the subsequent results 7.6 – 7.9.

Lemma 7.6. *If all the vector bundles parameterized by C are S-equivalent, then the degree of $\mathcal{O}_C(\Theta_C)$ is zero.*

Proof. Pick a $c \in C(k)$ and consider the semistable vector bundle $E := \mathrm{gr}(\mathcal{E}_c)$. Then there exists a line bundle $L \in \mathrm{Pic}^0(X)$ such that $H^*(X, E \otimes L) = 0$. Since $\mathcal{E}_c \otimes L$ can be filtered with quotients the direct summands of $E \otimes L$, we deduce that $H^*(\mathcal{E}_c \otimes L) = 0$. This holds for all $c \in C(k)$. Thus, by Proposition 4.1 the corresponding global section of $\mathcal{O}_C(\Theta_C)$ is nowhere vanishing. □

Lemma 7.7. *If the degree of $\mathcal{O}_C(\Theta_C)$ is zero, then there exists a vector bundle F on C such that $\mathcal{E}_x \cong F$ for all points $x \in X(k)$.*

Proof. We take a line bundle L of degree zero on X such that $H^*(\mathcal{E}_c \otimes L) = 0$ for a fixed $c \in C(k)$. Since the degree of $\mathcal{O}_C(\Theta_C)$ is zero we deduce from Proposition 4.1 that $H^*(\mathcal{E}_{c'} \otimes L) = 0$ for points $c' \in C(k)$, in other words $p_*(\mathcal{E} \otimes q^*L) = 0 = R^1p_*(\mathcal{E} \otimes q^*L)$.
We choose a point $x \in X(k)$. When applying $R^*p_*(\mathcal{E} \otimes q^*\_\_)$ to the short exact sequence

$$0 \to L \to L(x) \to (L(x) \otimes k(x) \cong k(x)) \to 0\,,$$

we deduce from $R^*p_*(\mathcal{E} \otimes q^*L) = 0$ that $\mathcal{E}_x \cong R^*p_*(\mathcal{E} \otimes q^*L(x))$. The function $X(k) \to \mathbb{N}$ which assigns $x' \mapsto h^1(\mathcal{E}_c \otimes L(x - x'))$ is upper semi-continuous. Since it is zero for $x' = x$, we have that for almost all $x' \in X(k)$ the cohomology

$H^*(\mathcal{E}_c \otimes L(x-x'))$ is zero. As before, this implies $p_*(\mathcal{E} \otimes q^*L(x-x')) = 0 = R^1p_*(\mathcal{E} \otimes q^*L(x-x'))$ for almost all x'. Now from this and the short exact sequence

$$0 \to L(x-x') \to L(x) \to k(x') \to 0$$

we deduce that $\mathcal{E}_{x'} \cong R^*p_*(\mathcal{E} \otimes q^*L(x))$. Thus, $\mathcal{E}_x \cong \mathcal{E}_{x'}$ for almost all $x' \in X(k)$. Since $X(k)$ is infinite, this eventually yields the asserted statement. □

Lemma 7.8. *If the C-vector bundle F of Lemma 7.7 is semistable, then there exists a vector bundle E on X such that $\mathcal{E}_c \cong E$ for all points $c \in C(k)$.*

Proof. We have by Theorem 5.1 a line bundle M on C such that $F \otimes L$ has no cohomology. Therefore, interchanging the role of X and C in Lemma 7.7, we obtain the result. □

Lemma 7.9. *If the C-vector bundle F of Lemma 7.7 is unstable, then there exist two line bundles L_1 and L_2 on X of degree $g_X - 1$ such that,* $\mathrm{gr}(\mathcal{E}_c) \cong L_1 \oplus L_2$.

Proof. Let $0 \to M_1 \to F \to M_2 \to 0$ be the short exact sequence with M_1 the maximal destabilizing line bundle. That is $\deg(M_1) > \deg(M_2)$. It follows that $q_*\mathcal{H}om(p^*M_1, \mathcal{E})$ is a line bundle. Thus we have by Proposition 7.1 a short exact sequence

$$0 \to M_1 \boxtimes L_1 \to \mathcal{E} \to M_2 \boxtimes L_2 \to 0$$

inducing the above sequence on all fibers of q.

Since $\mathcal{E}_c$ contains L_1 we conclude from semistability that $\deg(L_1) = g_X - 1 - a$, and $\deg(L_2) = g_X - 1 + a$ for some integer $a \geq 0$. Choosing a line bundle $L \in \mathrm{Pic}^0(X)$ such that $R^*p_*(\mathcal{E} \otimes q^*L) = 0$, we obtain from the above short exact sequence that $R^1p_*(M_2 \boxtimes (L_2 \otimes L)) = 0$ which implies $h^1(X, L_2 \otimes L) = 0$. Hence, $h^0(X, L_2 \otimes L) = a$. Analogously we obtain that $h^1(X, L_1 \otimes L) = a$. Applying $p_*(q^*L \otimes \_\_)$ to the above short exact sequence yields

$$p_*(\mathcal{E} \otimes q^*L) = 0 \to M_2^{\oplus a} \xrightarrow{\alpha} M_1^{\oplus a} \to 0 = R^1p_*(\mathcal{E} \otimes q^*L)\,.$$

Thus, α must be an isomorphism, which is only possible for $a = 0$. □

8. The construction

8.1. Constructing the moduli space of vector bundles

Now we have everything we need to perform the construction outlined at the beginning.

(1) We start with our nice overparameterizing family $\mathbb{P}(V)$ from Section 3 which parameterizes all semistable vector bundles E on X of rank two with $\det(E) \cong \omega_X$.

(2) On $\mathbb{P}(V)$ we have the Θ-line bundle $\mathcal{O}_{\mathbb{P}(V)}(\Theta)$ and for any $L \in \mathrm{Pic}^0(X)$ a global section $s_L \in H^0(\mathcal{O}_{\mathbb{P}(V)}(\Theta))$ with vanishing divisor

$$\Theta_L = \{[E] \in \mathbb{P}(V) \,|\, H^*(E \otimes L) \neq 0\}\,.$$

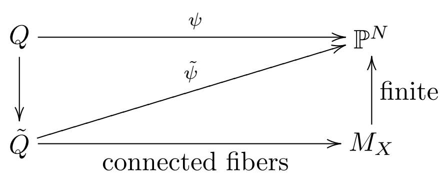
(3) The intersection $B := \cap_{L \in \mathrm{Pic}^0(X)} \Theta_L$ is the base locus of the linear system spanned by the divisors Θ_L. By Raynaud's Theorem 5.1 the base locus is given by

$$B = \{[E] \in \mathbb{P}(V) \,|\, [E] \text{ is not semistable}\}.$$

We can write B as a finite intersection $B = \cap_{i=0,\dots,N} \Theta_{L_i}$. We denote the complement of B by $Q := \mathbb{P}(V) \setminus B$. This is the semistable locus.

(4) The global sections $\{s_{L_i}\}_{i=0,\dots,N}$ define a morphism $\psi : Q \to \mathbb{P}^N$. The image of this morphism is a proper subset by Langton's Theorem 6.4. Thus, we do not change the image when passing to the blow up $\tilde{\psi} : \tilde{Q} = \mathrm{Bl}_B \mathbb{P}(V) \to \mathbb{P}^N$.

(5) A connected closed curve C is contracted by $\tilde{\psi}$ if and only if all vector bundles parameterized by C are S-equivalent. Thus, the moduli space M_X of all S-equivalence classes of semistable vector bundles of rank two and determinant ω_X is the Stein factorization of $\tilde{\psi}$.



Let us check that M_X is a coarse moduli space:

Points of M_X.

First of all each point of $M_X(k)$ corresponds by construction to the image of a bundle parameterized by Q, a semistable bundle. By Theorem 7.5 the S-equivalence class of this bundle is uniquely determined.

Functoriality

Let $\mathcal{E}_S$ be a family of semistable vector bundles on $S \times X$. By Proposition 3.4, we have a covering $S = \cup_{i=1,\dots,n} S_i$ and morphisms $S_i \to Q$ which induce $\mathcal{E}_{S_i}$. Since the locus $\mathbb{P}(V)_E \subset Q$ is connected (see Proposition 3.3), the invariant functions $S_i \to M_X$ glue along the intersections. Thus, we get a morphism $S \to M_X$.

8.2. Consequences from the construction

Let us illustrate some direct consequences for the moduli space $M_X = \mathrm{SU}_X(2, \omega_X)$.

M_X **is unirational:** Indeed, we have constructed M_X as the image of some open subset $Q \subset \mathbb{P}(V)$.

The dimension of M_X **is** $3g-3$**:** Again we use the morphism $Q \to M_X$. We have $\dim(Q) = \dim \mathbb{P}(V) = 5g-2$. The fibers are the subschemes $\mathbb{P}(V)_E$ which are open subsets in $\mathbb{P}(\mathrm{Hom}(L^{-1}, E)^\vee)$ and therefore of dimension $2g+1$.

The Θ-line bundle is ample and globally generated: This is a conclusion from Theorems 5.1 and 7.5. This result is the basis for the presented construction and provides us with a finite morphism $M_X \to \mathbb{P}^N$ given by the global section of $\mathcal{O}_{M_X}(\Theta)$.

8.3. Generalization to the case of arbitrary rank and degree

It is the hope of the author to show that the Θ-line bundle $\mathcal{O}_{M_X}(\Theta)$ together with the alternative criterion of Theorem 5.1 for semistability are the key ingredients for the construction. Indeed, even some well known results, like Langton's result (Theorem 6.4), allow new proofs in this context.

We will next point out how the construction can be generalized to arbitrary rank and degree. Note that we can either fix the determinant or only its degree. The numbers in the following table refer to the table from part 8.1.

(1) If we fix the determinant, then we can choose again a projective space as our starting point. For fixed degree we have to consider a projective bundle over the Picard torus $\mathrm{Pic}^d(X)$. Grothendieck's Quot scheme (see [9]) also works and was the classical origin. However, it has the disadvantage that it also parameterizes sheaves which are not vector bundles.

(2) Our construction of the Θ-line bundle is just the general construction of Drezet and Narasimhan (see [6]) specialized to our case.

(3) The generalization of Raynaud's Theorem 5.1 to vector bundles of higher rank and arbitrary determinant began in Faltings' article [8, Theorem 1.3]. Popa gave in [25] the best known explicit bounds. They depend only on the rank of the bundle in question. Thus, passing to a fixed multiple of the generalized Θ-divisor everything works fine.

(4) As in step (2), our proof of Langton's Theorem 6.4 was just a special version of Langton's original result from [18] adapted to the rank two case.

(5) This can be copied verbatim.

Beauville's articles [3] and [4] survey the theory of moduli spaces of vector bundles on algebraic curves with a focus to the generalized Θ-divisor.

9. Prospect to higher dimension

The generalization of Faltings' construction to moduli spaces of coherent sheaves on higher dimensional varieties started in 1977 when Barth showed in [2] that we can use a construction similar to the one in 8.1 to construct the moduli space of rank two bundles on $\mathbb{P}^2$ by considering jumping lines (see also Le Potier's presentation in [19]).

Let us first introduce the notation to describe Barth's approach. Let E be a rank two vector bundle on $\mathbb{P}^2$ with trivial determinant, and second Chern number $n = \int_{\mathbb{P}^2} c_2(E) \in \mathbb{Z}$. The vector bundle E is stable if $H^0(E) = 0$, and semistable when $H^0(E(-1)) = 0$. This easy description of (semi)stability is due to the fact that $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ and E is of rank two.

The equivalent of Raynaud's theorem (Theorem 5.1) is the following result:

Theorem 9.1. (Grauert-Mülich Theorem)
For a vector bundle E of rank two with $c_1(E) = 0$ on $\mathbb{P}^2$ we have an equivalence

$$E \text{ is semistable} \iff E|_H \cong \mathcal{O}_H^{\oplus 2} \text{ for a general linear hyperplane } H\,.$$

Proof. See [24] II Theorem, 2.1.4. □

Note that $E|_H \cong \mathcal{O}_H^{\oplus 2} \iff H^*(E(-1)|_H) = 0$. The linear hyperplanes H undertake the task of the line bundles L in the Picard torus $\mathrm{Pic}^0(X)$. Like the Picard torus they form a nice family — $\mathbb{P}_2$ - the dual projective space of lines in $\mathbb{P}^2$. If we suppose that there exists an overparameterizing scheme Q as before, we can define for any line $l \in \mathbb{P}_2$ a divisor Θ_l given by

$$\Theta_l = \{[E] \in Q \,|\, H^*(E(-1)|_l) \neq 0\}\,.$$

Since all lines are rationally equivalent the line bundle $\mathcal{O}_Q(\Theta) := \mathcal{O}_Q(\Theta_l)$ does not depend on the choice of $l \in \mathbb{P}_2$, and any line defines a global section $s_l \in H^0(\mathcal{O}_Q(\Theta))$ with vanishing divisor Θ_l. For a certain equivalence relation we obtain as before a finite morphism $M \to \mathbb{P}^N$ given by the global sections s_l. This morphism assigns a vector bundle E the divisor

$$\Theta_E := \{l \in \mathbb{P}_2 \,|\, H^*(E(-1)|_l) \neq 0\}\,.$$

This is called the *divisor of jumping lines of the vector bundle E*, because along those lines E does not have the expected behavior from Theorem 9.1. The degree of Θ_E is the second Chern number of E.

Replacing lines by conics Hulek obtained an analogous description of vector bundles on $\mathbb{P}^2$ with odd first Chern class in [16].

The author used in [11] restriction theorems and Raynaud's result 5.1 (as well as its generalizations to higher ranks) to obtain a similar construction for moduli spaces of vector bundles on algebraic surfaces. Here (in the rank two case) the lines were replaced by curves in the surfaces with a given line bundle on the curve. This ends in a finite morphism $M \to \mathbb{P}^N$ which was called the *Barth morphism.* In [21] J. Li gave a similar construction which should coincide with ours.

We call a pair (F, E) of coherent sheaves perpendicular when $\mathrm{Ext}^i(F, E) = 0$ for all integers i. If this is the case, then we write $F \perp E$ for such a pair. We can rephrase Proposition 2.7 as the implication: if for a sheaf E there exists a nonzero F such that $F \perp E$, then E is semistable. Theorem 5.1 shows that for a semistable vector bundles E of rank two and degree $2g - 2$ there exists a perpendicular sheaf $F = L^\vee$ which is a line bundle of degree zero. The fact, that semistable objects have perpendicular partners allows the definition of semistability in the derived category. For μ-semistability this is derived from restriction theorems and the existence of orthogonal objects on curves. For Gieseker semistability this is shown in [1, Theorem 7.2]. This approach was pursued in [14] and [15].

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