Abelian theorems for stochastic volatility models with application to the estimation of jump activity [1]

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Abstract

In this paper, we prove a kind of Abelian theorem for a class of stochastic volatility models (X, V) where both the state process X and the volatility process V may have jumps. Our results relate the asymptotic behavior of the characteristic function of X_Δ for some $\Delta > 0$ in a stationary regime to the Blumenthal-Getoor indexes of the Lévy processes driving the jumps in X and V. The results obtained are used to construct consistent estimators for the above Blumenthal-Getoor indexes based on low-frequency observations of the state process X. We derive convergence rates for the corresponding estimator and show that these rates can not be improved in general.

Keywords: affine stochastic volatility model, Abelian theorem, Blumenthal-Getoor index

1 Introduction

Consider a class of *affine stochastic volatility* (ASV) models with jumps both in a state process and in a volatility of the form:

$$dX_t = (a_X + b_X V_{t-})dt + \sqrt{V_{t-}}\, dW_{1,t} + dZ_{1,t}, \tag{1}$$

$$dV_t = (a_V - b_V V_{t-})dt + a_V \sigma \sqrt{V_{t-}}\, dW_{2,t} + dZ_{2,t}, \tag{2}$$

where $(W_{1,t}, W_{2,t})$ is a two-dimensional Wiener process such that $\operatorname{corr}(W_{1,t}, W_{2,t}) = \rho$, $(Z_{1,t}, Z_{2,t})$ is a two-dimensional pure jump Lévy process with an increasing or constant $Z_{2,t}$, a_X, b_X, b_V are real numbers, $\sigma > 0$ and a_V is a nonnegative real number. ASV models have got much attention in the past decade (see Keller-Ressel, 2008 for an overview). Such well-known stochastic volatility models as Heston, 1993, Bates, 1996 and Barndorff-Nielsen and Shephard, 2001 models are in the class of ASV models, and this fact allows one to treat all of them within one theoretical framework. The main reason for the popularity of ASV models is their analytic tractability: the conditional characteristic function of the vector (X_t, V_t) given (X_0, V_0) has, for any $t > 0$, an exponentially affine structure in (X_0, V_0) and can be efficiently computed via solving a system of ordinary differential equations. Various analytical properties of ASV models such as ergodicity

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or the existence of moments were extensively studied in the literature (see, e.g., Glasserman and Kim, 2010 and Keller-Ressel, 2011 for the most recent results). In this respect, one contribution of the current paper is the derivation of the so-called Abelian theorem relating the asymptotic behavior of the characteristic function of X_t for any $t > 0$ to the asymptotic behavior of the Lévy measure of the two-dimensional Lévy process (Z_1, Z_2) at the point $(0, 0)$. The latter behavior is closely connected to the notion of Blumenthal-Getoor index, which is the main object of our study. For a one-dimensional Lévy process $Z = (Z_t)_{t\geq 0}$ with a Lévy measure ν, the Blumenthal-Getoor index of Z is defined as

$$\mathtt{BG}(Z) = \inf\left\{ r > 0 : \int_{|x|\leq 1} |x|^r \nu(dx) < \infty \right\}.$$

The Blumenthal-Getoor (BG) index is a fundamental characteristic of the Lévy process Z that determines the activity of jumps in Z. If $\nu([-\varepsilon, \varepsilon]) < \infty$, then the process Z has finite activity of jumps and $\mathtt{BG}(Z) = 0$. If the Lévy measure $\nu((-\infty, -\varepsilon] \cup [\varepsilon, \infty))$ diverges near $\varepsilon = 0$ at a rate $\varepsilon^{-\alpha}$ for some $\alpha > 0$, then the BG index of Z is equal to α. From a practical point of view, the importance of the Blumenthal-Getoor index lies in the fact that it determines the smoothness properties of the marginal density of Z and has significant impact on the convergence of different approximation algorithms for Z (see, e.g., Dereich, 2011). One of the main results of our study states that the c.f. $\phi_\Delta(u)$ of increments $X_{t+\Delta} - X_t$ for some $\Delta > 0$ in a stationary regime has the following representation:

$$\log|\phi_\Delta(u)| = -\tau_1 u - \tau_2 u^\alpha (1 + r(u)), \quad |r(u)| \leq \tau_3 u^{-\varkappa}, \quad u > 1 \tag{3}$$

with some constants $\tau_1 \geq 0$, $\tau_2 > 0$, $\tau_3 \geq 0$, $\varkappa > 0$ and $\alpha \geq 0$ depending on the parameters of the model (1)-(2). The representation (3) reveals the essential difference in the asymptotic behavior of $\phi_\Delta(u)$ between the Heston-like ASV models ($a_V > 0$) and the Barndorff-Nielsen-Shephard-like ASV models ($a_V = 0$). While in the first case the leading term in the asymptotic of $\log|\phi_\Delta(u)|$ is given by $-\tau_1 u$, in the second case $\log|\phi_\Delta(u)|$ behaves like $-\tau_2 u^\alpha$ as u tends to infinity, where α is proportional to maximum of the BG indexes for the Lévy processes Z_1 and Z_2.

The representation (3) is not only of theoretical interest, it can be used to construct statistical procedures for estimating the Blumenthal-Getoor indexes for Z_1 and Z_2. Recently, the problem of estimation of the BG index from discrete observations of a Lévy process Z or some other processes based on Z has drawn much attention in the literature. Aït-Sahalia and Jacod, 2009 (see also Aït-Sahalia and Jacod, 2012) studied the problem of estimating the so-called jump activity index that is defined for any Itô semimartingale Y via

$$\mathtt{JAI}(Y) = \inf\left\{ r > 0 : \sum_{0\leq s\leq T} |\Delta Y_s|^r < \infty \right\},$$

where $\Delta Y_s = Y_s - Y_{s-}$ is the size of the jump at time s and T is a fixed time horizon. Note that $\mathtt{JAI}(Y)$ is a random quantity, which is to be determined pathwise. In the case of a Lévy process Y, $\mathtt{JAI}(Y)$ coincides with the Blumenthal-Getoor index of Y. Obviously, one can compute $\mathtt{JAI}(Y)$ if the whole path of the process Y up to time T is observed. In a more realistic situation when the process Y is observed on discrete grid $\{0, \Delta, \ldots, \Delta n\}$ with $\Delta n = T$ and $\Delta \to 0$ as $n \to \infty$ (*high-frequency* data), Aït-Sahalia and Jacod proposed a method which is able to consistently estimate $\mathtt{JAI}(Y)$ and is based on a statistics that counts the "big" increments of the process Y.

Turning to the case of *low-frequency* data, i.e., the case of fixed $\Delta > 0$ and $T \to \infty$, one may wonder if any kind of statistical inference is possible in this situation at all. Indeed, one challenge is that the transition density of X in ASV models is hardly ever known in closed form making the maximum-likelihood estimation difficult. Furthermore, the volatility process V is not directly observable leading to a kind of filtering problem, which requires elimination of V. The latter filtering problem is well understood in the case of high-frequency data and poses significant problems if Δ does not tend to 0. The first results showing that a consistent estimation of the BG index based on the low-frequency data is possible, were obtained in Belomestny, 2010 for the case of a Lévy process Z. The inference in Belomestny, 2010 relied on the kind of Abelian theorem that characterizes the decay of the c.f. of Z. Such Abelian theorems are well known in the literature: Bismut, 1983 showed that the tail integral $\nu\big((-\infty,-x)\cup(x,+\infty)\big)$ behaves asymptotically like $c_1 x^{-\gamma}$ as $x \to +0$ if and only if the characteristic exponent of the Lévy process Z with the Lévy measure ν behaves like $-c_2|u|^\gamma$ as $|u| \to \infty$ (here c_1, c_2, and γ are positive numbers). It turns out that ideas similar to ones in Belomestny, 2010 can be used to construct estimates for the BG indexes in the model (1)-(2) and the representation (3) shall play a crucial role in this construction.

The paper is organized as follows. In Section 2, we establish and discuss the representation (3). The estimation algorithm for the BG indexes of Z_1 and Z_2 is formulated and analyzed in Section 3. In particular, we derive convergence rates for the proposed estimate and discuss their optimality. In Section 4 some numerical examples are presented. Section 6 contains the proofs, which use the auxiliary results collected in Section 7. An exponential inequality for empirical characteristic function in the case of dependent data is given in Appendix.

2 Abelian theorems

Denote by ν_1 and ν_2 the Lévy measures of the Lévy processes Z_1 and Z_2, respectively. Assume that the following asymptotic relations hold:

(AN1)

$$\varepsilon^{\gamma_1}\int_{|x|>\varepsilon}\nu_1(dx) = \beta_{0,1}+\beta_{1,1}\varepsilon^{\chi_1}(1+O(\varepsilon)), \quad \varepsilon\to+0,$$

(AN2)

$$\varepsilon^{\gamma_2}\int_{y>\varepsilon}\nu_2(dy) = \beta_{0,2}+\beta_{1,2}\varepsilon^{\chi_2}(1+O(\varepsilon)), \quad \varepsilon\to+0$$

with $0<\chi_1\le\gamma_1<2$, $0<\chi_2\le\gamma_2<1$, $\beta_{0,1}>0,\beta_{0,2}>0$. The assumptions (AN1) and (AN2) imply that the Blumenthal-Getoor indexes of the Lévy processes Z_1 and Z_2 are equal to γ_1 and γ_2, respectively. Moreover, suppose that

(AE)

$$b_V>0, \quad a_V\sigma^2<2,$$

(AM)

$$\int_{|x|>1} |x|^{2+\delta} \nu_2(dx) < \infty$$

for some $\delta > 0$.

The conditions (AE) and (AM) ensure the existence and the uniqueness of the solution of (2) together with positive recurrence on $(0, \infty)$ (see Masuda, 2007). As a result, V admits the unique invariant distribution π and $V_t > 0$ almost surely, for all $t > 0$. If additionally:

(AS) V_0 has the distribution π,

then V_t is strictly stationary with the stationary distribution π. Then strict stationarity of V implies strict stationarity of the increment process $(X_{t+\Delta} - X_t)_{t\geq 0}$ for any $\Delta > 0$. Denote by ϕ_Δ the characteristic function of $X_{t+\Delta} - X_t$ in the stationary regime. The following theorem describes the asymptotic behavior of $\phi_\Delta(u)$ as $|u| \to \infty$.

Theorem 2.1. *Assume that the assumptions (AN1), (AN2), (AE) and (AM) are fulfilled. Then*

$$\log|\phi_\Delta(u)| = -\tau_1 u - \tau_2 u^\alpha (1 + r(u)), \quad |r(u)| \leq \tau_3 u^{-\varkappa}, \quad u > 1, \tag{4}$$

where $\tau_1 \geq 0$, $\tau_2 > 0$, $\tau_3 \geq 0$, $\alpha \geq 0$ *and* $\varkappa > 0$ *are some numbers depending on the parameters of the model* (1)-(2) *and* Δ. *In particular,*

- *if* $a_V > 0$, *then* τ_1 *is positive,* $\alpha = \max\{\gamma_1, \gamma_2\}$, *and*

$$\varkappa = \begin{cases} (\gamma_2 - \gamma_1) \wedge \chi_1, & \text{if } \gamma_1 < \gamma_2, \\ (\gamma_1 - \gamma_2) \wedge \chi_2, & \text{if } \gamma_1 > \gamma_2, \\ \chi_1 \wedge \chi_2, & \text{if } \gamma_1 = \gamma_2; \end{cases}$$

- *if* $a_V = 0$, *then* $\tau_1 = 0$, $\alpha = \max\{\gamma_1, 2\gamma_2\}$, *and*

$$\varkappa = \begin{cases} (2\gamma_2 - \gamma_1) \wedge 2\chi_2 \wedge 1, & \text{if } \gamma_1 < 2\gamma_2, \\ (\gamma_1 - 2\gamma_2) \wedge \chi_1, & \text{if } \gamma_1 > 2\gamma_2, \\ \chi_1 \wedge 2\chi_2 \wedge 1, & \text{if } \gamma_1 = 2\gamma_2. \end{cases}$$

Discussion It is easily seen that $\tau_1 > 0$ as long as $a_V > 0$ and $\tau_1 = 0$ if $a_V = 0$, meaning that the asymptotic behavior of $\phi_\Delta(u)$ changes markedly if we move from the Heston-like ASV models ($a_V > 0$) to the Barndorf-Nielsen-Shephard-like ASV models ($a_V = 0$). Furthermore, if $\gamma_2 \geq \gamma_1$ then the value of α is always proportional to the BG index of Z_2. Hence, in the latter case the problem of statistical inference on γ_2, which determines the jump activity of volatility, can be reformulated as the problem of estimating α in (4), which is considered in the next section.

Remark 2.2. The results of Theorem 2.1 can be extended to the case where one of the BG indexes γ_1 or γ_2 equals 0. This case however requires separate treatment and is therefore omitted.

3 Estimation of the Blumenthal-Getoor index

The estimation procedure below makes use of Theorem 2.1, which implies that for any $\theta > 1$:

$$\mathcal{Y}(u) := \log\left\{-\log\left[|\phi_\Delta(u)|^{2\theta}/\left|\phi_\Delta(\theta u)\right|^2\right]\right\} = \log(2\tau_\theta) + \alpha\log(u) + \log(R(u)), \tag{5}$$

where $\tau_\theta = \tau_2(\theta - \theta^\alpha)$ and $R(u) \to 1$ as $u \to +\infty$. Hence $\mathcal{Y}(u)$ is well defined and remains bounded for all finite u. Since $\mathcal{Y}(u)$ is, up to a reminder term $\log(R(u))$, a linear function of $\log(u)$ with α determining the slope, one can view (at least for large u) the estimation of α as a linear regression problem and apply the (weighted) least-squares approach.

Suppose that the discrete observations $X_0, X_\Delta, \ldots, X_{n\Delta}$ of the state process X are available for some fixed $\Delta > 0$. First, estimate $\phi_\Delta(u)$ by its empirical counterpart $\phi_n(u)$ defined as

$$\phi_n(u) = \frac{1}{n}\sum_{k=1}^{n} e^{\mathrm{i}u(X_{\Delta k} - X_{\Delta(k-1)})}. \tag{6}$$

Note that under the assumptions (AE), (AS) and (AM) the sequence $X_{\Delta k} - X_{\Delta(k-1)}$, $k = 1, \ldots, n$, is exponentially α-mixing and hence ergodic. Indeed,

$$X_{\Delta k} - X_{\Delta(k-1)} = \xi_{1,k} + \xi_{2,k}$$

with

$$\xi_{1,k} = \int_{\Delta(k-1)}^{\Delta k} (a_X + b_X V_{t-})\,dt + \int_{\Delta(k-1)}^{\Delta k} \sqrt{V_{t-}}\,dW_{1,t}$$

and $\xi_{2,k} = Z_{1,\Delta k} - Z_{1,\Delta(k-1)}$. Since the r.v. $(\xi_{2,k})$ are independent, it is enough to check that the sequence $(\xi_{1,k})$ is exponentially α-mixing. The latter fact can be derived from the α-mixing of the volatility process V_t in a rather standard way (see, e.g., Proposition 3.1 in Genon-Catalot *et al.* (2000)). The ergodicity of $(X_{\Delta k} - X_{\Delta(k-1)})$ implies that

$$\frac{1}{n}\sum_{k=1}^{n} e^{\mathrm{i}u(X_{\Delta k} - X_{\Delta(k-1)})} \xrightarrow{a.s.} \phi_\Delta(u), \quad n \to \infty$$

by the Birkhoff's ergodic theorem (see, e.g., Athreya and Lahiri, 2010). Fix some $\theta > 2$ such that $2\theta \in \mathbb{N}$ and consider the random process:

$$\mathcal{Y}_n(u) = \log\left\{-\log\left[|\phi_n(u)|^{2\theta}/\left|\phi_n(\theta u)\right|^2\right]\right\}, \quad u \in \mathbb{R}.$$

Furthermore, introduce a weighting function $w^{U_n}(u) = U_n^{-1} w^1(u/U_n)$, where U_n is a sequence of positive numbers tending to infinity, the function w^1 is supported on $[\varepsilon, 1]$, for some $\varepsilon > 0$, and satisfies

$$\int_\varepsilon^1 w^1(u)\,du = 0, \quad \int_\varepsilon^1 w^1(u)\log u\,du = 1. \tag{7}$$

Next, define an estimate for α in (4) by

$$\hat{\alpha}_n := \int_0^\infty w^{U_n}(u)\mathcal{Y}_n(u)\,du. \tag{8}$$

The estimate (52) can be alternatively defined as $\hat{\alpha}_n = l_{n,1}$ with

$$(l_{n,0}, l_{n,1}) := \operatorname*{argmin}_{(l_0,l_1)} \int_0^{U_n} w_\diamond^{U_n}(u)(\mathcal{Y}_n(u) - l_1 \log(u) - l_0)^2 \, du,$$

where $w_\diamond^{U_n}(u)$ is a suitable weighting function supported on $[\varepsilon U_n, U_n]$ (see Subsection 8.2 for more details). In order to see that $\hat{\alpha}_n$ is a reasonable estimate of α, we introduce a deterministic quantity:

$$\alpha_n = \int_0^\infty w^{U_n}(u)\mathcal{Y}(u) \, du.$$

Using Theorem 2.1 one can show (see Lemma 7.4 below) that for n large enough,

$$|\alpha - \alpha_n| \leq \frac{C\,\tau_3}{U_n^{\varkappa}(1-\theta^{\alpha-1})}, \tag{9}$$

with some constant C not depending on parameters of the underlying ASV model. Hence, α_n converges to α, provided $U_n \to \infty$ as $n \to \infty$. The next theorem shows that α_n is close to $\hat{\alpha}_n$ in probability.

Remark 3.1. The estimation procedure has several tuning parameters. Our numerical study indicates that, while estimation results are rather insensitive to the choice of θ, the cut-off parameter U_n plays a crucial role. On the one hand, if some prior bounds for the parameters of the model (1)-(2) are known, one can use them to get bounds on τ_1, τ_2, τ_3, α and hence on U_n via formulas given in the proof of Theorem 2.1. On the other hand, one can use an adaptive aggregation-based procedure of Belomestny, 2010 for the choice of U_n (see Section 4 for more details).

Theorem 3.2. *Consider a class of ASV models of the form* (1)-(2) *such that assumptions (AN1), (AN2), (AM) and (AE) are fulfilled. If* $a_V > 0$ $(\tau_1 > 0)$ *and the sequence* U_n *fulfills*

$$\varepsilon_{1,n} := \frac{\log n}{\sqrt{n}} \, e^{2\theta(\tau_1+\tau_2+\tau_2\tau_3)U_n} \to 0, \quad U_n \to \infty, \quad n \to \infty,$$

then

$$\mathrm{P}\left\{|\hat{\alpha}_n - \alpha_n| > C_2 \frac{\varepsilon_{1,n}}{\tau_\theta U_n^\alpha}\right\} \leq C_3 n^{-1-\delta} \tag{10}$$

for some constants $C_2 > 0$, $C_3 > 0$ *and* $\delta > 0$ *not depending on* α, τ_1, τ_2 *and* τ_3. *In the case* $a_V = 0$ $(\tau_1 = 0)$ *we get*

$$\mathrm{P}\left\{|\hat{\alpha}_n - \alpha_n| > C_2 \frac{\varepsilon_{2,n}}{\tau_\theta U_n^\alpha}\right\} \leq C_3 n^{-1-\delta},$$

provided

$$\varepsilon_{2,n} := \frac{\log n}{\sqrt{n}} \, e^{2\theta(\tau_2+\tau_2\tau_3)U_n^\alpha} \to 0, \quad U_n \to \infty, \quad n \to \infty.$$

Denote by $\mathcal{A}_H$ a class of ASV models (1) such that a_V is strictly positive, assumptions (AN1), (AN2), (AM) and (AE) are fulfilled, and additionally

$$
\min\{\tau_1, \tau_2\} \geq \underline{\tau} > 0, \quad \tau_3 \leq \bar{\tau} < \infty, \quad 0 < \alpha \leq \bar{\alpha}, \quad 0 < \varkappa \leq \bar{\varkappa} \tag{11}
$$

in the representation (4). As we will see in the proof of Theorem 2.1, all conditions in (11) can be reformulated in terms of parameters of the underlying ASV model (1)-(2). Combining (9) with (10) and taking $U_n = q\log(n) + p\log\log(n)$ in such a way that $U_n^{-\varkappa} \asymp \varepsilon_{1,n} U_n^{-\alpha}$, we arrive at

$$
\sup_{(X,V)\in\mathcal{A}_H} \mathrm{P}_{(X,V)} \left\{ |\alpha - \hat{\alpha}_n| > C_4 \log^{-\bar{\varkappa}} n \right\} \leq C_5 n^{-1-\delta}, \tag{12}
$$

where constants C_4 and C_5 depend on $\underline{\tau}$, $\bar{\tau}$ and $\bar{\alpha}$ only. Since

$$
\sum_{n=1}^{\infty} \mathrm{P}_{(X,V)}\{|\alpha - \hat{\alpha}_n| > C_4 \log^{-\bar{\varkappa}} n\} \leq C_5 \sum_{n=1}^{\infty} n^{-1-\delta} < \infty,
$$

for any $(X, V) \in \mathcal{A}_H$, it follows by Borel-Cantelli lemma that upper bound of the sequence of events $\{|\alpha - \hat{\alpha}_n| > C_4 \log^{-\bar{\varkappa}} n\}$, $n \in \mathbb{N}$, is of probability 0, i.e.,

$$
\mathrm{P}_{(X,V)} \left\{ |\alpha - \hat{\alpha}_n| > C_4 \log^{-\bar{\varkappa}} n \;\text{ for infinitely many n} \right\} = 0,
$$

or, equivalently,

$$
\mathrm{P}_{(X,V)} \left\{ \overline{\lim_{n\to\infty}} \left(\log^{\bar{\varkappa}} n \, |\alpha - \hat{\alpha}_n| \right) > C_4 \right\} = 0.
$$

In the case $a_V = 0$, i.e., $\tau_1 = 0$ in (4), one can define a class $\mathcal{A}_{BNS}$ with

$$
\tau_2 \geq \bar{\tau} > 0, \quad \tau_3 \leq \bar{\tau} < \infty \quad 0 < \alpha \leq \bar{\alpha}, \quad 0 < \varkappa \leq \bar{\varkappa} \tag{13}
$$

to get

$$
\sup_{(X,V)\in\mathcal{A}_{BNS}} \mathrm{P}_{(X,V)} \left\{ |\alpha - \hat{\alpha}_n| > C_4 \log^{-\bar{\varkappa}/\bar{\alpha}} n \right\} \leq C_5 n^{-1-\delta}. \tag{14}
$$

Discussion As can be seen, the rates of convergence of $\hat{\alpha}_n$ are logarithmic and depend on an upper bound $\bar{\alpha}$ for the BG index α. The latter feature can also be observed in the high-frequency setup of Aït-Sahalia and Jacod, 2009. Comparing the first part of Theorem 3.2 with the situation where a Lévy process Z with the BG index α is observed directly (see Belomestny, 2010, Theorem 6.7), we immediately realize that the convergence rates in both cases are of the same order, indicating that the problem of estimating $\alpha = \max\{\gamma_1, \gamma_2\}$ from the low-frequency observations of the process X has the same complexity as a similar problem of statistical inference based on the direct observations of the Lévy process Z. Moreover, the results of Belomestny, 2010 (Theorem 6.5) indicate that the convergence rates in (12) and (14) are optimal and can not be improved in general.

4 Numerical study

4.1 Heston-like model

Let us consider the so-called Bates model (see Bates, 1996) which is of the form (1)-(2) with $Z_2 = 0$. We take α-stable Lévy process $Z_{1,t}$ and put $\alpha = 0.8$, $a_X = b_X = 0$, $a_V = 1.4, b_V = 0.01$ and $a_V\sigma = 1.6$. First, we generate a discretized trajectory $X_0, X_\Delta, \ldots, X_{n\Delta}$ of the state process X with $\Delta = 0.1$ using `sde` package in R® with $V_0 \sim \Gamma(\sigma^2 a_V^2/2b_V, 2/(a_V\sigma^2))$. Next we estimate $\phi_\Delta(u)$ by its empirical counterpart $\phi_n(u)$ defined as

$$\phi_n(u) = \frac{1}{n}\sum_{k=1}^{n} e^{\mathrm{i}u(X_{\Delta k} - X_{\Delta(k-1)})}. \tag{15}$$

Then the function

$$\mathcal{Y}_n(u) = \log\left\{-\log\left[|\phi_n(u)|^{2\theta}/\,|\phi_n(\theta u)|^2\right]\right\}, \quad u \in \mathbb{R}$$

with $\theta = 3$ is computed. Finally we solve the optimization problem

$$(l_{n,0}, l_{n,1}) := \operatorname*{argmin}_{(l_0,l_1)} \int_\varepsilon^1 \left(\mathcal{Y}_n(u\,U_n) - l_1\log(u) - l_0\right)^2 du, \tag{16}$$

where U_n is a truncation level and define an estimate $\widehat{\alpha}_n = l_{n,1}$. The latter construction is based on Lemma 8.4 with $\tilde{w}^1(u) = \mathbf{1}_{\{\varepsilon\le u\le 1\}}$ for some $\varepsilon > 0$.

The truncation level U_n can be chosen using adaptive aggregation procedure of Belomestny, 2010. Alternatively one can use the following approach. We generate a discretized path of the process $\tilde{X}$, which follows the same ASV model as before but without jumps. By proceeding in the same way as before we construct the function $\tilde{\mathcal{Y}}_n(u)$ which is plotted in Figure 1. By inspecting the plot one locates an interval $[U_{n,low}, U_{n,up}]$ where the function $\tilde{\mathcal{Y}}_n(u)$ is flat. Usually such an interval can be easily identified. Then we put $U_n = U_{n,up}$ and $\varepsilon = U_{n,low}/U_n$. Figure 2 shows boxplots of the resulting estimate $\widehat{\alpha}_n = l_{n,1}$ as a function of n (based on 50 independent simulation runs).

4.2 OU-like model

Here we consider an ASV model such that the volatility process follows the Gamma-OU process:

$$dV_t = -\lambda V_t\,dt + dZ_{1,t}, \quad V_0 \sim \Gamma(a,b),$$

where $Z_{1,t}$ is a Gamma process with $Z_{1,1} \sim \Gamma(\lambda a, b)$. An observable state process X has the form:

$$X_t = \int_0^t V_s\,ds + Z_{2,t}$$

where $Z_{2,t}$ is an α-stable process independent of $Z_{1,t}$. We take $\lambda = a = b = 1$, $\alpha = 0.8$ and simulate a discretized trajectory $X_0, X_\Delta, \ldots, X_{n\Delta}$ of the state process X with $\Delta = 0.1$. Next we estimate $\phi_\Delta(u)$ by its empirical counterpart $\phi_n(u)$ defined as

$$\phi_n(u) = \frac{1}{n}\sum_{k=1}^{n} e^{\mathrm{i}u(X_{\Delta k} - X_{\Delta(k-1)})}. \tag{17}$$

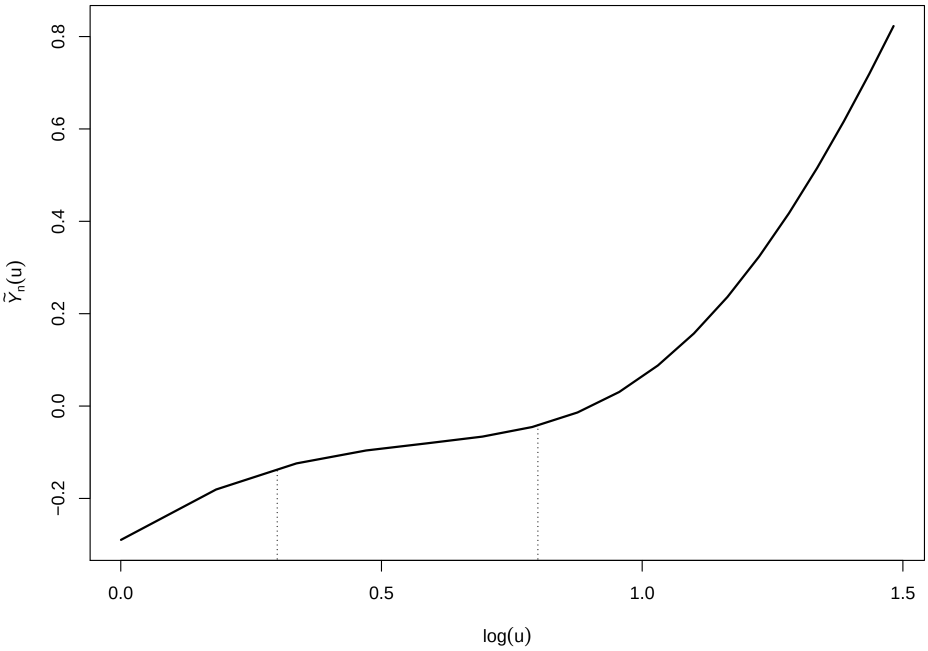

Figure 1: Function $\tilde{\mathcal{Y}}_n(u)$ on a log-scale. The abscises of the dotted lines show the chosen values of cut-off values $U_{n,low}$ and $U_{n,up}$.

Then we compute the function

$$\mathcal{Y}_n(u) = \log\left\{-\log\left[|\phi_n(u)|\right]\right\}, \quad u \in \mathbb{R}$$

and solve the optimization problem (16). It is known that the BG index of the Gamma process Z_2 is zero. Hence $\gamma_2 = 0$ and $\widehat{\alpha}_n = l_{1,n}$ is a natural estimate for $\gamma_1 = \alpha$. Let $U_1 > U_2 > \ldots > U_K$ be an exponentially decreasing sequence of cut-offs $U_{n,up}$ and let $\tilde{\alpha}_{n,1}, \ldots, \tilde{\alpha}_{n,K}$ be the corresponding sequence of estimates. Following an adaptive aggregation-based procedure introduced in Belomestny, 2010, we construct a sequence of aggregated estimates $\hat{\alpha}_{n,1}, \ldots, \hat{\alpha}_{n,K}$ using a triangle kernel and a set of critical values $\mathcal{V}_1, \ldots \mathcal{V}_K$ computed as described in Section 7 of Belomestny, 2010. Box plots of $\hat{\alpha}_n = \hat{\alpha}_{n,K}$ based on 50 independent trials for different n are shown in Figure 3.

5 Conclusion

In this article we study the problem of estimating the jump activity in affine stochastic volatility models (X, V) based on low-frequency observations of the state process X. The estimation procedure we propose relies on the so-called Abelian theorem connecting the large-argument asymptotic behavior of the marginal c.f. of X to the Blumenthal-Getoor indexes of Lévy processes driving the jumps in X and V. The Abelian theorem derived in the paper indicates,

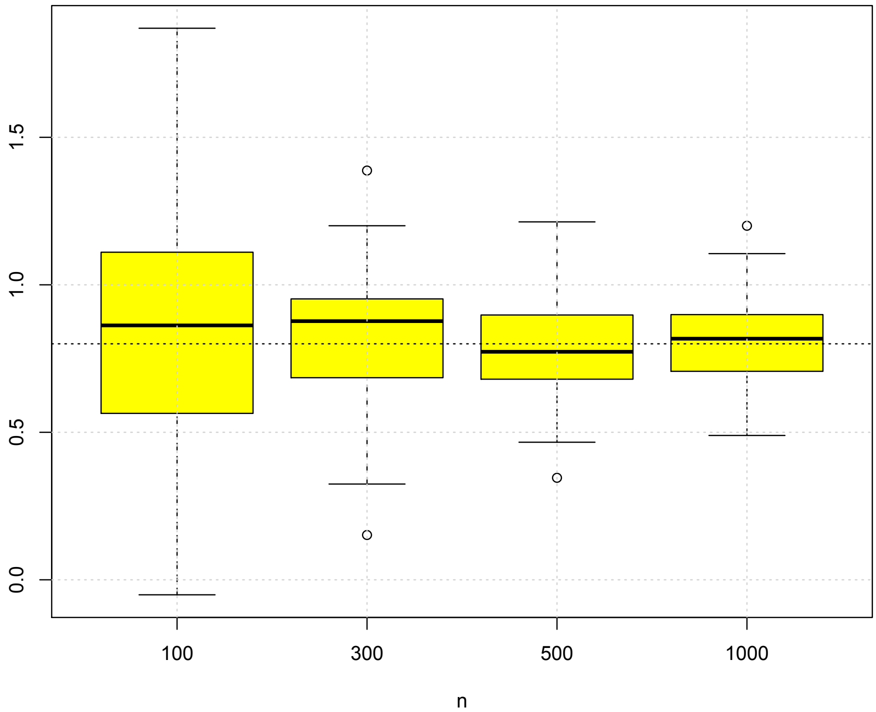

Figure 2: Boxplots of the BG index estimate $\widehat{\alpha}_n$ for different values of n based on 50 independent simulation runs for the Heston-like model.

for example, that the Heston stochastic volatility model and the Barndorf-Nielsen-Shephard stochastic volatility model lead to a qualitatively different behavior of the c. f. of X. Interestingly enough, this implies that the problem of statistical inference on the jump activity index of volatility is usually more difficult in the Barndorf-Nielsen-Shephard SV model, at least as far as the convergence rates are concerned.

6 Proofs

6.1 Proof of Theorem 2.1

It follows from the general results on affine processes (see, e.g., Duffie, Filipović and Schachermayer, 2003) that for any $s \leq t$

$$\begin{aligned} \phi(u, w, t-s|x, v) &= \mathbb{E}\left[e^{\mathrm{i}uX_t + \mathrm{i}wV_t} | X_s = x, V_s = v\right] \\ &= \exp\left\{\psi_0(u, w, t-s) + \mathrm{i}xu + v\psi_1(u, w, t-s)\right\}, \quad (u, v) \in \mathbb{R} \times \mathbb{R}_{\geq 0}, \end{aligned} \tag{18}$$

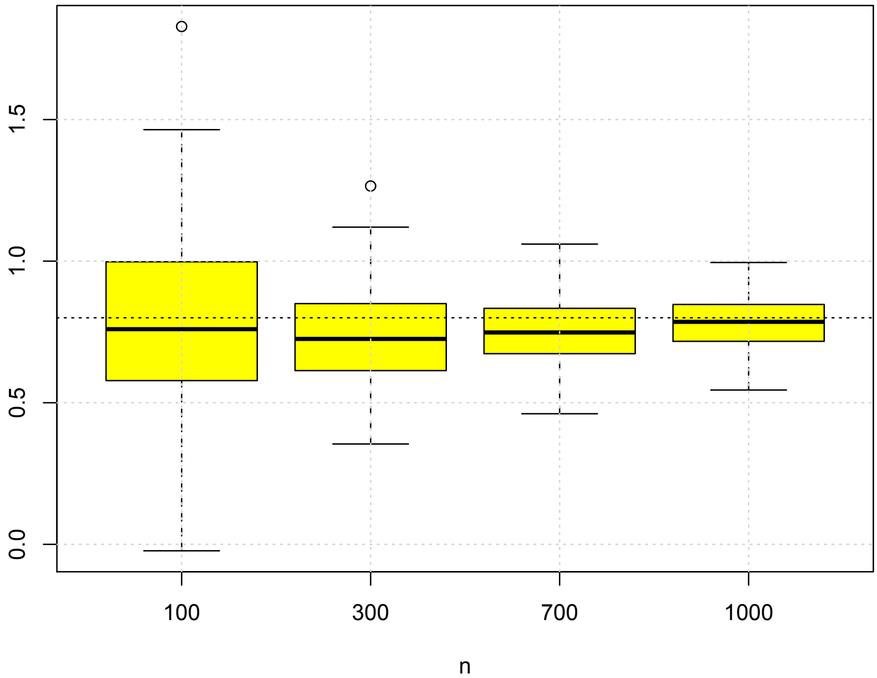

Figure 3: Box plots of the estimate $\hat{\alpha}_n$ in the stochastic volatility model with the OU-Gamma volatility process for different sample sizes.

where $\psi_0(u,w,t)$ and $\psi_1(u,w,t)$ are some complex-valued functions satisfying the system of nonlinear differential equations

$$(19)\quad \begin{cases} \frac{\partial \psi_1(u,w,t)}{\partial t} & = & \sigma^2 a_V^2 \psi_1^2(u,w,t) + (2 \cdot \mathrm{i}\, a_V \sigma \rho u - b_V)\, \psi_1(u,w,t) - \left(u^2 - \mathrm{i}\, b_X u\right), \\ \frac{\partial \psi_0(u,w,t)}{\partial t} & = & \mathrm{i}\, a_X u + a_V \psi_1(u,w,t) + \int_{-\infty}^{\infty}\int_0^{\infty} \left(e^{\mathrm{i}ux + \psi_1(u,w,t)y} - 1\right) \nu(dx,dy) \end{cases}$$

with the initial conditions

$$\psi_1(u,w,0) = \mathrm{i}w, \qquad \psi_0(u,w,0) = 0.$$

The following lemma easily follows from the standard results on ODEs.

Lemma 6.1. *The solution of the equation*

$$(20)\qquad \frac{\partial \psi(w,s)}{\partial s} = \Phi(\psi(w,s)), \quad \psi(w,0) = \mathrm{i}w$$

with

$$\Phi(z) = Az^2 + Bz - C,$$

where A, B and C are complex numbers is explicitly given by the formula

$$\psi(w,s) = -\frac{2C(\exp(\lambda s) - 1) - (\lambda(\exp(\lambda s) + 1) + B(\exp(\lambda s) - 1))(\mathrm{i} \cdot w)}{\lambda(\exp(\lambda s) + 1) - B(\exp(\lambda s) - 1) - 2A(\exp(\lambda s) - 1)(\mathrm{i} \cdot w)},$$

where $\lambda = \sqrt{B^2 + 4AC}$.

Lemma 6.1 implies that

$$(21)\qquad \psi_1(u,w,s) = -\frac{2C(\exp(\lambda s) - 1) - (\lambda(\exp(\lambda s) + 1) + B(\exp(\lambda s) - 1))(\mathrm{i} \cdot w)}{\lambda(\exp(\lambda s) + 1) - B(\exp(\lambda s) - 1) - 2A(\exp(\lambda s) - 1)(\mathrm{i} \cdot w)}$$

with

$$A = \sigma^2 a_V^2, \quad B = 2 \cdot \mathrm{i}\, a_V \sigma \rho u - b_V, \quad C = u^2 - \mathrm{i}\, b_X u, \quad \lambda = \sqrt{B^2 + 4AC},$$

and

$$(22)\qquad \begin{aligned} \psi_0(u,w,t) &= \mathrm{i}\, a_X u t + a_V \int_0^t \psi_1(u,w,s)\, ds \\ &+ \int_0^t \left[\int_{-\infty}^{\infty}\int_0^{\infty} \Big(\exp\{\mathrm{i}\, ux + \psi_1(u,w,s)y\} - 1\Big)\nu(dx,dy)\right] ds. \end{aligned}$$

Under assumptions (AE) and (AM), the process $(V_t)_{t\geq 0}$ and, consequently, $(X_{t+\Delta} - X_t)_{t\geq 0}$ is ergodic. Due to (18), the c.f. of the increments $X_{t+\Delta} - X_t$ in a stationary regime is given by

$$\phi_\Delta(u) = \mathbb{E}_\pi\left[e^{\mathrm{i}u(X_{t+\Delta} - X_t)}\right] = e^{\psi_0(u,0,\Delta)}\mathbb{E}_\pi\left[e^{V_t \psi_1(u,0,\Delta)}\right] = \exp\left\{\psi_0(u,0,\Delta) + l(\psi_1(u,0,\Delta))\right\},$$

where π is the invariant distribution of the volatility process V and l is the Laplace exponent of π, i.e.,

$$l(w) = \log\left[\int_0^\infty e^{wy}\, \pi(dy)\right] = \lim_{t\to\infty} \psi_0(0, -\mathrm{i}w, t).$$

As a result,

$$(23) \qquad l(w) = a_V \int_0^\infty \psi_1(0,-\mathrm{i}w,s)ds + \int_0^\infty \left[\int_0^\infty \left(e^{\psi_1(0,-\mathrm{i}w,s)y} - 1 \right) \nu_2(dy) \right] ds.$$

Our objective is now to infer on the asymptotic behavior of the function

$$(24) \qquad \log |\phi_\Delta(u)| \quad = \quad \mathrm{Re}\left\{ \psi_0(u,0,\Delta) \right\} + \mathrm{Re}\left\{ l(\psi_1(u,0,\Delta)) \right\}$$

as $u \to +\infty$, where ψ_1 is given by (21), ψ_0 - by (22), and l is in the form (23). Consider now two cases. Suppose first that $\min(\gamma_1,\gamma_2) > 0$.

Case $\mathbf{a_V = 0}$. We have $A = 0,\ B = -b_V,\ \lambda = b_V$, and formula (21) boils down to

$$\psi_1(u,w,s) \quad = \quad \frac{C}{b_V}(\exp(-b_V s) - 1) + (\mathrm{i}\cdot w)\exp(-b_V s).$$

Hence

$$\begin{aligned} \psi_1(0,w,s) &= \mathrm{i}e^{-b_V s}w, \\ \psi_1(u,0,s) &= B_s C = B_s(u^2 - \mathrm{i}b_X u) \end{aligned}$$

with $B_s = b_V^{-1}(\exp(-b_V s) - 1)$. Moreover,

$$l(w) = \int_0^\infty \left[\int_0^\infty \left(e^{e^{-b_V s}wy} - 1 \right) \nu_2(dy) \right] ds,$$

and

$$\psi_0(u,0,\Delta) = \mathrm{i}a_X u\Delta + \int_0^\Delta \left[\int_{-\infty}^\infty \int_0^\infty \left(e^{\mathrm{i}ux + B_\Delta(u^2 - \mathrm{i}b_X u)e^{-b_V s}y} - 1 \right) \nu(dx,dy) \right] ds.$$

Formula (24) yields

$$\begin{aligned} \log|\phi_\Delta(u)| &= \mathrm{Re}\left\{ \int_0^\Delta \left[\int_{-\infty}^\infty \int_0^\infty \left(e^{\mathrm{i}ux + B_\Delta(u^2 - \mathrm{i}b_X u)e^{-b_V s}y} - 1 \right) \nu(dx,dy) \right] ds \right\} \\ &\quad + \mathrm{Re}\left\{ \int_0^\infty \left[\int_0^\infty \left(e^{e^{-b_V s}B_\Delta(u^2 - \mathrm{i}b_X u)y} - 1 \right) \nu_2(dy) \right] ds \right\} \\ &=: W_1 + W_2. \end{aligned}$$

In what follows we derive asymptotic expansions (as $u \to +\infty$) for the terms W_1 and W_2. Set $c_\gamma = \Gamma(1-\gamma)$, $d_\gamma = \Gamma(1-\gamma)\sin((1-\gamma)\pi/2)$, and $e_\gamma = \Gamma(1-\gamma)\cos((1-\gamma)\pi/2)$ for any $\gamma \in \mathbb{R}$. For estimating the term W_1 we apply Lemma 7.3 with $\varrho = -B_\Delta e^{-b_V s}u^2$ and $\phi = -B_\Delta b_X e^{-b_V s}u$ to get

$$W_1 = -\int_0^\Delta \left[\beta_{0,2} c_{\gamma_2} \varrho^{\gamma_2} \left[1 + \mathcal{R}_1(\varrho,\phi)\right] + \mathcal{R}(u) \right] ds + O(1), \quad u \to +\infty,$$

where $\mathcal{R}_1(\varrho,\phi) = \bar{A}\varrho^{-\chi_2}\beta_{1,2}/\beta_{0,2} + \phi/\varrho$, $\mathcal{R}(u) = -u^{\gamma_1}\left(\beta_{0,1}d_{\gamma_1} + \beta_{1,1}d_{\gamma_1-\chi_1}u^{-\chi_1} \right)$ and $\bar{A}$ is some constant not depending on the parameters of the model (1)-(2) and Δ. This gives the expansion

$$W_1 \quad = \quad -\delta_{1,1}^{(1)}u^{\gamma_1} - \delta_{2,1}^{(1)}u^{\gamma_1-\chi_1} - \delta_{1,2}^{(1)}u^{2\gamma_2} - \delta_{2,2}^{(1)}u^{2\gamma_2-2\chi_2} - \delta_{3,2}^{(1)}u^{2\gamma_2-1} + O(1), \quad u \to +\infty$$

with the coefficients

$$
\begin{aligned}
\delta_{1,1}^{(1)} &= \beta_{0,1} d_{\gamma_1} \Delta, \\
\delta_{2,1}^{(1)} &= \beta_{1,1} d_{\gamma_1-\chi_1} \Delta, \\
\delta_{1,2}^{(1)} &= u^{-2\gamma_2} \int_0^\Delta \beta_{0,2} c_{\gamma_2} \varrho^{\gamma_2} ds = \beta_{0,2} c_{\gamma_2} (-B_\Delta)^{\gamma_2} \int_0^\Delta e^{-b_V s \gamma_2} ds \\
&= \beta_{0,2} c_{\gamma_2} (-B_\Delta)^{\gamma_2} \frac{1-e^{-b_V \Delta \gamma_2}}{b_V \gamma_2}, \\
\delta_{2,2}^{(1)} &= u^{-2(\gamma_2-\chi_2)} \int_0^\Delta c_{\gamma_2} \bar{A} \beta_{1,2} \varrho^{\gamma_2-\chi_2} ds = c_{\gamma_2} \bar{A} \beta_{1,2} \, (-B_\Delta)^{\gamma_2-\chi_2} \, \frac{1-e^{-b_V \Delta(\gamma_2-\chi_2)}}{b_V(\gamma_2-\chi_2)}, \\
\delta_{3,2}^{(1)} &= b_X \delta_{1,2}^{(1)}.
\end{aligned}
$$

Turn now to W_2. Making use of Lemma 7.1 with $\phi = -e^{-b_V s} B_\Delta b_X u$ and $\varrho = -e^{-b_V s} B_\Delta u^2$, we arrive at the asymptotic formula

$$
(25) \qquad W_2 \;=\; -\int_0^\infty \varrho^{\gamma_2} \left[\beta_{0,2} c_{\gamma_2} \left(1 + (\phi/\varrho)\right) + \beta_{1,2} c_{\gamma_2-\chi_2} \varrho^{-\chi_2} \right] ds + O(1), \quad u \to +\infty
$$

or, equivalently,

$$
(26) \qquad W_2 \;=\; -\delta_{1,2}^{(2)} u^{2\gamma_2} - \delta_{2,2}^{(2)} u^{2\gamma_2-2\chi_2} - \delta_{3,2}^{(2)} u^{2\gamma_2-1} + O(1),
$$

where

$$
\begin{aligned}
\delta_{1,2}^{(2)} &= u^{-2\gamma_2} \beta_{0,2} c_{\gamma_2} \int_0^\infty \varrho^{\gamma_2} ds = \frac{\beta_{0,2} c_{\gamma_2}}{\gamma_2 b_V} (-B_\Delta)^{\gamma_2}, \\
\delta_{2,2}^{(2)} &= u^{-2\gamma_2+2\chi_2} \beta_{1,2} c_{\gamma_2-\chi_2} \int_0^\infty \varrho^{\gamma_2-\chi_2} ds = \frac{\beta_{1,2} c_{\gamma_2-\chi_2}}{(\gamma_2-\chi_2) b_V} (-B_\Delta)^{\gamma_2-\chi_2}, \\
\delta_{3,2}^{(2)} &= u^{-2\gamma_2} \beta_{0,2} c_{\gamma_2} b_X \int_0^\infty \varrho^{\gamma_2} ds = \frac{\beta_{0,2} c_{\gamma_2} b_X}{\gamma_2 b_V} (-B_\Delta)^{\gamma_2}.
\end{aligned}
$$

Case $\mathbf{a_V > 0}$. In this case,

$$
(27) \qquad \psi_1(u,w,s) \;=\; -\frac{u(1+o(1/u))}{\sigma a_V (\sqrt{1-\rho^2} - \mathrm{i}\rho)}, \quad u \to +\infty,
$$

$$
(28) \qquad \psi_1(0,-\mathrm{i}w,s) \;=\; \frac{w e^{-b_V s}}{1 + wAB_s}
$$

with $B_s = b_V^{-1}(\exp(-b_V s) - 1)$. By (28), the function $l(w)$ remains bounded for all w such that $\mathrm{Re}\, w \le 0$. Therefore, we have $l(\psi_1(u,0,\Delta)) = O(1)$ as $u \to +\infty$. The asymptotic relation (27) implies

$$
\begin{aligned}
\mathrm{Re}\{\psi_0(u,0,\Delta)\} &= -a_V \left[u\sigma^{-1} a_V^{-1} \sqrt{1-\rho^2} \Delta \right] + \\
&\quad + \mathrm{Re} \left\{ \int_0^\Delta \left[\int_{-\infty}^\infty \int_0^\infty \left(e^{\mathrm{i}ux - [\sigma^{-1} a_V^{-1} (\sqrt{1-\rho^2} + \mathrm{i}\rho) u + o(1)] y} - 1 \right) \nu(dx,dy) \right] ds \right\}
\end{aligned}
$$

as $u \to +\infty$. Furthermore, Lemma 7.3 with $\varrho = u\sigma^{-1}a_V^{-1}\sqrt{1-\rho^2}$ and $\phi = u\sigma^{-1}a_V^{-1}\rho$ gives

$$\begin{aligned} \mathrm{Re}\{\psi_0(u,0,\Delta)\} &= -a_V\left[u\sigma^{-1}a_V^{-1}\sqrt{1-\rho^2}\,\Delta\right] + \\ &+ \int_0^\Delta \left[-\beta_{0,2}\, r_{\gamma_2}(a)\, \varrho^{\gamma_2}\left[1+\mathcal{R}_2(\varrho,\phi)\right] + \mathcal{R}(u)\right] ds + O(1), \quad u \to +\infty, \end{aligned}$$

where $a = \rho/\sqrt{1-\rho^2}$, $\mathcal{R}_2(\varrho,\phi) = (\bar{B}\beta_{1,2}/\beta_{0,2})\varrho^{-\chi_2}$, $\bar{B} = r_{\gamma_2-\chi_2}(a)/r_{\gamma_2}(a)$,

$$\mathcal{R}(u) = -u^{\gamma_1}\left(\beta_{0,1}d_{\gamma_1} + \beta_{1,1}d_{\gamma_1-\chi_1}u^{-\chi_1}\right)$$

and

$$r_{\gamma_2}(a) = \int_0^\infty \frac{e^{-y}}{y^{\gamma_2}}(\cos(ay) + a\sin(ay))\,dy.$$

Denote $\varsigma = \sigma a_V/\sqrt{1-\rho^2}$. Then the following relations hold

$$\begin{aligned} &a_V\left[u\sigma^{-1}a_V^{-1}\sqrt{1-\rho^2}\,\Delta\right] = a_V\varsigma^{-1}\Delta u, \\ &\int_0^\Delta \beta_{0,2}\, r_{\gamma_2}(\phi/\varrho)\, \varrho^{\gamma_2} ds = \beta_{0,2}\, r_{\gamma_2}(a)\, \left(\frac{u}{\varsigma}\right)^{\gamma_2} \Delta, \\ &\int_0^\Delta R(\varrho)\beta_{0,2}\, r_{\gamma_2}(\phi/\varrho)\, \varrho^{\gamma_2} ds = \beta_{0,2}\, r_{\gamma_2}(a)\, \bar{B}\frac{\beta_{1,2}}{\beta_{0,2}} \left(\frac{u}{\varsigma}\right)^{\gamma_2-\chi_2} \Delta, \\ &\int_0^\Delta \mathcal{R}_2(u) ds = -u^{\gamma_1}\left(\beta_{0,1}d_{\gamma_1} + \beta_{1,1}d_{\gamma_1-\chi_1}u^{-\chi_1}\right)\Delta + O(1), \quad u \to +\infty. \end{aligned}$$

Combining the last formulas, we arrive at the representation

$$\log|\phi(u)| = -\tau_1 u - \lambda_{1,1}u^{\gamma_1} - \lambda_{2,1}u^{\gamma_1-\chi_1} - \lambda_{1,2}u^{\gamma_2} - \lambda_{2,2}u^{\gamma_2-\chi_2} + O(1), \quad u \to +\infty, \tag{29}$$

with

$$\begin{aligned} \tau_1 &= a_V\varsigma^{-1}, \\ \lambda_{1,1} &= \beta_{0,1}d_{\gamma_1}, \\ \lambda_{2,1} &= \beta_{1,1}d_{\gamma_1-\chi_1}, \\ \lambda_{1,2} &= \beta_{0,2}r_{\gamma_2}(a)\varsigma^{-\gamma-2}, \\ \lambda_{2,2} &= \beta_{0,2}\, r_{\gamma_2}(a)\, \bar{B}\frac{\beta_{1,2}}{\beta_{0,2}}\, \varsigma^{\chi_2-\gamma_2}. \end{aligned}$$

This completes the proof of Theorem 2.1.

6.2 Proof of Theorem 3.2

We begin the proof with the following lemma.

Lemma 6.2. *Suppose that*

$$\widetilde{\varepsilon}_n := \left[\inf_{u\in[0,U_n]} |\phi(u)|\right]^{-2\theta} \frac{\log n}{\sqrt{n}} = o(1), \quad n\to\infty. \tag{30}$$

Then there exist positive constants D_1, D_2, and δ such that for any $n > 1$

$$\mathrm{P}\left\{|\hat{\alpha}_n - \alpha_n| > D_1\widetilde{\varepsilon}_n \int_0^{U_n} \left|w^{U_n}(u)\right| \left|\log^{-1}(\mathcal{G}(u))\right| du\right\} \le D_2 n^{-1-\delta}, \tag{31}$$

where $\mathcal{G}(u) = |\phi(u)|^{2\theta}/\,|\phi(u\theta)|^2$.

Remark 6.3. Note that $\mathcal{G}(u)$ is bounded away from 0 on any compact set due to (4).

Proof. We divide the proof into several steps.

1. Denote $\mathcal{G}_n(u) = |\phi_n(u)|^{2\theta}/\,|\phi_n(u\theta)|^2$. It holds

$$\begin{aligned}\mathcal{G}_n(u) - \mathcal{G}(u) &= \frac{|\phi_n(u)|^{2\theta} - |\phi(u)|^{2\theta}}{|\phi_n(u\theta)|^2} + \frac{|\phi(u)|^{2\theta}}{|\phi(u\theta)|^2}\,\frac{|\phi(u\theta)|^2 - |\phi_n(u\theta)|^2}{|\phi_n(u\theta)|^2} \\ &= \mathcal{G}(u)\left[\frac{\xi_{1,n}(u) + \xi_{2,n}(u)}{1-\xi_{2,n}(u)}\right] = \mathcal{G}(u)\Lambda_n(u)\end{aligned} \tag{32}$$

with

$$\xi_{1,n}(u) = \frac{|\phi_n(u)|^{2\theta} - |\phi(u)|^{2\theta}}{|\phi(u)|^{2\theta}} \qquad \text{and} \qquad \xi_{2,n}(u) = \frac{|\phi(u\theta)|^2 - |\phi_n(u\theta)|^2}{|\phi(u\theta)|^2}.$$

2. Lemma 7.5 shows that the event

$$\mathcal{W}_n = \left\{\sup_{u\in[0,U_n]} |\xi_{k,n}(u)| \le B_1\,\widetilde{\varepsilon}_n,\ k = 1,2\right\}$$

has a probability that tends to 1 as n tends to infinity. More precisely, it holds

$$\mathrm{P}(\overline{\mathcal{W}}_n) = \mathrm{P}\left(\sup_{u\in[0,U_n]} |\xi_{k,n}(u)| > B_1\widetilde{\varepsilon}_n\right) \le D_2 n^{-1-\delta}, \quad k = 1,2 \tag{33}$$

for some positive constants B_1, D_2, and δ.

3. For any $u \in [\varepsilon U_n, U_n]$, the Taylor expansion for the function $f(x) = \log(-\log(x))$ in the vicinity of the point $x = \mathcal{G}(u)$ yields

$$\mathcal{Y}_n(u) - \mathcal{Y}(u) = \chi_1(u)(\mathcal{G}_n(u) - \mathcal{G}(u)) + \chi_2(u)(\mathcal{G}_n(u) - \mathcal{G}(u))^2 \tag{34}$$

with

$$\chi_1(u) = \mathcal{G}^{-1}(u)\log^{-1}(\mathcal{G}(u)) \qquad \text{and} \qquad |\chi_2(u)| \le 2^{-1} \max_{z\in I_n(u)} \left[\frac{1 + |\log(z)|}{z^2\log^2(z)}\right], \tag{35}$$

where by $I_n(u)$ we denote the interval between $\mathcal{G}(u)$ and $\mathcal{G}_n(u)$. Due to (4),

$$\begin{aligned}\mathcal{G}(u) = \frac{|\phi(u)|^{2\theta}}{|\phi(\theta u)|^2} &= \exp\left\{2\tau_2 u^\alpha\left(-\theta\,(1 + r(u)) + \theta^\alpha\,(1 + r(\theta u))\right)\right\} \\ &\le \exp\left\{A_1 u^\alpha + A_2 u^{\alpha-\varkappa}\right\},\end{aligned}$$

where $A_1 = 2\tau_2 (\theta^\alpha - \theta) < 0$ and $A_2 = 2\tau_2\tau_3 (\theta^{\alpha-\varkappa} + \theta)$. Hence, $\mathcal{G}(u) \to 0$ as $u \to +\infty$. Moreover, the length of the interval $|I_n(u)| = \mathcal{G}(u)|\Lambda_n(u)|$ tends to 0 on the event $\mathcal{W}_n$, uniformly in $u \in [\varepsilon U_n, U_n]$. Thus, $I_n(u) \subset (0,1)$ on $\mathcal{W}_n$ for n large enough and the maximum on the right hand side of the inequality in (35) is attained at one of the endpoints of the interval $I_n(u)$.

4. Denote $Q(u) = \chi_2(u)(\mathcal{G}_n(u) - \mathcal{G}(u))^2$. Lemma 7.6 shows that there exist a positive constant B_3 such that for any $u \in [\varepsilon U_n, U_n]$ and for n large enough

$$\mathcal{W}_n \subset \left\{ |Q(u)| \le B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)) \left|\log^{-1}(\mathcal{G}(u))\right| \right\}. \tag{36}$$

5. The Taylor expansion (34) and previous discussion yield that on the set $\mathcal{W}_n$,

$$\begin{aligned} |\hat{\alpha}_n - \alpha_n| &= \left| \int_0^{U_n} w^{U_n}(u)(\mathcal{Y}_n(u) - \mathcal{Y}(u))\, du \right| \\ &\le \int_0^{U_n} |w^{U_n}(u)| \left(\frac{|\mathcal{G}_n(u) - \mathcal{G}(u)|}{|\mathcal{G}(u)|} \left|\log^{-1}(\mathcal{G}(u))\right| + |Q(u)| \right) du \\ &\le \int_0^{U_n} |w^{U_n}(u)| \log^{-1}\left(\mathcal{G}^{-1}(u)\right) \left(\frac{|\mathcal{G}_n(u) - \mathcal{G}(u)|}{|\mathcal{G}(u)|} + B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)) \right) du. \end{aligned}$$

By (32), expression in the brackets is equal to

$$P := \frac{|\mathcal{G}_n(u) - \mathcal{G}(u)|}{|\mathcal{G}(u)|} + B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)) = \frac{|\xi_{1,n}(u) + \xi_{2,n}(u)|}{|1 - \xi_{2,n}(u)|} + B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)),$$

and P can be upper bounded on the set $\mathcal{W}_n$ as follows (all supremums are taken over $[0, U_n]$):

$$\begin{aligned} P &\le \frac{\sup|\xi_{1,n}(u)| + \sup|\xi_{2,n}(u)|}{1 - \sup|\xi_{2,n}(u)|} + B_3\left((\sup|\xi_{1,n}(u)|)^2 + (\sup|\xi_{2,n}(u)|)^2\right) \\ &\le \frac{2B_1\widetilde{\varepsilon}_n}{1 - B_1\widetilde{\varepsilon}_n} + 2B_3B_1^2\widetilde{\varepsilon}_n^2 \le D_1\widetilde{\varepsilon}_n. \end{aligned}$$

This completes the proof. □

Now we proceed with the proof of Theorem 3.2. First, we get a lower bound for the infimum of the function $|\phi(u)|$ over $[0, U_n]$. Consider two cases (see Theorem 2.1):

1. $a_V > 0$ ($\tau_1 > 0$) In this case,

$$\begin{aligned} \inf_{u\in[0,U_n]} |\phi(u)| = \inf_{u\in[1,U_n]} |\phi(u)| &= \inf_{u\in[1,U_n]} \exp\left\{-\tau_1 u - \tau_2 u^\alpha (1 + r(u))\right\} \\ &\ge \inf_{u\in[1,U_n]} \exp\left\{-\tau_1 u - \tau_2 u^\alpha - \tau_2\tau_3 u^{\alpha-\varkappa}\right\} \\ &\ge \exp\left\{-(\tau_1 + \tau_2 + \tau_2\tau_3) U_n\right\}. \end{aligned}$$

2. $a_V = 0$ ($\tau_1 = 0$) Following the same lines, we arrive at

$$\begin{aligned} \inf_{u\in[0,U_n]} |\phi(u)| = \inf_{u\in[1,U_n]} |\phi(u)| &= \inf_{u\in[1,U_n]} \exp\left\{-\tau_2 u^\alpha - \tau_2\tau_3 u^{\alpha-\varkappa}\right\} \\ &\ge \exp\left\{-(\tau_2 + \tau_2\tau_3) U_n^\alpha\right\}. \end{aligned}$$

Thus, we conclude that $\widetilde{\varepsilon}_n \leq \varepsilon_{1,n}$ in the first case and $\widetilde{\varepsilon}_n \leq \varepsilon_{2,n}$ in the second one, and therefore the assumption of Lemma 6.2 is fulfilled in both cases. Next,

$$\left|\log^{-1}\left(\mathcal{G}(u)\right)\right| = \frac{1}{2\tau_\theta u^\alpha R(u)}$$

with $\tau_\theta = \tau_2(\theta - \theta^\alpha)$ and

$$R(u) = 1 + \frac{\theta r(u) - \theta^\alpha r(\theta u)}{\theta - \theta^\alpha}.$$

Hence

$$\int_0^{U_n} \left|w^{U_n}(u)\right| \left|\log^{-1}\left(\mathcal{G}(u)\right)\right| du \;=\; \frac{1}{2\tau_\theta U_n^\alpha} \int_\varepsilon^1 \frac{\left|w^1(u)\right|}{u^\alpha R(U_n u)} du \leq \frac{C_2}{\tau_\theta U_n^\alpha}$$

for some $C_2 > 0$ and the statement of the theorem follows.

7 Auxiliary results

Lemma 7.1. *Consider a Lévy measure ν on $\mathbb{R}_+$ that satisfies*

$$\Pi(\varepsilon) := \int_\varepsilon^\infty \nu(dy) = \varepsilon^{-\gamma}(\beta_0 + \beta_1 \varepsilon^\chi(1 + O(\varepsilon))), \quad \varepsilon \to +0, \tag{37}$$

with $0 < \gamma < 1$, $\chi > 0$ and $\beta_0 > 0$. Denote

$$\Phi(\rho, \phi) = \int_0^\infty \left(e^{-\varrho z}\cos(\phi z) - 1\right) \nu(dz),$$

then the following asymptotic relations hold.

(i) *As $\phi, \varrho \to \infty$,*

$$\Phi(\varrho, \phi) = \begin{cases} -\varrho^\gamma \left[\beta_0 c_\gamma \left(1 + \phi/\varrho\right) + \beta_1 c_{\gamma-\chi}\varrho^{-\chi}\right] + O\left(e^{-\phi}\right), & \varrho/\phi \to +\infty, \\ -\phi^\gamma \Big[\beta_0 d_\gamma + \beta_0 e_\gamma\left(\varrho/\phi\right) + \beta_1 (d_{\gamma-\chi} + e_{\gamma-\chi})\phi^{-\chi}\left(\varrho/\phi\right)\Big] + O\left(e^{-\varrho}\right), & \phi/\varrho \to +\infty, \end{cases}$$

where $c_\gamma = \Gamma(1-\gamma)$, $d_\gamma = \Gamma(1-\gamma)\sin((1-\gamma)\pi/2)$, and $e_\gamma = \Gamma(1-\gamma)\cos((1-\gamma)\pi/2)$.

(ii) *As $\phi, \varrho \to \infty$ and $\phi/\varrho = a$ for some constant $a > 0$,*

$$\Phi(\varrho, \phi) = -\varrho^\gamma \left[\beta_0 r_\gamma(a) + \beta_1 r_{\gamma-\chi}(a)\varrho^{-\chi}\right] + O\left(e^{-\varrho}\right)$$

with

$$r_\gamma(a) = \int_0^\infty \frac{e^{-y}}{y^\gamma}(\cos(ay) + a\sin(ay))\, dy.$$

Proof. **(i)** Here we present the proof only for the case $\phi/\varrho \to +\infty$. The case $\varrho/\phi \to +\infty$ can be treated in a similar way.

i1. Integrating by parts, we get

$$\begin{aligned}\int_0^\infty \left(e^{-\varrho z}\cos(\phi z)-1\right)\nu(dz) &= \int_0^\infty \left(e^{-y}\cos(\phi y/\rho)-1\right)\nu(d(y/\varrho))\\ &= -\left(e^{-y}\cos(\phi y/\rho)-1\right)\Pi(y/\varrho)\big|_0^\infty\\ &\quad -\int_0^\infty \Pi(y/\varrho)e^{-y}\Big(\cos(\phi y/\varrho)+\phi/\varrho\ \sin(\phi y/\varrho)\Big)dy.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^\infty \left(e^{-\varrho z}\cos(\phi z)-1\right)\nu(dz) &= -\varrho^\gamma\int_0^\infty (y/\varrho)^\gamma\Pi(y/\varrho)\frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy\\ &\quad -\phi\varrho^{\gamma-1}\int_0^\infty (y/\varrho)^\gamma\Pi(y/\varrho)\frac{e^{-y}}{y^\gamma}\sin(\phi y/\varrho)dy\\ &= -\varrho^\gamma I_1-\phi\varrho^{\gamma-1}I_2.\end{aligned}$$

i2. Take $H=\varrho^p$ with $0<p<1$, and represent I_1 as a sum of two integrals:

$$\begin{aligned}I_1=\int_0^\infty (y/\varrho)^\gamma\Pi(y/\varrho)\frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy &= \int_0^H (y/\varrho)^\gamma\Pi(y/\varrho)\frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy\\ &\quad +\int_H^\infty \rho^{-\gamma}\Pi(y/\varrho)e^{-y}\cos(\phi y/\varrho)dy.\end{aligned}$$

The function $\varrho^{-\gamma}\Pi(y/\varrho)$ is uniformly bounded for $y>H$ as $\varrho\to+\infty$. Indeed,

$$\begin{aligned}\varrho^{-\gamma}\,\Pi(y/\varrho) &\le \varrho^{-\gamma}\,\Pi(H/\varrho)\\ &= \varrho^{-p\gamma}\Big(\beta_0+\beta_1\varrho^{\chi(p-1)}\left(1+O(\varrho^{p-1})\right)\Big)\\ &= \beta_0\varrho^{-p\gamma}+\beta_1\varrho^{-\left(\chi+(\gamma-\chi)p\right)}\left(1+\varrho^{p-1}O(1)\right)\end{aligned}$$

and $\chi+(\gamma-\chi)p>0$. This boundeness of $\rho^{-\gamma}\Pi(y/\varrho)$ implies

$$\int_H^{+\infty}\rho^{-\gamma}\Pi(y/\varrho)e^{-y}\cos(\phi y/\varrho)dy=O(e^{-H}).$$

As a result,

$$I_1=\int_0^H (y/\varrho)^\gamma\Pi(y/\varrho)\frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy+O(e^{-H}).$$

i3. If $\rho\to\infty$ and $y<H$, the assumption (37) implies

$$\begin{aligned}I_1 &= \beta_0\int_0^H\frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy+\beta_1\varrho^{-\chi}\int_0^H\frac{e^{-y}}{y^{\gamma-\chi}}\cos(\phi y/\varrho)dy\\ &\quad +O\left(\varrho^{-\chi-1}\int_0^H\frac{e^{-y}}{y^{\gamma-\chi-1}}dy\right)+O(e^{-H}).\end{aligned}$$

Note now that

$$\begin{aligned}\int_0^H \frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy &= \int_0^\infty \frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy - \int_H^\infty \frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy \\ &= \int_0^\infty \frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy + O(e^{-H}H^{-\gamma}).\end{aligned}$$

Analogously,

$$\int_0^H \frac{e^{-y}}{y^{\gamma-\chi}}\cos(\phi y/\varrho)dy = \int_0^\infty \frac{e^{-y}}{y^{\gamma-\chi}}\cos(\phi y/\varrho)dy + O(e^{-H}H^{\chi-\gamma}),$$

and we conclude that

$$I_1 = \beta_0 \int_0^\infty \frac{e^{-y}}{y^\gamma}\cos(\phi y/\varrho)dy + \beta_1\varrho^{-\chi}\int_0^\infty \frac{e^{-y}}{y^{\gamma-\chi}}\cos(\phi y/\varrho)dy + T_1,$$

where

$$\begin{aligned}T_1 &= O\left(\varrho^{-\chi-1}\int_0^H \frac{e^{-y}}{y^{\gamma-\chi-1}}dy\right) + O(e^{-H}H^{-\gamma}) + O\left(\varrho^{-\chi}e^{-H}H^{\gamma-\chi}\right) + O(e^{-H}) \\ &= O\left(\varrho^{-\gamma}e^{-H}\right).\end{aligned}$$

i4. Since

$$\int_0^\infty \frac{e^{-y}}{y^\gamma}\cos(hy)dy \asymp e_\gamma h^{\gamma-1}, \quad h\to+\infty$$

with $e_\gamma = \Gamma(1-\gamma)\cos((1-\gamma)\pi/2)$, we get

$$\varrho^\gamma I_1 = \phi^\gamma\left[\beta_0 e_\gamma(\varrho/\phi) + \beta_1 e_{\gamma-\chi}\phi^{-\chi}(\varrho/\phi)\right] + O(e^{-H}), \quad \varrho,\phi\to\infty.$$

Similarly, using the fact that

$$\int_0^\infty \frac{e^{-y}}{y^\gamma}\sin(hy)dy \asymp d_\gamma h^{\gamma-1}, \quad h\to\infty$$

with $e_\gamma = \Gamma(1-\gamma)\sin((1-\gamma)\pi/2)$, we arrive at

$$\phi\varrho^{\gamma-1}I_2 = \phi^\gamma\left[\beta_0 d_\gamma + \beta_1 d_{\gamma-\chi}\phi^{-\chi}\right] + O(e^{-H}), \quad \varrho,\phi\to\infty.$$

(ii) The first three steps are the same as i1, i2 and i3.

ii4. Introduce

$$v_\gamma(a) = \int_0^\infty \frac{e^{-y}\cos(ay)}{y^\gamma}dy,$$

then

$$\varrho^\gamma I_1 = \varrho^\gamma\left[\beta_0 v_\gamma(a) + \beta_1 v_{\gamma-\chi}(a)\varrho^{-\chi}\right] + O\left(e^{-H}\right).$$

Analogously,

$$\phi\varrho^{\gamma-1}I_2 = a\varrho^\gamma I_2 = a\varrho^\gamma\left[\beta_0 w_\gamma(a) + \beta_1 w_{\gamma-\chi}(a)\varrho^{-\chi}\right] + O\left(e^{-H}\right)$$

with

$$w_\gamma(a) = \int_0^\infty \frac{e^{-y}\sin(ay)}{y^\gamma}\,dy.$$

It remains to note that

$$r_\gamma(a) = v_\gamma(a) + a w_\gamma(a).$$

□

Lemma 7.2. *Consider a Lévy measure ν on $\mathbb{R}\setminus\{0\}$ that fulfilles*

$$G(\varepsilon) := \int_{|x|>\varepsilon} \nu(dx) = \varepsilon^{-\gamma}(\beta_0 + \beta_1\varepsilon^\chi(1+O(\varepsilon))), \quad \varepsilon \to +0 \tag{38}$$

with $0<\chi<\gamma<2$, and $\beta_0>0$. Denote

$$V(u) \;=\; \int_{\mathbb{R}} \big(\cos(ux)-1\big)d\nu(x).$$

Then as $u\to+\infty$,

$$V(u) \;=\; -u^\gamma\Big(\beta_0 d_\gamma + \beta_1 d_{\gamma-\chi}u^{-\chi}\Big) + O(1).$$

Proof. We divide the proof into 3 steps.

1. First, apply integration by parts to get

$$\begin{aligned} V(u) &= -\int_0^{+\infty} \big(\cos(ux)-1\big)dG(x) \\ &= -\big(\cos(ux)-1\big)G(x)\big|_0^{+\infty} - u\int_0^{+\infty} \sin(ux)G(x)dx \\ &= -\int_0^{+\infty} \sin(x)G(x/u)dx. \end{aligned}$$

2. Take $H=u^p$ with $0<p<1$ such that $p\gamma>\chi$, and represent the last integral as a sum of tho integrals:

$$\begin{aligned} \int_0^{+\infty} \sin(x)G(x/u)dx &= \int_0^H \sin(x)G(x/u)dx + \int_H^{+\infty} \sin(x)G(x/u)dx \\ &= I_1 + I_2. \end{aligned}$$

The integral I_2 is bounded, because $G(x/u)$ is monotone, converges to 0 as $x\to\infty$ and the antiderivative of $\sin(x)$ is bounded.

3. Next, we apply (38) to I_1:

$$\begin{aligned} I_1 &= \int_0^H \sin(x)\,(x/u)^{-\gamma}\left(\beta_0+\beta_1\,(x/u)^{\chi}\,(1+O\,(x/u))\right)dx \\ &= \beta_0 u^{\gamma}\int_0^H \frac{\sin(x)}{x^{\gamma}}dx+\beta_1 u^{\gamma-\chi}\int_0^H \frac{\sin(x)}{x^{\gamma-\chi}}dx+\beta_1 u^{\gamma-\chi-1}\int_0^H \frac{\sin(x)}{x^{\gamma-\chi-1}}dx. \end{aligned}$$

Note that the integral $\int_0^H \sin(x)x^{-\gamma}dx$ can be represented in the following way:

$$\int_0^H \frac{\sin(x)}{x^{\gamma}}dx \;=\; \int_0^{\infty} \frac{\sin(x)}{x^{\gamma}}dx-\int_H^{\infty} \frac{\sin(x)}{x^{\gamma}}dx=d_{\gamma}+O(H^{-\gamma}).$$

Analogously,

$$\int_0^H \frac{\sin(x)}{x^{\gamma-\chi}}dx \;=\; d_{\gamma-\chi}+O(H^{-(\gamma-\chi)}).$$

Finally, we arrive at

$$I_1 \;=\; \beta_0 d_{\gamma}u^{\gamma}+\beta_1 d_{\gamma-\chi}u^{\gamma-\chi}+T_1,$$

where

$$T_1 \;=\; O(u^{(1-p)\gamma})+O(u^{(1-p)(\gamma-\chi)})+O(u^{(1-p)(\gamma-\chi-1)})=O(u^{(1-p)\gamma}).$$

□

Lemma 7.3. *Let ν be a two-dimensional Lévy measure on $\mathbb{R}\times\mathbb{R}_+$ with marginals ν_1 and ν_2, and assumptions (AN1) and (AN2) are fulfilled. Denote*

$$Q(u,\varrho,\phi)=\int_{-\infty}^{\infty}\int_0^{\infty}\left(\exp\Big\{\mathrm{i}ux-(\varrho+\mathrm{i}\phi)y\Big\}-1\right)\nu(dx,dy)$$

for any real numbers u, ϱ and ϕ. Then

$$\mathrm{Re}\{Q(u,\varrho,\phi)\}=\Phi(\rho,\phi)+\mathcal{R}(u)+O(1),\qquad u,\varrho,\phi\to+\infty$$

with

$$\Phi(\rho,\phi)=\int_0^{\infty}\left(e^{-\varrho y}\cos(\phi y)-1\right)\nu_2(dy)$$

and

$$\mathcal{R}(u) \;=\; -u^{\gamma_1}\left(\beta_{0,1}d_{\gamma_1}+\beta_{1,1}d_{\gamma_1-\chi_1}u^{-\chi_1}\right).$$

Moreover, the following asymptotic relations hold as $\varrho,\phi\to+\infty$

$$\begin{aligned} \mathrm{Re}\{Q(u,\varrho,\phi)\} &= -\beta_{0,2}c_{\gamma_2}\varrho^{\gamma_2}\left[1+\mathcal{R}_1(\varrho,\phi)\right]+\mathcal{R}(u)+O(1), \quad \varrho/\phi\to+\infty, \\ \mathrm{Re}\{Q(u,\varrho,\phi)\} &= -\beta_{0,2}r_{\gamma_2}(a)\varrho^{\gamma_2}\left[1+\mathcal{R}_2(\varrho,\phi)\right]+\mathcal{R}(u)+O(1), \quad \phi/\varrho=a, \end{aligned}$$

where

$$\mathcal{R}_1(\varrho,\phi)=\bar{A}\frac{\beta_{1,2}}{\beta_{0,2}}\varrho^{-\chi_2}+\frac{\phi}{\varrho},\quad \mathcal{R}_2(\varrho,\phi)=(\bar{B}\beta_{1,2}/\beta_{0,2})\varrho^{-\chi_2}$$

and $\bar{A}$, $\bar{B}$ are two absolute constants.

Proof. We have

$$\begin{aligned}
\operatorname{Re}[Q(u,\varrho,\phi)] &= \int_0^\infty (\exp(-\varrho y)\cos(\phi y)-1)\,\nu_2(dy) \\
&\quad + \int_{-\infty}^\infty \int_0^\infty (\cos(ux)-1)\cdot\exp(-\varrho y)\cos(\phi y)\nu(dx,dy) \\
&\quad + \int_{-\infty}^\infty \int_0^\infty \sin(ux)\sin(\phi y)\exp(-\varrho y)\nu(dx,dy) \\
&= \Phi(\varrho,\phi)+I_1(u,\varrho,\phi)+I_2(u,\varrho,\phi).
\end{aligned}$$

Consider for simplicity the case of the Lévy measure ν with independent components. In this case (see Cont, Tankov, 2004, Proposition 5.3),

$$I_1(u,\varrho,\phi)=\int_{-\infty}^\infty (\cos(ux)-1)\,\nu_1(dx), \qquad I_2(u,\varrho,\phi)=0.$$

The asymptotical behavior of the integral $I_1(u,\varrho,\phi)$ is given by Lemma 7.2. Other statements directly follow from Lemma 7.1. The constants $\bar{A}$ and $\bar{B}$ are equal to

$$\bar{A}=c_{\gamma_2-\chi_2}/c_{\gamma_2}, \qquad \bar{B}=r_{\gamma_2-\chi_2}(a)/r_{\gamma_2}(a).$$

This completes the proof. □

Lemma 7.4. *For any n large enough, it holds*

$$|\alpha-\alpha_n| \le \frac{C\,\tau_3}{U_n^\varkappa(1-\theta^{\alpha-1})} \tag{39}$$

with some constant C not depending on the parameters of the underlying ASV model.

Proof. Denote

$$R(u)=1+\frac{\theta r(u)-\theta^\alpha r(\theta u)}{\theta-\theta^\alpha},$$

then

$$\begin{aligned}
|\alpha-\alpha_n| &= \left|\alpha-\int_0^{U_n} w^{U_n}(u)\mathcal{Y}(u)du\right| = \left|\alpha-\int_0^{U_n} w^{U_n}(u)\log(2\tau_\theta u^\alpha R(u))du\right| = \\
&= \left|\alpha-\log(2\tau_\theta)\int_0^{U_n} w^{U_n}(u)du-\alpha\int_0^{U_n} w^{U_n}(u)\log u\,du-\int_0^{U_n} w^{U_n}(u)\log R(u)du\right| \\
&= \left|\int_0^{U_n} w^{U_n}(u)\log\Big(1+\frac{\theta r(u)-\theta^\alpha r(\theta u)}{\theta-\theta^\alpha}\Big)du\right| \\
&= \left|\int_0^1 w^1(s)\log\Big(1+\frac{\theta r(sU_n)-\theta^\alpha r(\theta sU_n)}{\theta-\theta^\alpha}\Big)ds\right|.
\end{aligned}$$

Since the function w^1 is supported on $[\varepsilon,1]$, the lower bound of the integral can be changed to ε. It follows from

$$|r(u)|\le \tau_3 u^{-\varkappa}, \quad u>1$$

that

$$\left|\frac{\theta r(sU_n) - \theta^\alpha r(\theta s U_n)}{\theta - \theta^\alpha}\right| \le \frac{\theta\tau_3 (sU_n)^{-\varkappa} + \theta^\alpha \tau_3(\theta s U_n)^{-\varkappa}}{\theta - \theta^\alpha} = \tau_3 U_n^{-\varkappa} s^{-\varkappa}\,\frac{\theta + \theta^{\alpha-\varkappa}}{\theta - \theta^\alpha}$$

for n large enough (more precisely, for n s.t. $\varepsilon U_n > 1$). Hence for n large enough

$$\left|\frac{\theta r(sU_n) - \theta^\alpha r(\theta s U_n)}{\theta - \theta^\alpha}\right| \le \frac{1}{2}$$

and

$$|\alpha - \alpha_n| \le \tau_3 U_n^{-\varkappa} \frac{\theta + \theta^{\alpha - \varkappa}}{\theta - \theta^\alpha} \int_\varepsilon^1 |w^1(s)| s^{-\varkappa} ds, \tag{40}$$

as $|\log(1+x)| \le 2|x|$ for any $|x| \le 1/2$. The observation that the integral on the right hand side of (40) is finite completes the proof. □

Lemma 7.5. *Let the assumptions (AM) and (AE) be fulfilled. Denote*

$$\xi_{1,n}(u) = \frac{|\phi_n(u)|^{2\theta} - |\phi(u)|^{2\theta}}{|\phi(u)|^{2\theta}}, \qquad \xi_{2,n}(u) = \frac{|\phi(u\theta)|^2 - |\phi_n(u\theta)|^2}{|\phi(u\theta)|^2}, \tag{41}$$

and

$$\widetilde{\varepsilon}_n = \left[\inf_{u\in[0,U_n]} |\phi(u)|\right]^{-2\theta} \frac{\log n}{\sqrt{n}}. \tag{42}$$

There exist some positive constants B_1, B_2, and δ such that

$$\mathrm{P}\left\{ \sup_{u\in[0,U_n]} |\xi_{k,n}(u)| > B_1 \widetilde{\varepsilon}_n \right\} \le B_2 n^{-1-\delta}, \quad k = 1, 2. \tag{43}$$

Proof. Denote

$$\begin{aligned} H_1 &= \left[\inf_{u\in[0,U_n]} |\phi(u)|\right]^{2\theta} \sup_{u\in[0,U_n]} \frac{\left||\phi_n(u)|^{2\theta} - |\phi(u)|^{2\theta}\right|}{|\phi(u)|^{2\theta}}, \\ H_2 &= \left[\inf_{u\in[0,U_n]} |\phi(u)|\right]^{2\theta} \sup_{u\in[0,U_n]} \frac{\left||\phi_n(u\theta)|^{2} - |\phi(u\theta)|^{2}\right|}{|\phi(u\theta)|^{2}}. \end{aligned}$$

Substituting (41) and (42) into (43), we obtain an equivalent formulation of the statement of the lemma:

$$\left\{ \begin{array}{lcl} \mathrm{P}\left\{\frac{\sqrt{n}}{\log n} H_1 > B_1\right\} & \le & B_2 n^{-1-\delta}, \\ \mathrm{P}\left\{\frac{\sqrt{n}}{\log n} H_2 > B_1\right\} & \le & B_2 n^{-1-\delta}. \end{array}\right. \tag{44}$$

Denote $w^*(u) = \log^{-1/2}(e+|u|)$. The quantity H_1 can be upper bounded as follows:

$$
\begin{aligned}
H_1 &\leq \left[\inf_{u\in[0,U_n]} |\phi(u)|\right]^{2\theta} \frac{\sup_{u\in[0,U_n]} \left||\phi_n(u)|^{2\theta} - |\phi(u)|^{2\theta}\right|}{\inf_{u\in[0,U_n]} |\phi(u)|^{2\theta}} \\
&\leq 2\theta \sup_{u\in[0,U_n]} |\phi_n(u) - \phi(u)| \\
&\leq 2\theta \sup_{u\in[0,U_n]} \left[\frac{w^*(u)}{\inf_{s\in[0,U_n]} w^*(s)} |\phi_n(u) - \phi(u)|\right] \\
&\leq 2\theta\sqrt{\log(e+U_n)} \sup_{u\in[0,U_n]} \left[w^*(u)\, |\phi_n(u) - \phi(u)|\right] \\
&\leq C_1\sqrt{\log n} \sup_{u\in[0,U_n]} \left[w^*(u)\, |\phi_n(u) - \phi(u)|\right] \\
&\leq C_1\sqrt{\log n} \sup_{u\in\mathbb{R}} \left[w^*(u)\, |\phi_n(u) - \phi(u)|\right],
\end{aligned}
$$

for some constant C_1. The quantity H_2 can be upper bounded in a similar way:

$$
\begin{aligned}
H_2 &\leq \left[\inf_{u\in[0,U_n]} |\phi(u)|\right]^{2\theta} \frac{\sup_{u\in[0,U_n\theta]} \left||\phi_n(u)|^{2} - |\phi(u)|^{2}\right|}{\inf_{u\in[0,U_n\theta]} |\phi(u)|^{2}} \\
&\leq \left[\inf_{u\in[0,U_n\theta]} |\phi(u)|\right]^{2\theta-2} \sup_{u\in[0,U_n\theta]} \left||\phi_n(u)|^{2} - |\phi(u)|^{2}\right| \\
&\leq 2 \sup_{u\in[0,U_n\theta]} |\phi_n(u) - \phi(u)| \\
&\leq C_2\sqrt{\log n} \sup_{u\in\mathbb{R}} \left[w^*(u)\, |\phi_n(u) - \phi(u)|\right].
\end{aligned}
$$

Note that under the assumptions (AE) and (AM) the sequence $X_{k\Delta} - X_{(k-1)\Delta}$, $k = 2, \ldots, n$, is strongly mixing and ergodic with exponentially decreasing mixing coefficients (see Masuda, 2007). By the Proposition 8.3, there exist positive constants $B_1^{(0)}$, B_2 and δ such that

$$
\mathrm{P}\left\{\sqrt{\frac{n}{\log n}} \sup_{u\in\mathbb{R}} \Big[w^*(u)\, |\phi_n(u) - \phi(u)|\Big] > C_1 B_1^{(0)}\right\} \leq B_2 n^{-1-\delta}.
$$

Combining this result with the upper bounds for H_1 and H_2, we arrive at

$$
\mathrm{P}\left\{\frac{\sqrt{n}}{\log n} H_1 > C_1 B_1^{(0)}\right\} \leq \mathrm{P}\left\{\sqrt{\frac{n}{\log n}} \sup_{u\in\mathbb{R}} \left[w^*(u)\, |\phi_n(u) - \phi(u)|\right] > B_1^{(0)}\right\} \leq B_2 n^{-1-\delta}
$$

and

$$
\mathrm{P}\left\{\frac{\sqrt{n}}{\log n} H_2 > C_2 B_1^{(0)}\right\} \leq \mathrm{P}\left\{\sqrt{\frac{n}{\log n}} \sup_{u\in\mathbb{R}} \left[w^*(u)\, |\phi_n(u) - \phi(u)|\right] > B_1^{(0)}\right\} \leq B_2 n^{-1-\delta}.
$$

Formulae (44) follow with $B_1 = B_1^{(0)} \cdot \max\{C_1, C_2\}$. □

Lemma 7.6. *Denote* $Q(u) = \chi_2(u)(\mathcal{G}_n(u) - \mathcal{G}(u))^2$ *and let* $\widetilde{\varepsilon}_n = o(1)$. *Then*

$$
\mathcal{W}_n := \left\{\sup_{v\in[0,U_n]} |\xi_{k,n}(v)| \leq B_1\, \widetilde{\varepsilon}_n,\ k = 1,2\right\} \subset \left\{|Q(u)| \leq B_3(\xi_{1,n}^2(u) + \xi_{2,n}^2(u)) \left|\log^{-1}(\mathcal{G}(u))\right|\right\}
$$

for some positive constant B_3, *n large enough, and all* $u \in [\varepsilon U_n, U_n]$.

Proof. Denote

$$S(u) = |Q(u)| \, \frac{|\log(\mathcal{G}(u))|}{\xi_{1,n}^2(u) + \xi_{2,n}^2(u)}.$$

By formula (32) and a trivial inequality $(a+b)^2 \le 2\,(a^2+b^2)$, we get

$$(\mathcal{G}_n(u) - \mathcal{G}(u))^2 = \mathcal{G}^2(u)\Lambda_n^2(u) \le 2\,\mathcal{G}^2(u) \frac{\xi_{1,n}^2(u) + \xi_{2,n}^2(u)}{(1 - \xi_{2,n}(u))^2}.$$

Hence

$$S(u) \le 2\,|\chi_2(u)| \, \frac{\mathcal{G}^2(u)\,|\log(\mathcal{G}(u))|}{(1 - \xi_{2,n}(u))^2}.$$

Let us now show that for n large enough

$$\mathcal{W}_n \subset \left\{ \omega : |\Lambda_n(u)| \le \frac{1}{2} \right\}.$$

In fact, we have on $\mathcal{W}_n$ for n large enough:

$$\begin{aligned} |\Lambda_n(u)| &= \frac{|\xi_{1,n}(u) + \xi_{2,n}(u)|}{|1 - \xi_{2,n}(u)|} \le \frac{\sup|\xi_{1,n}(u)| + \sup|\xi_{2,n}(u)|}{1 - \sup|\xi_{2,n}(u)|} \\ &\le \frac{2B_1\tilde{\varepsilon}_n}{1 - B_1\tilde{\varepsilon}_n} \le \frac{1}{2} \end{aligned}$$

because $\tilde{\varepsilon}_n = o(1)$. By (35), we get

$$|\chi_2(u)| \le 2^{-1} \max_{z \in I_1(u)} \left[\frac{1 + |\log(z\mathcal{G}(u))|}{z^2\mathcal{G}^2(u)\log^2(z\mathcal{G}(u))} \right],$$

where $I_1(u)$ is an interval between 1 and $1 + \Lambda_n(u)$. On the set $\mathcal{W}_n$, we have $I_1(u) \subset [1/2, 3/2]$. Therefore

$$\begin{aligned} |\chi_2(u)|\,\mathcal{G}^2(u)\,|\log(\mathcal{G}(u))| &\le 2^{-1} \max_{z \in [1/2,3/2]} \left[\frac{1 + |\log(z\mathcal{G}(u))|}{\log^2(z\mathcal{G}(u))} \right] |\log(\mathcal{G}(u))| \\ &\le 2^{-1} \frac{\left(1 + \left|\log(\frac{1}{2}\mathcal{G}(u))\right|\right) |\log(\mathcal{G}(u))|}{\left|\log(\frac{1}{2}\mathcal{G}(u))\right|^2}. \end{aligned}$$

Since $\sup_{u \in [\varepsilon U_n, U_n]} |\mathcal{G}(u)| \to 0$ as $n \to \infty$, the function $|\chi_2(u)|\,\mathcal{G}^2(u)\,|\log(\mathcal{G}(u))|$ is bounded on $[\varepsilon U_n, U_n]$ by a constant $\widetilde{C}$. So, we have proved that on $\mathcal{W}_n$,

$$S(u) \le \frac{2\,\widetilde{C}}{(1 - \xi_{2,n}(u))^2},$$

for u large enough. Moreover, it holds on $\mathcal{W}_n$

$$\begin{aligned} S(u) &\le \frac{C}{(1 - \xi_{2,n}(u))^2} \le \sup_{u \in [0,U_n]} \frac{C}{(1 - \xi_{2,n}(u))^2} \le \frac{C}{\left(1 - \sup_{u \in [0,U_n]} |\xi_{2,n}(u)|\right)^2} \\ &\le \frac{C}{(1 - B_1\widetilde{\varepsilon}_n)^2} \le B_3 \end{aligned}$$

for some B_3, $C = 2\widetilde{C}$ and n large enough. This completes the proof. □

8 Appendix

8.1 Exponential inequalities for dependent sequences and for empirical characteristic functions

The following theorem can be found in Merlevéde, Peligrad, and Rio, 2009.

Theorem 8.1. *Let $(Z_k,\, k \geq 1)$ be a strongly mixing sequence of centered real-valued random variables on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the mixing coefficients satisfying*

$$\alpha(n) \leq \bar{\alpha} \exp(-cn), \quad n \geq 1, \quad \bar{\alpha} > 0, \quad c > 0. \tag{45}$$

Assume that $\sup_{k\geq 1} |Z_k| \leq M$ a.s., then there is a positive constant C depending on c and $\bar{\alpha}$ such that

$$\mathrm{P}\left\{\sum_{i=1}^{n} Z_i \geq \zeta\right\} \leq \exp\left[-\frac{C\zeta^2}{nv^2 + M^2 + M\zeta \log^2(n)}\right].$$

for all $\zeta > 0$ and $n \geq 4$, where

$$v^2 = \sup_i \left(\mathbb{E}[Z_i]^2 + 2\sum_{j\geq i} \mathrm{Cov}(Z_i, Z_j)\right).$$

Corollary 8.2. *Denote*

$$\rho_j = \mathbb{E}\left[Z_j^2 \log^{2(1+\varepsilon)}\left(|Z_j|^2\right)\right], \quad j = 1, 2, \ldots,$$

with arbitrary small $\varepsilon > 0$ and suppose that all ρ_j are finite. Then

$$\sum_{j\geq i} \mathrm{Cov}(Z_i, Z_j) \leq C \max_j \rho_j$$

for some constant $C > 0$, provided (45) holds. Consequently the following inequality holds

$$v^2 \leq \sup_i \mathbb{E}[Z_i]^2 + C \max_j \rho_j.$$

Proof. Due to the Rio inequality (see, e.g., Bosq and Blanke, 2007, p. 296)

$$|\mathrm{Cov}(Z_i, Z_j)| \leq 2 \int_0^{\alpha(|j-i|)} Q_{Z_i}(u) Q_{Z_j}(u)\, du$$

where for any random variable X we denote by Q_X the quantile function of X. Define

$$\rho_X = \mathbb{E}\left[X^2 \log^{2(1+\varepsilon)}\left(|X|^2\right)\right].$$

The Markov inequality implies for small enough $u > 0$

$$\begin{aligned} \mathrm{P}\left(|X| > \frac{\rho_X^{1/2}}{u^{1/2} |\log(u)|^{(1+\varepsilon)}}\right) &\leq \mathbb{E}\left[X^2 \log^{2(1+\varepsilon)}\left(|X|^2\right)\right] \frac{\rho_X^{-1}}{u^{-1} \log^{-2(1+\varepsilon)}(u)} \\ &\quad \times \log^{-2(1+\varepsilon)}\left(\frac{\rho_X}{u \log^{2(1+\varepsilon)}(u)}\right) \\ &= u \log^{-2(1+\varepsilon)}\left(\rho_X \log^{-2(1+\varepsilon)}(u)\right) \leq u \end{aligned}$$

and therefore

$$Q_X(u) \le \frac{\rho_X^{1/2}}{u^{1/2}|\log(u)|^{(1+\varepsilon)}}.$$

Hence

$$|\mathrm{Cov}(Z_i, Z_j)| \le 2\int_0^{\alpha(|j-i|)} \frac{\sqrt{\rho_i\rho_j}}{u\log^{2(1+\varepsilon)}(u)}du \le 2\sqrt{\rho_i\rho_j}\log^{-1-2\varepsilon}(\alpha(|j-i|))$$

and

$$\sum_{j\ge i}\mathrm{Cov}(Z_i, Z_j) \le C\sqrt{\rho_i\rho_j}\sum_{j>i}\frac{1}{|j-i|^{1+2\varepsilon}}$$

with some constant $C > 0$ depending on $\bar{\alpha}$. □

Let Z_j, $j = 1, \ldots, n$, be a stationary sequence of random variables. Define

$$\phi_n(u) \quad = \quad \frac{1}{n}\sum_{j=1}^{n}\exp(\mathrm{i}uZ_j).$$

The function $\phi_n(u)$ is an empirical characteristic function of the corresponding stationary distribution. The following proposition provides the uniform probabilistic inequality for the empirical process $\phi_n - \phi$, where ϕ is the characteristic function of the stationary distribution. For similar results in i.i.d. case, see Reiß and Neumann (2009).

Proposition 8.3. *Suppose that the following assumptions hold:*

(AZ1) *The sequence Z_j, $j = 1, \ldots, n$, is strictly stationary and is α-mixing with mixing coefficients $(\alpha_Z(k))_{k\in\mathbb{N}}$ satisfying*

$$\alpha_Z(k) \le \bar{\alpha}_0\exp(-\bar{\alpha}_1 k), \quad k \in \mathbb{N}$$

for some $\bar{\alpha}_0 > 0$ and $\bar{\alpha}_1 > 0$.

(AZ2) *The r.v. Z_j possess finite absolute moments of order $p > 2$.*

Let w be a positive monotone decreasing Lipschitz function on $\mathbb{R}_+$ such that

$$0 < w(z) \le \log^{-1/2}(e + |z|), \quad z \in \mathbb{R}. \tag{46}$$

Then there is $\delta' > 0$ and $\zeta_0 > 0$, such that the inequality

$$\mathrm{P}\left\{\sqrt{\frac{n}{\log n}}\,\|\phi_n - \phi\|_{L_\infty(\mathbb{R},w)} > \zeta\right\} \quad \le \quad B\zeta^{-p}n^{-1-\delta'}. \tag{47}$$

holds for any $\zeta > \zeta_0$ and some positive constant B not depending on ζ and n.

Proof. Denote $\mathcal{W}_n(u) = \phi_n(u) - \mathbb{E}[\phi_n(u)]$. Consider the sequence $A_k = e^k$, $k \in \mathbb{N}$ and cover each interval $[-A_k, A_k]$ by $M_k = (\lfloor 2A_k/\gamma\rfloor + 1)$ disjoint small intervals $\Lambda_{k,1}, \ldots, \Lambda_{k,M_k}$ of the length γ. Let $u_{k,1}, \ldots, u_{k,M_k}$ be the centers of these intervals. We have for any natural $K > 0$

$$\begin{aligned}\max_{k=1,\ldots,K}\sup_{A_{k-1}<|u|\le A_k}|\mathcal{W}_n(u)| \quad &\le \quad \max_{k=1,\ldots,K}\max_{|u_{k,m}|>A_{k-1}}|\mathcal{W}_n(u_{k,m})| \\ &+ \max_{k=1,\ldots,K}\max_{1\le m\le M_k}\sup_{u\in\Lambda_{k,m}}|\mathcal{W}_n(u) - \mathcal{W}_n(u_{k,m})|.\end{aligned}$$

Hence

$$\text{(48)}\quad \mathrm{P}\left(\max_{k=1,\dots,K}\sup_{A_{k-1}<|u|\le A_k}|\mathcal{W}_n(u)|>\lambda\right)\le\sum_{k=1}^{K}\sum_{\{|u_{k,m}|>A_{k-1}\}}\mathrm{P}(|\mathcal{W}_n(u_{k,m})|>\lambda/2)+$$
$$\mathrm{P}\left(\sup_{|u-v|<\gamma}|\mathcal{W}_n(v)-\mathcal{W}_n(u)|>\lambda/2\right)$$

with $\lambda=\zeta n^{-1/2}\log^{1/2}n$. It holds for any $u,v\in\mathbb{R}$

$$\begin{aligned}|\mathcal{W}_n(v)-\mathcal{W}_n(u)| &\le 2|w(|v|)-w(|u|)|\\ &\quad+\frac{1}{n}\sum_{j=1}^{n}|\exp(\mathrm{i}vZ_j)-\exp(\mathrm{i}uZ_j)|+|\phi(v)-\phi(u)|\\ &\le (u-v)\left[L_w+\frac{1}{n}\sum_{j=1}^{n}|Z_j|+\mathbb{E}|Z|\right], \qquad (49)\end{aligned}$$

where L_ω is the Lipschitz constant of w. The Markov inequality implies

$$\mathrm{P}\left(\frac{1}{n}\sum_{j=1}^{n}[|Z_j|-\mathbb{E}|Z|]>c\right)\le c^{-p}n^{-p}\mathbb{E}\left|\sum_{j=1}^{n}[|Z_j|-\mathbb{E}|Z|]\right|^p$$

for any $c>0$. Using now Yokoyama, **?** inequality and taking into account the assumptions (AZ1)-(AZ2), we get

$$\mathbb{E}\left|\sum_{j=1}^{n}[|Z_j|-\mathbb{E}|Z|]\right|^p\le C_p(\bar\alpha)n^{p/2},$$

where $C_p(\bar\alpha_1)$ is some constant depending on $\bar\alpha=(\bar\alpha_0,\bar\alpha_1)$ and p from assumptions (AZ1) and (AZ2) respectively. Setting $\gamma=n^{-1/2}\log^{1/2}n$, we obtain from (49)

$$\text{(50)}\quad \mathrm{P}\left(\sup_{|u-v|<\gamma}|\mathcal{W}_n(v)-\mathcal{W}_n(u)|>\lambda/2\right)\le$$
$$\mathrm{P}\left(\frac{1}{n}\sum_{j=1}^{n}[|Z_j|-\mathbb{E}|Z|]>\zeta/2-L_w-2\mathbb{E}|Z|\right)\le B_1\zeta^{-p}n^{-p/2}$$

with some constant B_1 not depending on ζ and n, provided ζ is large enough. Let us turn now to the first term on the right-hand side of (48). If $|u_{k,m}|>A_{k-1}$, then it follows from Theorem 8.1 and Corollary 8.2

$$\mathrm{P}\left(|\operatorname{Re}[\mathcal{W}_n(u_{k,m})]|>\lambda/4\right)$$
$$\le B_2\exp\left(-\frac{B_3\lambda^2 n}{4w^2(A_{k-1})\log^{2(1+\varepsilon)}(w(A_{k-1}))+\lambda\log^2(n)w(A_{k-1})}\right),$$

$$
\begin{aligned}
&\mathrm{P}\left(|\,\mathrm{Im}\left[\mathcal{W}_n(u_{k,m})\right]|>\lambda/4\right)\\
&\qquad\qquad \leq B_4 \exp\left(-\frac{B_3\lambda^2 n}{4w^2(A_{k-1})\log^{2(1+\varepsilon)}(w(A_{k-1}))+\lambda\log^2(n)w(A_{k-1})}\right)
\end{aligned}
$$

with some constants B_2, B_3 and B_4 depending only on the characteristics of the process Z. Further we get

$$
\begin{aligned}
\sum_{\{|u_{k,m}|>A_{k-1}\}} &\mathrm{P}(|\mathcal{W}_n(u_{k,m})|>\lambda/2) \leq (\lfloor 2A_k/\gamma\rfloor+1)\\
&\times\exp\left(-\frac{B_3\lambda^2 n}{4w^2(A_{k-1})\log^{2(1+\varepsilon)}(w(A_{k-1}))+\lambda\log^2(n)w(A_{k-1})}\right)\\
&\lesssim A_k n^{1/2}\exp\left(-\frac{B\zeta^2\log(n)}{w^2(A_{k-1})\log^{2(1+\varepsilon)}(w(A_{k-1}))}\right)\log^{(r-1)/2}(n), \quad n\to\infty
\end{aligned}
$$

with $r=2(1+\varepsilon)$ and some constant $B>0$. Fix $\theta>0$ such that $B\theta>d$ and compute

$$
\begin{aligned}
\sum_{\{\|u_{k,m}\|>A_{k-1}\}}\mathrm{P}(|\mathcal{W}_n(u_{k,m})|>\lambda/2) &\lesssim e^{k-\theta B(k-1)}n^{1/2}\log^{(r-1)/2}(n)e^{-B(k-1)(\zeta^2\log n-\theta)}\\
&\lesssim e^{k(1-\theta B)}\log^{(r-1)/2}(n)e^{-B(k-1)(\zeta^2\log n-\theta)+\log(n)}.
\end{aligned}
$$

a If $\zeta^2\log n>\theta$ we get asymptotically

$$
\sum_{k=2}^{K}\sum_{\{\|u_{k,m}\|>A_{k-1}\}}\mathrm{P}(|\mathcal{W}_n(u_{k,m})|>\lambda/2) \lesssim \log^{(r-1)/2}(n)e^{-(B\zeta^2-1)\log(n)}.
$$

Taking large enough $\zeta>0$, we get (47). □

8.2 Equivalent definition of the estimate α_n

The next lemma shows that the estimate (52) can be alternatively defined as a solution of some optimization problem.

Lemma 8.4. *Let $\widetilde{w}^{U_n}(u)$ be a smooth positive function on $\mathbb{R}$ that can be represented in the form*

$$
\widetilde{w}^{U_n}(u)=\frac{1}{U_n}\,\widetilde{w}^1\left(\frac{u}{U_n}\right)
$$

with some function $\widetilde{w}^1$ supported on the interval $[\varepsilon,1]$. Consider an optimization problem

$$
(\beta_n,\gamma_n)=\arg\min_{\beta,\gamma}\int_0^\infty \widetilde{w}^{U_n}(u)(\mathcal{Y}_n(u)-\gamma\log(u)-\beta)^2\,du. \tag{51}
$$

Then γ_n is equal to

$$
\gamma_n=\int_0^\infty \bar{w}^{U_n}(u)\mathcal{Y}_n(u)\,du \tag{52}
$$

with a smooth function $\bar{w}^{U_n}(u)$, which can be also represented in the form $\bar{w}^{U_n}(u) = 1/U_n\ \bar{w}^1(u/U_n)$. The function $\bar{w}^1$ is supported on the interval $[\varepsilon, 1]$ and satisfies

$$\int_\varepsilon^1 \bar{w}^1(v)\,dv = 0, \qquad \int_\varepsilon^1 \bar{w}^1(v)\log v\,dv = 1.$$

Proof. To minimize the expression in (51), we calculate the first and the second derivatives with respect to β and γ:

$$(53)\quad \begin{cases} 0 = -\int_0^\infty \widetilde{w}^{U_n}(u)\mathcal{Y}_n(u)\,du & +\gamma\int_0^\infty \widetilde{w}^{U_n}(u)\log u\,du \quad +\beta\int_0^\infty \widetilde{w}^{U_n}(u)\,du, \\ 0 = -\int_0^\infty \widetilde{w}^{U_n}(u)\mathcal{Y}_n(u)\log u\,du & +\gamma\int_0^\infty \widetilde{w}^{U_n}(u)\log^2 u\,du \quad +\beta\int_0^\infty \widetilde{w}^{U_n}(u)\log u\,du. \end{cases}$$

Note that the solution to this system gives the point of minimum because by the Cauchy - Schwarz inequality,

$$\int \widetilde{w}^{U_n}(u)\,du \int \widetilde{w}^{U_n}(u)\log^2 u\,du - \left(\int \widetilde{w}^{U_n}(u)\log u\,du\right)^2 \geq 0.$$

The solution to the system (53) is (β_n, γ_n), where

$$\begin{aligned} \gamma_n &= \frac{\int \widetilde{w}^{U_n}(u)\mathcal{Y}_n(u)\,du\ \cdot\ \int \widetilde{w}^{U_n}(u)\log u\,du - \int \widetilde{w}^{U_n}(u)\mathcal{Y}_n(u)\log u\,du\ \cdot\ \int \widetilde{w}^{U_n}(u)\,du}{\left(\int \widetilde{w}^{U_n}(u)\log u\,du\right)^2 - \int \widetilde{w}^{U_n}(u)\log^2 u\,du\ \cdot\ \int \widetilde{w}^{U_n}(u)\,du} \\ &= \int_0^\infty \widetilde{w}^{U_n}(u)\frac{\int \widetilde{w}^{U_n}(s)\log s\,ds - \left(\int \widetilde{w}^{U_n}(s)\,ds\right)\log u}{\left(\int \widetilde{w}^{U_n}(s)\log s\,ds\right)^2 - \int \widetilde{w}^{U_n}(s)\log^2 s\,ds\ \cdot\ \int \widetilde{w}^{U_n}(s)\,ds}\mathcal{Y}_n(u)\,du. \end{aligned}$$

Then (52) follows with

$$\bar{w}^{U_n}(u) := \widetilde{w}^{U_n}(u)\frac{\int_0^\infty \widetilde{w}^{U_n}(s)\log s\,ds - \left(\int_0^\infty \widetilde{w}^{U_n}(s)\,ds\right)\log u}{\left(\int_0^\infty \widetilde{w}^{U_n}(s)\log s\,ds\right)^2 - \int_0^\infty \widetilde{w}^{U_n}(s)\log^2 s\,ds\ \cdot\ \int_0^\infty \widetilde{w}^{U_n}(s)\,ds}.$$

Note that the function $\bar{w}^{U_n}(u)$ can be represented in the following way:

$$\bar{w}^{U_n}(u) \;=\; \frac{1}{U_n}\,\widetilde{w}^1\left(\frac{u}{U_n}\right)\frac{\int_\varepsilon^1 \widetilde{w}^1(s)\log(s)\,ds - \left(\int_\varepsilon^1 \widetilde{w}^1(s)\,ds\right)\log\left(\frac{u}{U_n}\right)}{\left(\int_0^\infty \widetilde{w}^{U_n}(s)\log s\,ds\right)^2 - \int_0^\infty \widetilde{w}^{U_n}(s)\log^2 s\,ds\ \cdot\ \int_0^\infty \widetilde{w}^{U_n}(s)\,ds}.$$

To conclude the proof, it remains to define

$$(54)\qquad \bar{w}^1(v) := \widetilde{w}^1(v)\frac{\int_\varepsilon^1 \widetilde{w}^1(s)\log s\,ds - \left(\int_\varepsilon^1 \widetilde{w}^1(s)\,ds\right)\log v}{\left(\int_\varepsilon^1 \widetilde{w}^1(s)\log s\,ds\right)^2 - \int_\varepsilon^1 \widetilde{w}^1(s)\log^2 s\,ds\ \cdot\ \int_\varepsilon^1 \widetilde{w}^1(s)\,ds}.$$

□

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