SIAM J. FINANCIAL MATH.
Vol. xx, pp. x–x


# Pricing Bermudan options via multilevel approximation methods*

Denis Belomestny†, Fabian Dickmann‡, and Tigran Nagapetyan§

**Abstract.** In this article we propose a novel approach to reduce the computational complexity of various approximation methods for pricing discrete time American or Bermudan options. Given a sequence of continuation values estimates corresponding to different levels of spatial approximation, we propose a multilevel low biased estimate for the price of the option. It turns out that the resulting complexity gain can be of order $\varepsilon^{-1}$ with $\varepsilon$ denoting the desired precision. The performance of the proposed multilevel algoritheorems is illustrated by a numerical example.

**Key words.** Bermudan options, Multilevel Monte Carlo, mesh method, global regression, complexity analysis

**AMS subject classifications.** 65C30, 65C20, 65C05, 60H35

## 1. Introduction.

Pricing of an American option usually reduces to solving an optimal stopping problem that can be efficiently solved in low dimensions via dynamic programming algoritheorem. However, many problems arising in practice (see e.g. [11]) have high dimensions, and these applications have motivated the development of Monte Carlo methods for pricing American option. Pricing American style derivatives via Monte Carlo is a challenging task, because it requires the backwards dynamic programming algoritheorem that seems to be incompatible with the forward structure of Monte Carlo methods. In recent years much research was focused on the development of fast methods to compute approximations to the optimal exercise policy. Eminent examples include the functional optimization approach of [2], the mesh method of [5], the regression-based approaches of [7], [13], [14], [9] and [3]. The complexity of the fast approximations algoritheorems depends on the desired precision $\varepsilon$ in a quite nonlinear way that, in turn, is determined by some fine properties of the underlying exercise boundary and the continuation values (see, e.g., [3]). In some situations (e.g. in the case of the stochastic mesh method or local regression) this complexity is of order $\varepsilon^{-3}$, which is rather high. One way to reduce the complexity of the fast approximation methods is to use various variance reduction methods. However, the latter methods are often ad hoc and, more importantly, do not lead to provably reduced asymptotic complexity. In this paper we propose a generic approach which is able to reduce the order of asymptotic complexity and which is applicable to various fast approximation methods, such as global regression, local regression or stochastic mesh method. The main idea of the method is inspired by the path-breaking work of [10] that introduced a multilevel idea into stochastics. As similar to the recent work of [4], we consider not levels corresponding to different discretization steps, but levels related to different degrees of approximation of the continuation values. For example, in the case of the Longstaff-Schwartz algoritheorem, the latter degree is basically governed by

*The research by Denis Belomestny was made in IITP RAS and supported by Russian Scientific Foundation grant (project N 14-50-00150)

†Duisburg-Essen University and IITP RAS. (denis.belomestny@uni-due.de)
‡Duisburg-Essen University. (fabian.dickmann@uni-due.de)
§Weierstrass Institute of Applied Mathematics. (nagapetyan@wias-berlin.de)

the number of basis functions and in the case of the mesh method by the number of training paths used to approximate the continuation values. The new multilevel approach is able to significantly reduce the complexity of the fast approximation methods leading in some cases to the complexity gain of order $\varepsilon^{-1}$. The paper is organised as follows. In Section 2 the pricing problem is formulated, the main assumptions are introduced and illustrated. In Section 3 the complexity analysis of a generic approximation algoritheorem is carried out. The main multilevel Monte Carlo algoritheorem is introduced in Section 4 where also its complexity is studied. In Section 5 we numerically test our approach for the problem of pricing Bermudan max-call options via mesh method. The proofs are collected in Section 7.

**2. Main setup.** An American option grants its holder the right to select the time at which to exercise the option, and in this differs from a European option that may be exercised only at a fixed date. A general class of American option pricing problems can be formulated through an $\mathbb{R}^d$ Markov process $\{X_t, 0 \leq t \leq T\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathrm{P})$ . It is assumed that the process $(X_t)$ is adapted to $(\mathcal{F}_t)_{0\leq t\leq T}$ in the sense that each $X_t$ is $\mathcal{F}_t$ measurable. Recall that each $\mathcal{F}_t$ is a $\sigma$ -algebra of subsets of $\Omega$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \leq t$. We restrict attention to options admitting a finite set of exercise opportunities $0 = t_0 < t_1 < t_2 < \ldots < t_{\mathcal{J}} = T$, called Bermudan options. Then

$$Z_j := X_{t_j}, \quad j = 0, \ldots, \mathcal{J},$$

is a Markov chain. If exercised at time $t_j$, $j = 1, \ldots, \mathcal{J}$, the option pays $g_j(Z_j)$, for some known functions $g_0, g_1, \ldots, g_{\mathcal{J}}$ mapping $\mathbb{R}^d$ into $[0, \infty)$. Let $\mathcal{T}_j$ denote the set of stopping times taking values in $\{j, j+1, \ldots, \mathcal{J}\}$. A standard result in the theory of contingent claims states that the equilibrium price $V_j(z)$ of the Bermudan option at time $t_j$ in state $z$, given that the option was not exercised prior to $t_j$, is its value under the optimal exercise policy:

$$V_j^*(z) = \sup_{\tau \in \mathcal{T}_j} \mathrm{E}[g_\tau(Z_\tau)|Z_j = z], \quad z \in \mathbb{R}^d.$$

A common feature of all fast approximation algoritheorems is that they deliver estimates $C_{k,0}(z), \ldots, C_{k,\mathcal{J}-1}(z)$ for the so-called continuation values:

$$C_j^*(z) := \mathrm{E}[V_{j+1}^*(Z_{j+1})|Z_j = z], \quad j = 0, \ldots, \mathcal{J} - 1. \tag{2.1}$$

Here the index $k$ indicates that the above estimates are based on the set of "training" trajectories $(Z_0^{(i)}, \ldots, Z_{\mathcal{J}}^{(i)})$, $i = 1, \ldots, k$, all starting from one point, i.e., $Z_0^{(1)} = \ldots = Z_0^{(k)}$. In the case of the so-called regression methods and the mesh method, the estimates for the continuation values are obtained via the recursion (*dynamic programming principle*):

$$\begin{aligned} C_{\mathcal{J}}^*(z) &= 0, \\ C_j^*(z) &= \mathrm{E}[\max(g_{j+1}(Z_{j+1}), C_{j+1}^*(Z_{j+1}))|Z_j = z] \end{aligned}$$

combined with Monte Carlo: at $(\mathcal{J} - j)$ th step one estimates the expectation

$$\mathrm{E}[\max(g_{j+1}(Z_{j+1}), C_{k,j+1}(Z_{j+1}))|Z_j = z] \tag{2.2}$$

via regression (global or local) based on the set of paths

$$(Z_j^{(i)}, C_{k,j+1}(Z_{j+1}^{(i)})), \quad i = 1, \ldots, k,$$

where $C_{k,j+1}(z)$ is the estimate for $C_{j+1}^*(z)$ obtained in the previous step.

Given the estimates $C_{k,0}(z), \ldots, C_{k,\mathcal{J}-1}(z)$, we can construct a lower bound (low biased estimate) for $V_0^*$ using the (generally suboptimal) stopping rule:

$$\tau_k = \min\{0 \le j \le \mathcal{J} : g_j(Z_j) \ge C_{k,j}(Z_j)\}$$

with $C_{k,\mathcal{J}} \equiv 0$ by definition. Indeed, fix a natural number $n$ and simulate $n$ new independent trajectories of the process $Z$. A low-biased estimate for $V_0^*$ can be then defined as

$$V_0^{n,k} = \frac{1}{n} \sum_{r=1}^{n} g_{\tau_k^{(r)}}\big(Z_{\tau_k^{(r)}}^{(r)}\big) \tag{2.3}$$

with

$$\tau_k^{(r)} = \inf\Big\{0 \le j \le \mathcal{J} : g_j(Z_j^{(r)}) \ge C_{k,j}(Z_j^{(r)})\Big\}.$$

Thus any fast approximation approximation algoritheorem can be viewed as consisting of the following two steps.

**Step 1** Construction of the estimates $C_{k,j}$, $j = 1, \ldots, J$, on $k$ training paths.

**Step 2** Construction of the low-biased estimate $V_0^{n,k}$ by evaluating functions $C_{k,j}$, $j = 1, \ldots, J$, on each of new $n$ testing trajectories.

Let us now consider a generic family of the continuation values estimates $C_{k,0}(z), \ldots, C_{k,\mathcal{J}-1}(z)$ with the natural number $k$ determining the quality of the estimates as well as their complexity. In particular, we make the following assumptions.

**(AP)** For any $k \in \mathbb{N}$, the estimates $C_{k,0}(z), \ldots, C_{k,\mathcal{J}-1}(z)$ are defined on some probability space $(\Omega^k, \mathcal{F}^k, \mathrm{P}^k)$ which is independent of $(\Omega, \mathcal{F}, \mathrm{P})$.

**(AC)** For any $j = 1, \ldots, \mathcal{J}$, the cost of constructing the estimate $C_{k,j}$ on $k$ training paths, i.e., $C_{k,j}\big(Z_j^{(i)}\big)$, $i = 1, \ldots, k$, is of order $k \times k^{\varkappa_1}$ for some $\varkappa_1 > 0$ and the cost of evaluating $C_{k,j}(z)$ in a new point $z \notin \{Z_j^{(1)}, \ldots, Z_j^{(k)}\}$ is of order $k^{\varkappa_2}$ for some $\varkappa_2 > 0$.

**(AQ)** There is a sequence of positive real numbers $\gamma_k$ with $\gamma_k \to 0$, $k \to \infty$ such that

$$\mathrm{P}^k\left(\sup_z \big|C_{k,j}(z) - C_j^*(z)\big| > \eta\sqrt{\gamma_k}\right) < B_1 e^{-B_2\eta}, \quad \eta > 0$$

for some constants $B_1 > 0$ and $B_2 > 0$ not depending on $k$ and $\eta$.

*Discussion.*

- Given (AC), the overall complexity of the corresponding fast approximation algoritheorem is proportional to

$$k^{1+\varkappa_1} + n \times k^{\varkappa_2}, \tag{2.4}$$

where the first term in (2.4) represents the cost of constructing the estimates $C_{k,j}$, $j = 1, \ldots, J$, on training paths and the second one gives the cost of evaluating the estimated continuation values on $n$ testing paths.

- Additionally, one usually has to take into account the cost of paths simulation. If the process $X$ solves a stochastic differential equation and the Euler discretisation scheme with a time step $h$ is used to generate paths, then the term $k \times h^{-1} + n \times h^{-1}$ needs to be added to (2.4). In order to make the analysis more focused and transparent, we do not take here the path generation costs and discretisation errors into account.

Let us now illustrate the above assumptions for three well known fast approximation methods.

Example 2.1 (Global regression). *Fix a vector of real-valued functions $\psi = (\psi_1, \ldots, \psi_M)$ on $\mathbb{R}^d$. Suppose that the estimate $C_{k,j+1}$ is already constructed and has the form*

$$C_{k,j+1}(z) = \alpha^k_{j+1,1}\psi_1(z) + \ldots + \alpha^k_{j+1,M}\psi_L(z)$$

*for some $(\alpha^k_{j+1,1}, \ldots, \alpha^k_{j+1,M}) \in \mathbb{R}^M$. Let $\boldsymbol{\alpha}^k_j = (\alpha^k_{j,1}, \ldots, \alpha^k_{j,M})$ be a solution of the following least squares optimization problem:*

$$\operatorname*{arginf}_{\boldsymbol{\alpha}\in\mathbb{R}^M} \sum_{i=1}^{k} \left[\zeta_{j+1,k}(Z^{(i)}_{j+1}) - \alpha_1\psi_1(Z^{(i)}_j) - \ldots - \alpha_M\psi_M(Z^{(i)}_j)\right]^2 \tag{2.5}$$

*with $\zeta_{j+1,k}(z) = \max\{g_{j+1}(z), C_{k,j+1}(z)\}$. Define an approximation for $C^*_j$ via*

$$C_{k,j}(z) = \alpha^k_{j,1}\psi_1(z) + \ldots + \alpha^k_{j,M}\psi_M(z), \quad z \in \mathbb{R}^d.$$

*It is clear that all estimates $C_{k,j}$ are well defined on the cartesian product of $k$ independent copies of $(\Omega, \mathcal{F}, \mathrm{P})$. The complexity $\mathrm{comp}(\boldsymbol{\alpha}^k_j)$ of computing $\boldsymbol{\alpha}^k_j$ is of order $k \cdot M^2 + \mathrm{comp}(\boldsymbol{\alpha}^k_{j+1})$, since each $\boldsymbol{\alpha}^k_j$ is of the form $\boldsymbol{\alpha}^k_j = B^{-1}b$ with*

$$B_{p,q} = \frac{1}{k}\sum_{i=1}^{k} \psi_p(Z^{(i)}_j)\psi_q(Z^{(i)}_j)$$

*and*

$$b_p = \frac{1}{k}\sum_{i=1}^{k} \psi_p(Z^{(i)}_j)\zeta_{k,j+1}(Z^{(i)}_{j+1}),$$

*$p, q \in \{1, \ldots, M\}$. Iterating backwardly in time, we get $\mathrm{comp}(\boldsymbol{\alpha}^k_j) \sim (\mathcal{J} - j) \cdot k \cdot M^2$. Furthermore, it can be shown that the estimates $C_{k,0}(z), \ldots, C_{k,\mathcal{J}-1}(z)$ satisfy the assumption (AQ) with $\gamma_k = 1/k$, provided $M$ increases with $k$ at polynomial rate, i.e., $M = k^\rho$ for some $\rho > 0$ (see, e.g., [15]). Thus the parameters $\varkappa_1$ and $\varkappa_2$ in (AC) are given by $2\rho$ and $\rho$, respectively.*

*[12] argue that the number of paths $k$ must increase exponentially with $M$. In fact, the paper of Glasserman and Yu deals with rather special case, as they consider Black-Scholes model and use orthogonal Hermite polynomials to estimate the corresponding continuation values. In the latter situation the variance of the least squares estimator exponentially increases with the number of Hermite polynomials $M$ and this leads to the aforementioned requirement. In general, the variance increases polynomially in $M$ and this is the case studied in [15].*

Example 2.2 (Local regression).*Local polynomial regression estimates can be defined as follows. Fix some $j$ such that $0 \le j < \mathcal{J}$ and suppose that we want to compute the expectation*

$$\mathrm{E}[\zeta_{j+1,k}(Z_{j+1})|Z_j = z], \quad z \in \mathbb{R}^d$$

*with* $\zeta_{j+1,k}(z) = \max\{g_{j+1}(z), C_{k,j+1}(z)\}$. *For some* $\delta > 0$, $z \in \mathbb{R}^d$, *an integer* $l \geq 0$ *and a function* $K : \mathbb{R}^d \to \mathbb{R}_+$, *denote by* $q_{z,k}$ *a polynomial on* $\mathbb{R}^d$ *of degree* $l$ *(i.e. the maximal order of the multi-index is less than or equal to* $l$*) which minimizes*

$$\sum_{i=1}^{k} \left[\zeta_{j+1,k}(Z_{j+1}^{(i)}) - q(Z_j^{(i)} - z)\right]^2 K\left(\frac{Z_j^{(i)} - z}{\delta}\right) \tag{2.6}$$

*over the set of all polynomials* $q$ *of degree* $l$. *The local polynomial estimator of order* $l$ *for* $C_j^*(z)$ *is then defined as* $C_{k,j}(z) = q_{z,k}(0)$ *if* $q_{z,k}$ *is the unique minimizer of* (2.6) *and* $C_{k,j}(z) = 0$ *otherwise. The value* $\delta$ *is called a bandwidth and the function* $K$ *is called a kernel function. In [3], it is shown that the local polynomial estimates* $C_{k,0}(z), \ldots, C_{k,\mathcal{J}-1}(z)$ *of degree* $l$ *satisfy the assumption (AQ) with* $\gamma_k = k^{-2\beta/(2\beta+d)}$ *under* $\beta$*-Hölder smoothness of the continuation values* $C_0^*(z), \ldots, C_{\mathcal{J}-1}^*(z)$, *provided* $\delta = k^{-1/(2l+d)}$. *Since in general the summation in* (2.6) *runs over all* $k$ *paths (see Figure 2.1) we have* $\varkappa_1 = 1$ *and* $\varkappa_2 = 1$ *in (AC).*

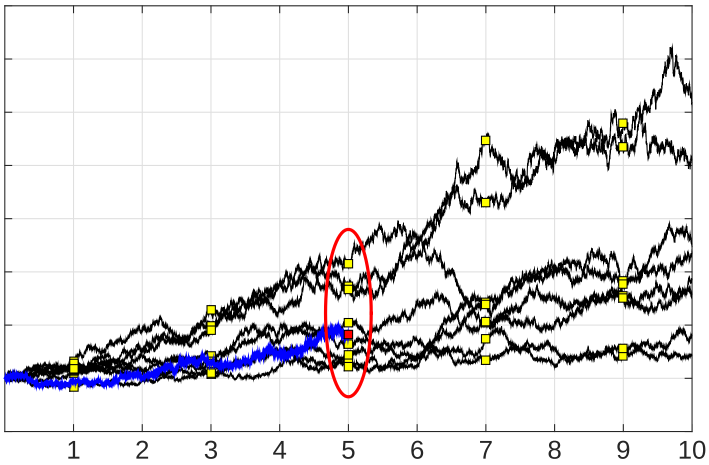


**Figure 2.1.** *Local regression and mesh methods: in order to compute the continuation value estimate* $C_{k,5}$ *in a point (red) lying on a testing path (blue), all* $k$ *points (yellow) on training paths at time* 5 *have to be used.*

Example 2.3 (Mesh Method). *In the mesh method of [6], the continuation value* $C_j^*$ *at a point* $z$ *is approximated via*

$$C_{k,j}(z) = \frac{1}{k}\sum_{i=1}^{k} \zeta_{k,j+1}(Z_{j+1}^{(i)}) \cdot w_{ij}(z), \tag{2.7}$$

*where* $\zeta_{k,j+1}(z) = \max\{g_{j+1}(z), C_{k,j+1}(z)\}$ *and*

$$w_{ij}(z) = \frac{p_j(z, Z_{j+1}^{(i)})}{\frac{1}{k}\sum_{l=1}^{k} p_j(Z_j^{(l)}, Z_{j+1}^{(i)})},$$

*where $p_j(x,\cdot)$ is the conditional density of $Z_{j+1}$ given $Z_j = x$. Again the summation in (2.7) runs over all $k$ paths. Hence $\varkappa_1 = 1$ in (AC) and for any $j = 0, \ldots, \mathcal{J}-1$, the complexity of computing $C_{k,j}(z)$ in a point $z$ not belonging to the set of training trajectories, is of order $k$ (see Figure 2.1), provided the transition density $p_j(x,y)$ is analytically known. For assumption (AQ) see, e.g., [1].*

**3. Complexity analysis of $V_0^{n,k}$ .** We shall use throughout the notation $A \lesssim B$ if $A$ is bounded by a constant multiple of $B$, independently of the parameters involved, that is, in the Landau notation $A = O(B)$. Equally $A \gtrsim B$ means $B \lesssim A$ and $A \sim B$ stands for $A \lesssim B$ and $A \gtrsim B$ simultaneously.

In order to carry out the complexity analysis of the estimate (2.3), we need the so-called "margin" or boundary assumption.

**(AM)** There exist constants $A > 0$, $\delta_0 > 0$ and $\alpha > 0$ such that

$$\mathrm{P}\left(|C_j^*(Z_j) - g_j(Z_j)| \leq \delta\right) \leq A\delta^\alpha$$

for all $j = 0, \ldots, \mathcal{J}$, and all $\delta > 0$.

Remark 3.1.*Assumption (AM) provides a useful characterization of the behavior of continuation values ($C_j^*$) and payoffs ($g_j$) near the exercise boundary $\partial\mathcal{E}$ with*

$$\mathcal{E} = \left\{(j,x) : g_j(x) \geq C_j^*(x)\right\}.$$

*In the situation when all functions $C_j^* - g_j$, $j = 0, \ldots, \mathcal{J}-1$, are smooth and have non-vanishing derivatives in the vicinity of the exercise boundary, we have $\alpha = 1$. Other values of $\alpha$ are possible as well, see [3].* Let us now turn to the properties of the estimate $V_0^{n,k}$. While the variance of the estimate $V_0^{n,k}$ is given by

$$\mathrm{Var}[V_0^{n,k}] = \mathrm{Var}[g_{\tau_k}(Z_{\tau_k})]/n, \tag{3.1}$$

its bias is analysed in the following theorem.

Theorem 3.2.*Suppose that (AP), (AM) and (AQ) hold with some $\alpha > 0$, then*

$$\left|V_0^* - \mathrm{E}[V_0^{n,k}]\right| \lesssim \gamma_k^{(1+\alpha)/2}, \quad k \to \infty.$$

The next theorem gives an upper estimate for the complexity of $V_0^{n,k}$.

Theorem 3.3. *Let assumptions (AP), (AC), (AQ) and (AM) hold with*

$$\gamma_k = k^{-\mu}, \quad k \in \mathbb{N}$$

*for some $\mu > 0$. Then for some constant $c > 0$ and any $\varepsilon > 0$ the choice*

$$k^* = c\varepsilon^{-\frac{2}{\mu(1+\alpha)}}, \quad n^* = c\varepsilon^{-2}$$

*leads to*

$$\mathrm{E}\left[V_0^{n^*,k^*} - V_0^*\right]^2 \leq \varepsilon^2, \tag{3.2}$$

*and the complexity of the estimate* $V_0^{n^*,k^*}$ *(i.e. the cost needed to achieve* (3.2)*) is bounded from above by* $\mathcal{C}_{n^*,k^*}(\varepsilon)$ *with*

$$\mathcal{C}_{n^*,k^*}(\varepsilon) \lesssim \varepsilon^{-2\cdot\max\left(\frac{\varkappa_1+1}{\mu(1+\alpha)},\,1+\frac{\varkappa_2}{\mu(1+\alpha)}\right)}, \quad \varepsilon \to 0. \tag{3.3}$$

*Discussion.* Theorem 3.3 implies that the complexity of $V_0^{n^*,k^*}$ is always larger than $\varepsilon^{-2}$. In the case $\varkappa_1 = 1$ and $\varkappa_2 = 1$ (mesh method or local regression) we get

$$\mathcal{C}_{n^*,k^*}(\varepsilon) \lesssim \varepsilon^{-2\cdot\max\left(\frac{2}{\mu(1+\alpha)},1+\frac{1}{\mu(1+\alpha)}\right)}. \tag{3.4}$$

Furthermore, in the most common case $\alpha = 1$, the bound (3.4) simplifies to

$$\mathcal{C}_{n^*,k^*}(\varepsilon) \lesssim \varepsilon^{-2\cdot\max\left(\frac{1}{\mu},1+\frac{1}{2\mu}\right)}.$$

Since for all regression methods and the mesh method $\mu \le 1$, the asymptotic complexity is always larger than $\varepsilon^{-3}$. In the next section, we present a multilevel approach that can reduce the asymptotic complexity to $\varepsilon^{-2}$ in some cases.

**4. Multilevel approach.** Fix some natural number $L$ and let $\mathbf{k} = (k_0, k_1, \ldots, k_L)$ and $\mathbf{n} = (n_0, n_1, \ldots, n_L)$ be two sequences of natural numbers, satisfying $k_0 < k_1 < \ldots < k_L$ and $n_0 > n_1 > \ldots > n_L$. Define

$$V_0^{\mathbf{n},\mathbf{k}} = \frac{1}{n_0}\sum_{r=1}^{n_0} g_{\tau_{k_0}^{(r)}}\left(Z^{(r)}_{\tau_{k_0}^{(r)}}\right) + \sum_{l=1}^{L}\frac{1}{n_l}\sum_{r=1}^{n_l}\left[g_{\tau_{k_l}^{(r)}}\left(Z^{(r)}_{\tau_{k_l}^{(r)}}\right) - g_{\tau_{k_{l-1}}^{(r)}}\left(Z^{(r)}_{\tau_{k_{l-1}}^{(r)}}\right)\right]$$

with

$$\tau_k^{(r)} = \inf\left\{0 \le j \le \mathcal{J} : g_j(Z_j^{(r)}) \ge C_{k,j}(Z_j^{(r)})\right\}, \quad k \in \mathbb{N},$$

where for any $l = 1, \ldots, L$, both estimates $C_{k_l,j}$ and $C_{k_{l-1},j}$ are based on one set of $k_l$ training trajectories, i.e. to estimate $C_{k_{l-1},j}$ we use a subset of $k_{l-1}$ trajectories from the set of $k_l$ trajectories used to approximate $C_{k_l,j}$. Note, that in all levels $l = 1, \ldots, L$, we use the same testing paths $Z^{(r)}$, $r = 1, \ldots, n_l$ in $g_{\tau_{k_l}^{(r)}}\left(Z^{(r)}_{\tau_{k_l}^{(r)}}\right)$ and $g_{\tau_{k_{l-1}}^{(r)}}\left(Z^{(r)}_{\tau_{k_{l-1}}^{(r)}}\right)$. Let us analyse the properties of the estimate $V_0^{\mathbf{n},\mathbf{k}}$. First note that its bias coincides with the bias of $g_{\tau_{k_L}}(Z_{\tau_{k_L}})$ corresponding to the finest approximation level. As to the variance of $V_0^{\mathbf{n},\mathbf{k}}$, it can be significantly reduced due the use of "good" continuation value estimates $C_{k_{l-1},j}$ and $C_{k_l,j}$ (that are both close to $C_j^*$) on the same set of testing trajectories in each level. In this way a "coupling" effect is achieved. The following theorem quantifies the above heuristics.

Theorem 4.1. *Let (AP), (AQ) and (AM) hold with some* $\alpha > 0$. *If*

$$M_p := \mathrm{E}\left[\left|\max_{l=0,\ldots,\mathcal{J}} g_l(Z_l)\right|^{2p}\right] < \infty$$

*for some* $p \ge 1$, *then*

$$\mathrm{E}\left[\left|g_{\tau_{k_l}}(Z_{\tau_{k_l}}) - g_{\tau_{k_{l-1}}}(Z_{\tau_{k_{l-1}}})\right|^2\right] \le C M_p^{1/p}\gamma_{k_{l-1}}^{\alpha/(2q)}$$

*for any* $l = 1, \dots, L$, *some absolute constant* $C > 0$ *and* $q$ *satisfying* $1/p + 1/q = 1$. As a corollary we get the following result.

Theorem 4.2. *Suppose that conditions of Theorem 4.1 are fulfilled, then the estimate* $V_0^{\mathbf{n},\mathbf{k}}$ *has the bias of order* $\gamma_{k_L}^{(1+\alpha)/2}$ *and the variance of order*

$$\frac{\operatorname{Var}[g(X_{\tau_{k_0}})]}{n_0} + \sum_{l=1}^{L} \frac{\gamma_{k_{l-1}}^{\alpha/(2q)}}{n_l}.$$

*Furthermore, under assumption (AC), the cost of* $V_0^{\mathbf{n},\mathbf{k}}$ *is bounded from above by a multiple of*

$$\sum_{l=0}^{L} (k_l^{\varkappa_1+1} + n_l \cdot k_l^{\varkappa_2})$$

Finally, the complexity of $V_0^{\mathbf{n},\mathbf{k}}$ is given in the following theorem.

Theorem 4.3. *Let assumptions (AP), (AC), (AQ) and (AM) hold with*

$$\gamma_{k_l} = k_l^{-\mu}, \quad k_l \in \mathbb{N}$$

*for some* $\mu > 0$. *Then under the choice* $k_l^* = k_0 \cdot \theta^l$, $l = 0, 1, \dots, L$, *with* $\theta > 1$,

$$L = c \left\lceil \frac{2}{\mu(1+\alpha)} \log_\theta \left( \varepsilon^{-1} \cdot k_0^{-\mu(1+\alpha)/2} \right) \right\rceil$$

*and*

$$n_l^* = c\varepsilon^{-2} \left( \sum_{i=1}^{L} \sqrt{k_i^{(\varkappa_2 - \mu\alpha/(2q))}} \right) \cdot \sqrt{k_l^{(-\varkappa_2 - \mu\alpha/(2q))}}$$

*for some constant* $c$ *not depending on* $\varepsilon$, *the estimate* (2.3) *fulfils*

$$\mathrm{E}\left[ V_0^{\mathbf{n}^*,\mathbf{k}^*} - V_0^* \right]^2 \le \varepsilon^2.$$

*As a result, the complexity of the estimate* (2.3) *is bounded up to a constant from above by*

$$\mathcal{C}_{\mathbf{n}^*,\mathbf{k}^*}(\varepsilon) \lesssim \begin{cases} \varepsilon^{-2\cdot\max\left(\frac{\varkappa_1+1}{\mu(1+\alpha)},1\right)}, & 2\cdot q\cdot\varkappa_2 < \mu\alpha \\ \varepsilon^{-2\cdot\frac{\varkappa_1+1}{\mu(1+\alpha)}}, & 2\cdot q\cdot\varkappa_2 = \mu\alpha \text{ and } \frac{\varkappa_1+1}{\mu(1+\alpha)} > 1 \\ \varepsilon^{-2}\cdot(\log\varepsilon)^2, & 2\cdot q\cdot\varkappa_2 = \mu\alpha \text{ and } \frac{\varkappa_1+1}{\mu(1+\alpha)} \le 1 \\ \varepsilon^{-2\cdot\max\left(\frac{\varkappa_1+1}{\mu(1+\alpha)},1+\frac{\varkappa_2-\mu\alpha/(2q)}{\mu(1+\alpha)}\right)}, & 2\cdot q\cdot\varkappa_2 > \mu\alpha \end{cases} \tag{4.1}$$

*Discussion.* Let us compare the complexities of the estimates $V_0^{n^*,k^*}$ and $V_0^{\mathbf{n}^*,\mathbf{k}^*}$. For the sake of clarity, we will assume that $\varkappa_1 = \varkappa_2 = \varkappa$ as in the mesh or local regression methods. Then (4.1) versus (3.3) can be written as

$$\begin{cases} \varepsilon^{-2\cdot\max\left(\frac{\varkappa+1}{\mu(1+\alpha)},1\right)}, & 2\cdot q\cdot\varkappa<\mu\alpha \\ \varepsilon^{-2\cdot\frac{\varkappa+1}{\mu(1+\alpha)}}, & 2\cdot q\cdot\varkappa=\mu\alpha \text{ and } \frac{\varkappa+1}{\mu(1+\alpha)}>1 \\ \varepsilon^{-2}\cdot(\log\varepsilon)^2, & 2\cdot q\cdot\varkappa=\mu\alpha \text{ and } \frac{\varkappa+1}{\mu(1+\alpha)}\le 1 \\ \varepsilon^{-2\cdot\max\left(\frac{\varkappa+1}{\mu(1+\alpha)},1+\frac{\varkappa-\mu\alpha/2}{\mu(1+\alpha)}\right)}, & 2\cdot q\cdot\varkappa>\mu\alpha. \end{cases} \quad\vee\quad \varepsilon^{-2\cdot\max\left(\frac{\varkappa+1}{\mu(1+\alpha)},\,1+\frac{\varkappa}{\mu(1+\alpha)}\right)}$$

Now it is clear that the multilevel algoritheorem will not be superior to the standard Monte Carlo algoritheorem as long as $\mu(1+\alpha)\le 1$. In the case $\mu(1+\alpha)>1$, the computational gain, up to a logaritheoremic factor, is given by

$$\begin{cases} \varepsilon^{-2\cdot\min\left(\frac{\varkappa}{\mu(1+\alpha)},1-\frac{1}{\mu(1+\alpha)}\right)}, & 2\cdot q\cdot\varkappa<\mu\alpha \\ \varepsilon^{-2\cdot\left(1-\frac{1}{\mu(1+\alpha)}\right)}, & 2\cdot q\cdot\varkappa=\mu\alpha \text{ and } \frac{\varkappa+1}{\mu(1+\alpha)}>1 \\ \varepsilon^{-2\cdot\frac{\varkappa}{\mu(1+\alpha)}}, & 2\cdot q\cdot\varkappa=\mu\alpha \text{ and } \frac{\varkappa+1}{\mu(1+\alpha)}\le 1 \\ \varepsilon^{-2\cdot\min\left(1-\frac{1}{\mu(1+\alpha)},\frac{\mu\alpha/(2q)}{\mu(1+\alpha)}\right)}, & 2\cdot q\cdot\varkappa>\mu\alpha \end{cases}$$

Taking into account the fact that $\alpha=1$ in the usual situation, we conclude that it is advantageous to use MLMC as long as $\mu>1/(2q)$.

## 5. Numerical example: Bermudan max calls on multiple assets.

Numerical experiments below aim to illustrate the potential of our algoritheorem as a general complexity reduction tool. Further computational savings can be achieved by using additional variance reduction techniques.

### 5.1. Computational problem.

Suppose that the price of the underlying asset $X=(X^1,\ldots,X^d)$ follows a Geometric Brownian Motion (GBM) under the risk-neutral measure, i.e.,

$$dX_t^i = (r-\delta)\,X_t^i\,dt + \sigma\,X_t^i\,dB_t^i, \tag{5.1}$$

where $r$ is the risk-free interest rate, $\delta$ the dividend rate, $\sigma$ is the volatility, and $B_t = (B_t^1,\ldots,B_t^d)$ is a $d$-dimensional Brownian motion. At any time $t\in\{t_0,\ldots,t_{\mathcal{J}}\}$ the holder of the option may exercise it and receive the payoff

$$h(X_t) = e^{-rt}(\max(X_t^1,\ldots,X_t^d)-S)^+. \tag{5.2}$$

We consider a benchmark example (see, e.g. [6], p. 462) with parameters $x_0=(90,\ldots,90)$, $\sigma=0.2$, $r=0.05$, $\delta=0.1$, $S=100$ $t_j=jT/\mathcal{J}$, $j=0,\ldots,\mathcal{J}$, with $T=3$ and $\mathcal{J}=9$.

### 5.2. MLMC algoritheorem implementation.

Let us first present a general version of the algoritheorem, which does not rely on the type of the continuation values estimation method. Write a multilevel estimate based on $L$ levels as

$$\hat{P}_L = \sum_{l=0}^{L}\hat{Y}_l = \frac{1}{n_0}\sum_{r=1}^{n_0} g_{\tau_{k_0}^{(r)}}\left(Z_{\tau_{k_0}^{(r)}}^{(r)}\right) + \sum_{l=1}^{L}\frac{1}{n_l}\sum_{r=1}^{n_l}\left[g_{\tau_{k_l}^{(r)}}\left(Z_{\tau_{k_l}^{(r)}}^{(r)}\right) - g_{\tau_{k_{l-1}}^{(r)}}\left(Z_{\tau_{k_{l-1}}^{(r)}}^{(r)}\right)\right]$$

with

$$\tau_k^{(r)} = \inf\left\{0 \le j \le \mathcal{J} : g_j(Z_j^{(r)}) \ge C_{k,j}(Z_j^{(r)})\right\}, \quad k \in \mathbb{N}.$$

Here $\hat{Y}_l$ is the Monte Carlo estimate for

$$Y_l = \begin{cases} \left[g_{\tau_{k_l}}\left(Z_{\tau_{k_l}}\right) - g_{\tau_{k_{l-1}}}\left(Z_{\tau_{k_{l-1}}}\right)\right], & l > 0 \\ g_{\tau_{k_0}}\left(Z_{\tau_{k_0}}\right), & l = 0 \end{cases}$$

1. Set $L = 2$.
2. For $l = 0, \dots, L$,
   - generate $k_l$ and $k_{l-1}$ training paths;
   - estimate continuation values $C_1^l, \dots, C_J^l$ and $C_1^{l-1}, \dots, C_J^{l-1}$;
   - generate $10^4$ testing paths and estimate the variance $\operatorname{Var} Y_l$.
3. Calculate $n_l$, $l = 0, \dots, L$, according to (5.3)

$$n_l = \left\lceil 3 \cdot \varepsilon^{-2} \cdot \left(\sum_{i=1}^{L} \sqrt{k_i^{\varkappa_2} \cdot \operatorname{Var} Y_i}\right) \cdot \sqrt{k_l^{-\varkappa_2} \cdot \operatorname{Var} Y_l}\right\rceil \tag{5.3}$$

4. Estimate/update $Y_0, \dots, Y_L$. If

$$\max\left(|\hat{Y}_{L-1}|/2, |\hat{Y}_L|\right) \le \varepsilon/\sqrt{3},$$

   then
   - set $L = L + 1$;
   - generate $k_L$ and $k_{L-1}$ training trajectories;
   - estimate continuation values $C_1^L, \dots, C_J^L$ and $C_1^{L-1}, \dots, C_J^{L-1}$;
   - generate $10^4$ testing paths and estimate the variance $\operatorname{Var} Y_L$;
   - go to step 3.
5. Return $P_L$.

While the cost of the standard MC algoritheorem is of order

$$k_L^{\varkappa_1+1} + 3 \cdot k_L^{\varkappa_2} \cdot \varepsilon^{-2} \cdot \sum_{l=0}^{L} \operatorname{Var} Y_l,$$

the cost of the above MLMC algoritheorem can be calculated as follows

$$\sum_{l=0}^{L} k_l^{\varkappa_1+1} + k_{l-1}^{\varkappa_1+1} + n_l \cdot (k_l^{\varkappa_2} + k_{l-1}^{\varkappa_2})$$

with $n_l$ given by (5.3).

Remark 5.1. *The constant 3 in (5.3) is motivated by the following decomposition*

$$\mathrm{E}\big[|\hat{P}_L - V_0^*|^2\big] = |\,\mathrm{E}[\hat{P}_L] - V_0^*|^2 + \operatorname{Var}\big[\mathrm{E}[\hat{P}_L|\mathcal{F}(k_1, \dots, k_l)]\big] \\ + \mathrm{E}\big[\operatorname{Var}[\hat{P}_L|\mathcal{F}(k_1, \dots, k_l)]\big],$$

*where $\mathcal{F}(k_1, \dots, k_l)$ is a $\sigma$-algebra generated by all training paths used to construct continuation values estimates on all levels.*

**5.3. Mesh method.** In this section, we report numerical results for a two dimensional case ($D = 2$), where the true price of the resulting Bermudan option is 8.08 (see [11], pp. 462). We shall use the mesh method combined with variance reduction to estimate the continuation values via

$$C_j^k(x) = \sum_{i=1}^{k} w_{ij}(x) \cdot \max\left(g_{j+1}(Z_{j+1}^{(i)}), C_{j+1}^k(Z_{j+1}^{(i)})\right) - b_j(x) \cdot \left(e^{-r(t_{\mathcal{J}}-t_j)} \max_{k=1,\ldots,d}\left(Z_{\mathcal{J}}^k - S\right)^+ - v_j(x)\right) \tag{5.4}$$

where $v_j(x)$ is the expectation of the control, i.e.,

$$v_j(x) := \mathrm{E}\left[e^{-r(t_{\mathcal{J}}-t_j)} \max_{k=1,\ldots,d}\left(Z_{\mathcal{J}}^k - S\right)^+ \middle| Z_j = x\right].$$

In fact, $v_j(x)$ can be obtained analytically, as the value of the corresponding European max-call option.

It is easy to see, that the conditions of Theorem 3.3 and Theorem 4.3 are fulfilled with $\gamma_k = 1/k$ in (AQ) and $\kappa_1 = \kappa_2 = 1$ in (AC) for the mesh method. Moreover, for the problem at hand, the assumption (AB) holds with $\alpha = 1$. We set the number of training paths for the mesh method to be

$$k_l = 20 \times 2^l.$$

and simulate independently $k_l$ training paths of the process $Z$ using the exact simulation formula

$$Z_j^{(i)} = Z_{j-1}^{(i)} \exp\left(\left[r - \delta - \frac{1}{2}\sigma^2\right](t_j - t_{j-1}) + \sigma\sqrt{(t_j - t_{j-1})} \cdot \xi_j^i\right),$$

where $\xi_j^i$, $i = 1, \ldots, k$, are i. i. d. standard normal random variables. The conditional density of $Z_j$ given $Z_{j-1}$ is explicitly given by

$$p_j(x, y) = \prod_{i=1}^{d} p_j(x_i, y_i), \quad x = (x_1, \ldots, x_d), \quad y = (y_1, \ldots, y_d),$$

where

$$p_j(x_i, y_i) = \frac{x_i}{y_i \sigma \sqrt{2\pi(t_j - t_{j-1})}} \times \exp\left(\frac{-\left(\log\left(\frac{y_i}{x_i}\right) - \left(r - \delta - \frac{1}{2}\sigma^2\right)(t_j - t_{j-1})\right)^2}{2\sigma^2(t_j - t_{j-1})}\right).$$

Using the above paths, we construct the sequence of estimates (training phase)

$$C_{k,0}(x), \ldots, C_{k,\mathcal{J}}(x)$$

as described in Example 2.3.

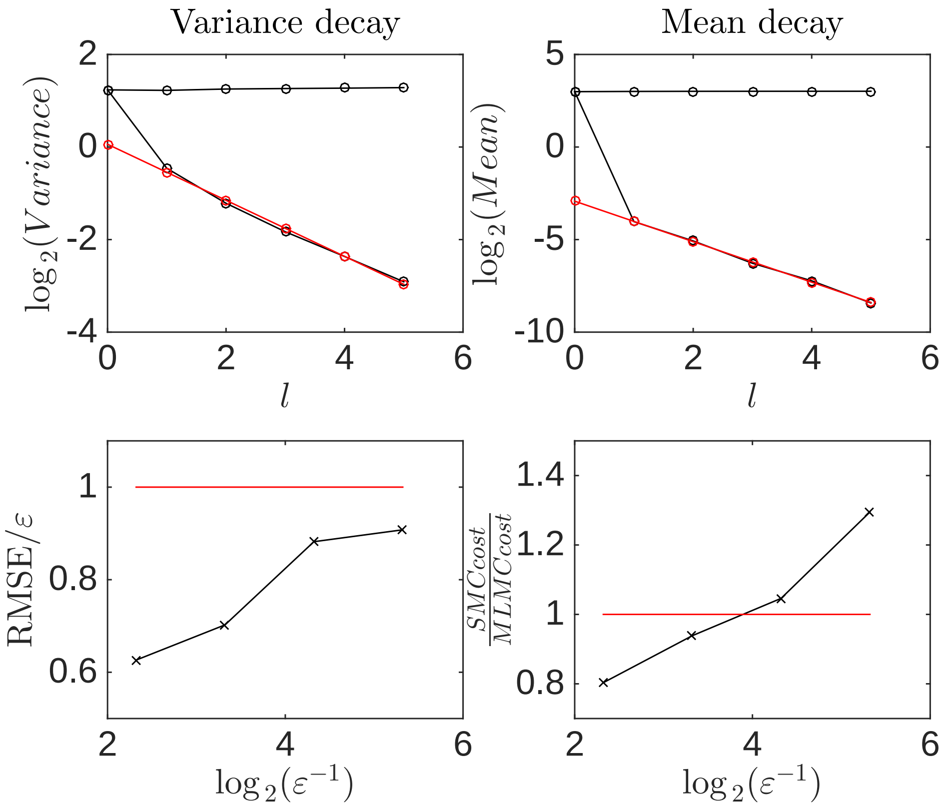


**Figure 5.1.** *Mesh method: level log-variances* $\mathrm{Var}[Y_l]$ *(estimated decay rate - 0.6, theoretical decay rate - 0.5), log absolute increments* $|Y_l|$ *(estimated decay rate - 1.09, theoretical decay rate - 1), RMSE and gain in cost.*

In Figure 5.1, the logaritheorems of the estimated level variances $\mathrm{Var}[Y_l]$ (top left plot) and the absolute values of the level means $|Y_l|$ (top right plot) are shown as functions of $l$, along with the resulting RMSE (bottom plot). The corresponding fitted lines $-0.6054 \cdot l + 0.0596$ and $-1.0941 \cdot l - 2.9253$ are in agreement with our theoretical analysis. The RMSE estimates are done for the range of precisions $\varepsilon = 0.2 \cdot 2^{-k},\ k = 0, 1, 2, 3.$

**5.4. Global regression.** Here a one dimensional situation is studied ($D = 1$) and a regression method with piecewise constant basis functions is used. For any natural number $m$, set $\Delta = 100/m$ and define

$$\psi_i(x) = \begin{cases} 0, & x - 50 > (i-1)\Delta, \\ 1, & \text{otherwise}, \\ 0, & x - 50 \leq i\Delta \end{cases}$$

for all $i = 1, \ldots, m$. Given a sequence of natural numbers $m(k)$, $k \in \mathbb{N}$, the continuation values estimates are

$$C_j^k(x) = \begin{cases} \mathcal{E}(x, j, \mathcal{J}), & x < 50, \\ \sum_{i=1}^{m(k)} \alpha_i \psi_i(x), & \text{otherwise} , \\ \mathcal{E}(x, j, j+1), & x > 150, \end{cases} \tag{5.5}$$

where

$$\mathcal{E}(x, j, l) = \mathrm{E}\left[ e^{-r(t_l - t_j)} \max_{k=1,\ldots,d} \left( Z_l^k - S \right)^+ \middle| Z_j = x \right]$$

for any $\mathcal{J} \geq l \geq j$. The number of training paths in each level is chosen to be

$$k_l = 31250 \times 2^l, \quad l = 1, \ldots, L,$$

while the sequence $m(k)$ is given by $m(k) = \lceil 7 \cdot k^\rho \rceil$ with $\rho = 0.5$ (see Example 2.1). We use the same control variates as in (5.4) while estimating the continuation values. This guarantees that the variance of continuation values estimates is small.

In Figure 5.2 the estimated level variances $\mathrm{Var}[Y_l]$ (top left plot) and the absolute values of the level means $|Y_l|$ (top right plot) are depicted along with the RMSE (bottom plot). The fitted lines for the log variances $(-0.4592 \cdot l - 1.4615)$ and for the log absolute means $(-0.9230 \cdot l - 4.9415)$ are in agreement with our theoretical result. The RMSE estimates are shown for $\varepsilon = 0.004 \cdot k$, $k = 1, 2, 3, 4, 5, 6$. Computational saving are greater, than in the mesh method case, but they can be even greater, if one combines our approach with a simple variance reduction technique, presented in [8].

## 6. Conclusion.

In this paper we introduced a novel MLMC algoritheorem for solving evaluation problem for Bermudan options. We developed a general framework, which can be used in combination with various fast approximation algoritheorems for continuation values estimation. As a matter of fact, the Bermudan option price evaluation problem is considered to be one of the most difficult and computationally demanding (see [11]) problems in financial mathematics and any gains are of importance. Our algoritheorem can be extended to other directions, see, e.g. a recent work [8], where additional computational gains were achieved by using a combination of our MLMC paradigm with a control variate technique. In the future we plan to develop a new version of our algoritheorems, which combines different levels of accuracy for continuation values estimation with different time discretization step sizes. Such an extension can be useful for much more difficult problem of American option pricing.

## 7. Proofs.

### 7.1. Proof of Theorem 3.2.

A family of stopping times $\left(\tau_j\right)_{j=0,\ldots,\mathcal{J}}$ w.r.t. the filtration $(\mathcal{F}_j)_{j=0,\ldots,\mathcal{J}}$ is called consistent if

$$j \leq \tau_j \leq \mathcal{J}, \quad \tau_{\mathcal{J}} = \mathcal{J}$$

and

$$\tau_j > j \quad \Longrightarrow \quad \tau_j = \tau_{j+1}.$$

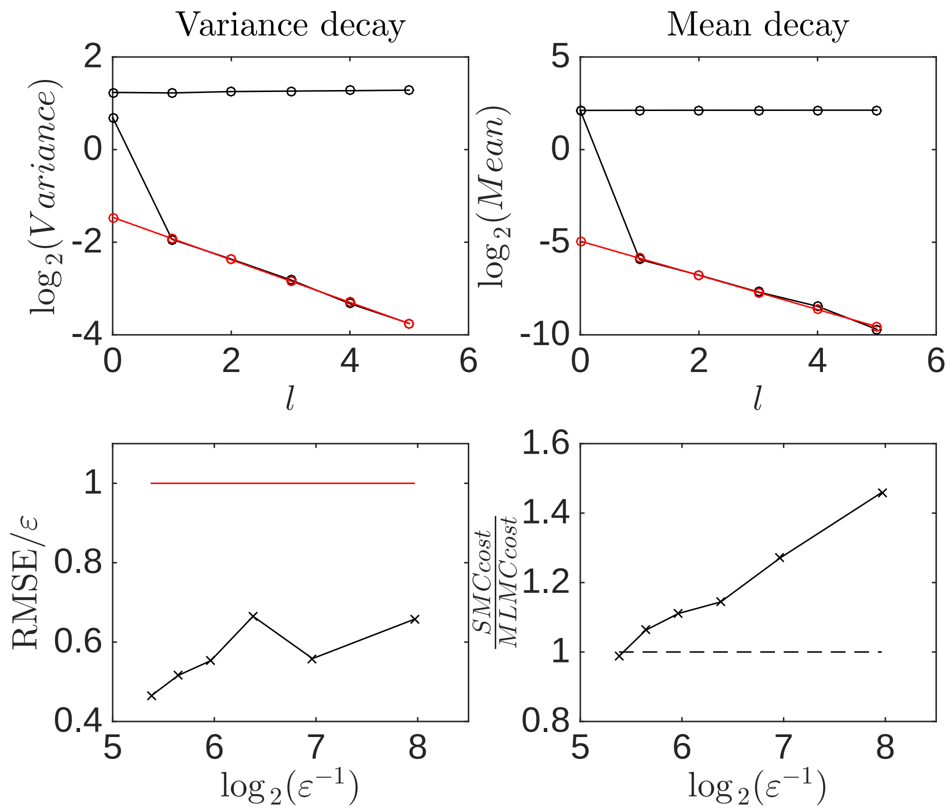


**Figure 5.2.** *Global regression method: variance decay (estimated decay rate - 0.4592, theoretical decay rate - 0.5), increments decay (estimated decay rate - 0.9230, theoretical decay rate - 1), RMSE and gain in cost.*

Lemma 7.1.*Let $(Y_j)$ be a process adapted to the filtration $(\mathcal{F}_j)$ and let $\left(\tau_j^1\right)$ and $\left(\tau_j^2\right)$ be two consistent families of stopping times. Then*

$$\begin{aligned}\mathrm{E}^{\mathcal{F}_j}\left[Y_{\tau_j^1}-Y_{\tau_j^2}\right] &= \left(Y_j-\mathrm{E}^{\mathcal{F}_j}\left[Y_{\tau_{j+1}^1}\right]\right)\left(1_{\{\tau_j^1=j,\tau_j^2>j\}}-1_{\{\tau_j^1>j,\tau_j^2=j\}}\right)\\ &\quad+\mathrm{E}^{\mathcal{F}_j}\left\{\sum_{l=j+1}^{\mathcal{J}-1}\left(Y_l-\mathrm{E}^{\mathcal{F}_l}\left[Y_{\tau_{l+1}^1}\right]\right)\left(1_{\{\tau_l^1=l,\tau_l^2>l\}}-1_{\{\tau_l^1>l,\tau_l^2=l\}}\right)1_{\{\tau_{l-1}^2>l-1\}}\right\}\end{aligned}$$

*and*

$$\mathrm{E}^{\mathcal{F}_j}\left[\left|Y_{\tau_j^1}-Y_{\tau_j^2}\right|^2\right]\leq\mathrm{E}^{\mathcal{F}_j}\left\{\sum_{l=j}^{\mathcal{J}-1}2^l\left(\mathrm{E}^{\mathcal{F}_l}\left[\left|Y_l-Y_{\tau_{l+1}^1}\right|^2\right]\right)\left(1_{\{\tau_l^1=l,\tau_l^2>l\}}+1_{\{\tau_l^1>l,\tau_l^2=l\}}\right)\right\}.$$

*for any $j=0,\ldots,\mathcal{J}-1$.*

*Proof.* We have

$$\begin{aligned}
Y_{\tau_l^1} - Y_{\tau_l^2} &= \left[Y_l - Y_{\tau_l^2}\right] 1_{\{\tau_l^1 = l, \tau_l^2 > l\}} + \left[Y_{\tau_l^1} - Y_l\right] 1_{\{\tau_l^1 > l, \tau_l^2 = l\}} \\
&\quad + \left[Y_{\tau_l^1} - Y_{\tau_l^2}\right] 1_{\{\tau_l^1 > l, \tau_l^2 > l\}} \\
&= \left[Y_l - Y_{\tau_{l+1}^1}\right] 1_{\{\tau_l^1 = l, \tau_l^2 > l\}} + \left[Y_{\tau_{l+1}^1} - Y_l\right] 1_{\{\tau_l^1 > l, \tau_l^2 = l\}} \\
&\quad + \left[Y_{\tau_{l+1}^1} - Y_{\tau_{l+1}^2}\right] 1_{\{\tau_l^1 = l, \tau_l^2 > l\}} + \left[Y_{\tau_{l+1}^1} - Y_{\tau_{l+1}^2}\right] 1_{\{\tau_l^1 > l, \tau_l^2 > l\}}.
\end{aligned}$$

Therefore it holds for $\Delta_l = \mathrm{E}^{\mathcal{F}_l}\left[Y_{\tau_l^1} - Y_{\tau_l^2}\right]$,

$$\Delta_l = \left[Y_l - \mathrm{E}^{\mathcal{F}_l}\left[Y_{\tau_{l+1}^1}\right]\right]\left(1_{\{\tau_l^1 = l, \tau_l^2 > l\}} - 1_{\{\tau_l^1 > l, \tau_l^2 = l\}}\right) + \mathrm{E}^{\mathcal{F}_l}\left\{\Delta_{l+1} 1_{\{\tau_l^2 > l\}}\right\}$$

with $\Delta_{\mathcal{J}} = 0$ and the first statement follows. To prove the second inequality note that

$$\begin{aligned}
\mathrm{E}^{\mathcal{F}_j}\left[\left|Y_{\tau_j^1} - Y_{\tau_j^2}\right|^2\right] &\leq 2\left(\mathrm{E}^{\mathcal{F}_j}\left[\left|Y_j - Y_{\tau_{j+1}^1}\right|^2\right]\right)\left(1_{\{\tau_j^1 = j, \tau_j^2 > j\}} + 1_{\{\tau_j^1 > j, \tau_j^2 = j\}}\right) \\
&\quad + 2\,\mathrm{E}^{\mathcal{F}_j}\left[\mathrm{E}^{\mathcal{F}_{j+1}}\left|Y_{\tau_{j+1}^1} - Y_{\tau_{j+1}^2}\right|^2\right].
\end{aligned}$$

∎

Introduce

$$\tau_{k,l} = \min\{l \leq j \leq \mathcal{J} : g_j(Z_j) \geq C_{k,j}(Z_j)\}, \quad l = 1, \ldots, \mathcal{J},$$

and note that $\tau_{k,l}$, $l = 1, \ldots, \mathcal{J}$, is a consistent family of stopping times. Taking into account that

$$C_l^*(Z_l) = \mathrm{E}^{\mathcal{F}_l}\left[g_{\tau_{l+1}^*}(Z_{\tau_{l+1}^*})\right] \leq g_l(Z_l)$$

on $\{\tau_l^* = l\}$ and

$$C_l^*(Z_l) > g_l(Z_l)$$

on $\{\tau_l^* > l\}$, we get from Lemma 7.1 with $\tau_j^1 = \tau_j^*$, $\tau_j^2 = \tau_{k,j}$ and $Y_j = g_j(Z_j)$ for $R := \mathrm{E}[V_0^{n,k}] - V_0^*$,

$$\begin{aligned}
|R| &= |\mathrm{E}\left[g_{\tau^*}(Z_{\tau^*}) - g_{\tau_k}(Z_{\tau_k})\right]| \\
&\leq \mathrm{E}\left[\sum_{l=0}^{\mathcal{J}-1} |C_l^*(Z_l) - g_l(Z_l)|\left(1_{\{\tau_l^* = l, \tau_{k,l} > l\}} + 1_{\{\tau_l^* > l, \tau_{k,l} = l\}}\right)\right].
\end{aligned}$$

Introduce

$$\mathcal{E}_{k,j} = \{g_j(Z_j) \geq C_j^*(Z_j),\ g_j(Z_j) < C_{k,j}(Z_j)\} \cup \{g_j(Z_j) < C_j^*(Z_j),\ g_j(Z_j) \geq C_{k,j}(Z_j)\},$$

$$\mathcal{A}_{k,j,0} = \left\{0 < \left|g_j(Z_j) - C_j^*(Z_j)\right| \leq \gamma_k^{1/2}\right\},$$

$$\mathcal{A}_{k,j,i} = \left\{2^{i-1}\gamma_k^{1/2} < \left|g_j(Z_j) - C_j^*(Z_j)\right| \leq 2^i \gamma_k^{1/2}\right\}$$

for $j=0,\ldots,\mathcal{J}-1$ and $i>0$. It holds

$$\begin{aligned}|R| &\leq \mathrm{E}\left[\sum_{l=0}^{\mathcal{J}-1}|C_l^*(Z_l)-g_l(Z_l)|\,1_{\{\mathcal{E}_{k,l}\}}\right]\\ &= \mathrm{E}\left[\sum_{i=0}^{\infty}\sum_{l=0}^{\mathcal{J}-1}|C_l^*(Z_l)-g_l(Z_l)|\,1_{\{\mathcal{E}_{k,l}\cap\mathcal{A}_{k,l,i}\}}\right]\\ &= \gamma_k^{1/2}\sum_{l=0}^{\mathcal{J}-1}\mathrm{P}\left(|g_l(Z_l)-C_l^*(Z_l)|\leq\gamma_k^{1/2}\right)\\ &\quad+\mathrm{E}\left[\sum_{i=1}^{\infty}\sum_{l=0}^{\mathcal{J}-1}|C_l^*(Z_l)-g_l(Z_l)|\,1_{\{\mathcal{E}_{k,l}\cap\mathcal{A}_{k,l,i}\}}\right].\end{aligned}$$

Using the fact that $|g_l(Z_l)-C_l^*(Z_l)|\leq|C_{k,l}(Z_l)-C_l^*(Z_l)|$ on $\mathcal{E}_{k,l}$, we derive

$$\begin{aligned}|R| &\leq \gamma_k^{1/2}\sum_{l=0}^{\mathcal{J}-1}\mathrm{P}\left(|g_l(Z_l)-C_l^*(Z_l)|\leq\gamma_k^{1/2}\right)\\ &\quad+\sum_{i=1}^{\infty}2^i\gamma_k^{1/2}\mathrm{E}\left[\sum_{l=0}^{\mathcal{J}-1}1_{\left\{|g_j(Z_l)-C_l^*(Z_l)|\leq 2^i\gamma_k^{1/2}\right\}}\mathrm{P}^k\left(|C_{k,l}(Z_l)-C_l^*(Z_l)|>2^{i-1}\gamma_k^{1/2}\right)\right]\\ &\leq A\mathcal{J}\gamma_k^{(1+\alpha)/2}+A\mathcal{J}\gamma_k^{(1+\alpha)/2}\sum_{i=1}^{\infty}2^{i(1+\alpha)}B_1\exp(-B_2 2^{i-1}),\end{aligned}$$

where we have made use of the assumptions (AQ) and (AM).

**7.2. Proof of Theorem 3.3.** Based on (3.1) we have the optimization problem

$$k^{\varkappa_1+1}+n\cdot k^{\varkappa_2}\to\min$$
$$\gamma_k^{(1+\alpha)/2}\lesssim\varepsilon$$
$$n\gtrsim\varepsilon^{-2}$$

It is clear that

$$\gamma_k^{(1+\alpha)/2}=k^{-\mu(1+\alpha)/2}\Rightarrow k\geq\varepsilon^{-\frac{2}{\mu(1+\alpha)}},$$

which immediately leads to the statement.

**7.3. Proof of Theorem 4.1.** We have

$$\begin{aligned}\mathrm{E}\left[g_{\tau_{k_l}}\left(Z_{\tau_{k_l}}\right)-g_{\tau_{k_{l-1}}}\left(Z_{\tau_{k_{l-1}}}\right)\right]^2 &\leq 2\,\mathrm{E}\left[g_{\tau^*}\left(Z_{\tau^*}\right)-g_{\tau_{k_{l-1}}}\left(Z_{\tau_{k_{l-1}}}\right)\right]^2\\ &\quad+2\,\mathrm{E}\left[g_{\tau_{k_l}}\left(Z_{\tau_{k_l}}\right)-g_{\tau^*}\left(Z_{\tau^*}\right)\right]^2.\end{aligned}$$

It follows from Lemma 7.1 that

$$\mathrm{E}\left[g_{\tau^*}\left(Z_{\tau^*}\right)-g_{\tau_{k_{l-1}}}\left(Z_{\tau_{k_{l-1}}}\right)\right]^2\leq$$
$$\mathrm{E}\left\{\sum_{s=0}^{\mathcal{J}-1}2^s\xi_s\left(1_{\{\tau_s^*=s,\tau_{k_{l-1},s}>s\}}+1_{\{\tau_s^*>s,\tau_{k_{l-1},s}>s\}}\right)\right\}$$

with $\xi_s = \mathrm{E}^{\mathcal{F}_s}\left[\left|g_s(Z_s) - V^*_{s+1}(Z_{s+1})\right|^2\right]$. By the Hölder inequality we get

$$
\begin{aligned}
\mathrm{E}\left[\xi_s\left(1_{\{\tau^*_s = s, \tau_{k_{l-1},s} > s\}} + 1_{\{\tau^*_s > s, \tau_{k_{l-1},s} > s\}}\right)\right] \\
\leq 4M_p^{1/p}\left[\mathrm{P}\left(\tau^*_s = s, \tau_{k_{l-1},s} > s\right) + \mathrm{P}\left(\tau^*_s > s, \tau_{k_{l-1},s} = s\right)\right]^{1/q} \\
= 4M_p^{1/p}\left[\mathrm{P}(\mathcal{E}_{k_{l-1},s})\right]^{1/q}
\end{aligned}
$$

with $1/p + 1/q = 1$, since

$$
\begin{aligned}
\mathrm{E}\left[|\xi_s|^p\right] &\leq \mathrm{E}\left[\left|g_s(Z_s) - V^*_{s+1}(Z_{s+1})\right|^{2p}\right] \\
&\leq 2^{2p-1}\mathrm{E}\left[\left|g_s(Z_s)\right|^{2p}\right] + 2^{2p-1}\mathrm{E}\left[\left|V^*_{s+1}(Z_{s+1})\right|^{2p}\right] \\
&\leq 2^{2p-1}\mathrm{E}\left[\left|g_s(Z_s)\right|^{2p}\right] + 2^{2p-1}\mathrm{E}\left[\left|\max_{k=s+1,\ldots,\mathcal{J}} g_k(Z_k)\right|^{2p}\right] \\
&\leq 2^{2p}M_p.
\end{aligned}
$$

Due to

$$
\mathrm{P}(\mathcal{E}_{k_{l-1},s}) = \sum_{i=0}^{\infty}\mathrm{P}(\mathcal{E}_{k_{l-1},s} \cap \mathcal{A}_{k_{l-1},s,i})
$$

we have

$$
\mathrm{E}\left[g_{\tau^*}\left(Z_{\tau^*}\right) - g_{\tau_{k_{l-1}}}\left(Z_{\tau_{k_{l-1}}}\right)\right]^2 \leq C_p^{1/p}\mathrm{E}\left[\sum_{i=0}^{\infty}\sum_{s=0}^{\mathcal{J}-1} 2^s\left[\mathrm{P}(\mathcal{E}_{k_{l-1},s} \cap \mathcal{A}_{k_{l-1},s,i})\right]^{1/q}\right],
$$

where

$$
\mathrm{P}(\mathcal{E}_{k_{l-1},s} \cap \mathcal{A}_{k_{l-1},s,i}) \leq \mathrm{P}\left(|g_s(Z_s) - C^*_s(Z_s)| \leq \gamma_{k_{l-1}}^{-1/2}\right) \leq A\gamma_{k_{l-1}}^{-\alpha/2}
$$

if $i = 0$ and

$$
\begin{aligned}
\mathrm{P}(\mathcal{E}_{k_{l-1},s} \cap \mathcal{A}_{k_{l-1},s,i}) &\leq \mathrm{E}\left[1_{\left\{|g_s(Z_s) - C^*_s(Z_s)| \leq 2^i\gamma_{k_{l-1}}^{-1/2}\right\}}\mathrm{P}^{k_{l-1}}\left(\left|C_{k_{l-1},s}(Z_s) - C^*_s(Z_s)\right| > 2^{i-1}\gamma_{k_{l-1}}^{-1/2}\right)\right] \\
&\leq A\gamma_{k_{l-1}}^{-\alpha/2}2^{i\alpha}B_1\exp(-B_2 2^{i-1})
\end{aligned}
$$

for $i > 0$. As a result

$$
\mathrm{E}\left[g_{\tau^*}\left(Z_{\tau^*}\right) - g_{\tau_{k_{l-1}}}\left(Z_{\tau_{k_{l-1}}}\right)\right]^2 \leq C\gamma_{k_{l-1}}^{-\alpha/(2q)}
$$

for some $C > 0$.

**7.4. Proof of Theorem 4.3.** Due to the monotone structure of the functional, we can consider the following optimization problem:

$$\sum_{l=0}^{L} k_l^{\varkappa_1+1} + n_l \cdot k_l^{\varkappa_2} \to \min \tag{7.1}$$

$$\gamma_{k_L}^{(1+\alpha)/2} = k_L^{-\mu(1+\alpha)/2} = \left(k_0 \cdot \theta^L\right)^{-\mu(1+\alpha)/2} \lesssim \varepsilon \tag{7.2}$$

$$\sum_{l=1}^{L} \frac{\gamma_{k_{l-1}}^{\alpha/(2q)}}{n_l} = k_0^{-\mu\alpha/2} \cdot \sum_{l=1}^{L} \frac{\theta^{-l\mu\alpha/(2q)}}{n_l} \lesssim \varepsilon^2. \tag{7.3}$$

$$n_0 \sim \varepsilon^{-2} \tag{7.4}$$

Now the Lagrange multiplier method with respect to $n_l$ gives us

$$k_l^{\varkappa_2} = -\lambda \frac{k_l^{-\mu\alpha/(2q)}}{n_l^2} \Rightarrow n_l = \sqrt{(-\lambda) \cdot k_l^{(-\varkappa_2-\mu\alpha/(2q))}}.$$

Now one can put the value of $n_l$ in (7.3):

$$\sum_{l=1}^{L} \frac{\gamma_{k_{l-1}}^{\alpha/(2q)}}{n_l} = \sum_{l=1}^{L} \frac{k_l^{-\mu\alpha/(2q)}}{\sqrt{(-\lambda) \cdot k_l^{(-\varkappa_2-\mu\alpha/(2q))}}} = \varepsilon^2 \Rightarrow$$

$$\sqrt{(-\lambda)} = \varepsilon^{-2} \cdot \sum_{l=1}^{L} \sqrt{k_l^{(\varkappa_2-\mu\alpha/(2q))}} \Rightarrow$$

$$n_l \sim \varepsilon^{-2} \left( \sum_{i=1}^{L} \sqrt{k_i^{(\varkappa_2-\mu\alpha/(2q))}} \right) \cdot \sqrt{k_l^{(-\varkappa_2-\mu\alpha/(2q))}}.$$

For total number of level we have from (7.2):

$$\left(k_0 \cdot \theta^L\right)^{-\mu(1+\alpha)/2} \lesssim \varepsilon \Rightarrow L \geq \frac{2}{\mu(1+\alpha)} \log_\theta \left( \varepsilon^{-1} \cdot k_0^{-\mu(1+\alpha)/2} \right).$$

Due to the special structure of the constraints and the functional, we can set

$$L = \left\lceil c \frac{2}{\mu(1+\alpha)} \log_\theta \left( \varepsilon^{-1} \cdot k_0^{-\mu(1+\alpha)/2} \right) \right\rceil$$

Now we can rewrite (7.1) as

$$\sum_{l=0}^{L} k_l^{\varkappa_1+1} + n_l \cdot k_l^{\varkappa_2} \sim k_L^{\varkappa_1+1} + \varepsilon^{-2} \cdot \left( \sum_{l=1}^{L} \sqrt{k_l^{(\varkappa_2-\mu\alpha/(2q))}} \right)^2 + \varepsilon^{-2} \cdot k_0^{\varkappa_2},$$

so we will have three cases.

Case 1. $2 \cdot q \cdot \varkappa_2 = \mu\alpha$.

$$
\begin{aligned}
k_L^{\varkappa_1+1} + \sum_{l=0}^{L} n_l \cdot k_l^{\varkappa_2} &\lesssim k_L^{\varkappa_1+1} + \varepsilon^{-2} \cdot L^2 \\
&\lesssim \varepsilon^{-\frac{2\cdot(\varkappa_1+1)}{\mu(1+\alpha)}} + \varepsilon^{-2} \cdot L^2
\end{aligned}
$$

Case 2. $2 \cdot q \cdot \varkappa_2 < \mu\alpha$.

$$
\begin{aligned}
k_L^{\varkappa_1+1} + \sum_{l=0}^{L} n_l \cdot k_l^{\varkappa_2} &\lesssim k_L^{\varkappa_1+1} + \varepsilon^{-2} \\
&\lesssim \varepsilon^{-\frac{2\cdot(\varkappa_1+1)}{\mu(1+\alpha)}} + \varepsilon^{-2}
\end{aligned}
$$

Case 3. $2 \cdot q \cdot \varkappa_2 > \mu\alpha$.

$$
\begin{aligned}
k_L^{\varkappa_1+1} + k_L^{\varkappa_1+1} + \sum_{l=0}^{L} n_l \cdot k_l^{\varkappa_2} &\lesssim k_L^{\varkappa_1+1} + \varepsilon^{-2} \cdot k_L^{\varkappa_2-\mu\alpha/(2q)} \\
&\lesssim \varepsilon^{-\frac{2\cdot(\varkappa_1+1)}{\mu(1+\alpha)}} + \varepsilon^{-2-\frac{2\varkappa_2-\mu\alpha/q}{\mu(1+\alpha)}}
\end{aligned}
$$

Combining all three cases one will get (4.1).

## REFERENCES


[1] Ankush Agarwal and Sandeep Juneja, *Comparing optimal convergence rate of stochastic mesh and least squares method for bermudan option pricing*, in Proceedings of the 2013 Winter Simulation Conference: Simulation: Making Decisions in a Complex World, IEEE Press, 2013, pp. 701–712.

[2] Leif BG Andersen, *A simple approach to the pricing of bermudan swaptions in the multi-factor libor market model*, Journal of Computational Finance, 3 (1999), pp. 5–32.

[3] Denis Belomestny, *Pricing bermudan options by nonparametric regression: optimal rates of convergence for lower estimates*, Finance and Stochastics, 15 (2011), pp. 655–683.

[4] Denis Belomestny, John Schoenmakers, and Fabian Dickmann, *Multilevel dual approach for pricing american style derivatives*, Finance and Stochastics, 17 (2013), pp. 717–742.

[5] Mark Broadie and Paul Glasserman, *Pricing american-style securities using simulation*, Journal of Economic Dynamics and Control, 21 (1997), pp. 1323–1352.

[6] ———, *A stochastic mesh method for pricing high-dimensional american options*, Journal of Computational Finance, 7 (2004), pp. 35–72.

[7] Jacques F Carriere, *Valuation of the early-exercise price for options using simulations and nonparametric regression*, Insurance: mathematics and Economics, 19 (1996), pp. 19–30.

[8] Fabian Dickmann and Nikolaus Schweizer, *Faster comparison of stopping times by nested conditional monte carlo*, arXiv preprint arXiv:1402.0243, (2014).

[9] Daniel Egloff et al., *Monte carlo algorithms for optimal stopping and statistical learning*, The Annals of Applied Probability, 15 (2005), pp. 1396–1432.

[10] Michael B Giles, *Multilevel monte carlo path simulation*, Operations Research, 56 (2008), pp. 607–617.

[11] Paul Glasserman, *Monte Carlo methods in financial engineering*, vol. 53, Springer Science & Business Media, 2003.

[12] Paul Glasserman, Bin Yu, et al., *Number of paths versus number of basis functions in american option pricing*, The Annals of Applied Probability, 14 (2004), pp. 2090–2119.

[13] FRANCIS A LONGSTAFF AND EDUARDO S SCHWARTZ, *Valuing american options by simulation: A simple least-squares approach*, Review of Financial studies, 14 (2001), pp. 113–147.

[14] JOHN N TSITSIKLIS AND BENJAMIN VAN ROY, *Regression methods for pricing complex american-style options*, Neural Networks, IEEE Transactions on, 12 (2001), pp. 694–703.

[15] DANIEL Z ZANGER, *Quantitative error estimates for a least-squares monte carlo algorithm for american option pricing*, Finance and Stochastics, 17 (2013), pp. 503–534.