THE FARGUES-FONTAINE CURVE WITH A VIEW TOWARDS LOCAL GEOMETRIC CLASS FIELD THEORY

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The aim of the seminar is to construct the Fargues-Fontaine curve ("the fundamental curve of p-adic Hodge theory"), to study its basic properties and to classify vector bundles on it. In the last part, we sketch how the curve is used to give a new proof of local class field theory [Fa17a]. The main reference for the seminar is the article of Fargues and Fontaine [FF17] which is the new version of [FF09].

The curve was discovered in context of p-adic Hodge theory, cf. Colmez's exposition in [FF17]. In the seminar, we are interested in the curve from the point of view of a conjectural geometrization of the local Langlands correspondence [Fa16], [Sch14].

0. Introduction

Let X be a smooth projective geometrically connected curve over a field k. There is the Abel-Jacobi map

$$\mathrm{AJ}:\ \mathrm{Div}(X) \to \mathrm{Pic}(X),\ D \mapsto \mathcal{O}_X(D),$$

where $\mathrm{Div}(X)$ is the space of divisors, and $\mathrm{Pic}(X)$ the space of line bundles. Geometric class field theory (CFT) states that there is an equivalence of categories

$$\mathrm{AJ}^*\,|_X:\ \mathrm{CharLoc}(\mathrm{Pic}(X)) \xrightarrow{\simeq} \mathrm{Loc}_1(X),$$

where $\mathrm{Loc}_1(X)$ are rank 1 local systems, and $\mathrm{CharLoc}(\mathrm{Pic}(X))$ are the character local systems, i.e. rank 1 local systems on $\mathrm{Pic}(X)$ compatible with the group structure. The key step in the construction of an inverse to $\mathrm{AJ}^*\,|_X$ is Deligne's observation [De74, §e] that the map from degree d divisors on X to degree d line bundles

$$\mathrm{AJ}^d:\ \mathrm{Div}(X)_d = X^d/S_d \longrightarrow \mathrm{Pic}(X)_d$$

is a locally trivial bundle in projective spaces if $d > 2g-2$ (by Riemann-Roch). In particular, the fibers are simply connected, and we use this to descend local systems: starting from a character φ of $\pi_1(\mathrm{Div}(X)_1)$ which corresponds to a rank 1 local system $\mathcal{E}_\varphi$ on $\mathrm{Div}(X)_1$, we form the local system

$$\mathcal{E}_\varphi^{\boxtimes d} \text{ on } X^d.$$

Due to the simply connectedness of the fibers for $d > 2g-2$, this descends along AJ^d to a local system $\mathcal{F}_{\varphi,d}$ on $\mathrm{Pic}(X)_d$. Using the group structure on $\mathrm{Pic}(X)$, we obtain the desired character local system $\mathcal{F}_\varphi$ on $\mathrm{Pic}(X)$. If k is a finite field, then (unramified) geometric CFT for X implies (unramified) CFT for the function field $E = k(X)$: we use that the fundamental group

$$\pi_1(X) = \mathrm{Gal}_E^{\mathrm{ur}}$$

is the unramified Galois group, and further Weil's adèlic interpretation of the isomorphism classes of line bundles $\mathrm{Pic}(X)(k) = E^\times \backslash \mathbb{A}_E^\times / \prod_{x\in|X|} \hat{\mathcal{O}}_{X,x}$. Then CFT is given by the association

$$\varphi \mapsto \mathrm{char}_{\mathcal{F}_\varphi},$$

Date: September 25th, 2017.

The author warmly thanks Jochen Heinloth and Jan Kohlhasse for helpful comments on a previous version.

where $\mathrm{char}_{\mathcal{F}_\varphi}$ denotes the character associated with $\mathcal{F}_\varphi$. By passing to open curves, we even obtain the full class field theory for E, cf. [De74, §e]. This is certainly very nice, but why should it be desirable to have such a geometric formulation?

When it comes to the study of higher dimensional representations of Gal_E in the sense of Langlands's conjectures for Gl_n (the case $n = 1$ is class field theory), we have seen in the last two seminars that moduli spaces of Drinfeld shtukas (i.e. vector bundles with a Frobenius connection) are extremely useful. Hence, we would like to have an analogous theory for number fields, and Scholze makes this precise when $E/\mathbb{Q}_p$ is a finite extension of the p-adic numbers, cf. [Sch14].

To explain Scholze's point of view, let us first consider the case $E = \mathbb{F}_p(\!(\pi)\!)$ is a local function field. Intuitively, we think of $\mathrm{Spec}(E)$ as a punctured open unit disc. This can be made precise using non-archimedean geometry: for all non-archimedean local fields $F/\mathbb{F}_p$, the product[1]

$$Y_{F,E} \overset{\mathrm{def}}{=} \mathrm{Spa}(F) \times_{\mathrm{Spa}(\mathbb{F}_p)} \mathrm{Spa}(E) \subset \mathbb{A}^1_F,$$

is the punctured open unit disc in the variable π over the field F. If F is perfect, we have an action $\varphi\colon Y_{F,E} \to Y_{F,E}$ given by $x \mapsto x^p$ on F and trivially on the coordinate π. This action is nice (totally discontinuous), and we form the quotient

$$X^{\mathrm{ad}}_{F,E} = Y_{F,E}/\varphi^{\mathbb{Z}}.$$

We think about φ as being the partial Frobenius Frob $\times$id on the product $\mathrm{Spa}(F) \times \mathrm{Spa}(E)$. The following theorem is a consequence of Theorem 2 ii) below, and can be understood as a local version of Drinfeld's lemma[2] which we already encountered in the shtuka seminar:

Theorem (Fargues-Fontaine, Weinstein)**.** *For the fundamental group*

$$\pi_1(X^{\mathrm{ad}}_{F,E}) = \mathrm{Gal}_F \times \mathrm{Gal}_E.$$

In particular, if the field $F/\mathbb{F}_p$ is algebraically closed, then $\pi_1(X^{\mathrm{ad}}_{F,E}) = \mathrm{Gal}_E$. In the case $E/\mathbb{Q}_p$, Scholze has a similar construction by making use of his tilting process which compares fields of characteristic 0 with fields of characteristic p, cf. Weinstein [We16].

0.1. **Basic properties of the curve.** Let E be a non-archimedean local field with uniformizer π and residue field $\mathbb{F}_q$, i.e. either $E/\mathbb{Q}_p$ is a finite extension or $E = \mathbb{F}_q(\!(\pi)\!)$ is a local function field. We fix an auxiliary field $F/\mathbb{F}_q$ which we assume to be algebraically closed and complete with respect to a non-trivial valuation. The amazing fact is that Fargues and Fontaine construct an E-scheme

$$X = X_{F,E},$$

which behaves like a complete curve, and which can be considered as a scheme version of $X^{\mathrm{ad}}_{F,E}$. See also Weinstein's post [We13] for another amazing description of $X^{\mathrm{ad}}_{F,E}$. Some important results which we aim to prove in the seminar are (cf. [FF17, Theorem 6.5.2]):

Theorem 1. *i) The scheme X is a complete curve, and all closed points are of degree 1.*

ii) The residue field at a closed point is a complete algebraically closed extension of E.

iii) The degree map is an isomorphism $\mathrm{Pic}(X) \simeq \mathbb{Z}$.

iv) One has $H^1(X, \mathcal{O}_X) = 0$.

Let us remark that the scheme X is not of finite type over E (nor any other field). The term "curve" means that X is Noetherian regular of dimension 1 and equipped with a degree map $\deg : |X| \to \mathbb{Z}_{\geq 0}$. A rational function $f \neq 0$ gives rise to a divisor $\mathrm{div}(f)$, and the term "complete" means that $\deg(\mathrm{div}(f)) = 0$.

[1] The product is formed in the category of Huber's adic spaces.

[2] The formula $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$ usually fails even for classical varieties in positive characteristic, and Drinfeld's lemma tells us how to repair the failure by introducing partial Frobenius actions, e.g. [Sch14, §17].

For any integer $d \in \mathbb{Z}$, there are line bundles $\mathcal{O}_X(d)$ on X. It is possible to generalize this definition to all $\lambda \in \mathbb{Q}$ to obtain vector bundles $\mathcal{O}_X(\lambda)$. The next result concerns the classification of vector bundles, and finite étale covers (cf. [FF17, Theorems 8.2.10, 8.6.1]):

Theorem 2. *i) The map*

$$\{(\lambda_1, \dots, \lambda_n) \in \mathbb{Q}^n \mid \lambda_1 \geq \dots \geq \lambda_n\} \longrightarrow \{\textit{isomorphism classes of vector bundles on } X\}$$
$$(\lambda_1, \dots, \lambda_n) \longmapsto \oplus_{i=1}^n \mathcal{O}_X(\lambda_i)$$

is a bijection.

ii) The curve $X_{\bar{E}}$ is simply connected, i.e. every finite étale cover is trivial.

In the last part of the seminar, we come back to geometric local class field theory via the curve X. Following the strategy explained at the beginning of the introduction, geometric local CFT for the field E reduces to the main result of [Fa17a]:

Theorem 3. *For $d > 2$, the Abel-Jacobi map*

$$\mathrm{AJ}^d : \mathrm{Div}(X)_d \longrightarrow \mathrm{Pic}(X)_d$$

has simply connected fibers.

0.2. **Organization of the talks.** In talks 2.1-2.3, we define a generalized notion of curves which becomes necessary because X is not of finite type. These talks are independent of the construction of X, and interesting in their own. The technical heart of the seminar is the construction and study of various rings in talks 3.1-3.4 which we use to define the E-scheme X. Here it is certainly helpful to have some experience with Witt vectors. In talks 4.1 and 4.2, we apply our study to prove Theorem 1 & 2 from the introduction. In Talk 5.1, we study classical geometric class field field theory as sketched above. This talk serves as a motivation for the last two talks, and is independent from the results proven so far. The last two talks 5.2 are concerned with the proof of Theorem 3, and its application to local class field theory.

1. Overview (Timo Richarz, Oct 12th)

This will be an introduction to the curve, and the presentation of the plan for the seminar.

2. Curves (Oct 26th - Nov 9th)

The reference is [FF17, §5]. We define a generalized notion of complete curves, and prove a classification result for vector bundles, cf. [FF17, Thm 5.6.26]. Later in the seminar, the results are applied to X which behaves in some respects similar to the projective line:

If $\infty \in |X|$ is a closed point, then the ring $\Gamma(X\backslash\{\infty\}, \mathcal{O}_X)$ is principal ("almost-Euclidean"), but not Euclidean in contrast to $\mathbb{A}^1 = \mathbb{P}^1\backslash\{\infty\}$.

There is a tautological line bundle $\mathcal{O}_X(1)$ on X. We have $H^1(X, \mathcal{O}_X(d)) = 0$ for $d \geq 0$ like on $\mathbb{P}^1$ but $H^1(X, \mathcal{O}_X(-1)) \neq 0$ in contrast to $\mathbb{P}^1$.

There are semi-stable vector bundles on X with non-integer slopes in contrast to $\mathbb{P}^1$.

2.1. **Vector bundles on curves. §5.1-5.4 (Oct 26th).** Define complete curves over $\mathbb{Z}$ and almost Euclidean rings. Prove Theorem 5.2.7, state the classification for vector bundles in 5.3, show Lemma 5.4.1 and Proposition 5.4.2.

2.2. **The Harder-Narasimhan formalism. §5.5-§5.6.3 (Nov 2nd).** Explain the Harder-Narasimhan formalism, define semi-stability, state Theorems 5.5.2, 5.5.3 and 5.5.4 without proof. Give example 5.5.2.1. Define Riemannian spheres, sketch the proof of Theorem 5.6.2, state Proposition 5.6.3, discuss finite étale morphisms and treat the rest of §5.6.3 as detailed as time permits.

2.3. **Generalized Riemannian spheres. §5.6.4 (Nov 9th).** Define generalized Riemannian spheres, give example [FF09, 4.21], prove Proposition 5.6.23, define pure vector bundles, give Proposition 5.6.25, show the classification Theorem 5.6.26 and state Corollary 5.6.28.

3. Holomorphic functions in the variable π (Nov 16th - Dec 07th)

The reference is [FF17, §1, §2]. We define a ring B which is a Frechet algebra of "holomorphic functions in the variable π". If $E = \mathbb{F}_q(\!(\pi)\!)$, then B is the algebra of rigid analytic functions on the punctured open unit disc in $\mathbb{A}^1_F$. Furthermore, for a function $f \in B$, the slopes of the Newton polygon $\mathrm{Newt}(f)$ are the valuations of the points y with $f(y) = 0$. If $E/\mathbb{Q}_p$, then we define a topological space $|Y|$ on which we can evaluate the elements of B, and characterize the zeros of elements in terms of the slopes of Newton polygons. We obtain various results which are similar to the case of complex analytic functions in one variable. We also recommend the overview articles [Fa17b, §1.1-1.2], [FF14, §1-2].

3.1. **The ring A and primitive elements. §1.1-1.3, §2.1-2.2 (Nov 16th).** Introduce Witt vectors (cf. [Se79]) with the standard operations F Frobenius, V Verschiebung, [-] Teichmüller. Fix notation as in §1.1 where we assume the field F to be algebraically closed throughout the seminar. For a perfect $\mathbb{F}_q$-algebra R define the ramified Witt vectors $W_{\mathcal{O}_E}(R)$ as $W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ and skip the rest of §1.2. Jump to §2.1, introduce (-)$^\flat$ (§2.1.1) and the morphism Θ (§2.1.2) where we assume $Q = X^q$ in the seminar (and drop it from the notation). Prove Proposition 2.1.7. Define $\mathbf{A} = W_{\mathcal{O}_E}(\mathcal{O}_F)$ if $E/\mathbb{Q}_p$ and $\mathbf{A} = \mathcal{O}_{\mathbf{F}}[\![\pi]\!]$ if $E = \mathbb{F}_q(\!(\pi)\!)$ (cf. Definition 1.3.2)[3], define primitive elements (Definition 2.2.1), give Example 2.2.2, sketch the proof of Corollary 2.2.8[4], deduce Corollary 2.2.9, 2.2.10, and explain Corollary 2.2.22. State Theorem 2.4.1 which we aim to prove in the sequel.

3.2. **The ring B and Newton polygons. §1.3-§1.6, §1.10 (Nov 23rd).** Introduce the rings $\mathscr{E}$ (Definition 1.3.1), B^b, $B^{b,+}$ (Definition 1.3.2) and the Frobenius $\varphi = F$ (Definition 1.3.3). Introduce the Gauss norms $|\text{-}|_\rho$ (Definition 1.4.1), state elementary properties, and explain Remark 1.4.6. Discuss the topology on $\mathbf{A}$ induced by the Gauss norms (§1.4.3), in particular Corollary 1.4.15. Define the Fréchet algebras B_I, B (Definition 1.6.2) and B_I^+, B^+ (Definition 1.10.2), give the examples, skip §1.6.2, state Proposition 1.6.15 and give the formulas preceding Proposition 1.10.7 (in particular the effect of φ). Define the Legendre transform and the Newton polygon $\mathrm{Newt}(b)$ first for elements $b \in B^b$ (§1.5.1, 1.5.2) and then for elements $b \in B_I$ (§1.6.3.2), state elementary properties of these and explain the analogy with the complex case (Remark 1.5.4). Sketch the proof of Proposition 1.6.25 and Proposition 1.10.7. Deduce part (1) of Theorem 2.5.1, and state the rest which we aim to prove in the next talk.

3.3. **The space $|Y|$ and Weierstraß factorizations. §2.3-2.4 (Nov 30th).** Define the set $|Y|$, its metric d, sketch Proposition 2.3.4. Prove the parametrization of degree 1 elements by a Lubin-Tate law, cf. Proposition 2.3.9, Proposition 2.3.12 (we assume $Q = X^q$). Sketch the analytic view point §2.3.3 and skip §2.3.4. Prove Theorem 2.4.1, state the Corollaries 2.4.2, 2.4.3, state Theorems 2.4.5, 2.4.6.

3.4. **Divisors of holomorphic functions on B. §2.4.1-2.5, §2.7 (Dec 07th).** Define the space $|Y_I|$ (§2.4.1), prove Theorem 2.4.10, prove Theorem 2.5.1 and Corollary 2.5.4. Define the ring B^+_{dR}[5], define the divisor $\mathrm{div}(f)$ associated with an element $f \in B_I \backslash \{0\}$ and prove Theorem 2.7.4.

[3] Funfact: Assume $E/\mathbb{Q}_p$. The ring $\mathbf{A}$ should be of Krull dimension 2. One can show that $\mathbf{A}$ is at least 3 dimensional, and it seems not to be known whether $\mathbf{A}$ is finite dimensional.

[4] This is the only place in the construction of the curve where the explicit structure of the ring of Witt vectors is needed.

[5] Funfact: Assume $E/\mathbb{Q}_p$. The ring B^+_{dR} is a DVR abstractly isomorphic to $C[\![t]\!]$ where $C = \hat{\bar{E}}$. It has a filtration and the associated graded is Galois equivariantly isomorphic to $\oplus_{i\geq 0} C(i)$ where $C(i)$ denotes the twist by the p-adic cyclotomic character. However, there is no Galois equivariant isomorphism $B^+_{\mathrm{dR}} \simeq C[\![t]\!]$.

4. Proof of Theorems 1 & 2 (Dec 14th - Dec 21st)

The reference is [FF17, §6, §8]. We apply our ring theoretic study to prove the basic properties of the curve (cf. Theorem 6.5.2), the classification of vector bundles (cf. Theorem 8.2.10) and the simple connectivity (cf. Theorem 8.6.1).

4.1. **Basic properties of the curve, §6.1-6.5 (Dec 14th).** Define the graded algebra $P_{F,E,\pi}$, and the curve X, and mention properties 6.1.2 and 6.1.3. Focus on the proof of Theorem 6.2.1. In particular, define $\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$, the sets $\mathbb{M}_d$, $\mathbb{M}$, the map Π and its relation to the Lubin-Tate logarithm. Show the fundamental exact sequence Theorem 6.4.1, state Corollary 6.4.3. Prove the main Theorem 6.5.2 and state Corollary 6.5.3. If time permits give the alternative definitions of X in §6.6, §6.7.

4.2. **Vector bundles on the curve, §8.2, 8.4 (Dec 21st).** Define $\mathcal{O}_X(d)$ for $d \in \mathbb{Z}$ (Definition 8.2.1), show the basic properties as in §8.2.1.1, 8.2.1.2 and give the geometric interpretation of the fundamental exact sequence §8.2.1.3. Define $\mathcal{O}_X(\lambda)$ for $\lambda \in \mathbb{Q}$ (Definition 8.2.2) and show Proposition 8.2.3, give Remark 8.2.4. Define isocrystals, the functor $\mathscr{E}\colon \varphi\text{-}\mathrm{Mod}_L \to \mathrm{Fib}_X$, show Proposition 8.2.8, give Remark 8.2.9. Prove Theorem 8.2.10 via Banach-Colmez spaces as in §8.4. Deduce Theorem 8.6.1.

5. Geometric Class Field Theory (Jan 11th - Jan 18th)

The reference is [Fa17a], [MO16]. In the last part of the seminar, we want to learn how the curve X is used to give a new proof of local class field theory.

5.1. **Classical GCFT (Jan 11th).** This concerns global fields of positive characteristic. Follow the notes of Bhatt's talk [MO16, §2]. Prove unramified global GCFT, sketch the proof of ramified global GCFT and the compatibility with local GCFT. The idea of the construction is also explained in [Fa17a, §1]. Details on the unramified case are in [La90], some details on the ramified case are in [De74, §e].

5.2. **Local GCFT via the curve I&II (Jan 18th).** Give an outline of Fargues's geometric proof of local class field theory using the curve X. This requires quite a lot of machinery which cannot be introduced in detail (e.g. perfectoid spaces and diamonds). Nonetheless, you should try to explain the analogy with the classical case and what kind of technical problems have to be solved in order to treat p-adic fields.

6. Program discussion (Jan 25th)

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